Matroid Optimization Problems with Monotone Monomials in the Objective

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August 23, 2017

Abstract

In this paper we investigate non-linear matroid optimization problems with polynomial objective functions where the monomials satisfy certain monotonicity properties. Indeed, we study problems where the set of non-linear monomials consists of all non-linear monomials that can be built from a given subset of the variables. Linearizing all non-linear monomials we study the respective polytope. We present a complete description of this polytope. Apart from linearization constraints one needs appropriately strengthened rank inequalities. The separation problem for these inequalities reduces to a submodular function minimization problem. These polyhedral results give rise to a new hierarchy for the solution of matroid optimization problems with polynomial objectives. Finally, we consider extensions of our results and give suggestions for future work.

Keywords: Polynomial 0-1 Programming, Polyhedral Combinatorics, Matroids, Complete Description, Hierarchy
MSC2010: 90C57, 90C27

1 Introduction

In the last twenty years there has been a growing interest in studying combinatorial optimization problems with non-linear objective functions, as this greatly expands the set of applications, see, e. g., [BL12] for a recent survey on non-convex mixed-integer nonlinear programming. An important subclass of non-linear objectives is *polynomial* objectives. A standard approach to solve such problems is to linearize the non-linear monomials by introducing a new variable for each of them and to investigate the properties of the enlarged polytope. Linearization variants for the unconstrained case were, e. g., presented in [BM84, For59, GW73, GW74, Pad89]. Since even the solution of unconstrained quadratic 0-1 problems is NP-hard, it is not surprising that one does not know complete descriptions of the associated linearized polytopes in general. In order to improve the standard linearization in these cases one tries to strengthen the relaxations by appropriate valid inequalities. A study of the structure of the so called Boolean quadric polytope is presented in [Pad89]. For the more general case of arbitrary sets of non-linear monomials strong cutting planes are provided in the unconstrained case, e. g., in [BCRH16, CRH17, DPK16, DPK17a, DPK17b].

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In the constrained case similar approaches are applied. Then one hopes that the knowledge of the polyhedral structure in the linear case helps in the polynomial case as well. So one tries to strengthen the known inequalities, see, e. g., [FH13, FF15, PGdC15], partially by problem specific lifting approaches. A standard way to do this is multiplying some of the original constraints by one of the variables as part of the RLT (reformulation linearization technique) [SA90, SA98].

Furthermore, papers such as Balas et al. [BCC93], Sherali and Adams [AS05, SA90, SA98], Lovász and Schrijver [LS91] and Lasserre [Las01] strengthen the relaxations of binary (and mixed-integer) optimization problems by introducing appropriate hierarchies. Then the convex hull of all integer points of the problem can be derived after a finite number of steps, getting better and better relaxations in each single step. An excellent overview of the different schemes is given in [Lau03]. We will later work with a separation oracle. In contrast to the other hierarchies mentioned, it is not known whether one can efficiently optimize over the Lasserre hierarchy at some constant level if one is given only a separation oracle in order to handle the often exponential number of constraints associated to the problem, see, e. g., [CT12].

Recently, Buchheim and Klein suggested a new way to derive strengthened inequalities [BK14, Kle14]. If we start with a polynomial-time solvable combinatorial optimization problem and add one additional quadratic monomial, then the resulting problem remains tractable and there is some hope that one finds a complete description of the linearized polytope by the well-known "optimization equals separation" result [GLS93]. Usually, the strengthened cutting planes remain valid for the problems with several linearized monomials and might so be used for strengthening them as well. In order to employ these cutting planes in branch-and-cut methods one aim is to derive more direct separators than using the ellipsoid method. Results for the spanning forest and the spanning tree problem with one quadratic monomial can be found in [BK13, BK14, FF13]. Furthermore, one studied the bipartite matching problem [Kle14, Wal16] and the general matching problem [HKL15, Kle14] with one quadratic monomial. [HKL15] stated that especially in the case of dense quadratic objective functions the interaction of different monomial terms should be taken into account in order to get significant reductions of the gap between the linear relaxation and the integer hull at the root node or during branch-and-cut.

[FFM17] considered larger sets of monomials for matroid optimization problems. They found a complete description of the linearization for a set of nested non-linear monomials of arbitrary degree. This paper was our starting point for the investigations in the current paper, where we will extend to a set of *non-nested* monomials when the monomials satisfy certain monotonicity properties (similar to the up- and downward completeness in [BR08]). We present a complete description of the linearized polytope for this class of matroid optimization problems. Our description combines linearization constraints with appropriately strengthened rank inequalities to be defined below. In contrast to [FFM17] some coefficients of these extended rank inequalities might be negative. The associated separation problem of our rank constraints reduces to a submodular function minimization problem, which enables us to use these constraints rather directly. In the case of three quadratic monomials on three original variables we derive a complete description as long as we also add the cubic monomial on these variables or this monomial is always zero (the associated set is dependent, see below). That allows us to take interactions of single monomials into account, which partially answers a research question in [HKL15]. If the number of variables in non-linear monomials is constant, then the associated optimization and separation problems can be combinatorially solved in polynomial time. This gives rise to a new hierarchy for solving general matroid optimization problems with polynomial objective functions. This hierarchy has the advantage that we only add larger monomials for elements that are contained in some non-linear monomial. For quadratic matroid optimization problems, the first step of our hierarchy is essentially the same as applying the polyhedral results for the one-monomial case to each single quadratic monomial separately. Solution approaches for different types of non-linear matroid optimization problem can be found, e. g., in [Onn04, BLMA$^+$08, LHLO10].

This paper is structured as follows. Section 1.1 covers the background we need on matroids and matroid polytopes. Afterwards, in Section 2 we propose our matroid optimization problem with polynomial objective where the monomials satisfy some monotonicity conditions. These monomials are linearized. We present a complete description of the linearized polytope and study the associated separation problems in Section 3. In Section 4 we present our new hierarchy for solving general *matroid* optimization problems with polynomial objectives. Finally, we present some extension of our work and give some suggestions for future work in Section 5. We further show that for matroid intersection problems the polyhedral structure can be much more complicated and so our results cannot easily be extended. Some rather technical proofs are deferred to the Appendix.

1.1 Preliminaries

Throughout this paper we will consider a matroid $M = (E, \mathcal{I})$ with finite ground set $E, |E| = n$, and a family of independent sets $\mathcal{I} \subseteq 2^E$, where 2^X denotes the power set of a finite set X. A matroid M satisfies the three properties

(M1) $\emptyset \in \mathcal{I}$,

(M2) $S, T \subseteq E, T \in \mathcal{I}, S \subseteq T \Rightarrow\ S \in \mathcal{I}$,

(M3) $S, T \in \mathcal{I}, |S| < |T| \Rightarrow\ \exists\, e \in T \setminus S$ with $S \cup \{e\} \in \mathcal{I}$.

In this paper we assume that $\{e\} \in \mathcal{I}$ for all $e \in E$ (otherwise we could consider the matroid $M' = (E \setminus \{e\}, \mathcal{I})$). The *rank function* of M reads $r\colon 2^E \to \mathbb{N}_0$ with $r(X) = \max\{|I|\colon I \in \mathcal{I}, I \subseteq X\}$ for each $X \in 2^E$. The rank function r of a matroid is well-known to be subcardinal, monotone and submodular, e. g., [Oxl92, Wel76]. In slight abuse of notation given $S, T \subseteq E, e \in E$ we often write $S + T, S - T$ and $S + e, S - e$ instead of $S \cup T, S \setminus T$ and $S \cup \{e\}, S \setminus \{e\}$, respectively. Let $T \subseteq E$. Then $B \in \mathcal{I}$ is a *basis* of T if $B \subseteq T$ and $B + e \notin \mathcal{I}$ for all $e \in T \setminus B$. Sets $C \subseteq E, C \notin \mathcal{I}$ with $C - e \in \mathcal{I}$ for all $e \in C$ are called *circuits*. The *closure* $\mathrm{cl}(T)$ of a set $T \subseteq E$ is $\mathrm{cl}(T) := T \cup \{e \in E\colon r(T) = r(T \cup \{e\})\}$. A set $T \subseteq E$ is called *closed* if $\mathrm{cl}(T) = T$. We often make use of the *strong basis exchange property* (equivalent to (M3)), see, also [Bru69]:

(M3′) B, B' bases of $T \subseteq E$ with $B \neq B' \Rightarrow$ for any $e \in B \setminus B'$ $\exists\, f \in B' \setminus B$ with $(B - e) + f$ and $(B' - f) + e$ bases of T.

For further results on matroids in general we refer the reader to the excellent books [Oxl92, Wel76].

We are mainly interested in the polytopes associated with some matroid. For these we need the following definition. For any $S \subseteq E$ we define the characteristic vector $\chi^S \in \{0,1\}^E$ by

$$\chi^S_e = \begin{cases} 1, & e \in S, \\ 0, & e \notin S. \end{cases}$$

(In slight abuse of notation, we often do not distinguish between the characteristic vector of an independent set $S \in \mathcal{I}$ and the set S itself.) Then the polytope associated with the matroid M is

$$P_M = \operatorname{conv}\{\chi^S \in \{0,1\}^E : S \in \mathcal{I}\}.$$

This polytope is well-understood.

Theorem 1 (Edmonds, [Edm70]). *Let M be a matroid with rank function $r\colon 2^E \to \mathbb{N}_0$, then*

$$P_M = \left\{ x \in \mathbb{R}_+^E : \sum_{e \in T} x_e \le r(T), T \subseteq E \right\}.$$

The inequalities

$$\sum_{e \in T} x_e \le r(T), \qquad\qquad T \subseteq E, \tag{1}$$

are often called *rank inequalities.* Together with the non-negativity of the variables they completely describe P_M. Edmonds further characterized which sets T lead to facets of P_M [Edm70], see also [Grö77].

2 Matroid Optimization Problems with a Set of Monotone Monomials

In this section we consider matroid optimization problems with polynomial objectives whose monomials satisfy certain monotonicity properties. We aim to develop a formulation and afterwards a complete description of the extended polytope that arises from linearizing the non-linear monomials. Later, in Section 3 we prove that the inequalities we develop here indeed form a complete description of that polytope.

Fix a matroid $M = (E, \mathcal{I})$. Note that each monomial $\prod_{e \in I} x_e$ is associated with a subset $I \subseteq E$. We say that a class of matroid optimization problems is *monotone* if there is a fixed independent set $\bar{E} := \{e_1, \dots, e_k\} \in \mathcal{I}$ (we globally define $k = |\bar{E}|$) such that the set of allowed monomials is associated with precisely all non-empty subsets of $\bar{E}$, plus linear terms for all $e \in E \setminus \bar{E}$. Thus our cost function is $c\colon E \cup (2^{\bar{E}} \setminus \emptyset) \to \mathbb{R}$, and our monotone matroid optimization problem is

$$\text{maximize} \left\{ \sum_{e \in E \setminus \bar{E}} c(e) \cdot x_e + \sum_{\emptyset \neq I \subseteq \bar{E}} c(I) \cdot \prod_{e \in I} x_e : x \in P_M \cap \{0,1\}^E \right\}. \tag{OPT}$$

Note that in general all coefficients can have arbitrary (possibly negative) values, and that this objective function is neither convex nor concave. In comparison to the definition of up- and downwards completeness of monomials in [BR08], our definition allows linear terms not belonging to any non-linear monomial. In contrast to the unconstrained case, including such terms might have an significant impact due to the matroid structure.

