Bounds on entanglement dimensions and quantum graph parameters via noncommutative polynomial optimization

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Abstract

In this paper we study bipartite quantum correlations using techniques from tracial polynomial optimization. We construct a hierarchy of semidefinite programming lower bounds on the minimal entanglement dimension of a bipartite correlation. This hierarchy converges to a new parameter: the minimal average entanglement dimension, which measures the amount of entanglement needed to reproduce a quantum correlation when access to shared randomness is free. For synchronous correlations, we show a correspondence between the minimal entanglement dimension and the completely positive semidefinite rank of an associated matrix. We then study optimization over the set of synchronous correlations by investigating quantum graph parameters. We unify existing bounds on the quantum chromatic number and the quantum stability number by placing them in the framework of tracial optimization. In particular, we show that the projective packing number, the projective rank, and the tracial rank arise naturally when considering tracial analogues of the Lasserre hierarchy for the stability and chromatic number of a graph. We also introduce semidefinite programming hierarchies converging to the commuting quantum chromatic number and commuting quantum stability number.

1 Introduction

1.1 Bipartite quantum correlations

One of the distinguishing features of quantum mechanics is quantum entanglement, which allows for nonclassical correlations between spatially separated parties. By performing a measurement on their part of an entangled system, the parties – who cannot communicate – can use such correlations to complete tasks that are impossible within classical mechanics. In this paper we consider the problems of quantifying the advantage entanglement can bring and quantifying the minimal amount of entanglement necessary for generating a given correlation. For this we use techniques from tracial polynomial optimization.

Quantum entanglement has been widely studied in the bipartite correlation setting. Here we have two parties, Alice and Bob, where Alice receives a question s from a finite set S and Bob receives a question t from a finite set T. The parties do not know each other's questions, and after receiving the questions they do not communicate. Then, according to some predetermined protocol, Alice returns an answer a from a finite set A and Bob returns an answer b from a finite set B. The probability that the parties answer (a, b) to questions (s, t) is given by

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a *bipartite correlation* $P(a,b|s,t)$, which satisfies $P(a,b|s,t) \geq 0$ for all $(a,b,s,t) \in \Gamma$ and $\sum_{a,b} P(a,b|s,t) = 1$ for all $(s,t) \in S \times T$. Throughout we set $\Gamma = A \times B \times S \times T$.

The bipartite correlations $P = (P(a,b|s,t)) \in \mathbb{R}^\Gamma$ depend on the additional resources that are available to the two parties Aice and Bob. As we discuss below, it is of fundamental importance in quantum information theory that quantum entanglement allows for correlations that are not possible in a classical setting.

If the parties do not have access to any additional resources, then the correlation will be *deterministic*, which means it is of the form $P(a,b|s,t) = P_A(a|s)\, P_B(b|t)$, with $P_A(a|s)$ and $P_B(b|t)$ taking values in $\{0,1\}$ and $\sum_a P_A(a|s) = \sum_b P_B(b|t) = 1$ for all s,t. If the parties have access to local randomness, then P_A and P_B take values in $[0,1]$. If the parties have access to shared randomness (they can draw from a shared random variable), then the resulting correlation will be a convex combination of deterministic correlations, and is said to be a *classical correlation*. The classical correlations form a polytope, denoted by $C_{loc}(\Gamma)$, and valid inequalities for it are known as Bell inequalities [Bel64].

We are interested in the quantum setting, where the parties have access to a shared quantum state upon which they can perform measurements. The quantum setting can be modeled in different ways, leading to the so-called tensor model and commuting model; see the discussion, e.g., in [Tsi06, NPA08, DLTW08].

In the *tensor model*, Alice and Bob each have access to one half of a finite dimensional *quantum state*, which is modeled by a unit vector $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$. Alice and Bob determine their answers by performing a measurement on their part of the state. Such a measurement is modeled by a positive operator valued measure (POVM), which consists of a set of $d \times d$ Hermitian positive semidefinite matrices labeled by the possible answers and summing to the identity matrix. If Alice uses the POVM $\{E_s^a\}_{a \in A}$ when she gets question $s \in S$ and Bob uses the POVM $\{F_t^b\}_{b \in B}$ when he gets question $t \in T$, then the probability of obtaining the answers (a,b) is given by

$$P(a,b|s,t) = \mathrm{Tr}((E_s^a \otimes F_t^b)\psi\psi^*) = \psi^*(E_s^a \otimes F_t^b)\psi. \tag{1}$$

If the state ψ cannot be written as a single tensor product $\psi_A \otimes \psi_B$, then ψ is said to be *entangled*, and this can lead to the above correlation P to be nonclassical.

A correlation of the above form (1) is called a *(tensor) quantum correlation*, and we say it is realizable in the tensor model in *local dimension* d or in *dimension* d^2. Let $C_q^d(\Gamma)$ be the set of quantum correlations realizable in local dimension d, denote the smallest dimension needed to realize the correlation $P \in C_q(\Gamma)$ in the tensor model by

$$D_q(P) = \min\{d^2 : d \in \mathbb{N},\, P \in C_q^d(\Gamma)\}, \tag{2}$$

and define the set

$$C_q(\Gamma) = \bigcup_{d \in \mathbb{N}} C_q^d(\Gamma).$$

The set $C_q(\Gamma)$ is convex, for if $P_1, P_2 \in C_q(\Gamma)$ with $P_i(a,b|s,t) = \psi_i^*(E_s^a(i) \otimes F_t^b(i))\psi_i$ for $i = 1,2$, and if $\lambda \in [0,1]$, then, with $\psi = \sqrt{\lambda}\psi_1 \oplus \sqrt{1-\lambda}\psi_2$, $E_s^a = E_s^a(1) \oplus E_s^a(2)$, and $F_t^b = F_t^b(1) \oplus F_t^b(2)$, we have $(\lambda P_1 + (1-\lambda)P_2)(a,b|s,t) = \psi^*(E_s^a \otimes F_t^b)\psi$, which shows $\lambda P_1 + (1-\lambda)P_2 \in C_q(\Gamma)$.

The set $C_q^1(\Gamma)$ contains the deterministic correlations, so by Carathéodory's theorem $C_{loc}(\Gamma)$ is contained in $C_q^c(\Gamma)$, where c is at most $|A||S| + |B||T| + 1$; that is, quantum entanglement can be used as an alternative to shared randomness. If A, B, S, and T all contain at least two elements, then Bell's theorem says the inclusion $C_{loc}(\Gamma) \subseteq C_q(\Gamma)$ is strict; that is, quantum entanglement can be used to obtain nonclassical correlations [Bel64].

The second model commonly used in quantum information theory to define quantum correlations is the *commuting model* (or *relativistic field theory model*). In this model a correlation $P \in \mathbb{R}^\Gamma$ is called a *commuting quantum correlation* if it is of the form

$$P(a,b|s,t) = \mathrm{Tr}(X_s^a Y_t^b \psi\psi^*) = \psi^*(X_s^a Y_t^b)\psi, \tag{3}$$

where $\{X_s^a\}_a$ and $\{Y_t^b\}_b$ are POVMs consisting of bounded operators on a separable Hilbert space H, satisfying $[X_s^a, Y_t^b] = X_s^a Y_t^b - Y_t^b X_s^a = 0$ for all $(a,b,s,t) \in \Gamma$, and where ψ is a unit vector in H. Such a correlation is said to be realizable in dimension $d = \dim(H)$ in the commuting model, and we denote the set of such correlations by $C_{qc}^d(\Gamma)$ and set $C_{qc}(\Gamma) = C_{qc}^\infty(\Gamma)$. We denote the smallest dimension needed to realize a quantum correlation $P \in C_{qc}(\Gamma)$ by

$$D_{qc}(P) = \min\big\{d \in \mathbb{N} \cup \{\infty\} : P \in C_{qc}^d(\Gamma)\big\}. \tag{4}$$

We have $C_q^d(\Gamma) \subseteq C_{qc}^{d^2}(\Gamma)$, which follows by setting $X_s^a = E_s^a \otimes I$ and $Y_t^b = I \otimes F_t^b$. This shows

$$D_{qc}(P) \leq D_q(P) \quad \text{for all} \quad P \in C_q(\Gamma).$$

The minimum Hilbert space dimension in which a given quantum correlation P can be realized in the tensor or commuting model quantifies the minimal amount of entanglement needed to represent P. Computing the parameter $D_q(P)$ is in fact an NP-hard problem [Sta15]. Hence a natural question is to find good lower bounds for the parameters $D_q(P)$ and $D_{qc}(P)$, and a main contribution of this paper is proposing a hierarchy of semidefinite programming lower bounds for these parameters. A lower bound for $D_q(P)$ based on the notion of fidelity is given in [SVW16].

As said above we have $C_q^d(\Gamma) \subseteq C_{qc}^{d^2}(\Gamma)$. Conversely, each finite dimensional commuting quantum correlation can be realized in the tensor model, although not necessarily in the same dimension [Tsi06] (see, e.g., [DLTW08] for a detailed proof). This shows

$$C_q(\Gamma) = \bigcup_{d \in \mathbb{N}} C_{qc}^d(\Gamma) \subseteq C_{qc}(\Gamma).$$

Whether the two sets $C_q(\Gamma)$ and $C_{qc}(\Gamma)$ coincide is known as Tsirelson's problem. In a recent breakthrough result Slofstra [Slo17] shows that if $|S| = 184$, $|T| = 235$, $|A| = 8$, and $|B| = 2$, then $C_q(\Gamma)$ is not closed. This implies the existence of a sequence $\{P_i\} \subseteq C_q(\Gamma)$ with $D_q(P_i) \to \infty$. Since $C_{qc}(\Gamma)$ is closed [Fri12, Prop. 3.4], this also implies the inclusion $C_q(\Gamma) \subseteq C_{qc}(\Gamma)$ is strict, thus settling Tsirelson's problem. Whether the closure of $C_q(\Gamma)$ equals $C_{qc}(\Gamma)$ is an open problem that is related to an important conjecture in operator theory: We have $\mathrm{cl}(C_q(\Gamma)) = C_{qc}(\Gamma)$ for all Γ if and only if Connes' embedding conjecture holds [JNP+11, Oza12].

Further variations on the above definitions are possible. For instance, we can consider a mixed state ρ (a Hermitian positive semidefinite matrix ρ with $\mathrm{Tr}(\rho) = 1$) instead of a pure state ψ, where we replace the rank 1 matrix $\psi\psi^*$ by ρ in the above definitions. By convexity this does not change the sets $C_q(\Gamma)$ and $C_{qc}(\Gamma)$, but the dimension parameters $D_q(P)$ and $D_{qc}(P)$ can be smaller when allowing mixed states. Another variation would be to use projection valued measures (PVMs) instead of POVMs, where the operators are projectors instead of positive semidefinite matrices. This again does not change the sets $C_q(\Gamma)$ and $C_{qc}(\Gamma)$ [NC00], but the dimension parameters can be larger when restricting to PVMs.

In the rest of the introduction we give a road map through the contents of the paper. We state the main results, which we number according to the section where they will be proved, and we will introduce the necessary background along the way.

1.2 From synchronous correlations to hierarchies

When the two parties have the same question sets ($S = T$) and the same answer sets ($A = B$), a bipartite correlation $P \in \mathbb{R}^\Gamma$ is called *synchronous* if $P(a, b|s, s) = 0$ for all s and $a \neq b$. The sets $C_{q,s}(\Gamma)$ and $C_{qc,s}(\Gamma)$ of synchronous correlations form particularly interesting subsets of bipartite correlations; The quantum graph parameters discussed in Section 1.4 will be defined through optimization problems over these sets. The sets of synchronous correlations are rich enough, so that the above mentioned result about Connes' embedding conjecture still holds when we restrict to synchronous correlations; that is, the conjecture holds if and only if $\mathrm{cl}(C_{q,s}(\Gamma)) = C_{qc,s}(\Gamma)$ for all Γ [DP16, Thm. 3.7].

We show that the minimal local dimension in which a synchronous quantum correlation P can be realized is given by the completely positive semidefinite rank of an associated matrix M_P, indexed by $A \times S$ and defined by

$$(M_P)_{(s,a),(t,b)} = P(a,b|s,t) \quad \text{for all} \quad (a,b,s,t) \in \Gamma.$$

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *completely positive semidefinite* if there exist $d \in \mathbb{N}$ and Hermitian positive semidefinite matrices $X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ such that $M_{ij} = \mathrm{Tr}(X_i X_j)$ for all $i, j \in [n]$. The minimal such d is called the *completely positive semidefinite rank* of M and denoted by $\text{cpsd-rank}_{\mathbb{C}}(M)$. Completely positive semidefinite matrices are investigated in [LP15], motivated by their use to model quantum graph parameters, and the completely positive semidefinite rank in [PSVW16, GdLL17b, PV17, GdLL17a]. To show the following result we combine proofs from [SV17] (see also [MR16]) and [PSS$^+$16]; the proof can be found in the Appendix.

Proposition A.1. *The smallest local dimension in which a synchronous quantum correlation P can be realized is given by* $\text{cpsd-rank}_{\mathbb{C}}(M_P)$.

In [GdLL17a] we use techniques from tracial polynomial optimization to define a semidefinite programming hierarchy of lower bounds $\{\xi_r^{\mathrm{cpsd}}(M)\}_{r\geq 1}$ on $\text{cpsd-rank}_{\mathbb{C}}(M)$. By the above result this hierarchy can be used to obtain lower bounds on the smallest local dimension in which a synchronous correlation can be realized in the tensor model. However, in [GdLL17a] we show that the hierarchy typically does not converge to $\text{cpsd-rank}_{\mathbb{C}}(M)$ but instead (under a certain flatness condition) to a parameter $\xi_*^{\mathrm{cpsd}}(M)$, which can be seen as a block-diagonal version of the completely positive semidefinite rank.

We will use similar techniques to construct a hierarchy $\{\xi_r^{\mathrm{q}}(P)\}_{r\geq 1}$ of lower bounds on the minimal dimension $D_q(P)$ of a quantum correlation $P \in C_q(\Gamma)$. This new hierarchy will have three advantages over the above approach. 1) It works for all correlations and not just for synchronous correlations. 2) The special structure of a quantum correlation allows us to add constraints that strengthen the lower bounds. 3) The hierarchy converges (under flatness) to $\xi_*^{\mathrm{q}}(P)$, and by using the extra constraints mentioned above we can show $\xi_*^{\mathrm{q}}(P)$ is equal to an interesting parameter $A_q(P) \leq D_q(P)$. This parameter describes the minimal average entanglement dimension of a correlation when the parties have free access to shared randomness; see the next section.

1.3 A hierarchy for the average entanglement dimension

We are interested in the minimal entanglement dimension needed to realize a given quantum correlation $P \in C_q(\Gamma)$. If P is deterministic or only uses local randomness, then $D_q(P) = D_{qc}(P) = 1$, but otherwise we have $D_q(P) \geq D_{qc}(P) > 1$. That is, the shared quantum state is used as a shared randomness resource. We define a new parameter $A_q(P) \leq D_q(P)$ that more closely measures the minimal entanglement dimension when the parties have free access to shared randomness, so that $A_q(P) = 1$ if and only if P is classical.

For this we assume that before the game starts the parties select a finite number of pure states ψ_i $(i \in I)$ (instead of a single one), in possibly different dimensions d_i, and POVMs $\{E_s^a(i)\}_a$, $\{F_t^b(i)\}_b$ for each $i \in I$ and $(s,t) \in S \times T$. As before, we assume that the parties cannot communicate after receiving their questions (s,t), but now they do have access to shared randomness, which they use to decide on which state ψ_i to use. The parties proceed to measure state ψ_i using POVMs $\{E_s^a(i)\}_a$, $\{F_t^b(i)\}_b$, so that the probability of answers (a,b) is given by the quantum correlation P_i. We want to know what the minimal average dimension of entanglement needed to reproduce a given correlation P is, which is obtained by minimizing the average dimension $\sum_{i\in I}\lambda_i d_i$ over all convex combinations $P=\sum_{i\in I}\lambda_i P_i$. Hence, in the tensor model the *minimal average entanglement dimension* is given by

$$A_q(P)=\inf\Big\{\sum_{i=1}^{I}\lambda_i D_q(P_i) : I\in\mathbb{N},\, \lambda\in\mathbb{R}_+^I,\, \sum_{i=1}^{I}\lambda_i=1,\, P=\sum_{i=1}^{I}\lambda_i P_i,\, P_i\in C_q(\Gamma)\Big\},$$

and, in the commuting model, $A_{qc}(P)$ is given by the same expression with $D_q(P_i)$ replaced by $D_{qc}(P_i)$. Observe that we need not replace $C_q(\Gamma)$ by $C_{qc}(\Gamma)$ since $D_{qc}(P)=\infty$ for any $P\in C_{qc}(\Gamma)\setminus C_q(\Gamma)$.

It follows by convexity that for the above definitions it does not matter whether we use pure or mixed states. In the following proposition we show that for the average minimal entanglement dimension it also does not matter whether we use the tensor or commuting model.

