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Robust Combinatorial Optimization under Convex and Discrete Cost Uncertainty

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Abstract In this survey, we discuss the state-of-the-art of robust combinatorial optimization under uncertain cost functions. We summarize complexity results presented in the literature for various underlying problems, with the aim of pointing out the connections between the different results and approaches, and with a special emphasis on the role of the chosen uncertainty sets. Moreover, we give an overview over exact solution methods for *NP*-hard cases. While mostly concentrating on the classical concept of strict robustness, we also cover more recent two-stage optimization paradigms.

Keywords Robust Optimization · Uncertainty · Combinatorial Optimization · Two-stage Robustness · K-Adaptability · Complexity

1 Introduction

Combinatorial optimization problems arise in many real-world applications, e.g., in the fields of economy, industry, or transport logistics. For many such problems, theoretically (or practically) fast algorithms have been developed under the assumption that all problem data is known precisely. However, the situation becomes more complex when considering uncertainty in the problem parameters. For example, the travel times for the shortest path problem or the vehicle routing problem can be subject to uncertainty, since we cannot predict the exact traffic situation in the future. One successful approach to tackle uncertainty in the input data is *robust optimization*: for a given set U containing all relevant scenarios, i.e., all sufficiently likely realizations of the uncertain parameters, a solution is

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sought that is feasible for every scenario in U and that is worst-case optimal under this constraint. This idea was first introduced by Soyster in [106]. The approach received increasing attention in the late 1990s. Kouvelis and Yu studied finite uncertainty sets U for several combinatorial optimization problems in [80]. Almost at the same time, Ben-Tal and Nemirovski studied robust convex problems with conic or ellipsoidal uncertainty sets [22,23]. Furthermore, El Ghaoui et al. applied the idea to semi-definite problems and least-squares problems [57,56]. Later, Bertsimas and Sim introduced budgeted uncertainty sets to reduce what they call the *Price of Robustness* [35]. A survey over robust optimization approaches for discrete and interval uncertainty can be found in [9]. The different uncertainty sets and their robust counterparts are intensively studied in [85].

Subsequently, new robust optimization paradigms were presented and studied in the literature, with the main objective of making the approach better applicable to practical problems. Besides various two-stage approaches [21,86,3], which we will discuss in detail in Section 4, several other paradigms have been investigated, e.g., min-max regret robustness [16,72,50,80,15,5,6,7] or the light robustness approach [61,59,103]. Surveys studying several of the different approaches can be found in [9,25,62,76,24,64,37], they also cover distributional robustness, which forms a connection between robust and stochastic optimization.

In the present survey, we consider general combinatorial optimization problems of the form

$$\min_{x \in X} \; c^\top x \tag{P}$$

where $X \subseteq \{0,1\}^n$ describes the certain set of feasible solutions and where only the cost-vector $c \in \mathbb{R}^n$ is subject to uncertainty. In particular, we assume that an uncertainty set $U \subseteq \mathbb{R}^n$ is given which contains all possible cost-vectors c. The classical robust counterpart of Problem (P) is then given by Problem

$$\min_{x \in X} \; \max_{c \in U} \; c^\top x \,. \tag{RP}$$

In contrast to other surveys on this topic, we aim at pointing out the differences between several common classes of uncertainty sets, with a focus on ellipsoidal uncertainty; see Section 2. In Section 3, we will sort and structure the complexity results for Problem (RP) achieved in the literature for several underlying combinatorial problems, again with a focus on the role of the chosen class of uncertainty set. Typical complexity results for Problem (RP) are illustrated for the most elementary case $X = \{0,1\}^n$, including sketches of the main proofs. Furthermore, we will discuss exact methods to solve Problem (RP) for the *NP*-hard cases, covering IP-based methods as well as oracle-based algorithms, which can be applied to every combinatorial problem (P) given by an optimization oracle. Finally, in Section 4, we will give an overview over various robust two-stage approaches presented in the literature and point out the connections between them.