It is also possible to think of starting with an instance of a matroid optimization problem with only some non-linear monomials, but with the property that the union of the elements in these non-linear monomials is an independent set (this restriction can be relaxed, see Corollary 13 below). Then we could consider such an instance to be a special case of monotone matroid optimization where many of the cost coefficients are zero.

Remark 2. In general, problem (OPT) is NP hard: Consider the case of the trivial matroid $M_{\text{triv}} = (E, 2^E)$ and zero costs for all monomials of degree at least three. Then the problem is equivalent to the unconstrained quadratic 0-1-problem, which is NP hard [SA98, BR08]. But if k is a fixed constant then we can solve the problem (OPT) with the help of the Greedy algorithm [Edm71]. For each $I \subseteq \bar{E}$ we use the Greedy algorithm to compute the 2^k values

$$z_I := \sum_{\emptyset \neq J \subseteq I} c(J) + \max\left\{ \sum_{e \in E \setminus \bar{E}} c(e) \cdot x_e \colon x \in P_M \cap \{0,1\}^E; x_e = \chi_e^I, e \in \bar{E} \right\},$$

and then an I that maximizes z_I is optimal.

Consider the monomial $\prod_{e \in I} x_e$ over index set $I \subseteq \bar{E}$. We linearize this monomial by introducing a new variable x_I. For each $S \in \mathcal{I}$ (so that χ^S is a feasible vector for (OPT)) we define the incidence vector of S w. r. t. all monomials (i. e., $I \subseteq \bar{E}$ with $I \neq \emptyset$) as

$$\chi_I^S = \begin{cases} 1, & I \subseteq S, \\ 0, & I \nsubseteq S. \end{cases}$$

We define $2_0^{\bar{E}} := 2^{\bar{E}} \setminus \{\emptyset\}$ as the non-empty subsets of $\bar{E}$. Then for $S \in \mathcal{I}$ we define $\chi^S_{2_0^{\bar{E}}}$ to be the incidence vector of all monomials over $\bar{E}$ except for the empty set.

In this paper we mainly focus on the study of the structure of the polytope

$$P_M^{\bar{E}} = \operatorname{conv}\left\{ \left(\chi^S_{E \setminus \bar{E}}, \chi^S_{2_0^{\bar{E}}} \right) \in \{0,1\}^{n-k+2^k-1} \colon S \in \mathcal{I} \right\}$$

that arises after linearization. Notice that when k is a constant, $P_M^{\bar{E}}$ has a polynomial number of dimensions, but when k is part of the input $P_M^{\bar{E}}$ has an exponential number of dimensions. For the formulation and complete description of $P_M^{\bar{E}}$ we need the following definitions.

Definition 3. Let $I \subseteq E, |I| \geq 2$, and $T \subseteq E$. Then we define

$$\beta_I(T) = |I \setminus T| + r(T) - r(T + I), \tag{2}$$

$$\alpha_I(T) = \sum_{\substack{J \subseteq I \\ |J| \geq 2}} (-1)^{|I|+|J|} \cdot \beta_J(T). \tag{3}$$

Let $I \subseteq E, |I| \geq 2$ and $T \subseteq E$. Since $r(T+I) \leq r(T) + |I \setminus T|$ we have $\beta_I(T) \geq 0$. If $\beta_I(T) > 0$, then some dependency occurs. The $\beta_.(.)$ can be interpreted as a generalization of the $\alpha_i(.)$ in [FFM17]. So similar properties are satisfied. Furthermore, $\alpha_I(T)$ is an inclusion-exclusion formula depending on the $\beta_.(T)$ where one tries to count "each dependency exactly once". We will later use the $\alpha_.(.)$ as the coefficients of the linearized monomials in appropriately strengthened rank inequalities. A visualization of $\alpha_.(T), \beta_.(T)$ for some $T \subseteq E$ on the example of the graphic matroid on a complete graph with six nodes can be found in Fig. 1. The edges of T are drawn solid and the edges of $\bar{E}$ are drawn with dotted lines.

Useful properties of $\alpha_I(.), \beta_I(.), I \subseteq \bar{E}$ are given in the next two lemmas.

Lemma 4. *Let $T \subseteq E$ and $e \in \operatorname{cl}(T) \setminus (T + \bar{E})$. Then $\beta_I(T) = \beta_I(T + e)$ for all $I \subseteq \bar{E}, |I| \geq 2$. So $\alpha_I(T) = \alpha_I(T + (\operatorname{cl}(T) \setminus (T + \bar{E})))$ for all $I \subseteq \bar{E}, |I| \geq 2$.*

$\beta_{\{e_i\}} = 0, i = 1, \dots, 4$
$\beta_{\{e_1,e_i\}} = 0, i = 2, 3, 4$
$\beta_{\{e_2,e_3\}} = \beta_{\{e_2,e_4\}} = \beta_{\{e_3,e_4\}} = 1$
$\beta_{\{e_1,e_2,e_3\}} = \beta_{\{e_1,e_2,e_4\}} = \beta_{\{e_1,e_3,e_4\}} = 1$
$\beta_{\{e_2,e_3,e_4\}} = 2$
$\beta_{\{e_1,e_2,e_3,e_4\}} = 2$
$\alpha_{\{e_2,e_3,e_4\}} = -1$
$\alpha_{\{e_1,e_2,e_3,e_4\}} = 0$

$\beta_{\{e_i\}} = 0, i = 1, \dots, 4$
$\beta_{\{e_1,e_i\}} = \beta_{\{e_4,e_i\}} = 0, i = 1, \dots, 4$
$\beta_{\{e_2,e_3\}} = 1$
$\beta_{\{e_1,e_i,e_4\}} = 0, i = 2, 3$
$\beta_{\{e_2,e_3,e_i\}} = 1, i = 1, 4$
$\beta_{\{e_1,e_2,e_3,e_4\}} = 1$
$\alpha_{\{e_2,e_3,e_4\}} = 0$
$\alpha_{\{e_1,e_2,e_3,e_4\}} = 0$

Figure 1: Illustration of $\alpha_.(T), \beta_.(T)$ for some set $T \subseteq E$ on the example of the graphic matroid on a complete graph $G = (V, E)$ with six nodes and $\bar{E} = \{e_1, e_2, e_3, e_4\}$. The edges of $T \subset E$ are drawn solid and the edges of $\bar{E}$ are drawn with dotted lines. All other edges are omitted in the picture for the sake of clarity. Note that $\alpha_I(T) = \beta_I(T)$ for sets $I \subseteq \bar{E}, |I| = 2$. Furthermore $\beta_{\{e_i\}}(T) = 0$ for all $i \in \{1, \dots, 4\}$ because T is closed.

Proof. Let $T \subseteq E, e \in \text{cl}(T) \setminus (T + \bar{E})$ and let $I \subseteq \bar{E}, |I| \geq 2$. Then $r(T+J+e) = r(T+J)$ for all $J \subseteq I$ because

$$r(T + J) + r(T) \leq r(T + J + e) + r(T + e) \leq r(T + J) + r(T + e) = r(T + J) + r(T)$$

by the monotonicity and submodularity of the rank function. Observation 5 in [FFM17] is similar to this result. Furthermore, $|J \setminus T| = |J \setminus (T + e)|$ for all $J \subseteq I$ because $e \notin \bar{E}$. So the statement immediately follows by the definition of $\alpha_I(T), \beta_I(T)$ in (2) and (3). □

The next result is a generalization of the proof of the feasibility of the extended rank inequalities in [FFM17] to sets of non-nested monomials.

Lemma 5. *Let $T \subseteq E$ and $I \in \mathcal{I}$. Then $|T \cap I| + \beta_{I \cap \bar{E}}(T) \leq r(T)$.*

Proof. Let $T \subseteq E$ and $I \in \mathcal{I}$. Then we get by the monotonicity of the rank function and by (M2) that

$$\begin{aligned}|T \cap I| + \beta_{I\cap\bar{E}}(T) &= |T \cap I| + |(I \cap \bar{E}) \setminus T| + r(T) - r(T + (I \cap \bar{E}))\\ &= r((T \cap I) + ((I \cap \bar{E}) \setminus T)) - r(T + (I \cap \bar{E})) + r(T) \leq r(T).\end{aligned}$$ □

Now we are ready to present a formulation for $P_M^{\bar{E}}$. Because we want to show that the integrality constraints are not needed we include some constraints as well that are redundant in the case of integral variables, but which will be needed for the complete description later on.

Lemma 6. *A formulation for* $P_M^{\bar{E}}$ *is given by*

$$-x_e \le 0, \qquad e \in E \setminus \bar{E}, \tag{4}$$

$$(-1)^{|I|+1} \sum_{\substack{J \subseteq \bar{E}:\\ I \subseteq J}} (-1)^{|J|} x_J \le 0, \qquad I \in 2_0^{\bar{E}}, \tag{5}$$

$$\sum_{I \in 2_0^{\bar{E}}} (-1)^{|I|+1} x_I \le 1, \tag{6}$$

$$\sum_{e \in T} x_e + \sum_{\substack{I \subseteq \bar{E}\\ |I| \ge 2}} \alpha_I(T) x_I \le r(T), \qquad T \subseteq E, \tag{7}$$

$$x_I \in \{0,1\}, \qquad I \in \left(2_0^{\bar{E}} \cup \bigcup_{e \in E \setminus \bar{E}} \{e\} \right). \tag{8}$$

Proof. Inequalities (4) are the non-negativity constraints and inequalities (5) and (6) give a kind of standard linearization for the linearized monomials. These constraints are variants of the ones presented in [BR08] where a complete description for linearized unconstrained polynomial optimization problems is given for the case that the monomials satisfy the so-called upward and downward completeness properties similar to our setting. Nonetheless, we prove the validity of these constraints because we use a different notation and in order to keep the presentation of the paper self-contained.

First, we consider the validity of (5). Let $I \in 2_0^{\bar{E}}$ be fixed and let $S \in \mathcal{I}$ be given. Because elements $e \in S \setminus \bar{E}$ do not influence the left-hand side of (5) we can assume that $S \subseteq \bar{E}$. Let $|S| = m$. Then the left-hand side is zero if $I \not\subseteq S$. If, otherwise, $I \subseteq S$, then

$$(-1)^{|I|+1} \sum_{\substack{J \subseteq \bar{E}:\\ I \subseteq J}} (-1)^{|J|} \chi_J^S = (-1)^{|I|+1} \sum_{i=0}^{m-|I|} (-1)^{|I|+i} \binom{m-|I|}{i} = (-1)(0)^{m-|I|} \le 0.$$

So (5) is satisfied with equality as long as $I \ne S$. Next, we prove the validity of (6). Let again $S \in \mathcal{I}, |S| = m$, be given and assume, w. l. o. g., $S \subseteq \bar{E}$. If $m \ge 1$, the left-hand side of (6) reads

$$\sum_{I \in 2_0^{\bar{E}}} (-1)^{|I|+1} \chi_I^S = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} = -(0-1) = 1.$$

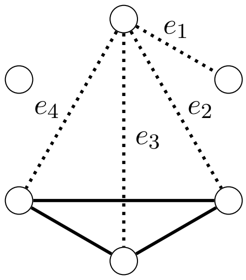
If $m = 0$, then the left-hand side is zero. So validity follows.