Proposition 2.1. *For any $P\in C_q(\Gamma)$ we have $A_q(P)=A_{qc}(P)$.*

We have $A_q(P)\le D_q(P)$ and $A_{qc}(P)\le D_{qc}(P)$ for $P\in C_q(\Gamma)$, with equality if P is an extreme point of $C_q(\Gamma)$. Hence, we have $D_q(P)=D_{qc}(P)$ if P is an extreme point of $C_q(\Gamma)$. We show that the parameter $A_q(\cdot)$ can be used to distinguish between classical and nonclassical correlations.

Proposition 2.2. *For a correlation $P\in\mathbb{R}^\Gamma$ we have $A_q(P)=1$ if and only if $P\in C_{loc}(\Gamma)$.*

As mentioned before, Slofstra showed the existence of Γ for which $C_q(\Gamma)$ is not closed, which implies the existence of a sequence $\{P_i\}\subseteq C_q(\Gamma)$ such that $D_q(P)\to\infty$. By the following proposition this also implies the existence of such a sequence with $A_q(P_i)\to\infty$.

Proposition 2.3. *If $C_q(\Gamma)$ is not closed, then there exists $\{P_i\}\subseteq C_q(\Gamma)$ with $A_q(P_i)\to\infty$.*

Using tracial polynomial optimization and building on the techniques from [GdLL17a] we construct a hierarchy of increasingly large optimization problems whose optimal values give increasingly good lower bounds $\{\xi_r^{\mathrm{q}}(P)\}_{r\ge 1}$ on $A_{qc}(P)$. For each $r\in\mathbb{N}$ this is a semidefinite program, and for $r=\infty$ it is an infinite dimensional semidefinite program. We further define a (hyperfinite) variation $\xi_*^{\mathrm{q}}(P)$ of $\xi_\infty^{\mathrm{q}}(P)$ by adding a finite rank constraint, so that

$$\xi_1^{\mathrm{q}}(P)\le\xi_2^{\mathrm{q}}(P)\le\ldots\le\xi_\infty^{\mathrm{q}}(P)\le\xi_*^{\mathrm{q}}(P)\le A_{qc}(P).$$

We do not know whether $\xi_\infty^{\mathrm{q}}(P)=\xi_*^{\mathrm{q}}(P)$ always holds; this question is related to Connes' embedding conjecture [KS08].

First we show that we imposed enough constraints in the bounds $\xi_r^{\mathrm{q}}(P)$ so that $\xi_*^{\mathrm{q}}(P)=A_{qc}(P)$.

Proposition 2.8. *For any $P\in C_q(\Gamma)$ we have $\xi_*^{\mathrm{q}}(P)=A_{\mathrm{qc}}(P)$.*

Then we show that the infinite dimensional semidefinite program $\xi_\infty^{\mathrm{q}}(P)$ is the limit of the finite dimensional semidefinite programs.

Proposition 2.9. *For any $P\in C_q(\Gamma)$ we have $\xi_r^{\mathrm{q}}(P)\to\xi_\infty^{\mathrm{q}}(P)$ as $r\to\infty$.*

Finally we give a criterion under which finite convergence $\xi_r^{\mathrm{q}}(P)=\xi_*^{\mathrm{q}}(P)$ holds. The definition of flatness follows later in the paper; here we only note that it is an easy to check criterion given the output of the semidefinite programming solver.

Proposition 2.10. *If $\xi_r^{\mathrm{q}}(P)$ admits a $(\lceil r/3\rceil+1)$-flat optimal solution, then $\xi_r^{\mathrm{q}}(P)=\xi_*^{\mathrm{q}}(P)$.*

1.4 Quantum graph parameters

Nonlocal games have been introduced in quantum information theory as abstract models to quantify the power of entanglement, in particular, in how much the sets $C_q(\Gamma)$ and $C_{qc}(\Gamma)$ differ from $C_{loc}(\Gamma)$. A *nonlocal game* is defined by a probability distribution $\pi\colon S \times T \to [0,1]$ and a function $f\colon A \times B \times S \times T \to \{0,1\}$, known as the *predicate* of the game, where $f(a,b,s,t) = 0$ means that the answer pair (a,b) is wrong for the question pair (s,t). Alice and Bob receive a question pair $(s,t) \in S \times T$ with probability $\pi(s,t)$. They know the game parameters π and f, but they do not know each other's questions, and they cannot communicate after they receive their questions. Their answers (a,b) are determined according to some correlation $P \in \mathbb{R}^\Gamma$, called their *strategy*, on which they may agree before the start of the game, and which can be classical or quantum depending on whether P belongs to $C_{loc}(\Gamma)$, $C_q(\Gamma)$, or $C_{qc}(\Gamma)$. Then their corresponding winning probability is given by

$$\sum_{(s,t)\in S\times T} \pi(s,t) \sum_{(a,b)\in A\times B} P(a,b|s,t) f(a,b,s,t). \tag{5}$$

A strategy P is called *perfect* if the above winning probability is equal to one, that is, if the probability of giving a wrong answer is zero: for all $(a,b,s,t) \in \Gamma$ we have

$$\pi(s,t) > 0 \quad \text{and} \quad f(a,b,s,t) = 0 \quad \Longrightarrow \quad P(a,b|s,t) = 0.$$

Computing the maximum winning probability of a nonlocal game is an instance of linear optimization over $C_{loc}(\Gamma)$ in the classical setting, and over $C_q(\Gamma)$ or $C_{qc}(\Gamma)$ in the quantum setting. Since the inclusion $C_{loc}(\Gamma) \subseteq C_q(\Gamma)$ can be strict, it is not surprising that the winning probability can be higher when the parties have access to entanglement. Perhaps more surprising is the existence of nonlocal games that can be won with probability 1 when using entanglement, but with optimal winning probability strictly less than 1 in the classical setting.

The quantum graph parameters $\alpha_q(G)$ and $\chi_q(G)$ (and the variants $\alpha_{qc}(G)$ and $\chi_{qc}(G)$) are quantum analogues of the classical *stability number* $\alpha(G)$, which is the size of a largest stable set in a graph G, and the *chromatic number* $\chi(G)$, which is the minimal number of colors needed to color the vertices of G such that no two adjacent vertices have the same color. These quantum graph parameters are defined through the coloring stability number games as described below. These nonlocal games use the set $[k]$ (whose elements are denoted as a,b) and the set V of vertices of G (whose elements are denoted as i,j) as question and answer sets.

In the *quantum coloring game*, introduced in [AHKS06, CMN$^+$07], we are given a graph $G = (V,E)$ and an integer $k \in \mathbb{N}$. We select $S = T = V$ as question sets and $A = B = [k]$ as answer sets. The distribution π is strictly positive for all elements of $V \times V$ (e.g., it is uniform) and the predicate f of the game is such that the players' answers have to be consistent with having a k-coloring of G, that is, $f(a,b,i,j) = 0$ precisely when ($i = j$ and $a \neq b$) or ($\{i,j\} \in E$ and $a = b$). This expresses the fact that if Alice and Bob receive the same vertex they should return the same color and if they receive adjacent vertices they should return distinct colors. A perfect classical strategy exists if and only if a perfect deterministic strategy exists, and a perfect deterministic strategy corresponds to a k-coloring of G. Hence the smallest number k of colors for which there exists a perfect classical strategy $P \in C_{loc}(\Gamma)$ is equal to the classical chromatic number $\chi(G)$. It is therefore natural to define the *quantum chromatic number* $\chi_q(G)$ (resp., the *commuting quantum chromatic number* $\chi_{qc}(G)$) as the smallest k for which there exists a perfect (resp., commuting) quantum strategy $P \in C_q(\Gamma)$ (resp., $P \in C_{qc}(\Gamma)$), where $\Gamma = [k]^2 \times V^2$. Note that such a strategy P is necessarily synchronous. In other words:

Definition 1.1. *The (commuting) quantum chromatic number $\chi_q(G)$ (resp., $\chi_{qc}(G)$) is the smallest integer $k \in \mathbb{N}$ for which there exists a synchronous correlation $P = (P(a,b|i,j))$ in $C_{q,s}([k]^2 \times V^2)$ (resp., $C_{qc,s}([k]^2 \times V^2)$) such that*

$$P(a,a|i,j) = 0 \quad \textit{for all} \quad a \in [k], \{i,j\} \in E.$$

In the *quantum stability number game*, introduced in [MR16, Rob13], we again have a graph $G = (V, E)$ and $k \in \mathbb{N}$, but now we use the question set $[k] \times [k]$ and the answer set $V \times V$. The distribution π is again strictly positive on the question set and now the predicate f of the game is such that the players' answers have to be consistent with having a stable set of size k, that is, $f(i, j, a, b) = 0$ precisely when ($a = b$ and $i \neq j$) or ($a \neq b$ and ($i = j$ or $\{i, j\} \in E$)). This expresses the fact that if Alice and Bob receive the same index $a = b \in [k]$ they should answer with the same vertex $i = j$ of G and if they receive distinct indices $a \neq b$ from $[k]$ they should answer with distinct nonadjacent vertices i and j of G. There is a perfect classical strategy precisely when there exists a stable set of size k, so that the largest integer k for which there exists a perfect classical strategy is equal to the stability number $\alpha(G)$. The largest integer k for which there exists a perfect quantum strategy $P \in C_q(\Gamma)$ (resp., $C_{qc}(\Gamma)$) is the *(commuting) quantum stability number* $\alpha_q(G)$ (resp., $\alpha_{qc}(G)$), where we now have $\Gamma = V^2 \times [k]^2$. Again, a perfect strategy P must be synchronous. In other words:

Definition 1.2. *The (commuting) stability number $\alpha_q(G)$ (resp., $\alpha_{qc}(G)$) is the largest integer $k \in \mathbb{N}$ for which there exists a synchronous correlation $P = (P(i, j|a, b))$ in $C_{q,s}(V^2 \times [k]^2)$ (resp., $C_{qc,s}(V^2 \times [k]^2)$) such that*

$$P(i, j|a, b) = 0 \quad \textit{whenever} \quad (i = j \textit{ or } \{i, j\} \in E) \textit{ and } a \neq b \in [k].$$

As is well known, the classical parameters $\chi(G)$ and $\alpha(G)$ are NP-hard to compute. The same holds for the quantum coloring number $\chi_q(G)$ [Ji13] and also for the quantum stability number $\alpha_q(G)$, in view of the following reduction to coloring shown in [MR16]:

$$\chi_q(G) = \min\{k \in \mathbb{N} : \alpha_q(G \square K_k) = |V|\}. \tag{6}$$

Here $G \square K_k$ is the Cartesian product of the graph $G = (V, E)$ and the complete graph K_k. Note that (6) extends to the quantum setting the analogous well-known reduction for the classical parameters. By construction we have $\chi_{qc}(G) \leq \chi_q(G) \leq \chi(G)$ and $\alpha(G) \leq \alpha_q(G) \leq \alpha_{qc}(G)$. Interestingly, the separation between $\chi_q(G)$ and $\chi(G)$, and between $\alpha_q(G)$ and $\alpha(G)$, can be exponentially large in the number of vertices; This is the case for the graphs G_n with vertex set $V = \{\pm 1\}^n$ for n a multiple of 4, where two vertices $x, y \in V$ are adjacent if they are orthogonal [AHKS06, MR16, MSS13].

By definition, the parameters $\alpha_q(G)$ and $\chi_q(G)$ involve synchronous quantum correlations, while the parameters $\alpha_{qc}(G)$ and $\chi_{qc}(G)$ involve synchronous commuting quantum correlations. It is not known whether there is a separation between the parameters $\chi_q(G)$ and $\chi_{qc}(G)$, and between $\alpha_q(G)$ and $\alpha_{qc}(G)$. A motivation for studying both versions of the games lies in the fact that it it is not known whether the two sets $C_{q,s}(\Gamma)$ and $C_{qc,s}(\Gamma)$ coincide, where $\Gamma = A^2 \times S^2$ for finite sets A and S. In the asynchronous setting, as already mentioned earlier, this has recently been settled by Slofstra [Slo17]: there exists a $\Gamma = A \times B \times S \times T$ for which $C_q(\Gamma) \neq C_{qc}(\Gamma)$.

A second motivation is the study of the following lower bounds on the (commuting) quantum chromatic number: the projective rank $\xi_f(G)$ [MR16] and the tracial rank $\xi_{tr}(G)$ [PSS$^+$16]. Recently it has been shown in [DP16, Cor. 3.10] that the projective rank and tracial rank coincide if Connes' embedding conjecture is true. In Section 3 we provide a hierarchy of semidefinite programming bounds $\{\xi_r^{\mathrm{col}}(G)\}_r$ that asymptotically converges to the tracial rank, and has finite convergence to the projective rank if a certain 'flatness' condition holds.

We now give an overview of the results of Section 3 and refer to that section for formal definitions. In Section 3.1.1 we reformulate the quantum graph parameters in terms of C^*-algebras, using a reformulation from [PSS$^+$16] for quantum synchronous correlations in terms of C^*-algebras. We then use this in Section 3.1.2 to express the quantum graph parameters in terms of positive tracial linear forms, which allows us to use techniques from tracial polynomial optimization to formulate bounds on the quantum graph parameters. In particular, we define a hierarchy $\{\gamma_r^{\rm col}(G)\}_{r\in\mathbb{N}\cup\{\infty\}}$ of semidefinite programming lower bounds on the commuting quantum chromatic number. We moreover define the parameter $\gamma_*^{\rm col}(G)$ as $\gamma_\infty^{\rm col}(G)$ with an additional rank constraint on the matrix variable. Similarly, we define a hierarchy $\{\gamma_r^{\rm stab}(G)\}_{r\in\mathbb{N}\cup\{\infty\}}$ of upper bounds on the commuting quantum stability number, and the corresponding parameter $\gamma_*^{\rm stab}(G)$. We show the following convergence results for these hierarchies.

Lemma 3.2. *Let G be a graph. There exists an $r_0 \in \mathbb{N}$ such that $\gamma_r^{\rm col}(G) = \chi_{qc}(G)$ and $\gamma_r^{\rm stab}(G) = \alpha_{qc}(G)$ for all $r \geq r_0$. Moreover, if $\gamma_r^{\rm col}(G)$ admits a flat optimal solution, then $\gamma_r^{\rm col}(G) = \chi_q(G)$, and similarly if $\gamma_r^{\rm stab}(G)$ admits a flat optimal solution, then $\gamma_r^{\rm stab}(G) = \alpha_q(G)$.*

Then, in Section 3.2, we use tracial analogues of Lasserre type bounds on $\alpha(G)$ and $\chi(G)$ to obtain hierarchies of semidefinite programming bounds for their quantum analogues, which are more economical than the bounds $\gamma_r^{\rm col}(G)$ and $\gamma_r^{\rm stab}(G)$ (since they use less variables) and also permit to recover some known bounds for the quantum parameters. The classical stability number $\alpha(G)$ has a natural formulation as a polynomial optimization problem. Applying the standard Lasserre hierarchy [Las01] to that problem gives a hierarchy $\{\text{las}_r^{\rm stab}(G)\}_{r\in\mathbb{N}\cup\{\infty\}}$ of upper bounds on the stability number. We define the tracial analogue $\xi_r^{\rm stab}(G)$ of $\text{las}_r^{\rm stab}(G)$ for $r \in \mathbb{N}\cup\{\infty\}$ and the corresponding parameter $\xi_*^{\rm stab}(G)$. We show that $\xi_*^{\rm stab}(G)$ coincides with the projective packing number $\alpha_p(G)$ and that $\xi_\infty^{\rm stab}(G)$ upper bounds $\alpha_{qc}(G)$.

Proposition 3.3. *We have $\xi_*^{\rm stab}(G) = \alpha_p(G) \geq \alpha_q(G)$ and $\xi_\infty^{\rm stab}(G) \geq \alpha_{qc}(G)$.*

Next, we consider the chromatic number. A Lasserre-type hierarchy $\{\text{las}_r^{\rm col}(G)\}_{r\in\mathbb{N}\cup\{\infty\}}$ of semidefinite programming lower bounds on the chromatic number $\chi(G)$ is defined in [GL08b]. We again consider the tracial analogue $\xi_r^{\rm col}(G)$ of $\text{las}_r^{\rm col}(G)$ for $r \in \mathbb{N}\cup\{\infty\}$ and the corresponding parameter $\xi_*^{\rm col}(G)$. The tracial hierarchy $\{\xi_r^{\rm col}(G)\}$ unifies two known bounds: the projective rank $\xi_f(G)$, a lower bound on the quantum chromatic number [MR16]; and the tracial rank $\xi_{tr}(G)$, a lower bound on the commuting chromatic number [PSS+16].

Proposition 3.5. *We have $\xi_*^{\rm col}(G) = \xi_f(G) \leq \chi_q(G)$ and $\xi_\infty^{\rm col}(G) = \xi_{tr}(G) \leq \chi_{qc}(G)$.*

After that we show $\xi_r^{\rm stab}(G)\xi_r^{\rm col}(G) \geq |V|$, with equality if G is vertex-transitive; this extends the corresponding known result for the commutative parameters (cf. Section 3.2.3). The bounds of order 1 correspond to the well-known theta number: $\xi_1^{\rm stab}(G) = \vartheta(G)$ and $\xi_1^{\rm col}(G) = \vartheta(\overline{G})$, and we point out the relation between $\xi_2^{\rm col}(G)$ and the semidefinite programming bound $\xi_{\rm SDP}(G)$ from [PSS+16] (cf. Section 3.2.4).