2 Common Uncertainty Sets

The choice of the uncertainty set U is crucial in the design of the robust counterpart (RP). On the one hand, this choice should reflect the situation given in the application and lead to a realistic model of the given uncertainty, including

the user's attitude towards risk. On the other hand, the choice of U influences the tractability of the resulting problem (RP). For this reason, many different types of uncertainty sets have been investigated in the literature, and are still being proposed. Roughly, most of these uncertainty sets can be classified as discrete, polyhedral, or ellipsoidal sets. For a study on the geometric relationship between the common uncertainty classes see [85].

2.1 Discrete Uncertainty

Discrete (or scenario-based) uncertainty sets are finite sets $U = \{c_1, \ldots, c_m\}$. They form the most intuitive case of uncertainty sets. In practice, uncertain problem data is often given as a finite list of scenarios observed in the past, e.g., the prices of stocks in portfolio optimization or the shipping volumes in the design of a transport network.

Unfortunately, in spite of their conceptual simplicity, assuming discrete uncertainty nearly always leads to intractable robust counterparts; see Section 3.1.1. In fact, for many well-studied underlying combinatorial problems, such as the shortest path problem or the spanning tree problem, the robust counterpart (RP) turns out to be weakly NP-hard if the number m of scenarios is fixed and strongly NP-hard if m is part of the input.

2.2 Polyhedral Uncertainty

Even if it seems natural in practice to define uncertain costs by a finite list of possible scenarios, in particular when only finitely many observations from the past are available, there is no reason to exclude convex combinations of these scenarios: if two scenarios are likely enough to appear, then why should the convex combination of them not be a likely scenario? This leads to the concept of polyhedral (or polytopal) uncertainty, as polytopes are exactly the convex hulls of finite sets.

For most models of robust optimization, including the robust counterpart defined in (RP), it is easy to prove that changing from U to its convex hull does not change the problem, as the worst case in the inner maximization problem will be attained in an extreme point anyway. This seems to suggest that discrete and polytopal uncertainty sets are equivalent. However, this is not true for all robust two-stage optimization paradigms; see Section 4. Moreover, even if the equivalence holds from an abstract point of view, it does not hold from an algorithmic or complexity-theoretic point of view: the convex hull of m points can have exponentially many facets in m, and, vice versa, the number of vertices of a polytope can be exponential in the number of its facets. In particular, complexity results do not necessarily carry over from the discrete to the polyhedral case or vice versa.

In fact, the number of vertices is exponential for one of the most widely used polyhedral uncertainty set, namely interval uncertainty. Here, every objective function coefficient can vary independently in an interval $[l_i, u_i]$, so that U is an axis-parallel box

$$U = \prod_{i=1}^{n} [l_i, u_i] = \{c \in \mathbb{R}^n \mid l \leq c \leq u\} =: [l, u] \ .$$

Note that the number of vertices is 2^n here, so that a reduction to the discrete case is not efficient. However, using interval uncertainty, the classical robust counterpart is as easy to solve as the underlying problem, since we can just replace every uncertain coefficient by u_i.

On the other hand, interval uncertainty leads to very conservative solutions, as it takes into account the possibility that every cost coefficient attains its worst case value independently. In an effort to mitigate this effect, the concept of budget uncertainty (also called Gamma-uncertainty) has been introduced [35]. Building on the interval uncertainty set, the idea is to allow only a fixed number Γ of coefficients to deviate from their mean values. This leads to the uncertainty set

$$U = \{c \in \mathbb{R}^n \mid l \leq c \leq u,\ c_i = \tfrac{l_i+u_i}{2} \forall i \notin I,\ I \subseteq \{1,\dots,n\},\ |I| \leq \Gamma\}\ .$$