It remains to consider the strengthened variant of the rank inequalities (7) visualized on some graphic matroid in Fig. 2 and on some uniform matroid in Example 7. Let $T \subseteq E$ be fixed. We consider an arbitrary $S \in \mathcal{I}$ and define $m = |S \cap \bar{E}|$. If $m \le 1$, then the validity follows from the validity of (1). So let $m \ge 2$. Then

$$\sum_{e \in T} \chi_e^S + \sum_{\substack{I \subseteq \bar{E}\\ |I| \ge 2}} \alpha_I(T) \chi_I^S = |S \cap T| + \sum_{\substack{I \subseteq S \cap \bar{E}\\ |I| \ge 2}} \alpha_I(T).$$

In order to simplify $\sum_{I \subseteq S \cap \bar{E}, |I| \ge 2} \alpha_I(T)$ we take definition (3) of the $\alpha_I(\cdot)$ into account. Indeed, given $K \subsetneq S \cap \bar{E}, |K| \ge 2, |K| = k$, we count how often $\beta_K(T)$ appears in the sum. This number equals

$$\sum_{i=0}^{m-k} \binom{m-k}{i} (-1)^i = 0,$$



- $\bar{E} = \{e_1, e_2\}$: $\sum_{e \in T} x_e \leq 2$
- $\bar{E} = \{e_2, e_3\}$: $\sum_{e \in T} x_e + x_{\bar{E}} \leq 2$
- $\bar{E} = \{e_2, e_3, e_4\}$: $\sum_{e \in T} x_e + x_{\{e_2,e_3\}} + x_{\{e_2,e_4\}} + x_{\{e_2,e_4\}} - x_{\bar{E}} \leq 2.$

Figure 2: Visualization of the extended rank inequalities (7) on the example of the graphic matroid on a complete graph $G = (V, E)$ with six nodes and $\bar{E} = \{e_1, e_2, e_3, e_4\}$. The edges of $T \subset E$ are drawn solid and the edges of $\bar{E}$ are drawn with dotted lines. All other edges are omitted in the picture for the sake of clarity, see Fig. 1.

in all cases because it is "contained" in all $\alpha_J(T)$ with $J \subseteq S \cap \bar{E}, K \subseteq J$ and there are exactly $\binom{m-k}{i}$ sets J with $|J| = i + k, i \in \{0, \ldots, m-k\}$. Surely, $\beta_{S \cap \bar{E}}(T)$ appears exactly once. So the left-hand side of (7) reduces to

$$|S \cap T| + \beta_{S \cap \bar{E}}(T) \overset{L.5}{\leq} r(T).$$

So feasibility of (7) follows.

Then the strengthened rank inequalities together with the linearization and the integrality conditions (8) are a formulation for $P_M^{\bar{E}}$ by [BR08] and Theorem 1. $\square$

Example 7. We consider a uniform matroid with $E = \{1, \ldots, 8\}, r(E) = 4, \bar{E} = \{1, 2, 3, 4\}$. Then inequalities (7) read:

$$T = \{5\}: \qquad x_5 + x_{\bar{E}} \leq 1$$

$$T = \{5, 6\}: \qquad x_5 + x_6 + \sum_{e \in \bar{E}} x_{\bar{E}-e} - 2x_{\bar{E}} \leq 2$$

$$T = \{5, 6, 7\}: \qquad x_5 + x_6 + x_7 + \sum_{I \subset \bar{E}, |I|=2} x_I - \sum_{e \in \bar{E}} x_{\bar{E}-e} + x_{\bar{E}} \leq 3$$

These inequalities show nicely the inclusion-exclusion form of the coefficients of the linearized monomials.

3 Complete Description of $P_M^{\bar{E}}$

In one of the main results of this paper we will even prove that the linearization constraints together with the extended rank inequalities (4)–(7) suffice to completely describe $P_M^{\bar{E}}$.

Theorem 8. *Inequalities (4)–(7) are a complete description for $P_M^{\bar{E}}$.*

To prove Theorem 8 we need several lemmas. The first lemma says that it suffices to consider the extended rank inequalities (7) for closed sets $T \subseteq E$.

Lemma 9. *Let $\bar{x} \in \mathbb{R}^{n-k+2^k-1}$ be a point satisfying (4)–(6) as well as all inequalities (7) for closed sets $T \subseteq E$, then the point also satisfies (7) for arbitrary sets $T \subseteq E$.*

Proof. Let $T \subseteq E$ be arbitrary, but fixed. We define $T' = T + (\mathrm{cl}(T) \setminus \bar{E})$ and $X = (\mathrm{cl}(T) \setminus T) \cap \bar{E} = \mathrm{cl}(T') \setminus T'$. Then $\mathrm{cl}(T) = \mathrm{cl}(T')$ and $\mathrm{cl}(T) = T' + X$. Furthermore, note that $\beta_I(T') = \beta_I(\mathrm{cl}(T')) + |I \cap X|$ for $I \subseteq \bar{E}, |I| \geq 2$, by (2) and that

$$\sum_{\substack{J \subseteq I \\ |J| \geq 2}} (-1)^{|J|} |J \cap X| = \sum_{\substack{J \subseteq I \\ |J| \geq 2}} \sum_{y \in J \cap X} (-1)^{|J|} = \sum_{y \in I \cap X} \sum_{\substack{J \subseteq I \\ |J| \geq 2 \\ y \in J}} (-1)^{|J|} = \sum_{y \in I \cap X} 1 = |I \cap X| \tag{9}$$

for some $I \subseteq \bar{E}, |I| \geq 2$. Then

$$\begin{aligned}
\sum_{e \in T} \bar{x}_e + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \alpha_I(T) \bar{x}_I &\overset{L.\ 4,(4)}{\leq} \sum_{e \in T'} \bar{x}_e + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \alpha_I(T') \bar{x}_I \\
&\overset{(3)}{=} \sum_{e \in T'} \bar{x}_e + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \sum_{\substack{J \subseteq I \\ |J| \geq 2}} (-1)^{|I|+|J|} (\beta_J(\mathrm{cl}(T')) + |J \cap X|) \bar{x}_I \\
&= \underbrace{\sum_{e \in \mathrm{cl}(T')} \bar{x}_e + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \sum_{\substack{J \subseteq I \\ |J| \geq 2}} (-1)^{|I|+|J|} \beta_J(\mathrm{cl}(T')) \bar{x}_I}_{=: \bar{x}'} \\
&\quad + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \sum_{\substack{J \subseteq I \\ |J| \geq 2}} (-1)^{|I|+|J|} |J \cap X| \bar{x}_I - \sum_{e \in X} \bar{x}_e \\
&\overset{(9)}{=} \bar{x}' + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} (-1)^{|I|} |I \cap X| \bar{x}_I - \sum_{e \in X} \bar{x}_e = \bar{x}' + \sum_{\substack{I \subseteq \bar{E} \\ X \cap I \neq \emptyset}} (-1)^{|I|} |I \cap X| \bar{x}_I \\
&= \bar{x}' + \sum_{e \in X} \sum_{\substack{I \subseteq \bar{E} \\ e \in I}} (-1)^{|I|} \bar{x}_I \overset{(5)}{\leq} \bar{x}' \leq r(\mathrm{cl}(T')) = r(T).
\end{aligned}$$

This proves the statement. □

Next we show that for facet-defining inequalities that are not positive multiples of the linearization constraints, the coefficients of all variables corresponding to linear and quadratic monomials are non-negative.

Lemma 10. *Let $a^T x \leq b$ be a facet-defining inequality of $P_M^{\bar{E}}$ that is not a positive multiple of (4)–(6). Then $a_e \geq 0$ for all $e \in E$ and $a_I \geq 0$ for all $I \subseteq \bar{E}, |I| = 2$.*

Proof. Let $a^T x \leq b$ be a facet-defining inequality of $P_M^{\bar{E}}$ that is not a positive multiple of (4)–(6). Then for all $e \in E \setminus \bar{E}$ there exists an $S \in \mathcal{I}$ such that $a^T \chi^S = b$ with $e \in S$ and so $\chi_e^S = 1 > 0$ because $a^T x \leq b$ is not a positive multiple of (4). By $(S - e) \in \mathcal{I}$ and $\chi_I^S = \chi_I^{S-e}$ for all $I \in 2_0^{\bar{E}}$ it follows that a_e is non-negative (otherwise $a^T x \leq b$ would not be feasible).

Next, let $e \in \bar{E}$. Then because $a^T x \leq b$ is not a positive multiple of (5) with $I = \{e\}$ there exists an $S \in \mathcal{I}$ with $a^T \chi^S = b$ as well as $S \cap \bar{E} = \{e\}$ and so $\sum_{\substack{J \subseteq \bar{E}: \\ e \in J}} (-1)^{|J|} \chi_J^S = -1 < 0$. With $(S - e) \in \mathcal{I}$ and $\chi_I^S = \chi_I^{S-e} = 0$ for all $I \in 2_0^{\bar{E}}, I \neq \{e\}$, we get by feasibility that $a_e \geq 0$.

Last, let $I = \{i, j\} \subseteq \bar{E}, i \neq j$. Then there exists an $S' \in \mathcal{I}$ with $a^T \chi^{S'} = b$ and $I = S' \cap \bar{E}$ and so $-\sum_{\substack{J \subseteq \bar{E}: \\ I \subseteq J}} (-1)^{|J|} \chi_J^{S'} = -1 < 0$ because $a^T x \leq b$ is not a positive multiple of (5) and there exists an $S \in \mathcal{I}$ with $a^T\chi^S = b$ and $S \cap \bar{E} = \emptyset$ and so $\sum_{I \in 2_0^{\bar{E}}} (-1)^{|I|+1}\chi_I^S = 0 < 1$ because $a^T x \leq b$ is not a positive multiple of (6). Let us assume that $|S \cap S'|$ is maximal. We consider three cases:

1. $r(S) < r(S')$: Then there exists an $e \in S' \setminus S$ with $(S+e) \in \mathcal{I}$. If $e \notin \bar{E}$, then $a_e = 0$, in contradiction to $|S \cap S'|$ maximal. If otherwise $e \in \bar{E}$, then $(S'-e) \in \mathcal{I}$ implies $-a_e - a_I \leq 0$ and $S+e \in \mathcal{I}$ implies $a_e \leq 0$ and so $a_e = 0$. Putting this together we get $a_I \geq 0$.

2. $r(S') < r(S)$: Then there exists an $e \in S \setminus S'$ with $(S'+e) \in \mathcal{I}$. But with $e \notin \bar{E}$ it follows $a_e = 0$, contradicting the maximality of $|S \cap S'|$.