In Section 3.3, we compare the hierarchies $\xi_r^{\rm col}(G)$ and $\gamma_r^{\rm col}(G)$, and the hierarchies $\xi_r^{\rm stab}(G)$ and $\gamma_r^{\rm stab}(G)$. For the coloring parameters, the analogue of reduction (6) applies to the semidefinite programming bounds.

Proposition 3.9. *For $r \in \mathbb{N}\cup\{\infty\}$ we have $\gamma_r^{\rm col}(G) = \min\{k : \xi_r^{\rm stab}(G\square K_k) = |V|\}$.*

An analogous statement holds for the stability parameters, when using the homomorphic graph product of K_k with the complement of G, denoted here as $K_k \star G$, and the following reduction shown in [MR16]:

$$\alpha_q(G) = \max\{k \in \mathbb{N} : \alpha_q(K_k \star G) = k\}.$$

We show the following result for the corresponding semidefinite programming bounds.

Proposition 3.10. *For $r \in \mathbb{N}\cup\{\infty\}$ we have $\gamma_r^{\rm stab}(G) = \max\{k : \xi_r^{\rm stab}(K_k \star G) = k\}$.*

Finally, we show that the hierarchies $\{\gamma_r^{\rm col}(G)\}$ and $\{\gamma_r^{\rm stab}(G)\}$ refine the hierarchies $\{\xi_r^{\rm col}(G)\}$ and $\{\xi_r^{\rm stab}(G)\}$.

Proposition 3.11. *For $r \in \mathbb{N}\cup\{\infty, *\}$ we have $\xi_r^{\rm col}(G) \leq \gamma_r^{\rm col}(G)$ and $\xi_r^{\rm stab}(G) \geq \gamma_r^{\rm stab}(G)$.*

1.5 Techniques from noncommutative polynomial optimization

To derive our bounds we use techniques from tracial polynomial optimization. This is a noncommutative extension of the widely used moment and sum-of-squares techniques from Lasserre [Las01] and Parrilo [Par00] in polynomial optimization, dealing with the problem of minimizing a multivariate polynomial function $f(x_1, \ldots, x_n)$ over a feasible region defined by polynomial inequalities $g(x_1, \ldots, x_n) \geq 0$ (for $g \in \mathcal{G} \subseteq \mathbb{R}[x_1, \ldots, x_n]$). These techniques have been adapted to the noncommutative setting in [NPA08] and [DLTW08] for approximating the set $C_{qc}(\Gamma)$ of commuting quantum correlations and the winning probability of nonlocal games over $C_{qc}(\Gamma)$ (and, more generally, computing Bell inequality violations). In [PNA10, NPA12] this approach has been extended to the general eigenvalue optimization problem, of the form

$$\inf\big\{\psi^* f(X_1, \ldots, X_n)\psi : d \in \mathbb{N},\, \psi \in \mathbb{C}^d \text{ unit vector},\, X_1, \ldots, X_n \in \mathbb{C}^{d\times d}, \\ g(X_1, \ldots, X_n) \succeq 0 \text{ for } g \in \mathcal{G}\big\}.$$

Here, the matrix variables X_i have free dimension $d \in \mathbb{N}$ and $\{f\} \cup \mathcal{G} \subseteq \mathbb{R}\langle x_1, \ldots, x_n\rangle$ is a set of symmetric polynomials in noncommutative variables. In tracial optimization, instead of minimizing the smallest eigenvalue of $f(X_1, \ldots, X_n)$, we minimize its normalized trace $\mathrm{Tr}(f(X_1, \ldots, X_n))/d$ (so that the identity matrix has trace one) [BK12, BCKP13, BKP16, KP16]. The moment approach for these problems relies on minimizing $L(f)$, where L is a linear functional on the space of noncommutative polynomials satisfying some necessary conditions, so that $L(f)$ models either $\psi^* f(X_1, \ldots, X_n)\psi$ or $\mathrm{Tr}(f(X_1, \ldots, X_n))/d$. By truncating the degrees one gets hierarchies of lower bounds for the original problem. By the GNS construction, the asymptotic limit of these bounds involves operators X_i on a Hilbert space (possibly with infinite dimension). In tracial optimization this leads to allowing solutions X_i in a C^*-algebra $\mathcal{A}$ equipped with a tracial state τ, so that $\tau(f(X_1, \ldots, X_n))$ is minimized.

In [PSS$^+$16] hierarchies of outer approximations $\{\mathcal{Q}_r(\Gamma)\}$ for the set $C_{qc}(\Gamma)$ of commuting quantum correlations are constructed and used to derive semidefinite programming bounds converging to the commuting quantum coloring number $\chi_{qc}(G)$. They are based on the eigenvalue optimization approach, applied to the formulation (3) of commuting quantum correlations. In this paper we construct new hierarchies of semidefinite programming bounds for $\chi_{qc}(G)$ and $\alpha_{qc}(G)$, exploiting the fact that these parameters are defined in terms of *synchronous* correlations and the fact (from [PSS$^+$16]) that such correlations admit a reformulation in terms of C^*-algebras with a tracial state. So our bounds are based on tracial optimization and they use less variables, roughly speaking they involve only the variables $\{x_s^a\}$ while the previous bounds of [PSS$^+$16] use the larger set of variables $\{x_s^a, y_t^b\}$.

An important feature in noncommutative optimization is the dimension independence: the optimization is over all possible matrix sizes $d \in \mathbb{N}$. In some applications one may want to restrict to optimizing over matrices with restricted size d. In [NV15, NFAV15] techniques are developed that allow to incorporate this dimension restriction by suitably selecting the linear functionals L in a specified space; this is used to give bounds on the maximum violation of a Bell inequality that can be achieved in a fixed dimension. A related natural problem is to decide what is the minimum dimension d needed to realize a given algebraically defined object, like a (commuting) quantum correlation P. We propose an approach based on tracial optimization: starting from the observation that the trace of the $d \times d$ identity matrix gives its size d, we consider the problem of minimizing $L(1)$ where L is a linear functional modeling the non-normalized matrix trace. This approach has been developed in the recent work [GdLL17a] for the problem of finding smallest matrix factorization ranks: Given a nonnegative matrix $M \in \mathbb{R}^{m\times n}$, the smallest dimension d for which there exist Hermitian positive semidefinite matrices X_i, Y_j so that $M = (\mathrm{Tr}(X_i Y_j))_{i\in[m], j\in[n]}$ is called the positive semidefinite rank of M; when $m = n$ and we restrict to using the same factors $X_i = Y_i$ the analogous parameter is called the completely positive semidefinite rank. Semidefinite programming bounds are constructed in [GdLL17a] for these matrix factorization ranks (and for their commutative analogues, where all factors are diagonal matrices: the nonnegative rank and the completely positive rank). Similar ideas are used here to derive semidefinite programming bounds for the minimum dimension parameters $D_q(P), D_{qc}(P)$ considered in this paper.

2 A hierarchy for the minimal entanglement dimension

2.1 The minimal average entanglement dimension

We start by showing that it does not matter whether we use the tensor or the commuting model when defining the average entanglement dimension.

Proposition 2.1. *For any $P \in C_q(\Gamma)$ we have $A_q(P) = A_{qc}(P)$.*

Proof. The easy inequality $A_{qc}(P) \leq A_q(P)$ follows from the identity $E_s^a \otimes F_t^b = (E_s^a \otimes I)(I \otimes F_t^b)$.

For the other inequality we suppose $P = \sum_{i=1}^I \lambda_i P_i$ is feasible for $A_{qc}(P)$. This means we have POVMs $\{X_s^a(i)\}_a$ and $\{Y_t^b(i)\}_b$ in $\mathbb{C}^{d_i \times d_i}$ with $[X_s^a(i), Y_t^b(i)] = 0$ and unit vectors $\psi_i \in \mathbb{C}^{d_i}$ such that $P_i(a,b|s,t) = \psi_i^* X_s^a(i) Y_t^b(i) \psi_i$ for all $(a,b,s,t) \in \Gamma$ and $i \in [I]$. We will construct a feasible solution to $A_q(P)$ with value at most $\sum_i \lambda_i d_i$, thus showing $A_q(P) \leq A_{qc}(P)$.

Fix some index $i \in [I]$. By Artin-Wedderburn theory applied to $\mathbb{C}\langle \{X_s^a(i)\}_{a,s}\rangle$, the $*$-algebra generated by the matrices $X_s^a(i)$ with $(a,s) \in A \times S$, there exists a unitary matrix U_i and integers K_i, m_k, n_k such that

$$U_i \mathbb{C}\langle \{X_s^a(i)\}_{a,s}\rangle U_i^* = \bigoplus_{k=1}^{K_i} (\mathbb{C}^{n_k \times n_k} \otimes I_{m_k}) \quad \text{and} \quad d_i = \sum_{k=1}^{K_i} m_k n_k.$$

By the commutation relations each matrix $Y_t^b(i)$ commutes with all matrices in $\mathbb{C}\langle \{X_s^a(i)\}_{a,s}\rangle$, and thus $U_i Y_t^b(i) U_i^*$ lies in the algebra $\bigoplus_k (I_{n_k} \otimes \mathbb{C}^{m_k \times m_k})$. Hence, we may assume

$$X_s^a(i) = \bigoplus_{k=1}^{K_i} E_s^a(i,k) \otimes I_{m_k}, \quad Y_t^b(i) = \bigoplus_{k=1}^{K_i} I_{n_k} \otimes F_t^b(i,k), \quad \psi_i = \bigoplus_{k=1}^{K_i} \psi_{i,k},$$

with $E_s^a(i,k) \in \mathbb{C}^{n_k \times n_k}$, $F_t^b(i,k) \in \mathbb{C}^{m_k \times m_k}$, and $\psi_{i,k} \in \mathbb{C}^{m_k n_k}$. Then we have

$$P_i(a,b|s,t) = \mathrm{Tr}(X_s^a(i) Y_t^b(i) \psi_i \psi_i^*) = \sum_k \|\psi_{i,k}\|^2 \underbrace{\mathrm{Tr}\left(E_s^a(i,k) \otimes F_t^b(i,k) \frac{\psi_{i,k}\psi_{i,k}^*}{\|\psi_{i,k}\|^2}\right)}_{Q_{i,k}(a,b|s,t)},$$

where $Q_{i,k} \in C_q(\Gamma)$. As $\sum_k \|\psi_{i,k}\|^2 = \|\psi_i\|^2 = 1$, $P_i = \sum_k \|\psi_{i,k}\|^2 Q_{i,k}$ is a convex combination.

We now show that $Q_{i,k} \in C_q^{\min\{m_k,n_k\}}(\Gamma)$. For this consider the Schmidt decomposition

$$\psi_{i,k}/\|\psi_{i,k}\| = \sum_{l=1}^{\min\{m_k,n_k\}} \lambda_{i,k,l}\, v_{i,k,l} \otimes w_{i,k,l},$$

where $\{v_{i,k,l}\}_{l=1}^{n_k} \subseteq \mathbb{C}^{n_k}$ and $\{w_{i,k,l}\}_{l=1}^{m_k} \subseteq \mathbb{C}^{m_k}$ are orthonormal bases, and $\lambda_{i,k,l} \geq 0$. Define unitary matrices $V_k \in \mathbb{C}^{n_k \times n_k}$ and $W_k \in \mathbb{C}^{m_k \times m_k}$ such that $V_k v_{i,k,l}$ is the lth unit vector in $\mathbb{R}^{n_k}$ and $W_k w_{i,k,l}$ is the lth unit vector in $\mathbb{R}^{m_k}$ for $l \leq \min\{m_k, n_k\}$. Let $E_s^a(i,k)'$ (resp., $F_t^b(i,k)'$) be the leading principal submatrices of $V_k E_s^a(i,k) V_k^*$ (resp., $W_k F_t^b(i,k) W_k^*$) of size $\min\{m_k, n_k\}$. Moreover, set $\phi_{i,k} = \sum_{l=1}^{\min\{m_k,n_k\}} \lambda_{i,k,l} e_l \otimes e_l$, where e_l is the lth unit vector in $\mathbb{R}^{\min\{m_k,n_k\}}$.

Then

$$\begin{aligned}Q_{i,k}(a,b|s,t) &= \mathrm{Tr}\left(E_s^a(i,k)\otimes F_t^b(i,k)\frac{\psi_{i,k}\psi_{i,k}^*}{\|\psi_{i,k}\|^2}\right)\\ &= \sum_{l,l'=1}^{\min\{m_k,n_k\}} \lambda_{i,k,l}\lambda_{i,k,l'}v_{i,k,l}^*E_s^a(i,k)v_{i,k,l'}w_{i,k,l}^*F_t^b(i,k)w_{i,k,l'}\\ &= \sum_{l,l'=1}^{\min\{m_k,n_k\}} \lambda_{i,k,l}\lambda_{i,k,l'}e_l^*E_s^a(i,k)'e_{l'}e_l^*F_t^b(i,k)'e_{l'}\\ &= \mathrm{Tr}((E_s^a(i,k)'\otimes F_t^b(i,k)')\phi_{i,k}\phi_{i,k}^*),\end{aligned}$$

thus showing $Q_{i,k}\in C_q^{\min\{m_k,n_k\}}(\Gamma)$. From the convex decomposition $P=\sum_{i,k}\lambda_i\|\psi_{i,k}\|^2Q_{i,k}$, we obtain

$$A_q(P)\le\sum_{i,k}\lambda_i\|\psi_{i,k}\|^2\min\{m_k,n_k\}^2\le\sum_{i,k}\lambda_i\min\{m_k,n_k\}^2\le\sum_{i,k}\lambda_im_kn_k=\sum_i\lambda_id_i,$$

which completes the proof. □

We now show that the parameter $A_q(\cdot)$ permits to characterize classical correlations.

Proposition 2.2. *For a correlation $P\in\mathbb{R}^\Gamma$ we have $A_q(P)=1$ if and only if $P\in C_{loc}(\Gamma)$.*

Proof. If $P\in C_{loc}(\Gamma)$, then P can be written as a convex combination of deterministic correlations (which are contained in $C_q^1(\Gamma)$), hence $A_q(P)=1$.

On the other hand, if $A_q(P)=1$, then there exist convex decompositions indexed by $l\in\mathbb{N}$:

$$P=\sum_{i\in I^l}\lambda_i^lP_i^l\quad\text{with}\quad\{P_i^l\}\subseteq C_q(\Gamma)\quad\text{and}\quad\lim_{l\to\infty}\sum_{i\in I^l}\lambda_lD_q(P_i^l)=1.$$

Decompose I^l as the disjoint union $I_-^l\cup I_+^l$ so that $D_q(P_i)$ is equal to 1 for $i\in I_-^l$ and strictly greater than 1 for $i\in I_+^l$. Let $\varepsilon>0$. For all l sufficiently large we have

$$\big(1-\sum_{i\in I_+^l}\lambda_i^l\big)+2\sum_{i\in I_+^l}\lambda_i^l\le\sum_{i\in I_-^l}\lambda_i^l+\sum_{i\in I_+^l}\lambda_i^lD_q(P_i^l)\le1+\varepsilon,$$

which shows that $\sum_{i\in I_+^l}\lambda_i^l\le\varepsilon$. This shows that P is the limit of convex combinations of deterministic correlations, which implies that $P\in C_{loc}(\Gamma)$. □

Proposition 2.3. *If $C_q(\Gamma)$ is not closed, then there exists $\{P_i\}\subseteq C_q(\Gamma)$ with $A_q(P_i)\to\infty$.*

Proof. Assume for contradiction that there exists an integer K such that $A_q(P)<K$ for all $P\in C_q(\Gamma)$. We will show this results in a uniform upper bound on $D_q(P)$ for $P\in C_q(\Gamma)$, which implies $C_q(\Gamma)$ is closed. For this we first observe that any $P\in C_q(\Gamma)$ can be decomposed as

$$P=\mu_1R_1+(1-\mu_1)Q_1, \tag{7}$$

where $R_1\in C_q(\Gamma)$, $Q_1\in\mathrm{conv}(C_q^K(\Gamma))$, and $\mu_1\le K/(K+1)$. Indeed, by assumption, P can be written as a convex combination

$$P=\sum_{i\in I}\lambda_iP_i\quad\text{with}\quad\{P_i\}\subseteq C_q(\Gamma)\quad\text{and}\quad\sum_{i\in I}\lambda_iD_q(P_i)\le K.$$

We can decompose I as the disjoint union $I_- \cup I_+$ so that $D_q(P_i)$ is at most K for $i \in I_-$ and at least $K+1$ for $i \in I_+$. Then,

$$(K+1)\sum_{i\in I_+}\lambda_i \leq \sum_{i\in I_+}\lambda_i D_q(P_i) \leq K,$$

and thus $\mu_1 := \sum_{i\in I_+}\lambda_i \leq K/(K+1)$. Hence (7) holds after setting $R_1 = (\sum_{i\in I_+}\lambda_i P_i)/\mu_1$ and $Q_1 = (\sum_{i\in I_-}\lambda_i P_i)/(1-\mu_1)$.