Dealing with a minimization problem and since $X \subseteq \{0,1\}^n$, we only need to consider positive deviations in the coefficients. For the classical robust counterpart (RP) we can equivalently consider the uncertainty set

$$U = \Big\{ c_0 + \sum_{i=1}^n \delta_i d_i e_i \mid \sum_{i=1}^n \delta_i \leq \Gamma, \delta \in \{0,1\}^n \Big\}\ , \tag{1}$$

as it was first introduced in [35], where $c_0 = \frac{1}{2}(u+l)$ is the center of the box $[l,u]$ and $d = \frac{1}{2}(u-l)$ are the maximum deviations from the center, and e_i denotes the i-th unit vector. Note that this set U is not a polytope in general, but when replacing it by its convex hull

$$\begin{aligned} \operatorname{conv}(U) =& \Big\{ c_0 + \sum_{i=1}^n \delta_i d_i e_i \mid \sum_{i=1}^n \delta_i \leq \Gamma, \delta \in [0,1]^n \Big\} \\ =& \Big\{ c \in \mathbb{R}^n \mid c_0 \leq c \leq c_0 + d, \sum_{i=1}^n \tfrac{1}{d_i}(c-c_0)_i \leq \Gamma \Big\}\ , \end{aligned} \tag{2}$$

assuming $\Gamma \in \mathbb{N}$, the problem becomes a special case of polyhedral uncertainty. Since the original set U is finite here, budget uncertainty could also be considered a special case of discrete uncertainty; see Section 2.1 above. However, if Γ is not fixed, the number of scenarios is exponential in n, so that this viewpoint may be problematic in terms of complexity. Note that all three versions of the budgeted uncertainty set are equivalent for the classical robust counterpart (RP) while in general all sets differ from each other and can lead to different solutions, e.g. in the case of two-stage robustness; see Section 4.

An alternative approach is to bound the absolute deviation of c from the mean c_0 by Γ; see Section 4.1. All these models have in common that, on the one hand, they cut off the extreme scenarios in the interval uncertainty set and thus lead to less conservative solutions, and on the other hand, they usually yield tractable robust counterparts, assuming that the underlying problem is tractable; see Section 3.2.1.

Several extensions of budgeted and general polyhedral uncertainty sets have been devised in the literature. In multi-band uncertainty sets, the single deviation band is partitioned into multiple sub-bands for which the concept of budgeted uncertainty is applied independently [43, 44, 53]. The concept of decision-dependent uncertainty sets was studied in [96, 83]. The authors consider uncertainty sets

$U(x)$ which depend on the decision x. This concept was also applied to budgeted uncertainty sets, assuming that the parameter Γ is not fixed but a function of the solution x [99, 105].

In general, for arbitrary polytopes U, the robust counterpart (RP) turns out to be strongly NP-hard, no matter whether U is given by an inner or by an outer description; see Section 3.1.2.

2.3 Ellipsoidal Uncertainty

Both discrete and polytopal uncertainty sets often depend on a collection of observed scenarios. These finitely many scenarios are only an approximation of the real distribution of the cost coefficients. In particular, when making the reasonable assumption that the cost coefficents are normally distributed, the confidence sets turn out to be ellipsoids of the form

$$U = \left\{ c \in \mathbb{R}^n \mid (c - c_0)^\top \Sigma^{-1} (c - c_0) \leq r^2 \right\}, \tag{3}$$

where $c_0 \in \mathbb{R}^n$ is the expected value of the uncertain objective vector c and Σ is the covariance matrix of the entries of c – for sake of simplicity, we assume $\Sigma \succ 0$ here. The parameter r describes the level of confidence, i.e., the risk the user is willing to take – a larger r leads to more conservative solutions. Given the set (3), it is easy to see, e.g., using conic duality, that (RP), can be rewritten as a non-linear optimization problem

$$\min_{x \in X} \; c_0^\top x + r \cdot \sqrt{x^\top \Sigma x} \; . \tag{4}$$

The existence of a smooth closed form expression of the objective function distinguishes ellipsoidal uncertainty from the uncertainty sets discussed above; it forms the basis of many solution approaches.

