Perturbation analysis of nonlinear semidefinite programming under Jacobian uniqueness conditions[1]

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Abstract. We consider the stability of a class of parameterized nonlinear semidefinite programming problems whose objective function and constraint mapping all have second partial derivatives only with respect to the decision variable which are jointly continuous. We show that when the Karush-Kuhn-Tucker (KKT) condition, the constraint nondegeneracy condition, the strict complementary condition and the second order sufficient condition (named as Jacobian uniqueness conditions here) are satisfied at a feasible point of the original problem, the perturbed problem also satisfies the Jacobian uniqueness conditions at some feasible point.

Key words. nonlinear semidefinite programming; Jacobian uniqueness conditions; KKT system; implicit function theorem.

AMS Subject Classifications.49K40,90C31,49J53

1 Introduction

Consider the optimization problem

$$(\text{OP}) : \left\{ \begin{array}{ll} \min\limits_{x\in X} & f(x) \\ \text{s.t.} & G(x) \in K, \end{array} \right. \tag{1.1}$$

where $f : X \to \Re$ and $G : X \to Y$ are twice continuously differentiable functions, X and Y are two finite dimensional real vector spaces, and K is a closed convex set in Y. The first order optimality condition, i.e., the KKT condition for (OP) takes the following form:

$$\nabla_x L(x, \mu) = 0, \mu \in \mathcal{N}_K(G(x)), \tag{1.2}$$

where the Lagrangian function $L : X \times Y \to \Re$ is defined by:

$$L(x, \mu) := f(x) + \langle \mu, G(x)\rangle, (x, \mu) \in X \times Y, \tag{1.3}$$

$\nabla_x L(x, \mu)$ denotes the gradient of $L(x, \mu)$ at (x, μ) with respect to $x \in X$, and $\mathcal{N}_K(y)$ is the normal cone of K at y in the sense of convex analysis (Rockafellar[15]) :

$$\mathcal{N}_K(y) = \left\{ \begin{array}{ll} \{d \in Y : \langle d, z - y\rangle \leq 0, \ \forall z \in K\} & , \ y \in K \\ \emptyset & , \ y \notin K \end{array} \right.$$

For any (x, μ) satisfying (1.2), we call x a stationary point and (x, μ) a KKT point of (OP), respectively.

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In the last four decades, considerable progress has been achieved towards stability analysis of solutions to the optimization problem (OP) (Bonnans and Shapiro[5], Rockafellar and Wets[16], etc). When K is a polyhedral set, this is exactly the conventional nonlinear programming (NLP) case, the stability theory is quite complete. Robinson[14] introduced the important concept of strong regularity for generalized equations, which include the KKT system as a special case, and defined a strong second order sufficient condition for (NLP). He proved that the strong second order sufficient condition and the linear independence constraint qualification imply the strong regularity of the solution to the KKT system. Interestingly, Jongen et al.[10], Bonnans and Sulem[6], Dontchev and Rockafellar[7] and Bonnans and Shapiro[5, Proposition 5.38] showed that the converse is also true.

When K is a nonpolyhedral set, relative conclusions are not easy to get, however, when K is $\mathcal{C}^2$-cone reducible in the sense of Bonnans and Shapiro[5, Definition 3.135], the situation is different, the research of sensitivity and stability of solutions for (OP) has been made a great progress(Bonnans et al.[2, 3], Bonnans and Shapiro[5, 4]). In fact, many kinds of sets are all $\mathcal{C}^2$-cone reducible, such as the polyhedral set, the second order cone and the cone of symmetric positive semidefinite matrices.

For nonlinear semidefinite programming problem with equality constraints, Sun[17] showed that for a locally optimal solution, the strong second order sufficient condition and constraint nondegeneracy, the nonsingularity of Clarke's Jacobian of the KKT system and the strong regularity are all equivalent under Robinson constraint qualification.

Recently, Ding, Sun and Zhang[8] considered the canonically perturbed optimization problem of (OP) and assumed that the set K in (OP) belongs to the class of $\mathcal{C}^2$-cone reducible sets, they showed that under the Robinson constraint qualification, the KKT solution mapping is robustly isolated calm if and only if both the strict Robinson constraint qualification and the second order sufficient condition hold.

Note that the strong regularity of the KKT system is directly related to the canonical parameterization which is a special case of $\mathcal{C}^2$-smooth parameterization. For instance, Robinson[14], Sun[17], Ding, Sun and Zhang[8] only analysed the stability of the canonical perturbed problem, and Bonnans and Shapiro[5] studied the stability of general $\mathcal{C}^2$-smooth parameterization. How to get the stability conclusions of the parameterized problems which are more general than $\mathcal{C}^2$-smooth parameterization is worth thinking about. In this aspect, Robinson[13] introduced a general perturbed nonlinear program under pertubations in the objective function and constraint mapping, he proved that when a solution of (NLP) satisfies Jacobian uniqueness conditions and the objective function and the constraint mapping of perturbed problem all have second partial derivatives with respect to the decision variable which are jointly continuous, the Jacobian uniqueness conditions also hold at a solution of perturbed problem. A natural question is asked, is this result still valid for the nonlinear semidefinite programming problem? We answer this question in this paper. The primary objective of this paper is to give the similar conclusions for the general nonlinear semidefinite programming problem as follows:

$$\text{(NLSDP)}: \begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0_m, \\ & q(x) \leq 0_l, \\ & g(x) \in S^p_+, \end{cases} \tag{1.4}$$

where $f : \Re^n \to \Re$, $h : \Re^n \to \Re^m$, $q : \Re^n \to \Re^l$ and $g : \Re^n \to S^p$ are twice continuously differentiable functions, S^p is the linear space of all $p \times p$ real symmetric matrices, and S^p_+ is the cone of all $p \times p$

positive semidefinite matrices. Note that (NLSDP) problem is a special case of (OP) with

$$G(x) = (h(x), q(x), g(x)), K = \{0\}_m \times \Re^l_- \times S^p_+, X = \Re^n, Y = \Re^m \times \Re^l \times S^p.$$

The organization of this paper is as follows. In Section 2, we introduce the notion of Jacobian uniqueness conditions for (NLSDP), it is shown that the Jacobian uniqueness conditions of a solution of (NLSDP) imply the nonsingularity of Jacobian of the KKT system. In Section 3, by using the implicit-function theorem, we show that when the Jacobian uniqueness conditions are satisfied at a solution of (NLSDP), the perturbed problem which are more general than $\mathcal{C}^2$-smooth parameterization also satisfies the Jacobian uniqueness conditions at some solution. We conclude this paper in Section 4.

2 Jacobian uniqueness conditions for problem (NLSDP)

We write $A \succeq 0$ and $A \succ 0$ to mean that A is a symmetric positive semidefinite matrix and a symmetric positive definite matrix, respectively. For any two matrices A and B in S^P, we write

$$\langle A, B\rangle := \mathrm{Tr}(A^T B),$$

for the inner product between A and B, where "Tr" denotes the trace of a matrix. In addition, if $\langle A, B\rangle = 0$, we write $A \perp B$. For any mapping $G : \Re^n \to Y$, we denote the derivative of G at $x \in \Re^n$ by $DG(x)$; when $Y = \Re^m$, $DG(x)$ means the Jacobian of G with respect to $x \in \Re^n$, we write $\mathcal{J}G(x)$.

By (1.2), the KKT condition for (NLSDP) is as follows:

$$\nabla_x L(x, \lambda, \mu, \Omega) = 0, h(x) = 0, \mu \in \mathcal{N}_{\Re^l_-}(q(x)), \Omega \in \mathcal{N}_{\Re^p_+}(g(x)), \tag{2.5}$$

namely,

$$\nabla_x L(x, \lambda, \mu, \Omega) = 0, h(x) = 0, 0 \leq \mu \perp q(x) \leq 0, 0 \preceq g(x) \perp \Omega \preceq 0, \tag{2.6}$$

where the Lagrangian function $L : \Re^n \times \Re^m \times \Re^l \times S^p \to \Re$ is defined by

$$L(x, \lambda, \mu, \Omega) = f(x) + \lambda^T h(x) + \mu^T q(x) + \langle \Omega, g(x)\rangle. \tag{2.7}$$

Let $\Lambda(x)$ denote the set of Lagrangian multipliers satisfying (2.5).

Definition 2.1 *We say that a feasible point $\bar{x}$ to* (NLSDP) *is constraint nondegenerate* [5, (4.172)] *if*

$$\begin{pmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}q(\bar{x}) \\ Dg(\bar{x}) \end{pmatrix} \Re^n + \begin{pmatrix} \{0\} \\ \mathrm{lin}\left(\mathcal{T}_{\Re^l_-}(q(\bar{x}))\right) \\ \mathrm{lin}\left(\mathcal{T}_{S^p_+}(g(\bar{x}))\right) \end{pmatrix} = \begin{pmatrix} \Re^m \\ \Re^l \\ S^p \end{pmatrix}. \tag{2.8}$$

We denote $\mathcal{T}_K(y)$ as the tangent cone of K at y:

$$\mathcal{T}_K(y) = \{d \in Y : dist(y + td, K) = o(t), t \geq 0\}, y \in K, K \subset Y.$$

Definition 2.2 *We say that the strict complementary condition* [5, Definition 4.74] *holds at a feasible point $\bar{x}$ to* (NLSDP) *if*

$$\exists(\mu,\Omega)\in\Lambda(\bar{x}), \text{s.t.} \mu_i - q_i(\bar{x}) > 0, \forall i=1,2,\cdots,l, g(\bar{x})-\Omega \succ 0. \tag{2.9}$$

Definition 2.3 *Let $\bar{x}$ be a feasible point to* (NLSDP). *The critical cone $C(\bar{x})$ of* (NLSDP) *at $\bar{x}$ is defined by*

$$C(\bar{x}) := \{d\in\Re^n : DG(\bar{x})d \in \mathcal{T}_K(G(\bar{x})), \mathcal{J}f(\bar{x})d \le 0\}. \tag{2.10}$$