By repeating the same argument for R_1 and iterating we obtain for each integer $k \in \mathbb{N}$ a decomposition

$$P = \mu_1\mu_2\cdots\mu_k R_k + \underbrace{(1-\mu_1)Q_1 + \mu_1(1-\mu_2)Q_2 + \ldots + \mu_1\mu_2\cdots\mu_{k-1}(1-\mu_k)Q_k}_{=(1-\mu_1\mu_2\cdots\mu_k)\hat{Q}_k},$$

where $R_k \in C_q(\Gamma)$, $\hat{Q}_k \in \mathrm{conv}(C_q^K(\Gamma))$ and $\mu_1\mu_2\cdots\mu_k \leq (K/(K+1))^k$. As the entries of R_k lie in $[0,1]$ we can conclude that $\mu_1\mu_2\cdots\mu_k R_k$ tends to 0 as $k\to\infty$. Hence the sequence $(\hat{Q}_k)_k$ has a limit $\hat{Q}$ and $P = \hat{Q}$ holds. As all $\hat{Q}_k$ lie in the compact set $\mathrm{conv}(C_q^K(\Gamma))$, we also have $P \in \mathrm{conv}(C_q^K(\Gamma))$. The extreme points of the compact convex set $\mathrm{conv}(C_q^K(\Gamma))$ lie in $C_q^K(\Gamma)$, so, by the Carathéodory theorem, $P \in \mathrm{conv}(C_q^K(\Gamma))$ is a convex combination of at most c elements from $C_q^K(\Gamma)$ where c is at most $|A||S|+|B||T|+1$. By a direct sum construction (see Section 1.1) we then obtain $D_q(P) \leq cK$. □

2.2 Setup of the hierarchy

We will now construct a hierarchy of lower bounds on the minimal entanglement dimension, using its formulation via $A_{qc}(P)$. Our approach is based on noncommutative polynomial optimization, thus similar to the approach in [GdLL17a] for bounding matrix factorization ranks.

We first need some notation. Set

$$\mathbf{x} = \{x_s^a : (a,s) \in A\times S\} \quad \text{and} \quad \mathbf{y} = \{y_t^b : (b,t) \in B\times T\},$$

and let $\langle \mathbf{x},\mathbf{y},z\rangle_r$ be the set of all words of length at most r in the $n = |S||A|+|T||B|+1$ symbols x_s^a, y_t^b, and z. Moreover, set $\langle \mathbf{x},\mathbf{y},z\rangle = \langle \mathbf{x},\mathbf{y},z\rangle_\infty$. We equip $\langle \mathbf{x},\mathbf{y},z\rangle_r$ with an involution $w\mapsto w^*$ that reverses the order of the symbols in the words and leaves the symbols x_s^a, y_t^b, z invariant; e.g., $(x_s^a z)^* = z x_s^a$. Let $\mathbb{R}\langle \mathbf{x},\mathbf{y},z\rangle_r$ be the vector space of all real linear combinations of the words of length (aka degree) at most r. The space $\mathbb{R}\langle \mathbf{x},\mathbf{y},z\rangle = \mathbb{R}\langle \mathbf{x},\mathbf{y},z\rangle_\infty$ is the $*$-algebra with Hermitian generators $\{x_s^a\}$, $\{y_t^b\}$, and z, and the elements in this algebra are called *noncommutative polynomials* in the variables $\{x_s^a\}, \{y_t^b\}, z$.

The hierarchy is based on the following idea: For any feasible solution to $A_{qc}(P)$, its objective value can be modeled as $L(1)$ for a certain tracial linear form L on the space of noncommutative polynomials (truncated to degree $2r$).

Indeed, assume $\{(P_i,\lambda_i)_i\}$ is a feasible solution to the program $A_{qc}(P)$ defined in Section 1.3, where $P_i(a,b|s,t) = \mathrm{Tr}\big(X_s^a(i)Y_t^b(i)\psi_i\psi_i^*\big)$ with $X_s^a(i), Y_t^b(i) \in \mathbb{C}^{d_i\times d_i}, \psi_i \in \mathbb{C}^{d_i}$, and $d_i = D_{qc}(P_i)$. Fix $r \in \mathbb{N}\cup\{\infty\}$, and consider the linear functional $L \in \mathbb{R}\langle \mathbf{x},\mathbf{y},z\rangle_{2r}^*$ defined by

$$L(p) = \sum_i \lambda_i \,\mathrm{Re}(\mathrm{Tr}(p(\mathbf{X}(i),\mathbf{Y}(i),\psi_i\psi_i^*))) \quad \text{for} \quad p \in \mathbb{R}\langle \mathbf{x},\mathbf{y},z\rangle_{2r}.$$

Here, for each index i, we set $\mathbf{X}(i) = (X_s^a(i) : (a,s) \in A\times S)$, $\mathbf{Y}(i) = (Y_t^b(i) : (b,t) \in B\times T)$, and replace the variables x_s^a, y_t^b, z by $X_s^a(i)$, $Y_t^b(i)$, and $\psi_i\psi_i^*$. Then $L(1) = \sum_i \lambda_i d_i$. That is, $L(1)$ is the objective value of the feasible solution $\{(P_i,\lambda_i)_i\}$ to $A_{qc}(P)$. We will now identify several computationally tractable properties that this linear functional L satisfies. Then the hierarchy consists of optimization problems where we minimize $L(1)$ over the set of linear functionals that satisfy these specified properties, which will result in a hierarchy of lower bounds on $A_{qc}(P)$.

First note that L is *symmetric*, that is, $L(w) = L(w^*)$ for all $w \in \langle \mathbf{x}, \mathbf{y}, z\rangle_{2r}$, and *tracial*, that is, $L(ww') = L(w'w)$ for all $w, w' \in \langle \mathbf{x}, \mathbf{y}, z\rangle$ with $\deg(ww') \leq 2r$.

For all $p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle_{r-1}$ we have

$$L(p^* x_s^a p) = \sum_i \lambda_i \operatorname{Re}(\operatorname{Tr}(M(i)^* X_s^a(i) M(i)), \quad \text{where} \quad M(i) = p(\mathbf{X}(i), \mathbf{Y}(i), \psi_i \psi_i^*).$$

Since $X_s^a(i)$ is a positive semidefinite matrix, $M(i)^* X_s^a(i) M(i)$ is positive semidefinite too, and thus we have $L(p^* x_s^a p) \geq 0$. In the same way we have $L(p^* y_t^b p) \geq 0$ and $L(p^* z p) \geq 0$. That is, if we set

$$\mathcal{G} = \big\{x_s^a : s \in S,\, a \in A\big\} \cup \big\{y_t^b : t \in T,\, b \in B\big\} \cup \{z\},$$

then L is nonnegative (denoted as $L \geq 0$) on the *truncated quadratic module*

$$\mathcal{M}_{2r}(\mathcal{G}) = \operatorname{cone}\Big\{p^* g p : p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle,\ g \in \mathcal{G} \cup \{1\},\ \deg(p^* g p) \leq 2r\Big\}. \tag{8}$$

Similarly, setting

$$\mathcal{H} = \big\{z - z^2\big\} \cup \Big\{\Big(1 - \sum_{a \in A} x_s^a\Big) : s \in S\Big\} \cup \Big\{\Big(1 - \sum_{b \in B} y_t^b\Big) : t \in T\Big\} \cup \big\{[x_s^a, y_t^b] : (s,t,a,b) \in \Gamma\big\},$$

we have $L = 0$ on the *truncated ideal*

$$\mathcal{I}_{2r}(\mathcal{H}) = \Big\{ph : p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle,\ h \in \mathcal{H},\ \deg(ph) \leq 2r\Big\}. \tag{9}$$

Moreover, we have $L(z) = \sum_i \lambda_i \operatorname{Re}(\operatorname{Tr}(\psi_i \psi_i^*)) = 1$. In addition, for any matrices $U, V \in \mathbb{C}^{d_i \times d_i}$ we have

$$\psi_i \psi_i^* U \psi_i \psi_i^* V \psi_i \psi_i^* = \psi_i \psi_i^* V \psi_i \psi_i^* U \psi_i \psi_i^*,$$

and therefore, in particular,

$$L(wzuzvz) = L(wzvzuz) \quad \text{for all} \quad u, v, w \in \langle \mathbf{x}, \mathbf{y}, z\rangle \quad \text{with} \quad \deg(wzuzvz) \leq 2r.$$

That is, we have $L = 0$ on $\mathcal{I}_{2r}(\mathcal{R}_r)$, where

$$\mathcal{R}_r = \big\{zuzvz - zvzuz : u, v \in u, v \in \langle \mathbf{x}, \mathbf{y}, z\rangle \text{ with } \deg(zuzvz) \leq 2r\big\}.$$

We get the idea of adding these last constraints from [NPA12], where this is used to study the mutually unbiased bases problem.

We call $\mathcal{M}(\mathcal{G}) = \mathcal{M}_\infty(\mathcal{G})$ the quadratic module generated by $\mathcal{G}$, and we call $\mathcal{I}(\mathcal{H} \cup \mathcal{R}_\infty) = \mathcal{I}_\infty(\mathcal{H} \cup \mathcal{R}_\infty)$ the ideal generated by $\mathcal{H} \cup \mathcal{R}_\infty$.

For $r \in \mathbb{N} \cup \{\infty\}$ we can now define the parameter:

$$\begin{aligned} \xi_r^{\mathrm{q}}(P) = \min\Big\{L(1) : \ & L \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle_{2r}^* \text{ tracial and symmetric}, \\ & L(z) = 1,\ L(x_s^a y_t^b z) = P(a,b|s,t) \text{ for all } (a,b,s,t) \in \Gamma, \\ & L \geq 0 \text{ on } \mathcal{M}_{2r}(\mathcal{G}),\ L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H} \cup \mathcal{R}_r)\Big\}. \end{aligned}$$

Additionally, we define $\xi_*^{\mathrm{q}}(P)$ by adding the constraint $\operatorname{rank}(M(L)) < \infty$ to $\xi_\infty^{\mathrm{q}}(P)$. By construction this gives a hierarchy of lower bounds for $A_{qc}(P)$:

$$\xi_1^{\mathrm{q}}(P) \leq \ldots \leq \xi_r^{\mathrm{q}}(P) \leq \xi_\infty^{\mathrm{q}}(P) \leq \xi_*^{\mathrm{q}}(P) \leq A_{qc}(P).$$

Note that for order $r = 1$ we get the trivial lower bound $\xi_1^{\mathrm{q}}(P) = 1$.

For each finite $r \in \mathbb{N}$ the parameter $\xi_r^{\mathrm{q}}(P)$ can be computed by semidefinite programming. Indeed, the condition $L \geq 0$ on $\mathcal{M}_{2r}(\mathcal{G})$ means that $L(p^* g p) \geq 0$ for all $g \in \mathcal{G} \cup \{1\}$ and all polynomials $p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle$ with degree at most $r - \lceil \deg(g)/2 \rceil$. This is equivalent to requiring that the matrices $(L(w^*w))$, indexed by all words w, w' with degree at most $r - \lceil \deg(g)/2 \rceil$, are positive semidefinite. To see this, write $p = \sum_w p_w w$ and let $\hat{p} = (p_w)$ denote the vector of coefficients, then $L(p^*gp) \geq 0$ is equivalent to $\hat{p}^{\mathsf{T}}(L(w^*gw'))\hat{p} \geq 0$. When $g = 1$, the matrix $(L(w^*w'))$ is indexed by the words of degree at most r, it is called the *moment matrix* of L and denoted by $M_r(L)$ (or $M(L)$ when $r = \infty$). The entries of the matrices $(L(w^*gw'))$ are linear combinations of the entries of $M_r(L)$, and the constraint $L = 0$ on $\mathcal{I}_{2r}(\mathcal{H} \cup \mathcal{R}_r)$ can be written as a set of linear constraints on the entries of $M_r(L)$. It follows that for finite $r \in \mathbb{N}$, the parameter $\xi_r^{\mathrm{q}}(P)$ is indeed computable by a semidefinite program.

2.3 Background on positive tracial linear forms

Before we show the convergence results we give some background on positive tracial linear forms, which we use again in Section 3. We state these results using the variables $x_1, \ldots, x_n$, where we use the notation $\langle \mathbf{x} \rangle = \langle x_1, \ldots, x_n \rangle$. The results stated below do not always appear in this way in the sources cited; we follow the presentation of [GdLL17a], where full proofs for these results are also provided.

First we need a few more definitions. A polynomial $p \in \mathbb{R}\langle \mathbf{x} \rangle$ is called symmetric if $p^* = p$, and we denote the set of symmetric polynomials by $\mathrm{Sym}\,\mathbb{R}\langle \mathbf{x} \rangle$. Given $\mathcal{G} \subseteq \mathrm{Sym}\,\mathbb{R}\langle \mathbf{x} \rangle$ and $\mathcal{H} \subseteq \mathbb{R}\langle \mathbf{x} \rangle$, the set $\mathcal{M}(\mathcal{G}) + \mathcal{I}(\mathcal{H})$ is called *Archimedean* if it contains the polynomial $R - \sum_{i=1}^n x_i^2$ for some $R > 0$. We will use the concept of a C^*-algebra, which for our purposes can be defined as a norm closed $*$-subalgebra of the space $\mathcal{B}(H)$ of bounded operators on a complex Hilbert space H. We say that $\mathcal{A}$ is *unital* if it contains the identity operator (denoted 1). An element $a \in \mathcal{A}$ is called *positive* if $a = b^*b$ for some $b \in \mathcal{A}$. A linear form τ on a unital C^*-algebra $\mathcal{A}$ is said to be a *state* if $\tau(1) = 1$ and τ is positive; that is, $\tau(a) \geq 0$ for all positive elements $a \in \mathcal{A}$. We say that a state τ is tracial if $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. See, for example, [Bla06] for more information on C^*-algebras.

The first result, which relates positive tracial linear forms to C^*-algebras, is due to [NPA12] in the noncommutative setting, and due to [BKP16] in the tracial setting.

Theorem 2.4. *Let $\mathcal{G} \subseteq \mathrm{Sym}\,\mathbb{R}\langle \mathbf{x} \rangle$ and $\mathcal{H} \subseteq \mathbb{R}\langle \mathbf{x} \rangle$ and assume that $\mathcal{M}(\mathcal{G}) + \mathcal{I}(\mathcal{H})$ is Archimedean. For a linear form $L \in \mathbb{R}\langle \mathbf{x} \rangle^*$, the following are equivalent:*

(1) L is symmetric, tracial, nonnegative on $\mathcal{M}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and $L(1) = 1$;

(2) there is a unital C^*-algebra $\mathcal{A}$ with tracial state τ and $\mathbf{X} \in \mathcal{A}^n$ such that $g(\mathbf{X})$ is positive in $\mathcal{A}$ for all $g \in \mathcal{G}$, and $h(\mathbf{X}) = 0$ for all $h \in \mathcal{H}$, with

$$L(p) = \tau(p(\mathbf{X})) \quad \text{for all} \quad p \in \mathbb{R}\langle \mathbf{x} \rangle. \tag{10}$$

The following can be seen as the finite dimensional analogue of the above result. The proof of the unconstrained case ($\mathcal{G} = \mathcal{H} = \emptyset$) can be found in [BK12], and for the constrained case in [BKP16]. Given a linear form $L \in \mathbb{R}\langle \mathbf{x} \rangle^*$, recall that the moment matrix $M(L)$ is given by $M(L)_{u,v} = L(u^*v)$ for $u, v \in \langle \mathbf{x} \rangle$.

Theorem 2.5. *Let $\mathcal{G} \subseteq \mathrm{Sym}\,\mathbb{R}\langle \mathbf{x} \rangle$ and $\mathcal{H} \subseteq \mathbb{R}\langle \mathbf{x} \rangle$. For $L \in \mathbb{R}\langle \mathbf{x} \rangle^*$, the following are equivalent:*

(1) L is a symmetric, tracial, linear form with $L(1) = 1$ that is nonnegative on $\mathcal{M}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and has $\mathrm{rank}(M(L)) < \infty$;

(2) there is a finite dimensional C^*-algebra $\mathcal{A}$ with a tracial state τ and $\mathbf{X} \in \mathcal{A}^n$ satisfying (10), with $g(\mathbf{X})$ positive in $\mathcal{A}$ for all $g \in \mathcal{G}$ and $h(\mathbf{X}) = 0$ for all $h \in \mathcal{H}$;

(3) L is a convex combination of normalized trace evaluations at tuples $\mathbf{X} = (X_1, \ldots, X_n)$ of Hermitian matrices that satisfy $g(\mathbf{X}) \succeq 0$ for all $g \in \mathcal{G}$ and $h(\mathbf{X}) = 0$ for all $h \in \mathcal{H}$.