Problems of the form (4) are also known as mean-risk optimization problems in the literature, as their objective function is a weighted sum of the mean $c_0^\top x$ and the risk $\sqrt{x^\top \Sigma x}$ of the chosen solution x. Often, the risk part is modeled as $x^\top \Sigma x$, which may lead to different optimal solutions. Mean-risk optimization is particularly popular in portfolio optimization, where the concept was introduced already in the 1950s by Henry Markowitz [87]. For a comprehensive overview over mathematical methods of portfolio optimization, see [54].

Another natural way to derive ellipsoidal uncertainty sets is by considering the so-called Value-at-Risk model. The objective (in the minimization case) is to find a feasible solution $x \in X$ and a value $z \in \mathbb{R}$ such that the probability of x having an objective value worse than z is at most a given $\varepsilon \in (0, 1)$. Under this condition, the aim is to minimize z. The resulting problem thus reads

$$\begin{aligned} \min \; & z \\ \text{s.t. } & \Pr(c^\top x \geq z) \leq \varepsilon \\ & x \in X \; . \end{aligned}$$

Assuming again that the entries of c are normally distributed, i.e., $c \sim \mathcal{N}(c_0, \Sigma)$, one can show that the constraint $\Pr(c^\top x \geq z) \leq \varepsilon$ is equivalent to

$$z \geq c_0^\top x + \Phi^{-1}(1 - \varepsilon) \sqrt{x^\top \Sigma x} \; ,$$

where Φ is the cumulative distribution function of the standard normal distribution. In summary, the above problem can be recast as (4) with $r := \Phi^{-1}(1-\varepsilon)$.

Usually, full information about the given distribution of c is not available in practice. However, one may approximate the mean c_0 and the covariance matrix Σ by means of a finite set of observations. This is often done in portfolio optimization; see, e.g., [49]. Arguably, the resulting normal distribution yields a more realistic model of the inherent uncertainty than the finite set of observations itself.

In general, ellipsoidal uncertainty sets lead to intractable counterparts (RP) again. However, in the special case of uncorrelated cost coefficients – or when correlations are ignored in the model – the complexity-theoretic situation becomes more interesting. We then have $\Sigma = \mathrm{Diag}(\sigma)$ for some $\sigma \in \mathbb{R}^n_+$ and, using binarity, we can rewrite Problem (4) as

$$\min_{x \in X} \; c_0^\top x + \sqrt{\sigma^\top x}. \tag{5}$$

Surprisingly, up to our knowledge, it is an open problem whether (5) is tractable for all X for which the underlying problem (P) is tractable. In particular, no tractable (certain) combinatorial optimization problem is known for which (5) turns out to be *NP*-hard. It is known however that an FPTAS exists as soon as the underlying problem admits an FPTAS [94]. Furthermore (5) can be solved in polynomial time if X is a matroid. We further discuss this in Section 3.2.2.

2.4 General Norms

Many of the uncertainty sets discussed above can be defined by means of a norm function $\|.\| \colon \mathbb{R}^n \to \mathbb{R}$. Indeed, it is a natural approach to assume that the cost function c can vary within a certain radius around the expected scenario c_0, with respect to a given norm. The resulting uncertainty set is thus of the form

$$U = \{c \in \mathbb{R}^n \mid \|c - c_0\| \leq r\} \ , \tag{6}$$

where r again models the risk-averseness of the user. Defining $\|c\| := \sqrt{c^\top \Sigma^{-1} c}$, we obtain ellipsoidal uncertainty as a special case, while the ∞-norm, after an approriate scaling, gives rise to interval uncertainty. The convex variants of budgeted uncertainty correspond to a combination of the ∞-norm with the 1-norm; the two latter norms give rise to polytopal uncertainty sets. In general, a closed form expression for the robust counterpart (RP) for the set in (6) is

$$\min_{x \in X} \; c_0^\top x + r\|x\|^* \ ,$$

where $\|.\|^*$ is the dual norm to $\|.\|$; see also [33].