We can rewrite this as

$$C(\bar{x}) = \{d\in\Re^n : \mathcal{J}h(\bar{x})d = 0, \mathcal{J}q(\bar{x})d \in \mathcal{T}_{\Re^l_-}(q(\bar{x})), Dg(\bar{x})d \in \mathcal{T}_{S^p_+}(g(\bar{x})), \mathcal{J}f(\bar{x})d \le 0\}. \tag{2.11}$$

If $\bar{x}$ is a stationary point of (NLSDP), namely $\Lambda(\bar{x}) \neq \emptyset$, then

$$C(\bar{x}) = \{d\in\Re^n : \mathcal{J}h(\bar{x})d = 0, \mathcal{J}q(\bar{x})d \in \mathcal{T}_{\Re^l_-}(q(\bar{x})), Dg(\bar{x})d \in \mathcal{T}_{S^p_+}(g(\bar{x})), \mathcal{J}f(\bar{x})d = 0\}. \tag{2.12}$$

Lemma 2.1 *Let $\bar{x}$ be a locally optimal solution to* (NLSDP)*, the constraint nondegenerate condition and the strict complementary condition hold at $\bar{x}$, then the critical cone of* (NLSDP) *at $\bar{x}$ can be written as*

$$C(\bar{x}) = \{d\in\Re^n : \mathcal{J}h(\bar{x})d = 0, \nabla q_i(\bar{x})^T d = 0, i\in I(\bar{x}), P_\gamma^T Dg(\bar{x}) d P_\gamma = 0\}. \tag{2.13}$$

where $I(\bar{x}) = \{i | q_i(\bar{x}) = 0, 1\le i\le l\}$, P_γ is a $p\times|\gamma|$ matrix whose columns form an orthonormal basis of the eigenvector space of $g(\bar{x})$ corresponding to its smallest eigenvalue 0.

Proof. Since $\bar{x}$ is a locally optimal solution to (NLSDP), then by(2.6), there exists $(\bar{\lambda},\bar{\mu},\bar{\Omega})\in\Lambda(\bar{x})$, such that

$$\nabla_x L(\bar{x},\bar{\lambda},\bar{\mu},\bar{\Omega}) = 0, h(\bar{x}) = 0, 0 \le \bar{\mu} \perp q(\bar{x}) \le 0, 0 \preceq g(\bar{x}) \perp \bar{\Omega} \preceq 0. \tag{2.14}$$

The constraint nondegeneracy condition is assumed to hold at $\bar{x}$, which imply $\Lambda(\bar{x})$ is a singleton[5, Proposition 4.75], i.e., $(\bar{\lambda},\bar{\mu},\bar{\Omega})$ is unique.

Let $A := g(\bar{x}) + \bar{\Omega}$ have the following spectral decomposition

$$A = P\Lambda P^T,$$

where Λ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors. Then by the strict complementary condition of $\bar{x}$, we can just assume

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 \\ 0 & \Lambda_\gamma \end{bmatrix}, P = [P_\alpha, P_\gamma], \tag{2.15}$$

and

$$g(\bar{x}) = P\begin{bmatrix} \Lambda_\alpha & \\ & 0 \end{bmatrix}P^T, \bar{\Omega} = P\begin{bmatrix} 0 & \\ & \Lambda_\gamma \end{bmatrix}P^T, \tag{2.16}$$

where α and γ are index sets of positive and negative eigenvalues of A, respectively, as

$$\alpha = \{i : \lambda_i(A) > 0\}, \gamma = \{i : \lambda_i(A) < 0\}.$$

Thus,

$$\begin{aligned} C(\bar{x}) &= \{d \in \Re^n : \mathcal{J}h(\bar{x})d = 0, \nabla q_i(\bar{x})^T d \leq 0, i \in I(\bar{x}), P_\gamma^T Dg(\bar{x})dP_\gamma \succeq 0, \mathcal{J}f(\bar{x})d = 0\} \\ &= \{d \in \Re^n : \mathcal{J}h(\bar{x})d = 0, \nabla q_i(\bar{x})^T d = 0, i \in I(\bar{x}), \langle \bar{\Omega}, Dg(\bar{x})d \rangle = 0, P_\gamma^T Dg(\bar{x})dP_\gamma \succeq 0\} \\ &= \{d \in \Re^n : \mathcal{J}h(\bar{x})d = 0, \nabla q_i(\bar{x})^T d = 0, i \in I(\bar{x}), P_\gamma^T Dg(\bar{x})dP_\gamma = 0\}. \end{aligned}$$

where the first equality follows from (2.12) and the forms of $\mathcal{T}_{\Re^l_-}(q(\bar{x}))$ and $\mathcal{T}_{S^p_+}(g(\bar{x}))$, the second equality follows from (2.14), and the third equality follows from (2.17) as below:

$$\langle \bar{\Omega}, Dg(\bar{x})d \rangle = \langle P^T \bar{\Omega} P, P^T Dg(\bar{x})dP \rangle = \langle \begin{bmatrix} 0 & \\ & \Lambda_\gamma \end{bmatrix}, P^T Dg(\bar{x})dP \rangle = \langle \Lambda_\gamma, P_\gamma^T Dg(\bar{x})dP_\gamma \rangle = 0. \quad (2.17)$$