3. $r(S) = r(S')$: By **(M3′)** there exists for each $e \in (S' \setminus S) \cap \bar{E} \neq \emptyset$ an $f \in S \setminus S'$ such that $(S'-e+f), (S-f+e) \in \mathcal{I}$. With $(S-f+e) \in \mathcal{I}$ we get $a_e \leq a_f$. For $(S'-e+f) \in \mathcal{I}$ feasibility implies $a_f \leq a_e + a_I$. So we conclude $a_I \geq 0$. □

In [FFM17] the coefficients of all linearized monomials are non-negative in constraints that do not correspond to the linearization. This situation changes here. Under the assumptions above, the coefficients of variables of linearized monomials of degree at least three might be negative, but then the sum of certain coefficients is non-negative.

Lemma 11. *Let $a^T x \leq b$ be a facet-defining inequality of $P_M^{\bar{E}}$ that is not a positive multiple of (4)–(6) and let $J \subseteq \bar{E}, |J| \geq 2$ with $\bar{e} \in J$ arbitrary, but fixed. Then $\sum_{I \subseteq J, \bar{e} \in I, |I| \geq 2} a_I \geq 0$.*

Proof. Before proving this result we introduce some notation. Let $I \subseteq \bar{E}$ and $i \in I$, then we write

$$\Sigma_a(I) := \sum_{\substack{I' \subseteq I \\ |I'| \geq 2}} a_I \qquad \text{and} \qquad \Sigma_a(I,i) := \sum_{\substack{I' \subseteq I \\ |I'| \geq 2 \\ i \in I'}} a_I.$$

Also note that $\Sigma_a(I,i) = \Sigma_a(I) - \Sigma_a(I-i)$.

Let $a^T x \leq b$ be a facet-defining inequality of $P_M^{\bar{E}}$ that is not a positive multiple of (4)–(6). Furthermore, let $J \subseteq \bar{E}, |J| \geq 2$, and let $e, f \in J, e \neq f$, be fixed. We set $J' := J \setminus \{e, f\}$. Then we prove this result by induction on $|J|$. For all $J \subseteq \bar{E}$ with $|J| = 2$ the statement is true by the previous lemma. So let us assume that the result is true for all sets $\bar{J} \subseteq \bar{E}$ with $|\bar{J}| < |J|$.

Because $a^T x \leq b$ is not a positive multiple of (5) there exists an $S \in \mathcal{I}$ with $a^T\chi^S = b$ and $S \cap \bar{E} = J - e - f$ as well as an $S' \in \mathcal{I}$ with $a^T\chi^{S'} = b$ and $S' \cap \bar{E} = J$. We assume that S, S' are chosen such that $|S \cap S'|$ is maximal. We consider three cases:

1. $r(S') < r(S)$: Then there exists a $g \in S \setminus S'$ with $g \notin \bar{E}$ such that $(S'+g) \in \mathcal{I}$. But this implies $a_g \leq 0$ and so $a_g = 0$ by Lemma 10 because $\chi_I^{S'+e} = \chi_I^{S'}$ for all $I \subseteq \bar{E}, |I| \geq 2$. But this is a contradiction to the maximality of $|S \cap S'|$.

2. $r(S) < r(S')$: Then there exists a $g \in S' \setminus S$ such that $(S+g) \in \mathcal{I}$. We consider two subcases. If $g \notin \bar{E}$, then by similar arguments as above, we get that $a_g = 0$ and that the value of all linearized monomials remains the same. This again contradicts the maximality of $|S \cap S'|$. Let, otherwise, $g \in \bar{E}$. Then $g \in \{e, f\}$. We define $h := \{e, f\} \setminus \{g\}$.
   - $(S+g) \in \mathcal{I}$: This implies $a_g + \Sigma_a(J'+g, g) \leq 0$.

- $(S' - g) \in \mathcal{I}$: This implies $a_g + \Sigma_a(J, g) \geq 0$.

Putting this together we get $\Sigma_a(J, g) \geq \Sigma_a(J' + g, g)$. By definition $J' + g = J - h$ and so $\Sigma_a(J, g) \geq \Sigma_a(J - h, g) \geq 0$ where the last inequality follows by induction. Furthermore, direct computations show that

$$\begin{aligned} 0 &\leq \Sigma_a(J, g) - \Sigma_a(J - h, g) \\ &= \Sigma_a(J) - \Sigma_a(J - g) - (\Sigma_a(J - h) - \Sigma_a(J - h - g)) \\ &= \Sigma_a(J) - \Sigma_a(J - h) - (\Sigma_a(J - g) - \Sigma_a(J - g - h)) \\ &= \Sigma_a(J, h) - \Sigma_a(J - g, h) \end{aligned}$$

By induction, we can conclude $\Sigma_a(J, h) \geq \Sigma_a(J - g, h) \geq 0$. With $e \in \{g, h\}$ the statement follows.

3. $r(S) = r(S')$: By **(M3′)** we know that for $e \in S' \setminus S$ there exists a $j \in S \setminus S'$ such that $(S - j + e), (S' - e + j) \in \mathcal{I}$.
   - $S - j + e$: This implies $a_e + \Sigma_a(J' + e, e) \leq a_j$.
   - $S' - e + j$: This implies $a_j \leq a_e + \Sigma_a(J, e)$.

   So we get $\Sigma_a(J, e) \geq \Sigma_a(J' + e, e) = \Sigma_a(J - f, e) \geq 0$ where the last inequality follows by induction. □

The previous lemmas allow us to prove our main result Theorem 8.

Proof (of Theorem 8). Let $a^T x \leq b$ be a facet-defining inequality of $P_M^{\bar{E}}$ that is not a positive multiple of (4)–(7). Then we can assume by Lemma 10 that $a_e \geq 0, e \in E$, and $a_I \geq 0, I \subseteq \bar{E}, |I| = 2$. Furthermore we can assume by Lemma 11 that for $J \subseteq \bar{E}, |J| \geq 2$ with $\bar{e} \in J$ we have $\sum_{I \subseteq J, \bar{e} \in I, |I| \geq 2} a_I \geq 0$.

We set $T' := \{e \in E \colon a_e > 0\}$ and define $T = \mathrm{cl}(T')$, see Lemma 9. Because $a^T x \leq b$ is not a positive multiple of (7) with the chosen T, there exists an $S \in \mathcal{I}$ with $a^T \chi^S = b$ such that

$$\underbrace{\sum_{e \in T} \chi_e^S}_{=|S \cap T|} + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} \alpha_I(T) \chi_I^S < r(T). \tag{10}$$

We consider two cases:

1. If $|S \cap \bar{E}| \leq 1$, we know that $\chi_I^S = 0$ for all $I \subseteq \bar{E}, |I| \geq 2$. This implies $r(S \cap T) < r(T)$. By the definition of T' we get

   $$r(S \cap T') \leq r(S \cap T) < r(T) = r(T').$$

   So there exists an $f \in T' \setminus (S \cap T')$ such that $((S \cap T') + f) \in \mathcal{I}$. But then

   $$\begin{aligned} b &= \sum_{e \in E} a_e \chi_e^S + \underbrace{\sum_{\substack{I \subseteq \bar{E}, \\ |I| \geq 2}} a_I \chi_I^S}_{=0} = \sum_{e \in E} a_e \chi_e^{S \cap T'} + \underbrace{\sum_{\substack{I \subseteq \bar{E}, \\ |I| \geq 2}} a_I \chi_I^{S \cap T'}}_{=0} \\ &< \sum_{e \in E} a_e \chi_e^{(S \cap T') + f} + \underbrace{\sum_{\substack{I \subseteq \bar{E}, \\ |I| \geq 2}} a_I \chi_I^{(S \cap T') + f}}_{\geq 0} \end{aligned}$$

   This contradicts the validity of $a^T x \leq b$.

2. If $|S \cap \bar{E}| \geq 2$, we set $\tilde{E} = S \cap \bar{E}$. We can assume that $S \subseteq T' \cup \tilde{E}$ (because all other coefficients are zero). We consider first the term $\sum_{I \subseteq \bar{E}, |I| \geq 2} \alpha_I(T)\chi_I^S$.

$$\sum_{I \subseteq \bar{E}, |I| \geq 2} \alpha_I(T)\chi_I^S = \sum_{I \subseteq \tilde{E}, |I| \geq 2} \alpha_I(T) = \beta_{\tilde{E}}(T) = |\tilde{E} \setminus T| + r(T) - r(\tilde{E} + T),$$

so we get

$$r(S \cap T) < -|\tilde{E} \setminus T| + r(\tilde{E} + T).$$

Furthermore this implies

$$r(S) = r(S \cap T) + |S \setminus T| < \underbrace{|S \setminus T| - |\tilde{E} \setminus T|}_{=|(S \setminus \tilde{E}) \setminus T| = 0} + r(\tilde{E} + T) = r(\tilde{E} + T) = r((S \cap \bar{E}) + T').$$

So we can conclude that there exists an $f \in T' \setminus S$ with $(S + f) \in \mathcal{I}$. We consider two subcases:

- $f \notin \bar{E}$: Then $\chi_I^S = \chi_I^{S+f}$ for all $I \subseteq \bar{E}, |I| \geq 2$, and $a_f > 0$ contradicts the validity of $a^T x \leq b$.
- $f \in ((T' \setminus S) \cap \bar{E})$: Then

$$\begin{aligned} b &= \sum_{e \in E} a_e \chi_e^S + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} a_I \chi_I^S \\ &= \sum_{e \in E} a_e \chi_e^{S+f} + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} a_I \chi_I^S + \sum_{\substack{I \subseteq \bar{E} \\ f \in I \\ |I| \geq 2}} a_I \chi_I^{S+f} - \underbrace{a_f}_{>0} - \underbrace{\sum_{\substack{I \subseteq \bar{E} \\ f \in I \\ |I| \geq 2}} a_I \chi_I^{S+f}}_{= \sum_{\substack{I \subseteq (\tilde{E}+f) \\ f \in I, |I| \geq 2}} a_I \geq 0} \\ &< \sum_{e \in E} a_e \chi_e^{S+f} + \sum_{\substack{I \subseteq \bar{E} \\ |I| \geq 2}} a_I \chi_I^{S+f}, \end{aligned}$$

  a contradiction to the validity of $a^T x \leq b$. ☐

So we know a complete description of $P_M^{\bar{E}}$. We now want to consider the associated separation problems. If k is fixed, violated linearization constraints can be determined by complete enumeration and the extended rank inequalities can be separated in polynomial time using the well-known optimization equals separation result [GLS93] and Remark 2. We next describe a more direct way to separate (7) in polynomial time if k is fixed.

Theorem 12. *The separation problem for inequalities (7) reduces to a submodular function minimization problem.*

For solution approaches and further information on submodular function minimization problems we refer to [Sch00, Fuj05, McC06].

Proof. Let $\bar{x} \in \mathbb{R}^{n-k+2^k-1}$ be a vector that satisfies (4)–(6). We consider the separation problem for (7). This reduces to determining the minimizer of the function $d\colon 2^E \to \mathbb{R}$

with

$$
\begin{aligned}
d(T) &:= r(T) - \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \alpha_I(T)\bar{x}_I - \sum_{e\in T}\bar{x}_e \\
&= r(T) + \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \left[\sum_{\substack{J\subseteq I\\ |J|\geq 2}} (-1)^{|I|+|J|}(r(T+J) - r(T) - |J\setminus T|) \right] \bar{x}_I - \sum_{e\in T}\bar{x}_e \\
&= \underbrace{r(T) + \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \left[\sum_{\substack{J\subseteq I\\ |J|\geq 2}} (-1)^{|I|+|J|}(r(T+J) - r(T)) \right] \bar{x}_I}_{=:d'(T)} \\
&\quad \underbrace{- \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \left[\sum_{\substack{J\subseteq I\\ |J|\geq 2}} (-1)^{|I|+|J|}|J\setminus T| \right] \bar{x}_I - \sum_{e\in T}\bar{x}_e}_{\text{modular}}
\end{aligned}
$$

So it remains to check whether the function $d' \colon 2^E \to \mathbb{R}$ is submodular or not.