In analogy to ellpsoidal uncertainty sets corresponding to the 2-norm, one can consider uncertainty sets based on the p-norm. Many results for the ellipsoidal case can be easily generalized to p-norms with $p \in (1, \infty)$, e.g., the tractability of the robust counterpart for uncorrelated costs in the case $X = \{0,1\}^n$ [71].

2.5 Unbounded Uncertainty Sets

Most uncertainty sets considered in the literature are bounded. Essentially, this can be assumed without loss of generality. Indeed, if an unbounded direction a of U exists, it is easy to see that the inner maximization problem yields an implicit linear constraint $a^\top x \leq 0$, since for all $x \in \mathbb{R}^n$ violating this constraint the inner maximum in (RP) is infinite. Adding this constraint explicitly allows to remove the unbounded direction from U. In other words, allowing unbounded directions in U means to allow to impose linear constraints on X. Note that if U is convex and unbounded, there always exists an unbounded direction. If uncertain constants are taken into account, as discussed in Section 2.6 below, it is even possible to model an affine linear constraint of the form $a^\top x \leq b$ by adding an unbounded direction $(a, -b)$ to U.

2.6 Uncertain Constants

For a certain problem of the form (P), adding a constant in the objective function does not have any affect on the set of optimal solutions, so that constants are usually not considered explicitly. However, this changes in the uncertain setting, as also the constant may be uncertain. Nevertheless, this is usually not covered in the literature. Clearly, the robust counterpart including a constant,

$$\min_{x \in X} \max_{(c,\bar{c}) \in U} c^\top x + \bar{c}\,, \tag{RPC}$$

is at least as hard as Problem (RP) for most classes of uncertainty sets, as the case of a certain constant can be modeled as a special case. It can be shown that also the reverse is true for most classical combinatorial optimization problems, i.e., including the uncertain constant does not increase the complexity of the problem [82]. Besides others, this is true for the shortest-path problem, the spanning-tree problem, the knapsack problem, and the unconstrained binary problem where $X = \{0,1\}^n$. On the other hand, allowing an uncertain constant often simplifies *NP*-hardness proofs, as we will see in the following section.

3 Strictly Robust Optimization

We consider the strictly robust counterpart (RP) of the underlying problem (P). We are mostly interested in the complexity of (RP), which of course depends both on the feasible set X and the uncertainty set U. We start by reviewing the complexity results for general discrete, polyhedral, and ellipsoidal uncertainty sets in Section 3.1. In Section 3.2, we will focus on uncertainty sets that often lead to tractable robust counterparts. In Section 3.3, we will survey possible solution approaches for *NP*-hard cases.

3.1 Complexity for General Sets

It turns out that the strictly robust counterpart (RP) is often *NP*-hard for general discrete, polyhedral, and ellipsoidal uncertainty sets, even in cases where the underlying problem (P) is tractable. Some of the main hardness results are collected

in the following subsections, where we distinguish between weakly and strongly *NP*-hard variants of (RP) and also cover the question whether polynomial time approximation schemes exist. For the convenience of the reader we will present proofs for the most elementary case $X = \{0,1\}^n$, which is usually not considered in the literature.

3.1.1 Discrete Uncertainty

The robust counterpart (RP) of many classical combinatorial optimization problems, including the shortest path and the spanning tree problem, is *NP*-hard even if U contains only two scenarios [80,6,7,19]. We now illustrate this by giving a short proof of *NP*-hardness for the most elementary case $X = \{0,1\}^n$. In particular, this shows that the hardness is not related to any combinatorial structure of X, but only stems from the integrality constraints.