A truncated linear functional $L \in \mathbb{R}\langle \mathbf{x} \rangle_{2r}$ is *δ-flat* if the principal submatrix $M_{r-\delta}(L)$ of $M_r(L)$ indexed by monomials up to degree $r-\delta$ has the same rank as $M_r(L)$. We call a truncated linear functional *flat* if it is δ-flat for some $\delta \geq 1$. The following result claims that any *flat* linear functional on a truncated polynomial space can be extended to a linear functional L on the full algebra of polynomials. It is due to Curto and Fialkow [CF96] in the commutative case and extensions to the noncommutative case can be found in [PNA10] (for eigenvalue optimization) and [BK12] (for trace optimization).

Theorem 2.6. *Let $1 \leq \delta \leq t < \infty$, $\mathcal{G} \subseteq \operatorname{Sym}\mathbb{R}\langle \mathbf{x} \rangle_{2r}$, and $\mathcal{H} \subseteq \mathbb{R}\langle \mathbf{x} \rangle_{2r}$. If $L \in \mathbb{R}\langle \mathbf{x} \rangle_{2r}^*$ is symmetric, tracial, δ-flat, nonnegative on $\mathcal{M}_{2r}(\mathcal{G})$, and zero on $\mathcal{I}_{2r}(\mathcal{H})$, then L extends to a symmetric, tracial, linear form on $\mathbb{R}\langle \mathbf{x} \rangle$ that is nonnegative on $\mathcal{M}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and whose moment matrix has finite rank.*

The following technical lemma, based on the Banach-Alaoglu theorem, is a well-known tool to show asymptotic convergence results in (tracial) polynomial optimization.

Lemma 2.7. *Let $\mathcal{G} \subseteq \operatorname{Sym}\mathbb{R}\langle \mathbf{x} \rangle$, $\mathcal{H} \subseteq \mathbb{R}\langle \mathbf{x} \rangle$, and assume $R-(x_1^2+\cdots+x_n^2) \in \mathcal{M}_{2d}(\mathcal{G})+\mathcal{I}_{2d}(\mathcal{H})$ for some $d \in \mathbb{N}$ and $R > 0$. For $r \in \mathbb{N}$ assume $L_r \in \mathbb{R}\langle \mathbf{x} \rangle_{2r}^*$ is tracial, nonnegative on $\mathcal{M}_{2r}(\mathcal{G})$ and zero on $\mathcal{I}_{2r}(\mathcal{H})$. Then we have $|L_r(w)| \leq R^{|w|/2} L_r(1)$ for all $w \in \langle \mathbf{x} \rangle_{2r-2d+2}$. In addition, if $\sup_r L_r(1) < \infty$, then $\{L_r\}_r$ has a pointwise converging subsequence in $\mathbb{R}\langle \mathbf{x} \rangle^*$.*

2.4 Convergence results

We first show equality $\xi_*^{\mathrm{q}}(P) = A_{qc}(P)$, and then we consider convergence properties of the bounds $\xi_r^{\mathrm{q}}(P)$ to the parameters $\xi_\infty^{\mathrm{q}}(P)$ and $\xi_*^{\mathrm{q}}(P)$.

Proposition 2.8. *For any $P \in C_q(\Gamma)$ we have $\xi_*^{\mathrm{q}}(P) = A_{\mathrm{qc}}(P)$.*

Proof. Since we know $\xi_*^{\mathrm{q}}(P) \leq A_{qc}(P)$, it remains to show $\xi_*^{\mathrm{q}}(P) \geq A_{qc}(P)$. For this let L be feasible for $\xi_*^{\mathrm{q}}(P)$, so that $L \geq 0$ on $\mathcal{M}(\mathcal{G})$ and $L = 0$ on $\mathcal{I}(\mathcal{H} \cup \mathcal{R}_\infty)$. By Theorem 2.5, there exist finitely many scalars $\lambda_i \geq 0$, Hermitian matrix tuples $\mathbf{X}(i) = (X_s^a(i))_{a,s}$ and $\mathbf{Y}(i) = (Y_t^b(i))_{b,t}$, and Hermitian matrices Z_i, so that $g(\mathbf{X}(i), \mathbf{Y}(i), Z_i) \succeq 0$ for all $g \in \mathcal{G}$, $h(\mathbf{X}(i), \mathbf{Y}(i), Z_i) = 0$ for all $h \in \mathcal{H} \cup \mathcal{R}_\infty$, and

$$L(p) = \sum_i \lambda_i \operatorname{Tr}(p(\mathbf{X}(i), \mathbf{Y}(i), Z_i)) \quad \text{for all} \quad p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z \rangle. \tag{11}$$

Here we may assume without loss of generality that, for each i, the algebra $\mathbb{C}\langle \mathbf{X}(i), \mathbf{Y}(i), Z_i \rangle$ is a full matrix algebra $\mathbb{C}^{d_i \times d_i}$. Indeed, if this is not the case, by the Artin–Wedderburn theorem there exists a unitary matrix U for which the algebra $U^*\mathbb{C}\langle \mathbf{X}(i), \mathbf{Y}(i), Z_i \rangle U$ can be block diagonalized into smaller blocks and thus we obtain another conic decomposition of L involving only full matrix algebras.

Since $h(\mathbf{E}(i), \mathbf{F}(i), Z_i) = 0$ for all $h \in R_\infty \cup \{z - z^2\}$, the commutator $\big[Z_i u Z_i, Z_i v Z_i\big]$ vanishes for all $u, v \in \langle \mathbf{E}(i), \mathbf{F}(i), Z_i \rangle$, and hence for all $u, v \in \mathbb{C}\langle \mathbf{E}(i), \mathbf{F}(i), Z_i \rangle$. This means that $[Z_i T_1 Z_i, Z_i T_2 Z_i] = 0$ for all $T_1, T_2 \in \mathbb{C}^{d_i \times d_i}$. Since Z_i is a projector, there exists a unitary matrix U_i such that

$$U_i Z_i U_i^* = \operatorname{Diag}(1, \ldots, 1, 0, \ldots, 0).$$

The above then implies that for all T_1 and T_2, the leading principal submatrices of size $\operatorname{rank}(Z_i)$ of $U_i T_1 U_i^*$ and $U_i T_2 U_i^*$ commute. This implies $\operatorname{rank}(Z_i) = 1$ and therefore $\operatorname{Tr}(Z_i) = 1$. Thus we have $1 = L(z) = \sum_i \lambda_i \operatorname{Tr}(Z_i) = \sum_i \lambda_i$.

For each index i define the correlation $P_i \in C_q(\Gamma)$ by

$$P_i(a, b|s, t) = \operatorname{Tr}\big(E_s^a(i) F_t^b(i) Z_i\big) \quad \text{for all} \quad (a, b, s, t) \in \Gamma.$$

Then, $P = \sum_i \lambda_i P_i$, so that (P_i, λ_i) forms a feasible solution to $A_{qc}(P)$ with objective value

$$\sum_i \lambda_i d_i = \sum_i \lambda_i \mathrm{Tr}(I_{d_i}) = L(1).$$

This shows $\xi_*^{\mathrm{q}}(P) \geq A_{qc}(P)$. □

The problem $\xi_r^{\mathrm{q}}(P)$ differs in two ways from a standard tracial optimization problem. It does not have the normalization condition $L(1) = 1$ (and instead minimizes $L(1)$), and it has the extra ideal constraints $L = 0$ on $\mathcal{I}_{2r}(\mathcal{R}_r)$, where $\mathcal{R}_r$ depends on r. The following proof is a straightforward adaptation of a similar proof for general tracial optimization problems from [KP16] and it relies on Lemma 2.7.

Proposition 2.9. *For any $P \in C_q(\Gamma)$ we have $\xi_r^{\mathrm{q}}(P) \to \xi_\infty^{\mathrm{q}}(P)$ as $r \to \infty$.*

Proof. First we observe that the polynomials $1 - z^2$, $1 - (x_s^a)^2$, and $1 - (y_t^b)^2$ lie in $\mathcal{M}_4(\mathcal{G} \cup \mathcal{H}_0)$, where $\mathcal{H}_0$ contains the symmetric polynomials in $\mathcal{H}$ (i.e., omitting the polynomials $[x_s^a, y_t^b]$). Indeed, we have $1 - z^2 = (1 - z)^2 + 2(z - z^2)$,

$$1 - (x_s^a)^2 = (1 - x_s^a)^2 + 2(1 - x_s^a)x_s^a(1 - x_s^a) + 2x_s^a\big((1 - \sum_{a'} x_s^{a'}) + \sum_{a' \neq a} x_s^{a'}\big)x_s^a,$$

and analogously for y_t^b. Hence $R - z^2 - \sum_{a,s}(x_s^a)^2 - \sum_{b,t}(y_t^b)^2 \in \mathcal{M}_4(\mathcal{G} \cup \mathcal{H}_0)$ for some $R > 0$. Fix $\varepsilon > 0$ and for each $r \in \mathbb{N}$ let L_r be feasible for $\xi_r^{\mathrm{q}}(P)$ with value $L_r(1) \leq \xi_r^{\mathrm{q}}(P) + \varepsilon$. As L_r is tracial and zero on $\mathcal{I}_{2r}(\mathcal{H}_0)$ it follows (using the identity $p^*gp = pp^*g + [p^*g, p]$) that $L = 0$ on $\mathcal{M}_{2r}(\mathcal{H}_0)$. Hence, $L_r \geq 0$ on $\mathcal{M}_{2r}(\mathcal{G} \cup \mathcal{H}_0)$. Since $\sup_r L_r(1) \leq A_q(P) + \varepsilon$, we can apply Lemma 2.7 and conclude that $\{L_r\}_r$ has a converging subsequence; denote its limit by $L_\varepsilon \in \mathbb{R}\langle \mathbf{x} \rangle^*$. Then one can verify that L_ε is feasible for $\xi_\infty^{\mathrm{q}}(P)$, and we have

$$\xi_\infty^{\mathrm{q}}(P) \leq L_\varepsilon(1) \leq \lim_{r \to \infty} \xi_r^{\mathrm{q}}(P) + \varepsilon \leq \xi_\infty^{\mathrm{q}}(P) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain that $\xi_\infty^{\mathrm{q}}(P) = \lim_{r \to \infty} \xi_r^{\mathrm{q}}(P)$. □

Recall that a feasible solution L of $\xi_r^{\mathrm{q}}(P)$ is said to be δ-flat if $\mathrm{rank}(M_r(L)) = \mathrm{rank}(M_{r-\delta}(L))$, where $M_{r-\delta}(L)$ is the principal submatrix of $M_r(L)$ whose rows and columns are indexed by $\langle \mathbf{e}, \mathbf{f}, z \rangle_{r-\delta}$. Since computing the rank of a matrix is easy, it is easy to check whether the solution given by the semidefinite programming solver is flat. In the following proposition we show that if $\xi_r^{\mathrm{q}}(P)$ admits a δ-flat optimal solution with $\delta = \lceil r/3 \rceil + 1$, then $\xi_r^{\mathrm{q}}(P) = \xi_*^{\mathrm{q}}(P)$. This proposition and its proof are a small extension of the flat extension result from [KP16] for tracial optimization, where now δ depends on r because the set $\mathcal{R}_r$ for the ideal constraint depends on r.

Proposition 2.10. *If $\xi_r^{\mathrm{q}}(P)$ admits a $(\lceil r/3 \rceil + 1)$-flat optimal solution, then $\xi_r^{\mathrm{q}}(P) = \xi_*^{\mathrm{q}}(P)$.*

Proof. Let $\delta = \lceil r/3 \rceil + 1$ and let L be a δ-flat optimal solution to $\xi_r^{\mathrm{q}}(P)$. We have to show $\xi_r^{\mathrm{q}}(P) \geq \xi_*^{\mathrm{q}}(P)$, which we do by constructing a feasible solution to $\xi_*^{\mathrm{q}}(P)$ with the same objective value. In the proof of Theorem 2.6 (see [GdLL17a, Thm. 2.3], and also [KP16, Prop. 6.1] for the original proof of this theorem), the linear form L is extended to a tracial symmetric linear form $\hat{L}$ on $\mathbb{R}\langle \mathbf{x}, \mathbf{y}, z \rangle$ that is nonnegative on $\mathcal{M}_{2r}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and satisfies $\mathrm{rank}(M(\hat{L})) < \infty$. To do this a subset W of $\langle \mathbf{x}, \mathbf{y}, z \rangle_{t-\delta}$ can be found such that we have the vector space direct sum

$$\mathbb{R}\langle \mathbf{x}, \mathbf{y}, z \rangle = \mathrm{span}(W) \oplus \mathcal{I}(N_r(L)),$$

where $N_r(L)$ is the vector space

$$N_r(L) = \big\{ p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z \rangle_r : L(qp) = 0 \text{ for all } q \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z \rangle_r \big\}.$$

It is moreover shown that $\mathcal{I}(N_r(L)) \subseteq N(\hat{L})$. For $p \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle$ we denote by r_p the unique element in $\mathrm{span}(W)$ such that $p - r_p \in \mathcal{I}(N_r(L))$.

Fix $u, v, w \in \mathbb{R}\langle \mathbf{x}, \mathbf{y}, z\rangle$. Then we have

$$\hat{L}(w(zuzvz - zvzuz)) = \hat{L}(wzuzvz) - \hat{L}(wzvzuz).$$

Since $\hat{L}$ is tracial and $u - r_u, v - r_v, w - r_w \in \mathcal{I}(N_r(L)) \subseteq N(\hat{L})$, we have

$$\hat{L}(wzuzvz) = \hat{L}(r_w z r_u z r_v z) \quad \text{and} \quad \hat{L}(wzvzuz) = \hat{L}(r_w z r_v z r_u z).$$

Since $\deg(r_u z r_v z r_w z) = \deg(r_v z r_u z r_w z) \le 2r$ we have

$$\hat{L}(r_w z r_u z r_v z) = L(r_w z r_u z r_v z) \quad \text{and} \quad \hat{L}(r_w z r_v z r_u z) = L(r_w z r_v z r_u z).$$

So $L \in \mathcal{I}_{2r}(\mathcal{R}_r)$ implies $\hat{L} \in \mathcal{I}(\mathcal{R}_\infty)$.

Since $\hat{L}$ extends L we have $\hat{L}(z) = L(z) = 1$ and $\hat{L}(x_s^a y_t^b z) = L(x_s^a y_t^b z) = P(a, b|s, t)$ for all a, b, s, t. So, $\hat{L}$ is feasible for $\xi_*^{\mathrm{q}}(P)$ and has the same objective value $\hat{L}(1) = L(1)$. ☐

3 Bounding quantum graph parameters

3.1 Hierarchies $\gamma_r^{\mathrm{col}}(G)$ and $\gamma_r^{\mathrm{stab}}(G)$ based on synchronous correlations

In Section 1.4 we introduced quantum chromatic numbers (Definition 1.1) and quantum stability numbers (Definition 1.2) in terms of the existence of synchronous quantum correlations satisfying certain linear constraints. We use this in Section 3.1.1 to reformulate these problems in terms of C^*-algebras, and then in Section 3.1.2 to reformulate this in terms of tracial optimization, which leads to the hierarchies $\gamma_r^{\mathrm{col}}(G)$ and $\gamma_r^{\mathrm{stab}}(G)$.

3.1.1 Graph parameters in terms of C^*-algebras

The following result from [PSS+16] allows us to write a synchronous quantum correlation in terms of C^*-algebras admitting a tracial state.

Theorem 3.1 ([PSS+16]). *Let $\Gamma = A^2 \times S^2$ and $P \in \mathbb{R}^\Gamma$. We have $P \in C_{qc,s}(\Gamma)$ (resp., $P \in C_{q,s}(\Gamma)$) if and only if there exists a unital (resp., finite dimensional) C^*-algebra $\mathcal{A}$ with a faithful tracial state τ and a set of projectors $\{X_s^a : s \in S, a \in A\} \subseteq \mathcal{A}$ satisfying $\sum_{a\in A} X_s^a = 1$ for all $s \in S$ and*

$$P(a, b|s, t) = \tau(X_s^a X_t^b) \quad \text{for all} \quad s, t \in S, a, b \in A.$$

Here we add the condition that τ is faithful, that is, $\tau(X^*X) = 0$ implies $X = 0$, since it follows from the GNS construction in the proof of [PSS+16]. This means that

$$0 = P(a, b|s, t) = \tau(X_s^a X_t^b) = \tau((X_s^a)^2 (X_t^b)^2) = \tau((X_s^a X_t^b)^* X_s^a X_t^b)$$

implies $X_s^a X_t^b = 0$.

It follows that $\chi_{qc}(G)$ is equal to the smallest $k \in \mathbb{N}$ for which there exists a C^*-algebra $\mathcal{A}$, a tracial state τ on $\mathcal{A}$, and a family of projectors $\{X_i^c : i \in V, c \in [k]\} \subseteq \mathcal{A}$ satisfying

$$\sum_{c\in[k]} X_i^c - 1 = 0 \quad \text{for all} \quad i \in V, \tag{12}$$

$$X_i^c X_j^{c'} = 0 \quad \text{if} \quad (c \neq c' \text{ and } i = j) \quad \text{or} \quad (c = c' \text{ and } \{i, j\} \in E). \tag{13}$$

The quantum chromatic number $\chi_q(G)$ is equal to the smallest $k \in \mathbb{N}$ for which there exists a finite dimensional C^*-algebra $\mathcal{A}$ with the above properties.