Let $\bar{J} \subseteq \bar{E}, |\bar{J}| \geq 2$. Then the coefficient of $r(T+\bar{J})$ equals

$$
\sum_{\substack{I\subseteq \bar{E}\\ \bar{J}\subseteq I}} (-1)^{|\bar{J}|+|I|}\bar{x}_I \overset{(5)}{\geq} 0.
$$

Last, we consider the coefficient of $r(T)$. This equals

$$
\begin{aligned}
1 - \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \left[\sum_{\substack{J\subseteq I\\ |J|\geq 2}} (-1)^{|I|+|J|} \right] \bar{x}_I &= 1 - \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} \left[\underbrace{\sum_{k=2}^{|I|} \binom{|I|}{k} (-1)^{|I|+k}}_{(-1)^{|I|}(|I|-1)} \right] \bar{x}_I \\
= 1 - \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} (-1)^{|I|}(|I|-1)\bar{x}_I &= 1 + \sum_{\substack{I\subseteq \bar{E}\\ |I|\geq 2}} (-1)^{|I|}(1-|I|)\bar{x}_I \qquad (11)
\end{aligned}
$$

Our statement follows if we can show that (11) is non-negative. For proving this, we consider inequalities (6) as well as (5) for all $I = \{e\}, e \in \bar{E}$. Adding up we get

$$
1 + \sum_{J\in 2_0^{\bar{E}}} (-1)^{|J|}\bar{x}_J - \sum_{e\in\bar{E}} \sum_{\substack{J\subseteq \bar{E}:\\ e\in J}} (-1)^{|J|}\bar{x}_J \geq 0.
$$

All $\bar{x}_e \in \bar{E}$ cancel out in the left-hand side and each $x_{J'}, J' \subseteq \bar{E}, |J'| \geq 2$, has the coefficient $(-1)^{|J'|} - |J'|(-1)^{|J'|} = (-1)^{|J'|}(1 - |J'|)$. So (11) is at least zero and $d'(T)$ as well as $d(T)$ are submodular functions (in T) by the submodularity of the rank function r. If $T' \subseteq E$ minimizing $d(T)$ has function value $d(T') \geq 0$, then all rank inequalities (7) are satisfied, if $d(T') < 0$, (7) with $T = T'$ is violated. □

Determining the exact minimum of the submodular function d in the previous theorem we even get a set T' which maximally violates (7). The previous result also implies that if k is a fixed constant, the separation problem for (7) can be solved in polynomial time even by purely combinatorial algorithms [Sch00]. For arbitrary matroids the result above is the best we could hope for. Further knowledge of the matroid structure might allow finding faster separation algorithms. For the spanning forest problem with exactly one quadratic monomial, [BK14] developed an adapted version of a separator for the classical subtour elimination constraints. Applying these separators to general quadratic spanning tree problems reduced the number of nodes in branch-and-cut and the running times in comparison to using only the standard linearization.

Let us at the end of this section consider an extension of our results. Given an arbitrary objective function of a polynomial matroid optimization problem we cannot assume that the union of all elements contained in non-linear monomials is actually independent, i. e., that $\bar{E} \notin \mathcal{I}$. In this case the approach presented above can still be applied. But then we have to fix the value of monomials that correspond to dependent sets explicitly to zero.

Corollary 13. *Let $M = (E, \mathcal{I})$ be a matroid and $\bar{E} \subseteq E$ arbitrary. A complete description of the associated polytope $P_M^{\bar{E}}$ is given by (4)–(7) and*

$$x_J = 0, \qquad\qquad J \in 2_0^{\bar{E}}, J \notin \mathcal{I}. \tag{12}$$

Note, if in contrast to our assumption there exists $e \in E$ with $\{e\} \notin \mathcal{I}$, then one also has to include $x_e = 0$.

Proof. The validity of all constraints is ensured by Lemma 6 and by setting all variables corresponding to dependent sets explicitly to zero. In the proofs leading to and of Theorem 8 we always use independent sets, so restricting to those shows the desired results. (Note that the coefficient of variables fixed to zero is unimportant. So it would suffice to consider sets $J \subseteq \bar{E}, |J| \geq 2, J \in \mathcal{I}$ in Lemma 11.) □

Setting some of the monomials to zero in (7) can lead to stronger inequalities for problems where only monomials of smaller degree are present. Consider, for instance, the uniform matroid with $E = \{1, 2, 3, 4\}, r(E) = 2$, and $\bar{E} = \{1, 2, 3\}$. Then

$$x_4 + x_{\{1,2\}} + x_{\{1,3\}} + x_{\{2,3\}} \leq 1$$

is valid and contains only linearized monomials of degree two.

4 A New Hierarchy for Solving Polynomial Matroid Optimization Problems

In this section we present a new hierarchy for solving general matroid optimization problems with polynomial objective. This hierarchy is a direct consequence of the previous results. Indeed, in each step of the hierarchy we enlarge the size of $\bar{E}$ to be considered for deriving strengthened inequalities according to (4)–(7) and (12).

Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{J} \subseteq 2_0^E$ be the index set of all monomials of the polynomial matroid optimization problem to be considered below (with $\mathcal{J}$ including the linear monomials) and let $\tilde{\mathcal{J}} = \mathcal{J} \setminus \{\{e\} \colon e \in E\}$ be the index set of all monomials with degree at least two. We denote by $\tilde{E} \subseteq E$ all elements of E that appear in monomials of degree at least two, i. e., $\tilde{E} := \{e \in E \colon \exists\, J \in \tilde{\mathcal{J}} \text{ with } e \in J\}$, and by $md(\mathcal{J})$ the maximum degree of a monomial in $\mathfrak{J}$. For the important case of quadratic objective functions $md(\mathfrak{J})$ equals two. The coefficients in the objective function are $c\colon \mathfrak{J} \to \mathbb{R}$. We consider matroid optimization problems of the form

$$\text{maximize}\left\{\sum_{J\in\mathfrak{J}} c(J)\prod_{j\in J} x_j\colon x\in P_M\cap\{0,1\}^E\right\}.$$

We are interested in the linearized polytope

$$\bar{P}_M^{\tilde{E}} = \operatorname{conv}\left\{\chi^S_{E+\tilde{\mathfrak{J}}}\colon S\in\mathfrak{I}\right\}$$

where $\chi^S_{E+\tilde{\mathfrak{J}}}$ corresponds to the incidence vector of a set $S\in\mathfrak{I}$ with a component for all linear monomials as well as for all non-linear monomials indexed by $\tilde{\mathfrak{J}}$. A relaxation of $\bar{P}_M^{\tilde{E}}$ with associated polytope $\bar{P}_M^{\tilde{E},1}$ is given by the constraints of the standard linearization of the non-linear monomials via

$$0\le x_J\le x_e, \qquad J\in\tilde{\mathfrak{J}}, e\in J, \tag{13}$$

$$\sum_{e\in J} x_e - x_J \le |J|-1, \qquad J\in\tilde{\mathfrak{J}}, \tag{14}$$

$$0\le x_e, \qquad e\in E\setminus\tilde{E}, \tag{15}$$

(see [GW74]) and the rank inequalities (1).

Our aim is to describe $\bar{P}_M^{\tilde{E}}$ by doing lift and project operations to $\bar{P}_M^{\tilde{E},1}$ that make it closer and closer to $\bar{P}_M^{\tilde{E}}$. For this we define for each $i\in\{2,\dots,|\tilde{E}|\}$ the lifted polytopes

$$\tilde{P}_M^{\tilde{E},i} = \left\{(x_{E\setminus\tilde{E}}, x_{2_0^{\tilde{E},i}+\tilde{\mathfrak{J}}})\colon (x_{E\setminus\bar{I}}, x_{2_0^{\bar{I}}})\in P_M^{\bar{I}} \text{ for each } \bar{I}\subseteq\tilde{E}, |\bar{I}|=i, x_{E+\tilde{\mathfrak{J}}}\in\bar{P}_M^{\tilde{E},1}\right\}$$

that contain in comparison to $\bar{P}_M^{\tilde{E}}$ additional components associated to the linearized monomials with degree between 2 and i. So these polytopes contain the entries associated to all $e\in E\setminus\tilde{E}$, to all non-linear monomials represented by $\tilde{\mathfrak{J}}$ and to all linearized monomials up to degree i (of elements in $\tilde{E}$). For all sets $\bar{I}\subseteq\tilde{E}, |\bar{I}|=i$, the components of a vector $x\in\tilde{P}_M^{\tilde{E},i}$ that correspond to $E\setminus\bar{I}$ and the monomials in $2_0^{\bar{I}}$ satisfy the constraints (4)–(7) where $\bar{I}$ has here the role of $\bar{E}$ in Sections 2 and 3. They additionally satisfy the constraints (12) if $\bar{I}\notin\mathfrak{I}$ and so some monomials are always zero in this case. Furthermore, the linearization constraints (13) and (14) are satisfied for the non-linear monomials in $\tilde{P}_M^{\tilde{E},i}$.

The projection of $\tilde{P}_M^{\tilde{E},i}$ onto the original variables in $\bar{P}_M^{\tilde{E}}$ is called $\bar{P}_M^{\tilde{E},i}$. Projecting to the space of original variables allows a comparison with $\bar{P}_M^{\tilde{E}}$ or $\bar{P}_M^{\tilde{E},1}$, respectively. So we use the complete description of the polytopes studied in Sections 2 and 3 to strengthen our description of $\bar{P}_M^{\tilde{E}}$. By construction our description of $\bar{P}_M^{\tilde{E}}$ improves in each step where for $i=|\tilde{E}|$ we have one variable for each subset of $\tilde{E}$. This implies the following result.

Theorem 14. *Let $M=(E,\mathfrak{I})$ be a matroid. Then*

$$\bar{P}_M^{\tilde{E}} = \bar{P}_M^{\tilde{E},|\tilde{E}|}\subseteq\bar{P}_M^{\tilde{E},|\tilde{E}|-1}\subseteq\ldots\subseteq\bar{P}_M^{\tilde{E},2}\subseteq\bar{P}_M^{\tilde{E},1}.$$

Assuming that $|\mathfrak{J}|$ is polynomially bounded and that the depth of the hierarchy i is a constant, one can optimize over $\tilde{P}_M^{\tilde{E},i}$ or its projection $\bar{P}_M^{\tilde{E},i}$ in polynomial time.

Note that the results from the previous sections give explicit descriptions of all polytopes $P_M^{\bar{I}}$, for each $\bar{I} \subseteq \bar{E}$ with $|\bar{I}| = i$ for some $i \in \{2, \ldots, |\tilde{E}|\}$.