Theorem 1 *The robust counterpart* (RP) *is* NP*-hard for* $X = \{0,1\}^n$ *if the uncertainty set* U *contains exactly two scenarios.*

Proof By the discussion in Section 2.6, it suffices to show the *NP*-hardness of Problem (RPC) for the case of two scenarios, i.e., $U = \{(c_1, \bar{c}_1), (c_2, \bar{c}_2)\}$. We describe a polynomial reduction of the (weakly) *NP*-hard Subset Sum problem to (RPC). Given integers $s_1, \dots, s_n$ and S, we have to decide whether there exists a subset $I \subseteq \{1, \dots, n\}$ with $\sum_{i \in I} s_i = S$. We construct a corresponding instance of (RPC) by setting $(c_1, \bar{c}_1) = (s, -S)$ and $(c_2, \bar{c}_2) = (-s, S)$. We then have

$$\min_{x \in \{0,1\}^n} \max\{c_1^\top x + \bar{c}_1, c_2^\top x + \bar{c}_2\} = \min_{x \in \{0,1\}^n} |s^\top x - S| \ .$$

It follows that there exists a set $I \subseteq \{1, \dots, n\}$ with $\sum_{i \in I} s_i = S$ if and only if the optimal value of the constructed instance of (RPC) is zero. □

Nevertheless, for several problems, e.g., the shortest path problem, the spanning-tree problem, the knapsack problem and the unconstrained binary problem, pseudopolynomial algorithms have been found under the assumption that the number m of scenarios is fixed [80,8,19]. Most of the latter algorithms are based on dynamic programming. As a simple example, we observe

Theorem 2 *For each fixed* m*, the robust counterpart* (RP) *admits a pseudopolynomial algorithm as well as a PTAS for* $X = \{0,1\}^n$ *if the uncertainty set* U *contains exactly* m *scenarios. It admits an FPTAS if* $m = 2$*.*

Proof The problem can be reduced to the solution of m multidimensional knapsack problems with $m-1$ constraints each [19]: given $U = \{c_1, \dots, c_m\}$, the i-th of these problems reads

$$\begin{aligned} \min \ & c_i^\top x \\ \text{s.t. } & c_i^\top x \geq c_j^\top x \text{ for all } j \in \{1, \dots, m\} \setminus \{i\} \\ & x \in \{0,1\}^n \ . \end{aligned}$$

The latter problems can be solved by pseudopolynomial algorithms and by a PTAS, if m is fixed, and by an FPTAS if $m = 2$; see, e.g., [78]. □

Note however that there exists no FPTAS for the bidimensional knapsack problem unless $P = NP$, so that the above construction does not directly yield an FPTAS for $m \geq 3$ scenarios. Nevertheless, it has been shown that the min-max versions of the shortest-path problem, the spanning-tree problem and the knapsack problem all admit an FPTAS [5] for a fixed number of scenarios.

Aissi et. al. prove that a pseudopolynomial algorithm for the min-max problem with a fixed number of scenarios always exists if the underlying search problem can be solved in polynomial time [8]. Here the underlying search problem is the problem of finding, for a given objective value and a given cost vector, a solution which attains the value with respect to the given cost vector, or returns that no such solution exists.

An interesting problem in its robust version is the min-cut problem. While the robust min s-t-cut problem is strongly NP-hard even if the number of scenarios is fixed, the robust min-cut problem can be solved in polynomial time [11, 7]. To the best of our knowledge it is still an open question whether the robust assignment problem is weakly or strongly NP-hard for a fixed number of scenarios.

When considering an unbounded number of scenarios, all of the mentioned problems become strongly NP-hard in their robust min-max versions [80, 6, 7]. Again, we include a proof for the unconstrained binary problem, as to the best of our knowledge this result has not been proved in the literature yet.