Analogously, $\alpha_{qc}(G)$ is equal to the largest integer $k \in \mathbb{N}$ for which there exists a C^*-algebra $\mathcal{A}$, a tracial state τ on $\mathcal{A}$, and a family of projectors $\{X_c^i : c \in [k], i \in V\} \subseteq \mathcal{A}$ satisfying

$$\sum_{i \in V} X_c^i - 1 = 0 \quad \text{for all} \quad c \in [k], \tag{14}$$

$$X_c^i X_{c'}^j = 0 \quad \text{if } (i \neq j \text{ and } c = c') \quad \text{or} \quad ((i = j \text{ or } \{i,j\} \in E) \text{ and } c \neq c'), \tag{15}$$

and the quantum stability number $\alpha_q(G)$ is equal to the largest $k \in \mathbb{N}$ for which there exists a finite dimensional C^*-algebra $\mathcal{A}$ with the above properties.

These reformulations of the parameters $\chi_q(G), \chi_{qc}(G), \alpha_q(G)$ and $\alpha_{qc}(G)$ can be obtained from [OP16, Thm. 4.7], where general quantum graph homomorphisms are considered; the formulations of $\chi_q(G)$ and $\chi_{qc}(G)$ are also made explicit in [OP16, Thm. 4.12].

By Artin-Wedderburn theory [Wed64, BEK78], a finite dimensional C^*-algebra is isomorphic to a matrix algebra. So the above reformulations of $\chi_q(G)$ and $\alpha_q(G)$ can be seen as feasibility problems of systems of equations in matrix variables of unspecified (but finite) dimension; such formulations are given in [CMN+07, MR16, SV17] and they also follow from the proof of Proposition A.1. If we restrict to scalar solutions (1×1 matrices) in these feasibility problems, then we recover the classical graph parameters $\chi(G)$ and $\alpha(G)$.

In [OP16] variations on the above parameters are considered where the C^*-algebras are not required to admit a tracial state.

3.1.2 Graph parameters in terms of positive tracial linear forms

Given a graph $G = (V, E)$ and an integer $k \in \mathbb{N}$, we let $\mathcal{H}_{G,k}^{\text{col}}$ and $\mathcal{H}_{G,k}^{\text{stab}}$ denote the set of polynomials corresponding to equations (12)–(13) and (14)–(15):

$$\mathcal{H}_{G,k}^{\text{col}} = \big\{1 - \sum_{c \in [k]} x_i^c : i \in V\big\} \cup \big\{x_i^c x_i^{c'} : (c \neq c' \text{ and } i = j) \text{ or } (c = c' \text{ and } \{i,j\} \in E)\big\},$$

$$\mathcal{H}_{G,k}^{\text{stab}} = \big\{1 - \sum_{i \in V} x_c^i : c \in [k]\big\} \cup \big\{x_c^i x_{c'}^j : (i \neq j \text{ and } c = c') \text{ or } ((i = j \text{ or } \{i,j\} \in E) \text{ and } c \neq c')\big\}.$$

We have $1 - (x_i^c)^2 \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}_{G,k}^{\text{col}})$, since $1 - (x_i^c)^2 = (1 - x_i^c)^2 + 2(x_i^c - (x_i^c)^2)$ and

$$x_i^c - (x_i^c)^2 = x_i^c\Big(1 - \sum_{c'} x_i^{c'}\Big) + \sum_{c' : c' \neq c} x_i^c x_i^{c'} \in \mathcal{I}_2(\mathcal{H}_{G,k}^{\text{col}}),$$

and the analogous statements hold for $\mathcal{H}_{G,k}^{\text{stab}}$. Hence, $\mathcal{M}(\emptyset) + \mathcal{I}(\mathcal{H}_k^{\text{col}})$ and $\mathcal{M}(\emptyset) + \mathcal{I}(\mathcal{H}_k^{\text{stab}})$ are Archimedean and we can apply Theorems 2.4 and 2.5 to express the quantum graph parameters in terms of positive tracial linear functionals. Namely,

$$\chi_{qc}(G) = \min\big\{k \in \mathbb{N} : L \in \mathbb{R}\langle\{x_i^c : i \in V, c \in [k]\}\rangle^* \text{ symmetric, tracial, positive,} \\ L(1) = 1,\, L = 0 \text{ on } \mathcal{I}(\mathcal{H}_{G,k}^{\text{col}})\big\},$$

and $\chi_q(G)$ is obtained by adding the constraint $\text{rank}(M(L)) < \infty$. Likewise,

$$\alpha_{qc}(G) = \min\big\{k \in \mathbb{N} : L \in \mathbb{R}\langle\{x_c^i : c \in [k], i \in V\}\rangle^* \text{ symmetric, tracial, positive,} \\ L(1) = 1,\, L = 0 \text{ on } \mathcal{I}(\mathcal{H}_{G,k}^{\text{stab}})\big\},$$

and $\alpha_q(G)$ is given by the same program with the additional constraint $\text{rank}(M(L)) < \infty$.

Starting from these formulations it is natural to define a hierarchy $\{\gamma_r^{\text{col}}(G)\}$ of lower bounds on $\chi_{qc}(G)$ and a hierarchy $\{\gamma_r^{\text{stab}}(G)\}$ of upper bounds on $\alpha_{qc}(G)$, where the bounds of order $r \in \mathbb{N}$ are obtained by truncating L to polynomials of degree at most $2r$ and truncating the ideal to degree $2r$. Then, if we define $\gamma_*^{\text{col}}(G)$ and $\gamma_*^{\text{stab}}(G)$ by adding the constraint $\text{rank}(M(L)) < \infty$ to $\gamma_\infty^{\text{col}}(G)$ and $\gamma_\infty^{\text{stab}}(G)$, it follows by definition that

$$\gamma_\infty^{\text{col}}(G) = \chi_{qc}(G), \quad \gamma_\infty^{\text{stab}}(G) = \alpha_{qc}(G), \quad \gamma_*^{\text{col}}(G) = \chi_q(G), \quad \text{and} \quad \gamma_*^{\text{stab}}(G) = \alpha_q(G).$$

The optimization problems $\gamma_r^{\text{col}}(G)$, for $r \in \mathbb{N}$, can be computed by semidefinite programming and binary search on k, since the positivity condition on L can be expressed by requiring that its truncated moment matrix $M_r(L) = (L(w^*w'))$ (indexed by words with degree at most r) is positive semidefinite. If there is an optimal solution (k, L) to $\gamma_r^{\text{col}}(G)$ with L flat, then, by Theorem 2.6, we have equality $\gamma_r^{\text{col}}(G) = \chi_q(G)$. Since $\{\gamma_r^{\text{col}}(G)\}_{r\in\mathbb{N}}$ is a monotone nondecreasing sequence of lower bounds on $\chi_q(G)$, there exists an r_0 such that for all $r \geq r_0$ we have $\gamma_r^{\text{col}}(G) = \gamma_{r_0}^{\text{col}}(G)$, which is equal to $\gamma_\infty^{\text{col}}(G) = \chi_{qc}(G)$ by Lemma 2.7. The analogous statements hold for the parameters $\gamma_r^{\text{stab}}(G)$. Hence, we have shown the following result.

Lemma 3.2. *Let G be a graph. There exists an $r_0 \in \mathbb{N}$ such that $\gamma_r^{\text{col}}(G) = \chi_{qc}(G)$ and $\gamma_r^{\text{stab}}(G) = \alpha_{qc}(G)$ for all $r \geq r_0$. Moreover, if $\gamma_r^{\text{col}}(G)$ admits a flat optimal solution, then $\gamma_r^{\text{col}}(G) = \chi_q(G)$, and similarly if $\gamma_r^{\text{stab}}(G)$ admits a flat optimal solution, then $\gamma_r^{\text{stab}}(G) = \alpha_q(G)$.*

Going back to the reformulation of synchronous commuting quantum correlations in Theorem 3.1 we can obtain in the same way a hierarchy of semidefinite programming based outer approximations for the set $C_{qc,s}(\Gamma)$: Define $\mathcal{Q}_{r,s}(\Gamma)$ as the set of $P \in \mathbb{R}^\Gamma$ for which there exists a symmetric, tracial, positive linear form $L \in \mathbb{R}\langle\{x_s^a : (a,s) \in A \times S\}\rangle_{2r}^*$ such that $L(1) = 1$ and $L = 0$ on the ideal generated by the polynomials $x_s^a - (x_s^a)^2$ ($(a,s) \in A \times S$) and $1 - \sum_{a\in A} x_s^a$ ($s \in S$), truncated at degree $2r$. Then we have

$$C_{qc,s}(\Gamma) = \mathcal{Q}_{\infty,s}(\Gamma) = \bigcap_{r\in\mathbb{N}} \mathcal{Q}_{r,s}(\Gamma).$$

Compared to the approximation $\mathcal{Q}_r(\Gamma)$ from [PSS+16], only one set of variables $\{x_s^a\}$ is used to define $\mathcal{Q}_{r,s}$ in the synchronous case while two sets of variables $\{x_s^a, y_t^b\}$ are used to define $\mathcal{Q}_r(\Gamma)$. The synchronous value of a nonlocal game is defined in [DP16] as the maximum value of the objective function (5) over the set $C_{qc,s}(\Gamma)$. By maximizing the objective (5) over the relaxations $\mathcal{Q}_{r,s}(\Gamma)$ we get a hierarchy of semidefinite programming upper bounds that converges to the synchronous value.

We will now present other hierarchies of bounds for the quantum parameters, inspired by existing results on the classical parameters $\alpha(G)$ and $\chi(G)$, and more economical since they involve variables indexed only by the vertices of G. These hierarchies capture existing bounds like projective packing, projective rank and tracial rank and are in fact tightly linked to the bounds $\gamma_r^{\text{col}}(\cdot)$ and $\gamma_r^{\text{stab}}(\cdot)$ via suitable graph products.

3.2 Hierarchies $\xi_r^{\text{col}}(G)$ and $\xi_r^{\text{stab}}(G)$ based on Lasserre type bounds

Here we revisit some known Lasserre type hierarchies for the classical stability number $\alpha(G)$ and chromatic number $\chi(G)$ and we show that their tracial noncommutative analogues can be used to recover known parameters such as the projective packing number $\alpha_p(G)$, the projective rank $\xi_f(G)$, and the tracial rank $\xi_{\text{tr}}(G)$. Compared to the hierarchies defined in the previous section, these Lasserre type hierarchies use less variables (they only use variables indexed by the vertices of the graph G), but they also do not converge to the (commuting) quantum chromatic or stability number.

Given a graph $G = (V, E)$, define the set of polynomials

$$\mathcal{H}_G = \{x_i - x_i^2 : i \in V\} \cup \{x_i x_j : \{i,j\} \in E\}$$

in the variables $\mathbf{x} = (x_i : i \in V)$ (which are commutative or noncommutative depending on the context). Note that $1 - x_i^2 \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}_G)$ for all $i \in V$, so $\mathcal{M}(\emptyset) + \mathcal{I}(\mathcal{H}_G)$ is Archimedean.

3.2.1 Semidefinite programming bounds on the projective packing number

We first recall the Lasserre hierarchy of bounds for the classical stability number $\alpha(G)$. Starting from the formulation of $\alpha(G)$ via the polynomial optimization problem

$$\alpha(G) = \sup\Big\{ \sum_{i\in V} x_i : x \in \mathbb{R}^n,\ h(x) = 0 \text{ for } h \in \mathcal{H}_G \Big\},$$

the r-th level of the Lasserre hierarchy for $\alpha(G)$ (introduced in [Las01, Lau03]) is defined by

$$\text{las}_r^{\text{stab}}(G) = \sup\Big\{ L\big(\sum_{i\in V} x_i\big) : L \in \mathbb{R}[\mathbf{x}]_{2r}^* \text{ positive},\ L(1) = 1,\ L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H}_G) \Big\}.$$

Then $\text{las}_{r+1}^{\text{stab}}(G) \le \text{las}_r^{\text{stab}}(G)$, the first bound is Lovász' theta number: $\text{las}_1^{\text{stab}}(G) = \vartheta(G)$, and finite convergence to $\alpha(G)$ is shown in [Lau03]: $\text{las}_{\alpha(G)}^{\text{stab}}(G) = \alpha(G)$.

Roberson [Rob13] introduces the *projective packing number*:

$$\begin{aligned}\alpha_p(G) &= \sup\Big\{ \frac{1}{d} \sum_{i\in V} \text{rank}\, X_i : d \in \mathbb{N},\ X_1,\ldots,X_n \in \mathcal{S}^d \text{ projectors},\ X_iX_j = 0 \text{ for } \{i,j\} \in E \Big\} \\ &= \sup\Big\{ \text{Tr}\Big(\sum_{i\in V} X_i\Big)/d : d \in \mathbb{N},\ \mathbf{X} \in (\mathcal{S}^d)^n,\ h(\mathbf{X}) = 0 \text{ for } h \in \mathcal{H}_G \Big\} \qquad (16)\end{aligned}$$

as an upper bound for the quantum stability number $\alpha_q(G)$; the inequality $\alpha_q(G) \le \alpha_p(G)$ also follows from Proposition 3.3 below. In view of (16), the parameter $\alpha_p(G)$ can be seen as a noncommutative analogue of $\alpha(G)$.

For $r \in \mathbb{N} \cup \{\infty\}$ we define the noncommutative analogue of the parameter $\text{las}_r^{\text{stab}}(G)$ by

$$\begin{aligned}\xi_r^{\text{stab}}(G) = \sup\Big\{ L\Big(\sum_{i\in V} x_i\Big) : L \in \mathbb{R}\langle\mathbf{x}\rangle_{2r}^* &\text{ tracial, symmetric, and positive,} \\ &L(1) = 1,\ L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H}_G) \Big\},\end{aligned}$$

and define $\xi_*^{\text{stab}}(G)$ by adding the constraint $\text{rank}(M(L)) < \infty$ to the definition of $\xi_\infty^{\text{stab}}(G)$.

In view of Theorems 2.4 and 2.5, both $\xi_\infty^{\text{stab}}(G)$ and $\xi_*^{\text{stab}}(G)$ can be reformulated in terms of C^*-algebras: $\xi_\infty^{\text{stab}}(G)$ (resp., $\xi_*^{\text{stab}}(G)$) is the largest value of $\tau(\sum_{i\in V} X_i)$, where $\mathcal{A}$ is a (resp., finite-dimensional) C^*-algebra with tracial state τ and $X_1,\ldots,X_n \in \mathcal{A}$ are projectors satisfying $X_iX_j = 0$ for all $\{i,j\} \in E$. Moreover, as we now see, the parameter $\xi_*^{\text{stab}}(G)$ coincides with the projective packing number and the parameters $\xi_*^{\text{stab}}(G)$ and $\xi_\infty^{\text{stab}}(G)$ upper bound the quantum stability numbers.

Proposition 3.3. *We have* $\xi_*^{\text{stab}}(G) = \alpha_p(G) \ge \alpha_q(G)$ *and* $\xi_\infty^{\text{stab}}(G) \ge \alpha_{qc}(G)$.

Proof. By the formulation (16), $\alpha_p(G)$ is the largest value of $L(\sum_{i\in V} x_i)$ over linear functionals L that are normalized trace evaluations at projectors $\mathbf{X} \in (\mathcal{S}^d)^n$ (for some $d \in \mathbb{N}$) with $X_iX_j = 0$ for $\{i,j\} \in E$. By convexity the optimum value remains unchanged when considering a convex combination of such trace evaluations. Now in view of Theorem 2.5(3), this optimum value is precisely the parameter $\xi_*^{\text{stab}}(G)$. This shows equality $\alpha_p(G) = \xi_*^{\text{stab}}(G)$.