□

Lemma 2.2 *Define* $\mathcal{A} : \Re^n \to \mathcal{L}(\Re^n, \Re^m \times \Re^{|I(x)|} \times S^{|\gamma|})$ *by*

$$\mathcal{A}(x)d = (\mathcal{J}h(x)d; \nabla q_i(x)^T d, i \in I(x); P_\gamma(x)^T Dg(x)dP_\gamma(x)), d \in \Re^n, \quad (2.18)$$

where $I(x) = \{i | q_i(x) = 0, 1 \leq i \leq l\}$, $P_\gamma(x)$ *is a* $p \times |\gamma|$ *matrix whose columns form an orthonormal basis of the eigenvector space of* $g(x)$ *corresponding to its smallest eigenvalue 0. Then the constraint nondegenerate condition at* x *is equivalent to* $\mathcal{A}(x)$ *is onto.*

Recall that for any set $D \subseteq Y$, the support function of the set D is defined as

$$\sigma(y, D) := \sup_{z \in D} \langle z, y \rangle, \ \ y \in Y.$$

Let $\kappa = (\lambda, \mu, \Omega) \in \Re^m \times \Re^l \times S^p$ for any $\kappa \in \Lambda(\bar{x})$. Then for $\kappa \in \Lambda(\bar{x})$ and $d \in C(\bar{x})$ the "sigma term"can be written as

$$\begin{aligned} \sigma(\kappa, \mathcal{T}^2_K(G(\bar{x}), \mathcal{J}G(\bar{x})d) &= \sigma(\lambda, \mathcal{T}^2_{\{0\}}(h(\bar{x}), \mathcal{J}h(\bar{x})d) + \sigma(\mu, \mathcal{T}^2_{\Re^l_-}(q(\bar{x}), \mathcal{J}q(\bar{x})d) + \sigma(\Omega, \mathcal{T}^2_{S^P_+}(g(\bar{x}), Dg(\bar{x})d) \\ &= 0 + 0 + \sigma(\Omega, \mathcal{T}^2_{S^P_+}(g(\bar{x}), Dg(\bar{x})d) \\ &= \sigma(\Omega, \mathcal{T}^2_{S^P_+}(g(\bar{x}), Dg(\bar{x})d), \end{aligned}$$

combining this and [5, P.487 and Theorem 5.89], we can state in the following definition the second order sufficient condition for (NLSDP).

Definition 2.4 *Let* $\bar{x}$ *be a stationary point to* (NLSDP)*, we say that the second order sufficient condition holds at* $\bar{x}$ *if*

$$\sup_{(\lambda,\mu,\Omega) \in \Lambda(\bar{x})} \left\{ d^T \nabla^2_{xx} L(\bar{x}, \lambda, \mu, \Omega)d - 2\langle \Omega, Dg(\bar{x})d[g(\bar{x})]^\dagger Dg(\bar{x})d \rangle \right\} > 0, \forall d \in C(\bar{x}) \backslash \{0\}, \quad (2.19)$$

where $[g(\bar{x})]^\dagger$ *is the Moore-Penrose pseudo-inverse of* $g(\bar{x})$*.*

Finally, we give the definition of Jacobian uniqueness conditions for (NLSDP):

Definition 2.5 *Let $\bar{x}$ be a feasible point to* (NLSDP)*, we say that the Jacobian uniqueness conditions hold at $\bar{x}$ if*

i)the point $\bar{x}$ is a stationary point of (NLSDP)*;*

ii)the constraint nondegenerate condition holds at $\bar{x}$;

iii)the strict complementary condition holds at $\bar{x}$;

iv)the second order sufficient condition holds at $\bar{x}$.

Since, from Eaves[9],

$$\Omega \in \mathcal{N}_{S_+^p}(g(x)) \Longleftrightarrow g(x) = \Pi_{S_+^p}(g(x) + \Omega),$$

then the KKT condition for (NLSDP) is equivalent to

$$F(x, \lambda, \mu, \Omega) := \begin{bmatrix} \nabla_x L(x, \lambda, \mu, \Omega) \\ h(x) \\ \mathcal{M}q(x) \\ g(x) - \Pi_{S_+^p}(g(x) + \Omega) \end{bmatrix} = 0, \tag{2.20}$$

where $\mathcal{M} := diag(\mu_i)$, $i = 1, 2, \cdots, l$.

The following result plays an important role in our subsequent analysis.

Lemma 2.3 *Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$ be a feasible point to* (NLSDP) *at which the Jacobian uniqueness conditions are satisfied, then the Jacobian of F is nonsingular at $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$.*

Proof. By Lemma 2.1, the critical cone of (NLSDP) at $\bar{x}$ can be written as (2.13), without loss of generality, we assume that $g(\bar{x})$ and $\bar{\Omega}$ have the spectral decomposition as in (2.16), then $g(\bar{x}) + \bar{\Omega}$ is nonsingular which imply that $\Pi_{S_+^p}(\cdot)$ is F-differentiable at $g(\bar{x}) + \bar{\Omega}$ [12, Corollary 10], for simplicity of expression, we denote $D\Pi_{S_+^p}(g(\bar{x}) + \bar{\Omega})$ by V, by [17, Proposition 2.2], we obtain

$$V(H) = P \begin{bmatrix} P_\alpha^T H P_\alpha & U_{\alpha\gamma} \circ P_\alpha^T H P_\gamma \\ P_\gamma^T H P_\alpha \circ U_{\alpha\gamma}^T & 0 \end{bmatrix} P^T, \tag{2.21}$$

where the matrix $U \in S^p$ with entries

$$U_{ij} := \frac{\max\{\lambda_i, 0\} + \max\{\lambda_j, 0\}}{|\lambda_i| + |\lambda_j|}, i, j = 1, \cdots, p,$$

and $0/0$ is defined to be 1, λ_i is the i-th eigenvalue of $g(\bar{x}) + \bar{\Omega}$.