Theorem 3 *The robust counterpart* (RP) *is strongly* NP*-hard for* $X = \{0,1\}^n$ *if the uncertainty set* U *is finite but unbounded.*

Proof Again, it suffices to show NP-hardness of Problem (RPC) containing an uncertain constant, for a finite set $U = \{(c_1, \bar{c}_1), \ldots, (c_m, \bar{c}_m)\}$. For the reduction, we use the strongly NP-hard Set Cover problem: for $k \in \mathbb{N}$, a given set of elements $I = \{1, \ldots, m\}$, and a set of subsets $J \subseteq 2^I$, the problem is to decide if there exists a set of at most k subsets contained in J, such that each $i \in I$ is contained in at least one of the subsets. We define an instance of the robust unconstrained binary problem (with uncertain constant) as follows: we set $X = \{0,1\}^{|J|}$ and define for each $i \in I$ a scenario $(c_i, 0) \in \mathbb{R}^{|J|+1}$ where $(c_i)_j = 1$ if element i is contained in the j-th subset of J and 0 otherwise. Furthermore, we add another scenario $(-M\mathbf{1}, Mk + 1)$, where $\mathbf{1}$ is the all-one vector and M is big enough. If U is defined as the set of all constructed scenarios, there exists a solution for the set cover problem if and only if problem

$$\max_{x \in \{0,1\}^{|J|}} \min_{(c,\bar{c}) \in U} c^\top x + \bar{c}$$

has an optimal value greater or equal to 1. □

An overview over the complexity of min-max problems under discrete uncertainty sets can be found in [9].

3.1.2 Polyhedral Uncertainty

For general polyhedral uncertainty, Problem (RP) is NP-hard for most of the classical combinatorial problems, since we can easily reduce the two-scenario case by choosing U as the convex hull of the two scenarios. Note however that this does not settle the question whether the problem is weakly or strongly NP-hard.

Even if the case of discrete uncertainty with unbounded number m of scenarios is strongly NP-hard, this does not imply strong NP-hardness for the general polyhedral uncertainty case, since the number of facets of the convex hull of the given m scenarios might be exponential in m. In other words, we cannot construct an outer description of U in an efficient way. Furthermore, considering polyhedra U with a fixed number of facets is not reasonable in general, since this implies unboundedness of U for $n \geq m$; see Section 2.5. However, for an unbounded number of facets, the problem turns out to be strongly NP-hard for $X = \{0,1\}^n$. To the best of our knowledge, this result has not been proved in the literature yet.

Theorem 4 *The robust counterpart* (RP) *is strongly* NP*-hard for* $X = \{0,1\}^n$ *and a polytope* U *(given by an outer description).*

Proof For the reduction, we use the strongly NP-hard Bin Packing problem. Given positive integers $a_1, \dots, a_t$ and C, the problem can be formulated as

$$\begin{aligned} \min \ & \sum_{j=1}^{t} z_j \\ \text{s.t.} \ & \sum_{i=1}^{t} a_i x_{ij} \leq z_j \cdot C \text{ for all } j = 1, \dots, t \\ & \sum_{j=1}^{t} x_{ij} \geq 1 \text{ for all } i = 1, \dots, t \\ & x_{ij} \in \{0,1\} \text{ for all } i, j = 1, \dots, t \\ & z_j \in \{0,1\} \text{ for all } j = 1, \dots, t \,. \end{aligned}$$

In short, this problem can be written as

$$\begin{aligned} \min \ & d^\top y \\ \text{s.t.} \ & Ay \geq b \\ & y \in \{0,1\}^n \,, \end{aligned} \tag{7}$$

where $n := t^2 + t$, $d \in \{0,1\}^n$, $A \in \mathbb{Z}^{m \times n}$, and $b \in \mathbb{Z}^m$ with $m := 2t$. Now consider the polytope

$$U := \{(d,0)\} + (n+1)\operatorname{conv}\left(\{0\} \cup \{(-a_i, b_i) \mid i = 1, \dots m\}\right) \,,$$

where a_i denotes the i-th row of A. One can easily verify that the vectors $(-a_i, b_i)$ are linearly independent. This implies that an outer description of U can be computed efficiently. Moreover, for each fixed $y \in \{0,1\}^n$, we have

$$\max_{(c,\bar{c}) \in U} \ c^\top y + \bar{c} \begin{cases} = d^\top y \leq n & \text{if } Ay \geq b \\ \geq n+1 & \text{otherwise.} \end{cases}$$