Consider a C^*-algebra $\mathcal{A}$ with tracial state τ and projectors $X_c^i \in \mathcal{A}$ ($i \in V,\ c \in [k]$) satisfying (14)-(15). Then, setting $X_i = \sum_{c\in[k]} X_c^i$ for $i \in V$, we obtain projectors $X_i \in \mathcal{A}$ that satisfy $X_iX_j = 0$ if $\{i,j\} \in E$. Moreover, $\tau(\sum_{i\in V} X_i) = \sum_{c\in[k]} \tau(\sum_{i\in V} X_c^i) = k$. This shows $\xi_\infty^{\text{stab}}(G) \ge \alpha_{qc}(G)$ and, when restricting $\mathcal{A}$ to be finite dimensional, $\xi_*^{\text{stab}}(G) \ge \alpha_q(G)$. □

Using Lemma 2.7 one can verify that $\xi_r^{\text{stab}}(G)$ converges to $\xi_\infty^{\text{stab}}(G)$ as $r \to \infty$, and for $r \in \mathbb{N} \cup \{\infty\}$ the infimum in $\xi_r^{\text{stab}}(G)$ is attained. Moreover, by Theorem 2.6, if $\xi_r^{\text{stab}}(G)$ admits a flat optimal solution, then $\xi_r^{\text{stab}} = \xi_*^{\text{stab}}(G)$. Also, the first bound $\xi_1^{\text{stab}}(G)$ coincides with the theta number, since $\xi_1^{\text{stab}}(G) = \text{las}_1^{\text{stab}}(G) = \vartheta(G)$. Summarizing we have $\alpha_{qc}(G) \le \xi_\infty^{\text{stab}}(G)$ and the following chain of inequalities

$$\alpha_q(G) \le \alpha_p(G) = \xi_*^{\text{stab}}(G) \le \xi_\infty^{\text{stab}}(G) \le \ldots \le \xi_r^{\text{stab}}(G) \le \ldots \le \xi_1^{\text{stab}}(G) = \vartheta(G).$$

3.2.2 Semidefinite programming bounds on the projective rank and tracial rank

We now turn to the (quantum) chromatic numbers. First recall the definition of the fractional chromatic number:

$$\chi_f(G) := \min\Big\{ \sum_{S\in\mathcal{S}} \lambda_S : \lambda \in \mathbb{R}_+^{\mathcal{S}},\ \sum_{S\in\mathcal{S}: i\in S} \lambda_S = 1 \text{ for all } i \in V \Big\},$$

where $\mathcal{S}$ is the set of stable sets of G. Clearly, $\chi_f(G) \le \chi(G)$. The following Lasserre type lower bounds for the classical chromatic number $\chi(G)$ are defined in [GL08b]:

$$\text{las}_r^{\text{col}}(G) = \inf\big\{ L(1) : L \in \mathbb{R}[\mathbf{x}]_{2r}^* \text{ positive, } L(x_i) = 1 \ (i \in V),\ L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H}_G) \big\}.$$

By viewing $\chi_f(G)$ as minimizing $L(1)$ over linear functionals $L \in \mathbb{R}[\mathbf{x}]^*$ that are conic combinations of evaluations at characteristic vectors of stable sets, we see that $\text{las}_r^{\text{col}}(G) \le \chi_f(G)$ for all $r \ge 1$. In [GL08b] it is shown that finite convergence to $\chi_f(G)$ holds: $\text{las}_{\alpha(G)}^{\text{col}}(G) = \chi_f(G)$. Moreover, the order 1 bound coincides with the theta number: $\text{las}_1^{\text{col}}(G) = \vartheta(\overline{G})$.

The following parameter $\xi_f(G)$, called the *projective rank* of G, was introduced in [MR16] as a lower bound on the quantum chromatic number $\chi_q(G)$:

$$\xi_f(G) := \inf\Big\{ \frac{d}{r} : d, r \in \mathbb{N},\ X_1, \ldots, X_n \in \mathcal{S}^d,\ \text{Tr}(X_i) = r \ (i \in V), \\ X_i^2 = X_i \ (i \in V),\ X_i X_j = 0 \ (\{i,j\} \in E) \Big\}.$$

Proposition 3.4 ([MR16])**.** *For any graph G we have $\xi_f(G) \le \chi_q(G)$.*

Proof. Set $k = \chi_q(G)$. It is shown in [CMN+07] that in the definition of $\chi_q(G)$ from (12)–(13), one may assume w.l.o.g. that all matrices X_i^c have the same rank, say, r. Then, for any given color $c \in [k]$, the matrices X_i^c $(i \in V)$ provide a feasible solution to $\xi_f(G)$ with value d/r. Finally, $d/r = k$ holds since by (12)–(13) we have $d = \text{rank}(I) = \sum_{c=1}^k \text{rank}(X_i^c) = kr$. □

Paulsen et al. [PSS+16, Prop. 5.11] show that the projective rank $\xi_f(G)$ can equivalently be defined as

$$\xi_f(G) = \inf\big\{ \lambda : \mathcal{A} \text{ is a finite dimensional } C^*\text{-algebra with tracial state } \tau, \\ X_i \in \mathcal{A} \text{ projector } (i \in V),\ X_i X_j = 0 \ (\{i,j\} \in E),\ \tau(X_i) = 1/\lambda \ (i \in V) \big\}.$$

They also define the *tracial rank* $\xi_{tr}(G)$ of G as the parameter obtained by omitting in the above definition of $\xi_f(G)$ the restriction that $\mathcal{A}$ has to be finite dimensional. The motivation for the parameter $\xi_{tr}(G)$ is that it lower bounds the *commuting* quantum chromatic number [PSS+16, Thm. 5.11]:

$$\xi_{tr}(G) \le \chi_{qc}(G).$$

In view of Theorems 2.4 and 2.5, we obtain the following reformulations for $\xi_f(G)$ and $\xi_{tr}(G)$:

$$\xi_f(G) = \inf\big\{ L(1) : L \in \mathbb{R}\langle \mathbf{x} \rangle^* \text{ tracial, symmetric, positive, } \text{rank}(M(L)) < \infty, \\ L(x_i) = 1 \ (i \in V),\ L = 0 \text{ on } \mathcal{I}(\mathcal{H}_G) \big\},$$

and $\xi_{tr}(G)$ is obtained by the same program without the restriction $\text{rank}(M(L)) < \infty$. In addition, using Theorem 2.5(3), we see that in this last definition of $\xi_f(G)$ we can equivalently optimize over all L that are conic combinations of trace evaluations at projectors $X_i \in \mathcal{S}^d$ (for some $d \in \mathbb{N}$) satisfying $X_i X_j = 0$ for all $\{i,j\} \in E$. If we restrict the optimization to *scalar* evaluations ($d = 1$) we obtain the fractional chromatic number $\chi_f(G)$. This shows that the projective rank $\xi_f(G)$ can be seen as the noncommutative analogue of the fractional chromatic number $\chi_f(G)$, as was already observed in [MR16, PSS+16].

The above formulations of the parameters $\xi_{tr}(G)$ and $\xi_f(G)$ in terms of linear functionals also show that they fit within the following hierarchy $\{\xi_r^{\rm col}(G)\}_{r\in\mathbb{N}\cup\{\infty\}}$, defined as the noncommutative tracial analogue of the hierarchy $\{{\rm las}_r^{\rm col}(G)\}_r$:

$$\begin{aligned}\xi_r^{\rm col}(G) = \inf\big\{L(1) : \; & L \in \mathbb{R}\langle \mathbf{x}\rangle_{2r}^* \text{ tracial, symmetric, and positive,} \\ & L(x_i) = 1 \; (i \in V), \; L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H}_G)\big\}.\end{aligned}$$

Again, define $\xi_*^{\rm col}(G)$ as the parameter obtained by adding the constraint rank $M(L) < \infty$ to the program defining $\xi_\infty^{\rm col}(G)$. By the above discussion the following holds.

Proposition 3.5. *We have $\xi_*^{\rm col}(G) = \xi_f(G) \le \chi_q(G)$ and $\xi_\infty^{\rm col}(G) = \xi_{tr}(G) \le \chi_{qc}(G)$.*

Using Lemma 2.7 one can verify that the parameters $\xi_r^{\rm col}(G)$ converge to $\xi_\infty^{\rm col}(G)$. Moreover, by Theorem 2.6, if $\xi_r^{\rm col}(G)$ admits a flat optimal solution, then $\xi_r^{\rm col} = \xi_*^{\rm col}(G)$. Also, the parameter $\xi_1^{\rm col}(G)$ coincides with ${\rm las}_1^{\rm col}(G) = \vartheta(\overline{G})$. Summarizing we have $\xi_\infty^{\rm col}(G) = \xi_{tr}(G) \le \chi_{qc}(G)$ and the following chain of inequalities

$$\vartheta(\overline{G}) = \xi_1^{\rm col}(G) \le \ldots \le \xi_r^{\rm col}(G) \le \ldots \le \xi_\infty^{\rm col}(G) = \xi_{tr}(G) \le \xi_*^{\rm col}(G) = \xi_f(G) \le \chi_q(G).$$

Observe that the bounds ${\rm las}_r^{\rm col}(G)$ and $\xi_r^{\rm col}(G)$ remain below the fractional chromatic number $\chi_f(G)$, since $\xi_f(G) = \xi_*^{\rm col}(G) \le {\rm las}_*^{\rm col}(G) = \chi_f(G)$. Hence, these bounds are weak if $\chi_f(G)$ is close to $\vartheta(\overline{G})$ and far from $\chi(G)$ or $\chi_q(G)$. In the classical setting this is the case, e.g., for the class of Kneser graphs $G = K(n, r)$, with vertex set the set of all r-subsets of $[n]$ and having an edge between any two disjoint r-subsets. By results of Lovász [Lov78, Lov06], the fractional chromatic number is $\chi_f(K(n, r)) = n/r$, which is known to be equal to $\vartheta(\overline{K}(n, r))$, while the chromatic number is $\chi(K(n, r)) = n - 2r + 2$. In [GL08b] this was used as a motivation to define a new hierarchy of lower bounds $\{\Lambda_r(G)\}$ on the chromatic number that can go beyond the fractional chromatic number. In Section 3.3 we recall this approach and show that its extension to the tracial setting recovers the hierarchy $\{\gamma_r^{\rm col}(G)\}$ introduced earlier in Section 3.1.2. We also show how a similar technique can be used to recover the hierarchy $\{\gamma_r^{\rm stab}(G)\}$.

3.2.3 A link between $\xi_r^{\rm stab}(G)$ and $\xi_r^{\rm col}(G)$

In [GL08b, Thm. 3.1] it is shown that, for any $r \ge 1$, the bounds ${\rm las}_r^{\rm stab}(G)$ and ${\rm las}_r^{\rm col}(G)$ satisfy

$${\rm las}_r^{\rm stab}(G){\rm las}_r^{\rm col}(G) \ge |V|,$$

with equality if G is vertex-transitive, which extends a well-known property of the theta number (case $r = 1$). The same holds for the noncommutative analogues $\xi_r^{\rm stab}(G)$ and $\xi_r^{\rm col}(G)$.

Lemma 3.6. *For any graph $G = (V, E)$ and $r \in \mathbb{N} \cup \{\infty, *\}$ we have $\xi_r^{\rm stab}(G)\xi_r^{\rm col}(G) \ge |V|$, with equality if G is vertex-transitive.*

Proof. Let L be feasible for $\xi_r^{\rm col}(G)$. Then $\tilde{L} = L/L(1)$ provides a solution to $\xi_r^{\rm stab}(G)$ with value $\tilde{L}\big(\sum_{i\in V} x_i\big) = |V|/L(1)$, implying $\xi_r^{\rm stab}(G) \ge |V|/L(1)$ and thus $\xi_r^{\rm stab}(G)\xi_r^{\rm col}(G) \ge |V|$.

Assume G is vertex-transitive. Let L be a feasible solution for $\xi_r^{\rm stab}(G)$. As G is vertex-transitive we may assume (after symmetrization) that $L(x_i)$ is constant, set $L(x_i) =: 1/\lambda$ for all $i \in V$, so that the objective value of L for $\xi_r^{\rm stab}(G)$ is $|V|/\lambda$. Then $\tilde{L} = \lambda L$ provides a feasible solution for $\xi_r^{\rm col}(G)$ with value λ, implying $\xi_r^{\rm col}(G) \le \lambda$. This implies $\xi_r^{\rm col}(G)\xi_r^{\rm stab}(G) \le |V|$. ☐

When G is vertex-transitive the inequality $\xi_f(G)\alpha_q(G) \le |V|$ was shown in [MR16, Lem. 6.5]; it can be recovered from the case $r = *$ of Lemma 3.6 and the inequality $\alpha_q(G) \le \alpha_p(G)$.

3.2.4 Comparison to existing semidefinite programming bounds

Observe that by adding the inequalities $L(x_ix_j) \geq 0$ for all $i, j \in V$ to $\xi_1^{\mathrm{col}}(G)$ we obtain the strengthened theta number $\vartheta^+(\overline{G})$ (considered in [Sze94]). Moreover, if we add the constraints

$$L(x_ix_j) \geq 0 \qquad \text{for } i \neq j \in V, \tag{17}$$

$$\sum_{j\in C} L(x_ix_j) \leq 1 \qquad \text{for } i \in V, \tag{18}$$

$$L(1) + \sum_{i\in C, j\in C'} L(x_ix_j) \geq |C| + |C'| \qquad \text{for } C, C' \text{ distinct cliques in } G \tag{19}$$

to the program defining the parameter $\xi_1^{\mathrm{col}}(G)$, then we obtain the parameter $\xi_{\mathrm{SDP}}(G)$, which is introduced in [PSS+16, Thm. 7.3] as a lower bound on $\xi_{\mathrm{tr}}(G)$. We will now show that the inequalities (17)–(19) are in fact valid for $\xi_2^{\mathrm{col}}(G)$, which implies

$$\xi_2^{\mathrm{col}}(G) \geq \xi_{\mathrm{SDP}}(G) \geq \vartheta^+(\overline{G}).$$

For this, given a clique C in G, we define the polynomial

$$g_C := 1 - \sum_{i\in C} x_i \in \mathbb{R}\langle \mathbf{x}\rangle.$$

Then the inequalities (18) and (19) can be reformulated as $L(x_ig_C) \geq 0$ and $L(g_Cg_{C'}) \geq 0$, respectively, using the fact that $L(x_i) = L(x_i^2) = 1$ for all $i \in V$. Hence, in order to see that any feasible L for $\xi_2^{\mathrm{col}}(G)$ satisfies the constraints (17)-(19), it suffices to show Lemma 3.7 below. Recall that a commutator is a polynomial of the form $[p, q] = pq - qp$ with $p, q \in \mathbb{R}\langle \mathbf{x}\rangle$. We denote the set of linear combinations of commutators $[p, q]$ with $\deg(pq) \leq r$ by Θ_r.

Lemma 3.7. *Let C and C' be cliques in a graph G and let $i, j \in V$. Then we have*

$$g_C \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}_G), \ \text{and} \ \ x_ix_j,\ x_ig_C,\ g_Cg_{C'} \in \mathcal{M}_4(\emptyset) + \mathcal{I}_4(\mathcal{H}_G) + \Theta_4.$$

Proof. The claim $g_C \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}_G)$ follows from the identity

$$g_C = \underbrace{\Big(1 - \sum_{i\in C} x_i\Big)^2}_{g_C} + \underbrace{\sum_{i\in C}(x_i - x_i^2) + \sum_{i\neq j\in C} x_ix_j}_{h} = g_C^2 + h, \tag{20}$$

where $h \in \mathcal{I}_2(\mathcal{H}_G)$. We also have

$$x_ix_j = x_ix_j^2x_i + x_j(x_i - x_i^2) + x_i^2(x_j - x_j^2) + [x_i, x_ix_j^2] + [x_i - x_i^2, x_j],$$
$$x_ig_C = x_ig_C^2x_i + g_C^2(x_i - x_i^2) + [x_i - x_i^2, g_C^2] + [x_i, x_ig_C^2],$$

and, writing analogously $g_{C'} = g_{C'}^2 + h'$ with $h' \in \mathcal{I}_2(\mathcal{H}_G)$, we have

$$g_Cg_{C'} = g_Cg_{C'}^2g_C + [g_C, g_Cg_{C'}^2] + [h, g_{C'}^2] + g_C^2h' + hh' + g_{C'}^2h.$$

□

Using the bound $\xi_{\mathrm{SDP}}(G)$, it is shown in [PSS+16, Thm. 7.4] that for the odd cycle C_{2n+1}, the tracial rank satisfies $\xi_\infty^{\mathrm{col}}(C_{2n+1}) = (2n + 1)/n$. Combining this with Lemma 3.6 gives $n = \xi_\infty^{\mathrm{stab}}(C_{2n+1}) \geq \alpha_{qc}(C_{2n+1})$. Equality holds since $\alpha_{qc}(C_{2n+1}) \geq \alpha(C_{2n+1}) = n$.

3.3 Links between the bounds $\gamma_r^{\text{col}}(G)$, $\xi_r^{\text{col}}(G)$, $\gamma_r^{\text{stab}}(G)$, and $\xi_r^{\text{stab}}(G)$

In this last section we make the link between the hierarchies $\{\xi_r^{\text{stab}}(G)\}$ and $\{\xi_r^{\text{col}}(G)\}$ from Section 3.2 and the hierarchies $\{\gamma_r^{\text{stab}}(G)\}$ and $\{\gamma_r^{\text{col}}(G)\}$ introduced in Section 3.1. The key fact is the interpretation of the coloring and stability numbers in terms of certain graph products.

We start with the (quantum) coloring number. For an integer k, recall that the Cartesian product $G\square K_k$ is the graph with vertex set $V \times [k]$, where the vertices (i,c) and (j,c') are adjacent if $(\{i,j\} \in E$ and $c = c')$ or $(i = j$ and $c \neq c')$. The following is a well-known reduction of the chromatic number $\chi(G)$ to the stability number of the Cartesian product $G\square K_k$:

$$\chi(G) = \min\big\{k \in \mathbb{N} : \alpha(G\square K_k) = |V|\big\}.$$

It was used in [GL08b] to define the following lower bounds on the chromatic number:

$$\Lambda_r(G) = \min\big\{k \in \mathbb{N} : \text{las}_r^{\text{stab}}(G\square K_k) = |V|\big\},$$

where it was also shown that $\text{las}_r^{\text{col}}(G) \leq \Lambda_r(G) \leq \chi(G)$ for all $r \geq 1$, with equality

$$\Lambda_{|V|}(G) = \chi(G).$$

Hence the bounds $\Lambda_r(G)$ may go beyond the fractional chromatic number. This is the case for the above mentioned Kneser graphs; see [GL08a] for other graph instances.