Let $d = (d_x, d_\lambda, d_\mu, d_\Omega) \in \Re^n \times \Re^m \times \Re^l \times S^p$ be such that

$$DF(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d = \begin{bmatrix} \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x + \mathcal{J}h(\bar{x})^T d_\lambda + \mathcal{J}q(\bar{x})^T d_\mu + Dg(\bar{x})^* d_\Omega \\ \mathcal{J}h(\bar{x})d_x \\ \mathcal{M}\mathcal{J}q(\bar{x})d_x + diag(q_i(\bar{x}))d_\mu \\ -Dg(\bar{x})d_x + V(Dg(\bar{x})d_x + d_\Omega) \end{bmatrix} = 0. \tag{2.22}$$

By the strict complementary condition (2.9), we can assume that

$$q_1(\bar{x}), \cdots, q_r(\bar{x}) = 0, q_{r+1}(\bar{x}), \cdots, q_l(\bar{x}) < 0,$$

and

$$\mu_1, \cdots, \mu_r > 0, \mu_{r+1}, \cdots, \mu_l = 0.$$

Hence, by the third equation of (2.22), we obtain that

$$\nabla q_i(\bar{x})^T d_x = 0, i = 1, \cdots, r. \tag{2.23}$$

By the fourth equation of (2.22), we denote it shortly as $Dg(\bar{x})d_x := \Delta B$, then

$$V(\Delta B + d_\Omega) = P \left[\begin{array}{cc} \Delta\widetilde{B}_{\alpha\alpha} + \widetilde{d}_{\Omega_{\alpha\alpha}} & U_{\alpha\gamma} \circ (\Delta\widetilde{B}_{\alpha\gamma} + \widetilde{d}_{\Omega_{\alpha\gamma}}) \\ (\Delta\widetilde{B}_{\alpha\gamma} + \widetilde{d}_{\Omega_{\alpha\gamma}})^T \circ U_{\alpha\gamma}^T & 0 \end{array} \right] P^T = \Delta B, \tag{2.24}$$

where $\Delta\widetilde{B} := P^T \Delta B P$. So we have

$$P_\gamma^T \Delta B P_\gamma = 0. \tag{2.25}$$

From the second equation of (2.22), (2.23) and (2.25) we know that

$$d_x \in C(\bar{x}).$$

By the first and second equations of (2.22), we obtain that

$$\begin{aligned} 0 &= \langle d_x, \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x + \mathcal{J}h(\bar{x})^T d_\lambda + \mathcal{J}q(\bar{x})^T d_\mu + Dg(\bar{x})^* d_\Omega \rangle \\ &= \langle d_x, \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x \rangle + \langle d_x, \mathcal{J}h(\bar{x})^T d_\lambda \rangle + \langle d_x, \mathcal{J}q(\bar{x})^T d_\mu \rangle + \langle d_x, Dg(\bar{x})^* d_\Omega \rangle \\ &= \langle d_x, \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x \rangle + \langle d_\lambda, \mathcal{J}h(\bar{x}) d_x \rangle + \langle d_\mu, \mathcal{J}q(\bar{x}) d_x \rangle + \langle d_\Omega, Dg(\bar{x}) d_x \rangle \\ &= \langle d_x, \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x \rangle + \langle d_\Omega, Dg(\bar{x}) d_x \rangle, \end{aligned}$$

which, together with the fourth equation of (2.22) and [17, Proposition 2.3], implies that

$$0 \geq \langle d_x, \nabla_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})d_x \rangle - 2\langle \Omega, Dg(\bar{x})d[g(\bar{x})]^\dagger Dg(\bar{x})d \rangle. \tag{2.26}$$

Hence, we can conclude from $d_x \in C(\bar{x})$ and the second order sufficient condition (2.19) that

$$d_x = 0.$$

Thus, (2.22) reduces to

$$\left[\begin{array}{c} \mathcal{J}h(\bar{x})^T d_\lambda + \mathcal{J}q(\bar{x})^T d_\mu + Dg(\bar{x})^* d_\Omega \\ diag(q_i(\bar{x}))d_\mu \\ V(d_\Omega) \end{array} \right] = 0. \tag{2.27}$$

From (2.21) and $V(d_\Omega) = 0$, we have

$$P_\alpha^T d_\Omega P_\alpha = 0, P_\alpha^T d_\Omega P_\gamma = 0. \tag{2.28}$$