Proof. By Theorem 8 we get that $\bar{P}_M^{\tilde{E}} = \bar{P}_M^{\tilde{E},|\tilde{E}|}$. Let $i \in \{2, \ldots, |\tilde{E}|\}$. We show that $\bar{P}_M^{\tilde{E},i} \subseteq \bar{P}_M^{\tilde{E},i-1}$. For this, let $\bar{x} \in \bar{P}_M^{\tilde{E},i}$. By definition $\bar{x}$ can be extended to some $\tilde{x} \in \tilde{P}_M^{\tilde{E},i}$ with $\bar{x} = \tilde{x}_{E+\tilde{\mathfrak{J}}}$. Obviously, $\tilde{x}$ satisfies $\tilde{x}_{E+\tilde{\mathfrak{J}}} \in \bar{P}_M^{\tilde{E},1}$, which does not depend on i. Now consider an arbitrary $\bar{I} \subseteq \tilde{E}$ with $|\bar{I}| = i-1$. Choosing an arbitrary $e \in \tilde{E} \setminus \bar{I}$, we get $(\tilde{x}_{E\setminus(\bar{I}+e)}, \tilde{x}_{2_0^{\bar{I}+e}}) \in P_M^{\bar{I}+e}$. Because $P_M^{\bar{I}+e}$ is a convex polytope, $(\tilde{x}_{E\setminus(\bar{I}+e)}, \tilde{x}_{2_0^{\bar{I}+e}})$ is a convex combination of vertices of $P_M^{\bar{I}+e}$ (which are integral by definition), i. e. $(\tilde{x}_{E\setminus(\bar{I}+e)}, \tilde{x}_{2_0^{\bar{I}+e}}) = \sum_{j=1}^m \alpha_j \tilde{x}^{(j)}$, $\alpha_j \geq 0$, $\sum_{j=1}^m \alpha_j = 1$ for some integral vertices $\tilde{x}^{(j)}$ of $P_M^{\bar{I}+e}$. Restricting to the monomials not containing e we get integral points $(\tilde{x}^{(j)}_{E\setminus\bar{I}}, \tilde{x}^{(j)}_{2_0^{\bar{I}}}) \in P_M^{\bar{I}}$ with $(\tilde{x}_{E\setminus\bar{I}}, \tilde{x}_{2_0^{\bar{I}}}) = \sum_{j=1}^m \alpha_j (\tilde{x}^{(j)}_{E\setminus\tilde{E}}, \tilde{x}^{(j)}_{2_0^{\bar{I}}}) \in P_M^{\bar{I}}$. Because $\bar{I}$ was arbitrary this implies $(\tilde{x}_{E\setminus\tilde{E}}, \tilde{x}_{2_0^{\tilde{E},i-1}+\tilde{\mathfrak{J}}}) \in \tilde{P}_M^{\tilde{E},i-1}$ and because $\bar{P}_M^{\tilde{E},i-1}$ is the projection of $\tilde{P}_M^{\tilde{E},i-1}$ onto the original monomials $\bar{x} \in \bar{P}_M^{\tilde{E},i-1}$ follows.

For the second part of the theorem, let the level of the hierarchy be a fixed constant as well as $|\tilde{J}|$ be polynomially bounded. Then we can explicitly separate the linearization constraints as there are only polynomially many and we can separate the extended rank inequalities (7) using submodular function minimization, see Theorem 12, on problems of polynomial size leading to some polynomial algorithm. ☐

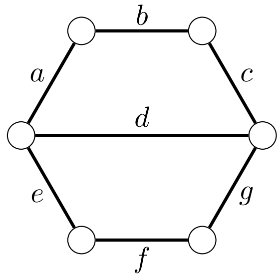
Let us consider the hierarchy in more detail.

Remark 15.
1. An advantage of our new hierarchy in comparison to other hierarchies such as Sherali-Adams is that for the lifted polytopes, new monomials are *only* introduced for variables that appear in some non-linear monomial. So, if $|\tilde{E}|$ is small in comparison to E, our hierarchy might pay off.
2. The extended rank inequalities (7) and the associated separator (Theorem 12) can as well be used in general branch-and-cut approaches for solving polynomial matroid optimization problems or polynomial problems with some underlying matroid structure. For just improving a current formulation, one can also use only some of the new constraints.

In the next remark we compare our new hierarchy to the hierarchy of Sherali-Adams [SA90], which is the weakest of the ones treated in [Lau03], and mention some properties of our hierarchy.

Remark 16.
1. The structure of the extended rank inequalities is rather special. All linear monomials as well as all linearized monomials of degree two have non-negative coefficients, see Lemma 10. Apart from this, for small levels of the hierarchy, the extended rank inequalities are somewhat sparse with respect to the linearized monomials, i. e., in level i at most $2^i - i - 1$ monomials with non-zero coefficients appear in these constraints. In contrast to this in the hierarchy of Sherali and Adams for each constraint most of the linear terms are replaced by terms corresponding to linearized monomials of degree two even in level two [SA90].
2. Due to the sparsity with respect to the number of added monomials it does not seem to be fair to compare the hierarchy of Sherali-Adams on level l_{SA} with our hierarchy at the same level i (we apply the Sherali-Adams procedure to the matroid polytope

and so get variables corresponding to quadratic monomials for $l_{SA} \geq 2$). So we set $l_{SA} + 1 = i$. We consider the graphical matroid on the graph



Using the following objective functions for the matroid maximization problem

$$f_1(x) := x_d + x_{ab} + x_{ac} + x_{bc} + x_{ef} + x_{eg} + x_{fg} - \tfrac{1}{2}x_{ag} - \tfrac{1}{2}x_{bf} - \tfrac{1}{2}x_{ce}$$
$$f_2(x) := f_1(x) - \tfrac{1}{2}x_{bd} - \tfrac{1}{2}x_{df}$$

we get the following optimal solution values (rounded) if we neglect the integrality condition (in both cases the integral optimal solution value equals 3) [GO16]

	Sherali-Adams at $l_{SA} = 2$	our approach at level $i = 3$
f_1	3.7857	3.8333
f_2	3.7857	3.7500

Note that the smallest circuit contains four elements so we do not need equations of type (12). These computational results show that the associated polyhedra are not comparable.

Apart from this observation one can easily show that the Sherali-Adams hierarchy for $l_{SA} = 2$ dominates our hierarchy with $i = 2$ on graphical matroids. In this level at most one linearized monomial might be added to some rank inequality. Indeed, let $G = (V, E)$ be a graph. In this case one of the two forms of the strengthened rank inequalities turns out to be (see [BK14, FF13])

$$\sum_{\substack{e=\{i,j\}\in E:\\ i,j\in S}} x_e + x_{\{\{u,v\},\{v,w\}\}} \leq |S| - 1, \quad S \subsetneq V, u, w \in S, v \in V \setminus S.$$

For Sherali-Adams, this inequality can be obtained by multiplying

$$\sum_{\substack{e=\{i,j\}\in E:\\ i,j\in S}} x_e \leq |S| - 1 \text{ with } (1 - x_{\{u,v\}}), \qquad \sum_{\substack{e=\{i,j\}\in E:\\ i,j\in S\cup\{v\}}} x_e \leq |S| \text{ with } x_{\{u,v\}},$$

replacing the products by the corresponding linearized variables and x_e^2 by x_e, summing up as well as using the non-negativity of the variables. The second form of the strengthened rank inequalities can be handled similarly.

Let us finally have a look at the important special case of matroid optimization problems with quadratic objective. Using our hierarchy at level three (with monomials of degree two and three) we present in this paper a way to take interactions of different (non-nested) monomials into account, see the research question in [HKL15]. Applying it at level four, we even can handle interactions between quadratic monomials that do not share a common variable, e. g., x_1x_2, x_3x_4.

4.1 Matroid Optimization Problems with Monomials Corresponding to the Bases of a Circuit

In the following we present some example where monotonicity of the monomials is not needed in order to derive a complete description. We consider a set of monomials that correspond to the bases of a circuit. Let $M = (E, \mathcal{I})$ be a matroid and let $C = \{c_1, \dots, c_{|C|}\} \subseteq E$, $|C| \geq 3$, be a circuit. We define $C_i := C \setminus \{c_i\}, i \in \{1, \dots, |C|\}$. By definition $C_i \in \mathcal{I}$ for all $i = 1, \dots, |C|$, and $(C_i \cup C_j) \notin \mathcal{I}$ for all $i, j \in \{1, \dots, |C|\}, i \neq j$. For cost functions $c\colon E \to \mathbb{R}$ and $\bar{c}\colon \{1, \dots, |C|\} \to \mathbb{R}$ we consider the optimization problem

$$\text{maximize} \left\{ \sum_{e \in E} c(e) \cdot x_e + \sum_{i=1}^{|C|} \bar{c}(i) \prod_{e \in C_i} x_e \colon x \in P_M \cap \{0,1\}^E \right\}. \tag{16}$$

So $\tilde{E} = C$ in this case, i. e., all elements of C appear in non-linear monomials.

Remark 17. A solution of (16) can be obtained by solving $|C|$ matroid optimization problems as well as one matroid intersection problem. First, we determine for each $i \in \{1, \dots, |C|\}$ an $I_i \in \mathcal{I}$ with $C_i \subseteq I_i$ maximizing the sum $\sum_{e \in I_i \setminus C} c(e)$ (the total objective value of I_i equals $c(I_i) + \bar{c}(i)$). In a second step, we determine an optimal solution $\bar{I}$ for the intersection of matroid M and matroid $M' = (E, \mathcal{I}')$ with $\mathcal{I}' := \{S \subset E \colon |S \cap C| \leq |C| - 2\}$ (the total objective value of $\bar{I}$ equals $c(\bar{I})$). Finally, we take a solution with total maximal objective value as a solution of (16).

We are interested in the structure of the linearized polytope

$$P_M^C := \operatorname{conv} \left\{ (\chi^S, \chi^S_{C_1}, \dots, \chi^S_{C_{|C|}}) \in \{0,1\}^{E + |C|} \colon S \in \mathcal{I} \right\}.$$

Similarly to above we provide a complete description of P_M^C that is our main result in this subsection.

Theorem 18. *The inequalities*

$$-x_{C_i} \leq 0, \qquad i \in \{1, \dots, |C|\}, \tag{17}$$

$$\sum_{i=1}^{|C|} x_{c_i} - \sum_{i=1}^{|C|} x_{C_i} \leq |C| - 2, \tag{18}$$

$$\sum_{j=1, j \neq i}^{|C|} x_{C_j} - x_{c_i} \leq 0, \qquad i \in \{1, \dots, |C|\}, \tag{19}$$

$$-x_e \leq 0, \qquad e \in E \setminus C, \tag{20}$$

$$\sum_{e \in T} x_e + \sum_{i=1}^{|C|} \beta_{C_i}(T) x_{C_i} \leq r(T), \qquad T \subseteq E, \tag{21}$$

are a complete description of P_M^C.

In order to prove this result we need several lemmas whose proofs are deferred to the Appendix. The structure of the results is very similar to the structure in Section 3.

First we provide a formulation for P_M^C.

Lemma 19. *The inequalities (17)–(21) together with*

$$x \in \{0,1\}^{E + |C|}, \tag{22}$$

are a formulation for P_M^C.