The above reduction from coloring to stability number has been extended to the quantum setting by [MR16], where it is shown that

$$\chi_q(G) = \min\{k \in \mathbb{N} : \alpha_q(G\square K_k) = |V|\}.$$

It is therefore natural to use the upper bounds $\xi_r^{\text{stab}}(G\square K_k)$ on $\alpha_q(G\square K_k)$ in order to get the following lower bounds on the quantum coloring number:

$$\min\{k : \xi_r^{\text{stab}}(G\square K_k) = |V|\}, \tag{21}$$

which are thus the noncommutative analogues of the bounds $\Lambda_r(G)$. Observe that, for any integer $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty, *\}$, we have $\xi_r^{\text{stab}}(G\square K_k) \leq |V|$, which follows from Lemma 3.7 and the fact that the cliques $C_i = \{(i,c) : c \in [k]\}$, for $i \in V$, cover all vertices in $G\square K_k$. Let

$$\mathcal{C}_{G\square K_k} = \big\{g_{C_i} : i \in V\big\}, \quad \text{where} \quad g_{C_i} = 1 - \sum_{c\in[k]} x_i^c,$$

denote the set of polynomials corresponding to these cliques. We now show that the parameters (21) coincide in fact with $\gamma_r^{\text{col}}(G)$ for all $r \in \mathbb{N} \cup \{\infty\}$. For this observe first that the quadratic polynomials in the set $\mathcal{H}_{G,k}^{\text{col}}$ correspond precisely to the edges of $G\square K_k$, and the projector constraints are included in $\mathcal{I}_2(\mathcal{H}_{G,k}^{\text{col}})$ (see Section 3.1.2), so that

$$\mathcal{I}_{2r}(\mathcal{H}_{G,k}^{\text{col}}) = \mathcal{I}_{2r}(\mathcal{H}_{G\square K_k} \cup \mathcal{C}_{G\square K_k}).$$

We will also use the following result.

Lemma 3.8. *Let $r \in \mathbb{N} \cup \{\infty, *\}$ and assume L is feasible for $\xi_r^{\text{stab}}(G\square K_k)$. Then, we have $L(\sum_{i\in V, c\in[k]} x_i^c) = |V|$ if and only if $L = 0$ on $\mathcal{I}_{2r}(\mathcal{C}_{G\square K_k})$.*

Proof. One direction is easy: If $L = 0$ on $\mathcal{I}_{2r}(\mathcal{C}_{G\square K_k})$, then $0 = \sum_{i\in V} L(g_{C_i}) = |V| - L(\sum_{i,c} x_i^c)$.

Conversely assume that

$$0 = L\Big(\sum_{i\in V, c\in[k]} x_i^c\Big) - |V| = \sum_{i\in V} L(g_{C_i}).$$

We will show $L = 0$ on $\mathcal{I}_{2r}(\mathcal{C}_{G\square K_k})$. For this we first observe that $g_{C_i} - (g_{C_i})^2 \in \mathcal{I}_2(\mathcal{H}_{G\square K_k})$ by (20). Hence $L(g_{C_i}) = L(g_{C_i}^2) \geq 0$, which, combined with $\sum_i L(g_{C_i}) = 0$, implies $L(g_{C_i}) = 0$ for all $i \in V$. Next we show $L(wg_{C_i}) = 0$ for all words w with degree at most $2r-1$, using induction on $\deg(w)$. The base case $w = 1$ holds by the above. Assume now $w = uv$, where $\deg(v) < \deg(u) \leq r$. Using the positivity of L, the Cauchy-Schwarz inequality gives

$$|L(uvg_{C_i})| \leq L(u^*u)^{1/2} L(v^* g_{C_i}^2 v)^{1/2}.$$

Note that it suffices to show $L(v^*g_{C_i}v) = 0$ since, using again (20), this implies $L(v^*g_{C_i}^2 v) = 0$ and thus $L(uvg_{C_i}) = 0$. Using the tracial property of L and the induction assumption, we see that $L(v^*g_{C_i}v) = L(vv^*g_{C_i}) = 0$ since $\deg(vv^*) < \deg(w)$. □

Proposition 3.9. *For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\mathrm{col}}(G) = \min\{k : \xi_r^{\mathrm{stab}}(G\square K_k) = |V|\}$.*

Proof. Let L be a linear functional certifying $\gamma_r^{\mathrm{col}}(G) \leq k$. Then L is feasible for $\xi_r^{\mathrm{stab}}(G\square K_k)$ and, as $L = 0$ on $\mathcal{I}_{2r}(\mathcal{C}_{G\square K_k})$, we can conclude using Lemma 3.8 that $L(\sum_{i,c} x_i^c) = |V|$. This shows $\xi_r^{\mathrm{stab}}(G\square K_k) = |V|$ and thus $\min\{k : \xi_r^{\mathrm{stab}}(G\square K_k) = |V|\} \leq k$.

Conversely, assume $\xi_r^{\mathrm{stab}}(G\square K_k) = |V|$. Since the optimum is attained, there exists a linear functional L feasible for $\xi_r^{\mathrm{stab}}(G\square K_k)$ with $L(\sum_{i,c} x_i^c) = |V|$. Using Lemma 3.8 we can conclude that L is zero on $\mathcal{I}_{2r}(\mathcal{C}_{G\square K_k})$ and thus also on $\mathcal{I}_{2r}(\mathcal{H}_{G,k}^{\mathrm{col}})$. This shows $\gamma_r^{\mathrm{col}}(G) \leq k$. □

Note that the proof of Proposition 3.9 also works in the commutative setting; this shows that the sequence $\Lambda_r(G)$ corresponds to the usual Lasserre hierarchy for the feasibility problem defined by the equations (12)–(13), which is another way of showing $\Lambda_\infty(G) = \chi(G)$.

We now turn to the (quantum) stability number. For an integer k, consider the graph product $K_k \star G$, with vertex set $[k] \times G$ and with an edge between (c, i) and (c', j) when $(c \neq c', i = j)$ or $(c = c', i \neq j)$ or $(c \neq c', \{i, j\} \in E)$. The product $K_k \star G$ coincides with the homomorphic product $K_k \ltimes \overline{G}$ used in [MR16, Sec. 4.2], where it is shown that

$$\alpha_q(G) = \max\big\{k \in \mathbb{N} : \alpha_q(K_k \star G) = k\big\}.$$

This suggests naturally to use the upper bounds $\xi_r^{\mathrm{stab}}(K_k \star G)$ on $\alpha_q(K_k \star G)$ to define the following upper bounds on $\alpha_q(G)$:

$$\max\big\{k \in \mathbb{N} : \xi_r^{\mathrm{stab}}(K_k \star G) = k\big\}. \tag{22}$$

For each $c \in [k]$, the set $C^c = \{(c, i) : i \in V\}$ is a clique in $K_k \star G$ and we let

$$\mathcal{C}_{K_k \star G} = \big\{g_{C^c} : c \in [k]\big\}, \quad \text{where} \quad g_{C^c} = 1 - \sum_{i \in V} x_c^i,$$

denote the set of polynomials corresponding to these cliques. Since these k cliques cover the vertex set of $K_k \star G$, we can use Lemma 3.7 to conclude $\xi_r^{\mathrm{stab}}(K_k \star G) \leq k$ for all $r \in \mathbb{N} \cup \{\infty, *\}$. Again, observe that the quadratic polynomials in the set $\mathcal{H}_{G,k}^{\mathrm{stab}}$ correspond precisely to the edges of $K_k \star G$ and that we have

$$\mathcal{I}_{2r}(\mathcal{H}_{G,k}^{\mathrm{stab}}) = \mathcal{I}_{2r}(\mathcal{H}_{K_k \star G} \cup \mathcal{C}_{K_k \star G}).$$

Based on this, one can show the analogue of Lemma 3.8: If L is feasible for the program $\xi_r^{\mathrm{stab}}(K_k \star G)$, then we have $L(\sum_{i,c} x_c^i) = k$ if and only if $L = 0$ on $\mathcal{I}_{2r}(\mathcal{C}_{K_k \star G})$, which implies the following result.

Proposition 3.10. *For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\mathrm{stab}}(G) = \max\{k : \xi_r^{\mathrm{stab}}(K_k \star G) = k\}$.*

We do not know whether the results of Propositions 3.9 and 3.10 hold for $r = *$, since we do not know whether the supremum is attained in the parameter $\xi_*^{\text{stab}}(\cdot) = \alpha_p(\cdot)$ (as was already observed in [Rob13, p. 120]). Hence we can only claim the inequalities

$$\gamma_*^{\text{col}}(G) \geq \min\{k : \xi_*^{\text{stab}}(G \Box K_k) = |V|\} \quad \text{and} \quad \gamma_*^{\text{stab}}(G) \leq \max\{k : \xi_*^{\text{stab}}(K_k \star G) = k\}.$$

As mentioned above, we have $\text{las}_r^{\text{col}}(G) \leq \Lambda_r(G)$ for any $r \in \mathbb{N}$ [GL08b, Prop. 3.3]. This result extends to the noncommutative setting and the analogous result holds for the stability parameters. In other words the hierarchies $\{\gamma_r^{\text{col}}(G)\}$ and $\{\gamma_r^{\text{stab}}(G)\}$ refine the hierarchies $\{\xi_r^{\text{col}}(G)\}$ and $\xi_r^{\text{stab}}(G)\}$.

Proposition 3.11. *For $r \in \mathbb{N} \cup \{\infty, *\}$ we have $\xi_r^{\text{col}}(G) \leq \gamma_r^{\text{col}}(G)$ and $\xi_r^{\text{stab}}(G) \geq \gamma_r^{\text{stab}}(G)$.*

Proof. We may restrict to $r \in \mathbb{N}$ since we have seen earlier that the inequalities hold for $r \in \{\infty, *\}$. The proof for the coloring parameters is similar to the proof of [GL08b, Prop. 3.3] in the classical case and thus omitted. We now show the inequality $\xi_r^{\text{stab}}(G) \geq \gamma_r^{\text{stab}}(G)$. Set $k = \gamma_r^{\text{stab}}(G)$ and, using Proposition 3.10, let $L \in \mathbb{R}\langle x_c^i : i \in V, c \in [k]\rangle_{2r}^*$ be optimal for $\xi_r^{\text{stab}}(K_k \star G) = k$. That is, L is tracial, symmetric, positive, and satisfies $L(1) = 1$, $L(\sum_{i,c} x_c^i) = k$, and $L = 0$ on $\mathcal{I}(\mathcal{H}_{K_k \star G})$. It suffices now to construct a tracial symmetric positive linear form $\hat{L} \in \mathbb{R}\langle x_i : i \in V\rangle_{2r}^*$ such that $\hat{L}(1) = 1$, $\hat{L}(\sum_{i \in V} x_i) = k$, and $\hat{L} = 0$ on $\mathcal{I}_{2r}(\mathcal{H}_G)$, since this will imply $\xi_r^{\text{stab}}(G) \geq k$. For this, for any word $x_{i_1} \cdots x_{i_t}$ with degree $1 \leq t \leq 2r$, we define

$$\hat{L}(x_{i_1} \cdots x_{i_t}) := \sum_{c \in [k]} L(x_c^{i_1} \cdots x_c^{i_t}).$$

Also, we set $\hat{L}(1) = L(1) = 1$. Then, we have $\hat{L}(\sum_{i \in V} x_i) = k$. Moreover, one can easily check that $\hat{L}$ is indeed tracial, symmetric, positive, and vanishes on $\mathcal{I}_{2r}(\mathcal{H}_G)$. □

A Synchronous quantum correlations

We prove the following by combining proofs from [SV17] (see also [MR16]) and [PSS+16].

Proposition A.1. *The smallest local dimension in which a synchronous quantum correlation P can be realized is given by* $\text{cpsd-rank}_{\mathbb{C}}(M_P)$.

Proof. Suppose first that (ψ, E_s^a, F_t^b) is a realization of P in local dimension d. We will show $\text{cpsd-rank}_{\mathbb{C}}(A_P) \leq d$.

The Schmidt decomposition gives scalars $\{\lambda_i\}$ and orthonormal bases $\{u_i\}$ and $\{v_i\}$ of $\mathbb{C}^d$ such that $\psi = \sum_{i=1}^d \sqrt{\lambda_i}\, u_i \otimes v_i$. We can replace ψ by $\sum_{i=1}^d \sqrt{\lambda_i}\, v_i \otimes v_i$ and E_s^a by $UE_s^aU^*$, where U is the unitary matrix for which $u_i = Uv_i$ for all i, such that (ψ, E_s^a, F_t^b) still realizes P and is of the same dimension.

Given such a realization $(\sum_{i=1}^d \sqrt{\lambda_i}\, v_i \otimes v_i, E_s^a, F_t^b)$ of P, we define the matrices

$$K = \sum_{i=1}^d \sqrt{\lambda_i}\, v_i v_i^*, \quad X_s^a = K^{1/2} E_s^a K^{1/2}, \quad Y_t^b = K^{1/2} F_t^b K^{1/2}.$$

By using the identities $\text{vec}(K) = \psi$ and

$$\text{vec}(K)^*(E_s^a \otimes F_t^b)\text{vec}(K) = \text{Tr}(KE_s^aKF_t^b) = \text{Tr}(K^{1/2}E_s^aK^{1/2}K^{1/2}F_t^bK^{1/2}),$$

we see that

$$P(a, b|s, t) = \langle X_s^a, Y_t^b\rangle \quad \text{for all} \quad a, b, s, t, \tag{23}$$

and

$$\langle K, K\rangle = 1, \quad \sum_a X_s^a = \sum_b Y_t^b = K \quad \text{for all} \quad s,t. \tag{24}$$

For each s, the Cauchy–Schwarz inequality gives

$$\begin{aligned}
1 = \sum_a P(a,a|s,s) &= \sum_a \langle X_s^a, Y_s^a\rangle \le \sum_a \langle X_s^a, X_s^a\rangle^{1/2}\langle Y_s^a, Y_s^a\rangle^{1/2} \\
&\le \Big(\sum_a \langle X_s^a, X_s^a\rangle\Big)^{1/2}\Big(\sum_a \langle Y_s^a, Y_s^a\rangle\Big)^{1/2} \\
&\le \Big\langle \sum_a X_s^a, \sum_a X_s^a\Big\rangle^{1/2}\Big\langle \sum_a Y_s^a, \sum_a Y_s^a\Big\rangle^{1/2} = \langle K, K\rangle = 1.
\end{aligned}$$

Thus all inequalities above are equalities. The first inequality being an equality shows that there exist $\alpha_{s,a}$ such that $X_s^a = \alpha_{s,a} Y_s^a$ for all a, s. The second inequality being an equality shows that there exist β_s such that $\|X_s^a\| = \beta_s \|Y_s^a\|$ for all s. Hence,

$$\beta_s \|Y_s^a\| = \|X_s^a\| = \|\alpha_{s,a} Y_s^a\| = \alpha_{s,a}\|Y_s^a\| = \alpha_{s,a}\|Y_s^a\| \quad \text{for all} \quad s,a,$$

which shows $X_s^a = \beta_s Y_s^a$ for all s. Since $\sum_a X_s^a = K = \sum_a Y_s^a$, we have $\beta_s = 1$ for all s. Thus $X_s^a = Y_s^a$ for all a, s. Therefore,

$$(A_P)_{(s,a),(t,b)} = \langle X_s^a, X_t^b\rangle \quad \text{for all} \quad a,b,s,t,$$

which shows $\text{cpsd-rank}_{\mathbb{C}}(A_P) \le d$.

For the other direction we suppose $\{X_s^a\}$ are smallest possible Hermitian positive semidefinite matrices such that $(A_P)_{(s,a),(t,b)} = \langle X_s^a, X_t^b\rangle$ for all a, s, t, b. Then,

$$1 = \sum_{a,b} P(a,b|s,t) = \sum_{a,b}\langle X_s^a, X_t^b\rangle = \Big\langle \sum_a X_s^a, \sum_b X_t^b\Big\rangle \quad \text{for all} \quad s,t,$$

which shows the existence of a matrix K such that $K = \sum_a X_s^a$ for all s. We have $\langle K, K\rangle = 1$ so that $\text{vec}(K)$ is a unit vector, and since the factorization is smallest possible, K is invertible. Set $E_s^a = K^{-1/2} X_s^a K^{-1/2}$ for all s, a, so that $\sum_a E_s^a = I$ for all s. Then,

$$P(a,b|s,t) = (A_P)_{(s,a),(t,b)} = \langle X_s^a, X_t^b\rangle = \text{vec}(K)^*(E_s^a \otimes E_t^b)\text{vec}(K),$$

which shows P has a realization of local dimension $\text{cpsd-rank}_{\mathbb{C}}(A_P)$. □

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