By the constraint nondegeneracy condition (2.8), there exist a vector $d \in \Re^n$, a matrix $S \in \text{lin}(\mathcal{T}_{S_+^p} g(\bar{x}))$, and a vector $d_1 \in \text{lin}(\mathcal{T}_{\Re_-^l} q(\bar{x}))$ such that

$$\mathcal{J}h(\bar{x})d = d_\lambda, \mathcal{J}q(\bar{x})d + d_1 = d_\mu, Dg(\bar{x})d + S = d_\Omega. \tag{2.29}$$

where

$$\text{lin}(\mathcal{T}_{S_+^p} g(\bar{x})) = \{H \in S^p : P_\gamma^T H P_\gamma = 0\},$$

$$\mathrm{lin}(\mathcal{T}_{\Re^l_-} q(\bar{x})) = \{h \in \Re^l : h_i = 0, i \in I(\bar{x})\}, I(\bar{x}) = \{i | q_i(\bar{x}) = 0, 1 \leq i \leq l\}.$$

Therefore, by (2.28), (2.29) and the first equation of (2.27), we have

$$\begin{aligned}
&\langle d_\lambda, d_\lambda\rangle + \langle d_\mu, d_\mu\rangle + \langle d_\Omega, d_\Omega\rangle \\
=&\langle \mathcal{J}h(\bar{x})d, d_\lambda\rangle + \langle \mathcal{J}q(\bar{x})d, d_\mu\rangle + \langle d_1, d_\mu\rangle + \langle Dg(\bar{x})d + S, d_\Omega\rangle \\
=&\langle d, \mathcal{J}h(\bar{x})^T d_\lambda\rangle + \langle d, \mathcal{J}q(\bar{x})^T d_\mu\rangle + \langle d, Dg(\bar{x})^* d_\Omega\rangle + \langle S, d_\Omega\rangle + \langle d_1, d_\mu\rangle \\
=&\langle d, \mathcal{J}h(\bar{x})^T d_\lambda + \mathcal{J}q(\bar{x})^T d_\mu + Dg(\bar{x})^* d_\Omega\rangle + \langle S, d_\Omega\rangle + \langle d_1, d_\mu\rangle \\
=&\langle S, d_\Omega\rangle + \langle d_1, d_\mu\rangle \\
=&\langle P^T S P, P^T d_\Omega P\rangle + \langle d_1, d_\mu\rangle \\
=&0,
\end{aligned}$$

namely,

$$d_\lambda = d_\mu = d_\Omega = 0.$$

Thus, together with $d_x = 0$, we obtain that $DF(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$ is nonsingular. □

Remark 2.1 *The above proof is similar to that of Proposition 3.2 in Sun[17], here we need to consider the inequality constraints.*

3 Perturbation analysis of problem (NLSDP)

Consider the parameterized nonlinear semidefinite programming problem in the form

$$(P_u) \begin{cases} \min & f(x, u) \\ \text{s.t.} & h(x, u) = 0_m, \\ & q(x, u) \leq 0_l, \\ & g(x, u) \in S^p_+, \end{cases} \tag{3.30}$$

depending on the parameter vector u varying in a Banach space $\mathcal{U}$. Here $f : \Re^n \times \mathcal{U} \to \Re$, $h : \Re^n \times \mathcal{U} \to \Re^m$, $q : \Re^n \times \mathcal{U} \to \Re^l$, and $g : \Re^n \times \mathcal{U} \to S^p$, S^p is the linear space of all $p \times p$ real symmetric matrices, and S^p_+ is the cone of all $p \times p$ positive semidefinite matrices. We assume that for a given value $\bar{u}$ of the parameter vector, problem $(P_{\bar{u}})$ coincides with the unperturbed problem (NLSDP).

Now, we are ready to state the main result of this paper.

Theorem 3.1 *Suppose that $f(x, u)$, $h(x, u)$, $q(x, u)$ and $g(x, u)$ in (P_u) all have second partial derivatives with respect to x which are jointly continuous. Let $\bar{u} \in \mathcal{U}$, and let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$ be a feasible point to the $(P_{\bar{u}})$ problem at which the Jacobian uniqueness conditions are satisfied.*

Then, there exist open neighborhoods $M(\bar{u}) \subset \mathcal{U}$ and $N(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega}) \subset \Re^n \times \Re^m \times \Re^l \times S^p$, and a continuous function $\mathcal{Z} : M \to N$, such that $\mathcal{Z}(\bar{u}) = (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$, and for each $u \in M$, $\mathcal{Z}(u)$ is both the unique KKT point of (P_u) in N and the unique zero in N of the function $F(\cdot, \cdot, \cdot, \cdot, u)$. Further, if $\mathcal{Z}(u) := (x(u), \lambda(u), \mu(u), \Omega(u))$, then for each $u \in M$, $x(u)$ is an isolated local minimizer of (P_u) at which the Jacobian uniqueness conditions are satisfied.