Next, we prove that we can restrict to extended rank inequalities (21) for closed sets $T \subseteq E$.

Lemma 20. *A point $\bar{x} \in \mathbb{R}^{E+|C|}$ that satisfies (17)–(20) as well as (21) for closed sets $T = \mathrm{cl}(T) \subseteq E$, satisfies (21) for arbitrary sets $T \subseteq E$.*

Facet-defining inequalities of P_M^C that are not positive multiples of the linearization constraints have non-negative coefficients, see also Lemma 10.

Lemma 21. *Each facet defining inequality $a^T x \leq b$ of P_M^C that is not a positive multiple of one of (17)–(20) satisfies $a \geq 0$.*

The previous lemmas allow us to prove our main result on the structure of P_M^C.

Proof (of Theorem 18). Let $a^T x \leq b$ be a facet defining inequality of P_M^C that is not a positive multiple of (17)–(20). Then Lemma 21 implies $a \geq 0$. Let $T' := \{e \in E : a_e > 0\}$ and $T := \mathrm{cl}(T')$. We consider the constraint (21) associated with T. Because $a^T x \leq b$ is not a positive multiple of this constraint, there is an $S \in \mathcal{I}$ with $\sum_{e \in T} \chi_e^S + \sum_{i=1}^{|C|} \beta_{C_i}(T) \chi_{C_i}^S < r(T)$ and $a^T \chi^S = b$. In particular, $r(S \cap T) = \sum_{e \in T} \chi_e^S$ implies

$$r(S \cap T) + \sum_{i=1}^{|C|} (r(T) + |C_i \setminus T| - r(T \cup C_i)) \chi_{C_i}^S < r(T). \tag{23}$$

We may assume $S \subseteq T' \cup C$ (otherwise use $\bar{S} := S \cap (T' \cup C)$).

a) If $C_i \subseteq S$ for some $i \in \{1, \ldots, |C|\}$, then $\chi_{C_i}^S = 1$ and $\chi_{C_j}^S = 0$ for all $j \in \{1, \ldots, |C|\} \setminus \{i\}$ implying $r(S \cap T) < r(T \cup C_i) - |C_i \setminus T|$ (see (23)). Therefore $r(S) = r(S \cap T) + |S \setminus T| < r(T \cup C_i) - |C_i \setminus T| + |S \setminus T| = r(T \cup C_i) = r(T' \cup C_i)$. So there exists an $e \in (T' \cup C_i) \setminus S = T' \setminus S$ with $S + e \in \mathcal{I}$, contradicting the validity of $a^T x \leq b$.

b) If $C_i \nsubseteq S$ for all $i \in \{1, \ldots, |C|\}$, then $\chi_{C_i}^S = 0$ for all $i \in \{1, \ldots, |C|\}$, and with $S' := S \cap T'$ we know $r(S') = r(S \cap T') \leq r(S \cap T) < r(T) = r(T')$. So there exists an $e \in T' \setminus S'$ with $S' + e \in \mathcal{I}$. By the choice of T', $a^T \chi^{S'} = b$, too, and $S' + e$ contradicts the validity of $a^T x \leq b$ by Lemma 21. ☐

Considering the graphic matroid, we derive exactly the known complete description of the so called Boolean quadric polytope for complete graphs on three nodes [LL04].

Similarly to above, the separation problem for P_M^C simplifies to a submodular function minimization problem and explicit enumeration of the linearization constraints.

Observation 22. *The separation problem of (21) reduces to a submodular function minimization problem.*

Remark 23. The result in Observation 22 can be further generalized. We consider matroid optimization problems with a set of non-linear monomials in the objective function such that all these monomials are in pairwise conflict. Extended rank inequalities for $T \subseteq E$ (similar to (7) and (21)) can be derived that consist of the sum of the variables of the elements in T, each with coefficient one, and the variables corresponding to the linearized monomials, each with coefficient $\beta_K(T) = r(T) + |K \setminus T| - r(T + K)$ for the monomials corresponding to $K \subseteq E$. Using the fact that at most one of these monomials can be present, validity easily follows. The associated separation problem can be solved via submodular function minimization as well. Unfortunately, the inequalities of that type combined with linearization constraints and one packing type constraint (stating that at most one of the monomials can be present) are usually not sufficient for a complete description of such linearized polytopes. For this consider the uniform matroid with $E = \{1, 2, 3, 4\}$ and $r(E) = 2$. Then $x_3 + x_4 - x_{\{3,4\}} + x_{\{1,2\}} \leq 1$ defines a facet of the associated polytope with two linearized monomials [LR15].

5 Future Work

In this paper we studied matroid optimization problems with polynomial objective functions where the monomials satisfy certain monotonicity properties. We derived a complete description of the linearized polytope that was the basis for a new hierarchy for solving polynomial matroid optimization problems. We have presented a case in which the monotonicity of the monomials is not necessary in order to derive a complete description for the linearized polytope using our framework. It remains for future work to detect more cases in which weaker types of monotonicity work. For such extensions, a good knowledge of the unconstrained case is essential. Furthermore one could compare our new hierarchy in more detail with the existing ones with respect to the quality of the bounds in the different hierarchy levels and the computational effort. For these tests on certain matroids it seems essential to develop problem-specific separation algorithms that do not rely on submodular function minimization.

Improving the understanding of the linearizations of more general monomial structures seems worthwhile in both the unconstrained (see, e. g., [CRH17, DPK17b]) and the constrained case. One starting point could be the study of matroid optimization problems with some, not necessarily related, quadratic monomials without adding further monomials. Additionally, we could try to extend our results to other combinatorial optimization problems. In principle our approach of combining some combinatorial optimization problem with some non-linear monomials which are linearized and studying the associated polyhedron should be possible for all polynomial-time solvable problems where the polyhedral structure in the linear case is well understood. But determining the concrete descriptions might be quite challenging for non-matroid problems. One possibility is to consider the intersection of two matroids. Here Walter [Wal16] proved a conjecture of [Kle14] on the complete description of the bipartite matching problem with one (linearized) quadratic monomial. But an extension to arbitrary matroid intersection problems might be complicated because the structure of the constraints that do not correspond to the linearization is different to our current setting and differs in general from the structure in the bipartite matching case (intersection of two partition matroids). Indeed, even in the case with only one quadratic monomial, the coefficient of the new variable can be negative in constraints not related to the linearization, see also [Kle14, Wal16]. Furthermore, the coefficient of the two original variables that belong to the quadratic monomial might be two in such constraints. One example is the braching problem. For a complete digraph on four nodes, the constraint $2x_{(1,2)} + x_{(1,4)} + x_{(2,1)} + x_{(2,3)} + x_{(2,4)} + x_{(3,2)} + x_{(3,4)} - x_{(1,2),(2,3)} \leq 3$ is facet-defining for the associated polytope [LR15] where we linearize the monomial $x_{(1,2)} \cdot x_{(2,3)}$ (the x-variables correspond to the arcs of the complete digraph on the nodes $\{1, 2, 3, 4\}$). In our extended rank inequalities and in the ones in [FFM17, Kle14, Wal16] the coefficients of variables corresponding to monomials of degree one are always zero or one.

Appendix

Proof (of Lemma 19). The integrality constraints (22) together with the linearization constraints (17)–(20) ensure the dependencies between the variables $x_e, e \in E$, and the linearized monomials $x_{C_i}, i = 1, \dots, |C|$. Indeed, at most one of $x_{C_i}, i \in \{1, \dots, |C|\}$ can be one. So, the desired result follows directly from Theorem 1 if we can prove the validity of (21). If $S \in \mathfrak{I}$ with $|C \cap S| \le |C| - 2$, then $\chi^S_{C_i} = 0$ for all $i \in \{1, \dots, |C|\}$ and so the validity of the extended rank inequalities is clear. If, w. l. o. g., $C_1 \subset S \in \mathfrak{I}$ and so $C_i \not\subset S$ for all $i \in \{2, \dots, |C|\}$, then the coefficient of x_{C_1} equals $r(T) + |C_1 \setminus T| - r(T + C_1) = \beta_{C_1}(T)$ and so Lemma 5 proves the result. □

Proof (of Lemma 20). Let $T \subseteq E$ be arbitrary, but fixed, and $\bar{x}$ be a point with the desired properties. We set $T' = T + (\mathrm{cl}(T) \setminus \bar{E})$. Then

$$\sum_{e \in T} \bar{x}_e + \sum_{i=1}^{|C|} \beta_{C_i}(T)\bar{x}_{C_i} \overset{(20),\mathrm{L.4}}{\le} \sum_{e \in T'} \bar{x}_e + \sum_{i=1}^{|C|} \beta_{C_i}(T')\bar{x}_{C_i}$$
$$\overset{(19)}{\le} \sum_{e \in \mathrm{cl}(T)} \bar{x}_e + \sum_{i=1}^{|C|} \beta_{C_i}(\mathrm{cl}(T))\bar{x}_{C_i} \le r(\mathrm{cl}(T)) = r(T). \qquad □$$

Proof (of Lemma 21). Let $a^T x \le b$ be a facet defining inequality of P^C_M that is not a positive multiple of (17)–(20). We consider three cases. For all $e \in E \setminus C$ there exists an $S \in \mathfrak{I}$ with $a^T \chi^S = b$ and $e \in S$ because $a^T x \le b$ is not a positive multiple of (20). By $(S - e) \in \mathfrak{I}$ and $\chi^S_{C_i} = \chi^{S-e}_{C_i}$ for all $i \in \{1, \dots, |C|\}$, it follows that $a_e \ge 0$ by feasibility of $a^T x \le b$. Next, let $c_i \in C$. Then, because $a^T x \le b$ is not a positive multiple of (19), there exists some $S \in \mathfrak{I}$ with $a^T \chi^S = b$ and $\chi^S_{C_j} = 0$ for all $j \in \{1, \dots, |C|\} \setminus \{i\}$, $c_i \in S$. Then $(S - c_i) \in \mathfrak{I}$ and $\chi^S_{C_j} = \chi^{S-c_i}_{C_j} = 0$ for all $j \in \{1, \dots, |C|\}$ imply $a_{c_i} \ge 0$ by feasibility.

It remains to prove $a_{C_i} \ge 0$ for all $i \in \{1, \dots, |C|\}$. By the previous cases we may assume $a_e \ge 0$ for all $e \in E$. Because $a^T x \le b$ is not a positive multiple of (18) there exists an $S \in \mathfrak{I}$ with $|C \cap S| \le |C| - 3$ and $a^T \chi^S = b$.

1. Let $i \in \{1, \dots, |C|\}$ with $c_i \in C \setminus S$. Because $a^T x \le b$ is not a positive multiple of (17), there exists $S'_i \in \mathfrak{I}$ with $a^T \chi^{S'_i} = b$ and $C_i \subseteq S'_i$. We set $T_i := \{e \in E \setminus C_i \colon a_e > 0\} \cup (S \cap C_i)$. We may assume $S \subseteq T_i$ and $r(S) = r(T_i)$ as well as $S'_i \subseteq T_i + C_i$ and $r(S'_i) = r(T_i + C_i)$. We distinguish two cases.