Proof. By Lemma 2.3, the Jacobian of F is nonsingular at $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$, then, by the implicit-function theorem[11, Theorems 1-2(4.XVII)], there exist open neighborhoods $M_0(\bar{u}) \subset \mathcal{U}$ and $N_0(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega}) \subset \Re^n \times \Re^m \times \Re^l \times S^p$, and a continuous function $\mathcal{Z} : M_0 \to N_0$ such that $\mathcal{Z}(\bar{u}) = (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$, and for each $u \in M_0$, $\mathcal{Z}(u)$ is the unique zero of $F(\cdot, \cdot, \cdot, \cdot, u)$ in N_0.

In addition, there are some open neighborhoods $M_1(\bar{u})$ and $N_1(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$ such that for $(x, \lambda, \mu, \Omega) \in N_1$ and $u \in M_1$, we have for $\forall 1 \le i \le l$ and $\forall 1 \le j \le p$

$$q_i(\bar{x}, \bar{u}) < 0 \Longrightarrow q_i(x, u) < 0, \tag{3.31a}$$

$$\bar{\mu}_i > 0 \Longrightarrow \mu_i > 0, \tag{3.31b}$$

$$\lambda_j(g(\bar{x}, \bar{u})) > 0 \Longrightarrow \lambda_j(g(x, u)) > 0, \tag{3.31c}$$

$$\lambda_j(\bar{\Omega}) < 0 \Longrightarrow \lambda_j(\Omega) < 0. \tag{3.31d}$$

$\lambda_j(\cdot)$ means the j-th eigenvalue function. Let $N := N_0 \cap N_1$, then since $\mathcal{Z}$ is continuous and $\mathcal{Z}(\bar{u}) = (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$, it is possible to find an open neighborhood $M_2(\bar{u}) \subset M_1 \cap M_0$ such that $u \in M_2$ implies $\mathcal{Z}(u) \in N$. Because $F(\mathcal{Z}(u), u) = 0$, then for each $u \in M_2$, $\mathcal{Z}(u)$ satisfies the KKT condition for (P_u). Let $\mathcal{Z}(u) = (\tilde{x}, \tilde{\lambda}, \tilde{\mu}, \tilde{\Omega})$, and let i be chosen with $i : 1 \le i \le l$. If $\bar{\mu}_i > 0$, then since $\mathcal{Z}(u) \in N_1$ we have also $\tilde{\mu}_i > 0$, so $q_i(\tilde{x}, u) = 0$; on the other hand, if $q_i(\bar{x}, \bar{u}) < 0$, then $q_i(\tilde{x}, u) < 0$, so $\tilde{\mu}_i = 0$. Since strict complementary condition holds at $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\Omega})$, for each i, one of these two cases is applicable. Hence $\tilde{\mu} \ge 0$ and $q(\tilde{x}, u) \le 0$, similarly, we have $\tilde{\Omega} \preceq 0$ and $g(\tilde{x}, u) \succeq 0$, which means that $\tilde{x}$ is a feasible point of (P_u), so $\mathcal{Z}(u)$ is the KKT point of (P_u), it is the only such point in N because it is the only zero of $F(\cdot, \cdot, \cdot, \cdot, u)$ there. Further, if $\mathcal{Z}(u) := (x(u), \lambda(u), \mu(u), \Omega(u))$, we note that (3.31) implies that for each $u \in M_2$, precisely the same inequality constraints will be active at $x(u)$, as were active at $x(\bar{u})$, and that strict complementary condition will hold for $\mathcal{Z}(u)$.

Since the constraint nondegenerate condition holds at $x(\bar{u})$, by Lemma 2.2, the operator $\mathcal{A}(x(\bar{u}), \bar{u})$ is onto, here $\forall d \in \Re^n$,

$$\mathcal{A}(x(u), u)d = (\mathcal{J}_x h(x(u), u)d; \nabla_x q_i(x(u), u)^T d, i \in I(x(u), u); P_\gamma(u)^T D_x g(x(u), u) d P_\gamma(u)), \tag{3.32}$$

where $\forall u \in M_2$, $P_\gamma(u)$ is a $p \times |\gamma|$ matrix whose columns form an orthonormal basis of the eigenvector space of $g(x(u), u)$ corresponding to its smallest eigenvalue 0. Since the eigenvectors in $P_\gamma(u)$ are not uniquely defined and $P_\gamma(u)$ is not a continuous function of u near $\bar{u}$ unless 0 is a simple eigenvalue of $g(x(\bar{u}), \bar{u})$. In order to overcome this difficulty we proceed as follows.