1.1 $r(T_i) < r(T_i + C_i)$: Then there exists an $e \in C_i \setminus T_i$ with $r(T_i + e) > r(T_i) = r(S)$, so $S + e \in \mathfrak{I}$. This implies $a_e \le 0$ and so $a_e = 0$. Furthermore with $e \in C_i \subseteq S'_i$ we get $b = a^T \chi^{S'_i} = a^T \chi^{S'_i - e} + a_e + a_{C_i} \le b + a_{C_i}$ and so $a_{C_i} \ge 0$.

1.2 $r(T_i) = r(T_i + C_i)$: By assumption S, S'_i are both bases of $T_i + C_i$. We may assume that $|S \cap S'_i|$ is maximal. By **(M3′)** there exists for each $f \in (S'_i \setminus S) \cap C_i$ an $e \in S \setminus S'_i, e \notin C$, so that $S - e + f, S'_i + e - f \in \mathfrak{I}$. Because $a^T \chi^{S-e+f} \le b = a^T \chi^S$ we get $a_e - a_f \ge 0$. Then $a^T \chi^{S'_i + e - f} \le b = a^T \chi^{S'_i} = a^T \chi^{S'_i + e - f} - a_e + a_f + a_{C_i}$ implying $a_{C_i} \ge 0$.

2. Let $i \in \{1, \dots, |C|\}$ with $c_i \in C \cap S$. Similarly to above, there exists $S'_i \in \mathfrak{I}$ with $a^T \chi^{S'_i} = b$ and $C_i \subseteq S'_i$. Case 1.1 and case 1.2 with $f \in C_i$ and $e \ne c_i$ can be treated analogously without any changes. It remains to consider $e = c_i$ and $f = c_j$ for some $j \in \{1, \dots, |C|\} \setminus \{i\}$. Then we get $b \ge a^T \chi^{S'_i + e - f} = a^T \chi^{S'_i} - a_f + a_e + a_{C_j} - a_{C_i}$ and $b \ge a^T \chi^{S-e+f} = a^T \chi^S - a_e + a_f$. This implies $a_f \le a_e$ and $a_{C_i} - a_{C_j} \ge 0$. By the choice of f we have $c_j = f \in S'_i \setminus S$ with $c_j \in C \setminus S$ and so $a_{C_j} \ge 0$ by case 1. This implies $a_{C_i} \ge a_{C_j} \ge 0$ in this case. □

Proof (of Observation 22). Let $\bar{x} \in \mathbb{R}^{E+|C|}$ be a vector that satisfies (17)–(20). We consider the separation problem for (21). This reduces to determining the minimizer of the function $d\colon 2^E \to \mathbb{R}$ with

$$\begin{aligned} d(T) &:= r(T) - \sum_{e \in E} \bar{x}_e - \sum_{i=1}^{|C|} (r(T) + |C_i \setminus T| - r(T + C_i)) \bar{x}_{C_i} \\ &= \left(1 - \sum_{i=1}^{|C|} \bar{x}_{C_i}\right) r(T) + \sum_{i=1}^{|C|} \underbrace{\bar{x}_{C_i}}_{\geq 0} r(T + C_i) \underbrace{- \sum_{e \in E} \bar{x}_e - \sum_{i=1}^{|C|} |C_i \setminus T| \bar{x}_{C_i}}_{modular} \end{aligned}$$

So in order to show the submodularity of $d(.)$ it remains to prove that $1 - \sum_{i=1}^{|C|} \bar{x}_{C_i} \geq 0$. This formula can be derived by adding (18) and (19) for each $i \in \{1, \ldots, |C|\}$ and scaling both sides by $|C| - 2$. ☐

References

[AS05] Warren P. Adams and Hanif D. Sherali. A hierarchy of relaxations leading to the convex hull representation for general discrete optimization problems. *Annals of Operations Research*, 140(1):21–47, 2005.

[BCC93] Egon Balas, Sebastián Ceria, and Gérard Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0–1 programs. *Mathematical Programming*, 58(1):295–324, 1993.

[BCRH16] Christoph Buchheim, Yves Crama, and Elisabeth Rodríguez-Heck. Berge-acyclic multilinear 0-1 optimization problems. Preprint, 2016.

[BK13] Christoph Buchheim and Laura Klein. The spanning tree problem with one quadratic term. In Kamiel Cornelissen, Ruben Hoeksma, Johann Hurink, and Bodo Manthey, editors, *CTW*, volume WP 13-01 of *CTIT Workshop Proceedings*, pages 31–34, 2013.

[BK14] Christoph Buchheim and Laura Klein. Combinatorial optimization with one quadratic term: Spanning trees and forests. *Discrete Applied Mathematics*, 177:34–52, 2014.

[BL12] Samuel Burer and Adam N. Letchford. Non-convex mixed-integer nonlinear programming: A survey. *Surveys in Operations Research and Management Science*, 17(2):97–106, 2012.

[BLMA+08] Yael Berstein, Jon Lee, Hugo Maruri-Aguilar, Shmuel Onn, Eva Riccomagno, Robert Weismantel, and Henry Wynn. Nonlinear matroid optimization and experimental design. *SIAM Journal on Discrete Mathematics*, 22(3):901–919, 2008.

[BM84] Egon Balas and Joseph B. Mazzola. Nonlinear 0–1 programming: I. linearization techniques. *Mathematical Programming*, 30(1):1–21, 1984.

[BR08] Christoph Buchheim and Giovanni Rinaldi. Efficient reduction of polynomial zero-one optimization to the quadratic case. *SIAM Journal on Optimization*, 18(4):1398–1413, 2008.

[Bru69] Richard A. Brualdi. Comments on bases in dependence structures. *Bulletin of the Australian Mathematical Society*, 1(2):161–167, 8 1969.

[CRH17] Yves Crama and Elisabeth Rodríguez-Heck. A class of valid inequalities for multilinear 0–1 optimization problems. *Discrete Optimization*, 25:28–47, 2017. Available online.

[CT12] Eden Chlamtac and Madhur Tulsiani. Convex relaxations and integrality gaps. In Miguel F. Anjos and Jean B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, pages 139–169. Springer US, Boston, MA, 2012.

[DPK16] Alberto Del Pia and Aida Khajavirad. The multilinear polytope for gamma-acyclic hypergraphs. Preprint on Optimization online., 2016.

[DPK17a] Alberto Del Pia and Aida Khajavirad. On decomposability of multilinear sets. *Mathematical Programming*, May 2017.

[DPK17b] Alberto Del Pia and Aida Khajavirad. A polyhedral study of binary polynomial programs. *Mathematics of Operations Research*, 42(2):389–410, 2017.

[Edm70] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. *Combinatorial Structures and Their Applications*, pages 69–87, 1970.

[Edm71] Jack Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1(1):127–136, 1971.

[FF13] Anja Fischer and Frank Fischer. Complete description for the spanning tree problem with one linearised quadratic term. *Operations Research Letters*, 41(6):701–705, 2013.

[FF15] Anja Fischer and Frank Fischer. An extended approach for lifting clique tree inequalities. *Journal of Combinatorial Optimization*, 30(3):489–519, 2015.

[FFM17] Anja Fischer, Frank Fischer, and S.Thomas McCormick. Matroid optimisation problems with nested non-linear monomials in the objective function. *Mathematical Programming A*, 2017.

[FH13] Anja Fischer and Christoph Helmberg. The symmetric quadratic traveling salesman problem. *Mathematical Programming A*, 142(1):205–254, 2013.

[For59] R. Fortet. L'Algèbre de Boole et ses applications en recherche opérationnelle. *Cahiers du Centre d'Etudes de Recherche Opérationelle*, 4:5–36, 1959.

[Fuj05] Satoru Fujishige. *Submodular Functions and Optimization.* Elsevier, 2005.

[GLS93] Martin Grötschel, Lászlo Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics.* Springer, second corrected edition, 1993.

[GO16] Inc. Gurobi Optimization. Gurobi optimizer reference manual, 2016.

[Grö77] Martin Grötschel. *Polyedrische Charakterisierungen kombinatorischer Optimierungsprobleme*, volume 36 of *Mathematical Systems in Economics.* Verlag Anton Hain, Meisenheim am Glan, 1977.

[GW73] Fred Glover and Eugene Woolsey. Further reduction of zero-one polynomial programming problems to zero-one linear programming problems. *Operations Research*, 21(1):156–161, 1973.

[GW74] Fred Glover and Eugene Woolsey. Converting the 0-1 polynomial programming problem to a 0-1 linear program. *Operations Research*, 22(1):180–182, 1974.

[HKL15] Lena Hupp, Laura Klein, and Frauke Liers. An exact solution method for quadratic matching: The one-quadratic-term technique and generalisations. *Discrete Optimization*, 18:193–216, 2015.

[Kle14] Laura Klein. *Combinatorial optimization with one quadratic term*. PhD thesis, TU Dortmund, 2014.

[Las01] Jean B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. In *Proceedings of the 8th International IPCO Conference on Integer Programming and Combinatorial Optimization*, pages 293–303, London, UK, UK, 2001. Springer-Verlag.

[Lau03] Monique Laurent. A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0–1 programming. *Mathematics of Operations Research*, 28(3):470–496, July 2003.

[LHLO10] Jesús A. De Loera, David C. Haws, Jon Lee, and Allison O'Hair. Computation in multicriteria matroid optimization. *Journal of Experimental Algorithmics*, 14:8:1.8–8:1.33, January 2010.

[LL04] Jon Lee and Janny Leung. On the Boolean quadric forest polytope. *INFOR*, 42(2):125–141, 2004.

[LR15] Stefan Lörwald and Gerhard Reinelt. PANDA: a software for polyhedral transformations. *EURO Journal on Computational Optimization*, 3(4):297–308, 2015.

[LS91] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0–1 optimization. *SIAM Journal on Optimization*, 1(2):166–190, 1991.

[McC06] S. Thomas McCormick. Submodular function minimization. In K. Aardal, G. Nemhauser, and R. Weismantel, editors, *Handbook on Discrete Optimization*, pages 321–391. Elsevier, 2006.

[Onn04] Shmuel Onn. Convex matroid optimization. *SIAM Journal on Discrete Mathematics*, 17(2):249–253, February 2004.

[Oxl92] James G. Oxley. *Matroid theory*. Oxford University Press, 1992.

[Pad89] Manfred Padberg. The Boolean quadric polytope: Some characteristics, facets and relatives. *Mathematical Programming*, 45(1-3):139–172, 1989.

[PGdC15] Dilson Lucas Pereira, Michel Gendreau, and Alexandre Salles da Cunha. Lower bounds and exact algorithms for the quadratic minimum spanning tree problem. *Computers & Operations Research*, 63:149–160, 2015.

[SA90] Hanif D. Sherali and Warren P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal on Discrete Mathematics*, 3(3):411–430, 1990.

[SA98] Hanif D. Sherali and Warren P. Adams. *Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems.* Springer, 1998.

[Sch00] Alexander Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory Ser. B*, 80(2):346–355, November 2000.

[Wal16] Matthias Walter. Complete description of matching polytopes with one linearized quadratic term for bipartite graphs. ArXiv e-prints, 2016.

[Wel76] Dominic J. A. Welsh. *Matroid theory.* Academic Press, London, 1976.