For arbitrary $u \in M_2$, we denote $g(u) := g(x(u), u)$, $g(\bar{u}) := g(x(\bar{u}), \bar{u})$. Let $L(g(u))$ the eigenspace corresponding to the smallest eigenvalue 0 of $g(u)$, and let $P(g(u))$ be the orthogonal projection matrix onto $L(g(u))$, also let P_γ be (fixed) $p \times |\gamma|$ matrix whose columns are orthonormal and span the space $L(g(\bar{u}))$. It is known that $P(g(u))$ is a continuously differentiable function of $g(u)$ in a sufficiently small neighborhood of $g(\bar{u})$, consequently, $E(g(u)) := P(g(u))P_\gamma$ is also a continuously differentiable function of $g(u)$ in a neighborhood of $g(\bar{u})$, and moreover $E(g(\bar{u})) = P_\gamma$. It follows that for all $g(u)$ sufficiently close to $g(\bar{u})$, the rank of $E(g(u))$ is $|\gamma|$, i.e., its column vectors are linearly independent. Let $S(g(u))$ be the matrix whose columns are obtained by applying the Gram-Schmidt orthonormalization procedure to the columns of $E(g(u))$. The matrix $S(g(u))$ is well defined and continuously differentiable near $g(\bar{u})$, and moreover satisfies the following conditions: $S(g(\bar{u})) = P_\gamma$, the column space of $S(g(u))$ coincides with the column space of $P_\gamma(u)$, and $S(g(u))^T S(g(u)) = I_{|\gamma|}$. Hence, in a sufficiently small neighborhood of $g(\bar{u})$, $\mathcal{A}(x(u), u)$ can be defined in the form $\forall d \in \Re^n$,

$$\mathcal{A}(x(u), u)d = (\mathcal{J}_x h(x(u), u)d; \nabla_x q_i(x(u), u)^T d, i \in I(x(u), u); S(g(u))^T D_x g(x(u), u) d S(g(u))), \tag{3.33}$$

by the continuity of $\mathcal{Z}$, $\mathcal{A}(x(u), u)$ will be continuous as function of u in a sufficiently small neighborhood of $\bar{u}$. So there exist an open neighborhood $M_3(\bar{u}) \subset M_2$, $\mathcal{A}(x(u), u)$ is also onto, i.e., $\forall u \in M_3$, the constraint nondegenerate condition holds at $x(u)$.

Now we shall show that the second order sufficient condition also holds at $\mathcal{Z}(u)$. We observe first that $\mathcal{Z}(u)$ is KKT point by each $u \in M_3$, if the set of active gradients (the dimension of $\mathcal{A}(x(u), u)$) contains n vectors, the second order sufficient condition is trivially satisfied by $\mathcal{Z}(u)$ for each $u \in M_3$, and there is nothing more to prove. We may therefore assume with no loss of generality that there are fewer than n active gradients. For each $u \in M_3$, consider the set-valued function

$$\Gamma(u) := \{d \in \Re^n : \mathcal{A}(x(u), u)d = 0\}.$$

Obviously, the graph of Γ is closed; thus, if we let B be the unit sphere in $\Re^n$, the function $\Gamma(u) \cap B$ is upper semicontinuous on M_3, by our assumption about the number of active gradients, $\Gamma(u) \cap B$ is also nonempty. Then consider the functional

$$\delta(u) := \min_{d \in \Gamma(u) \cap B} \left\{ d^T \nabla^2_{xx} L(x(u), \lambda(u), \mu(u), \Omega(u))d - 2\langle \Omega(u), Dg(u)d[g(u)]^\dagger Dg(u)d \rangle \right\}.$$

Let $T \in S^p$ have the following spectral decomposition: $T = Q\Lambda(T)Q^T$, and let $\varphi : \Re \to \Re$ be a scalar function as follows:

$$\varphi(t) = \begin{cases} \dfrac{1}{t} & , t > 0, \\ 0 & , t = 0, \end{cases}$$

the corresponding Löwner operator $\Phi(T) : S^p \to S^p$ is defined by

$$\Phi(T) := \sum_{i=1}^{p} \varphi(\lambda_i(T)) q_i {q_i}^T, T \in S^p,$$

so $[g(u)]^\dagger$ can be seen as the Löwner operator about $\varphi(\cdot)$, i.e., $[g(u)]^\dagger = \Phi(g(u))$. By the continuity and the strict complementary condition of $\mathcal{Z}(u)$, we can know that $\varphi(\lambda_i(g(u)))$ is continuous in u on M_3, then $[g(u)]^\dagger$ is also continuous in u on M_3. Thus, the quantity being minimized is jointly continuous in u and d, and since the constraint set $\Gamma(u) \cap B$ is upper semicontinuous in u, follows from [1, Theorem 2,p.116] that δ is lower semicontinuous in u for $u \in M_3$. Also since the second order sufficient condition holds for $\mathcal{Z}(\bar{u})$, we have $\delta(\bar{u}) > 0$; hence, there exist an open neighborhood $M(\bar{u}) \subset M_3$ such that for each $u \in M$, we also have $\delta(u) > 0$, i.e., for each $u \in M$, $\mathcal{Z}(u)$ satisfies the second order sufficient condition. □

4 conclusion

In this paper, we consider the stability of a class of perturbed problems of nonlinear semidefinite programming problems which are more general than the $\mathcal{C}^2$-smooth parameterization. $\mathcal{C}^2$-smooth parameterization require that the objective function and the constraint mapping of perturbed problem are twice continuously differentiable, but we assume that they all have second partial derivatives only with respect to the decision variable which are jointly continuous. We showed that when a feasible point to the unperturbed problem satisfies the Jacobian uniqueness conditions, the Jacobian of the KKT system is nonsingular at the KKT point, then by using implicit-function theorem, we proved that the locally optimal solution to the perturbed problem is isolated and can be formulate by parameter vector. Finally, we showed that the Jacobian uniqueness conditions also hold at some feasible point to the perturbed problem.

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