In summary, Problem (7) reduces to

$$\min_{y \in \{0,1\}^n} \ \max_{(c,\bar{c}) \in U} \ c^\top y + \bar{c} \,,$$

which is of the form (RPC). Finally, the uncertain constant can again be eliminated, as discussed in Section 2.6. □

In this case, the hardness essentially comes from the fact that we can use the polytope U to model linear constraints on X; see also the discussion in Section 2.5. However, the crucial step is to move from an inner to an outer description of U, which in general is not efficient.

On the other hand, if U is originally given by an inner description, i.e., U is the convex hull of a set of vectors $U = \operatorname{conv}(\{c_1, \ldots, c_m\})$, then we can easily reduce the robust min-max problem with unbounded number of scenarios, again by choosing U as the convex hull of the scenarios. Therefore, also in this case Problem (RP) is strongly *NP*-hard for most of the classical problems.

3.1.3 Ellipsoidal Uncertainty

For general ellipsoidal uncertainty, Problem (RP) is *NP*-hard as well for most of the classical problems. This can again be proved by reducing the two-scenario problem, by choosing U as the convex hull of the two scenarios [105]. Note that the resulting ellipsoid is only one-dimensional, but also a reduction from the two-scenario problem to the min-max problem with full-dimensional ellipsoidal uncertainty, as defined in Section 2.3, is possible [71]. In [19] it has been proved that the unconstrained binary min-max problem with ellipsoidal uncertainty is strongly *NP*-hard. For convenience of the reader, we sketch the proof here.

Theorem 5 *The robust counterpart* (RP) *is strongly* NP*-hard for* $X = \{0,1\}^n$ *and an ellipsoid* U *(given by the covariance matrix* Σ *and the mean vector* c_0*).*

Proof We describe a polynomial reduction from Binary Quadratic Programming, which is a strongly *NP*-hard problem (equivalent to the Maximum-Cut Problem). We are thus given a problem of the form

$$\min_{x \in \{0,1\}^n} \frac{1}{2} x^\top Q x + L^\top x \ , \tag{8}$$

where $Q \in \mathbb{Z}^{n \times n}$ is any symmetric matrix and $L \in \mathbb{Z}^n$. Using the binary of x, we may assume that Q is positive definite, so that also

$$A := \begin{pmatrix} Q & L \\ L^\top & c \end{pmatrix}$$

is positive definite for $c := L^\top L + 1 \in \mathbb{Z}$. Now (8) can be reduced to

$$\begin{aligned} \min \ & \sqrt{y^\top A y} \\ \text{s.t. } & y_{n+1} = 1 \\ & y \in \{0,1\}^{n+1} \ . \end{aligned}$$

For $M := \sum_{ij} |A_{ij}| + 1 \in \mathbb{Z}$, we can in turn rewrite this as

$$\begin{aligned} M + \min \ & (-M) y_{n+1} + \sqrt{y^\top A y} \\ \text{s.t. } & y \in \{0,1\}^{n+1} \ . \end{aligned}$$

The latter problem is of the form (4). □

3.2 Tractable Uncertainty Sets

As discussed in the previous sections, Problem (RP) is *NP*-hard for most classical combinatorial optimization problems even if U only contains two scenarios or is a general polytope or ellipsoid. Therefore, in order to obtain positive complexity results, it is generally necessary to restrict oneself to more specific uncertainty sets. In particular, one may expect tractable robust counterparts in the case of interval or budgeted uncertainty, see Section 3.2.1, or for uncorrelated ellipsoidal uncertainty, see Section 3.2.2.

3.2.1 Interval and Budgeted Uncertainty

As already mentioned before, using interval uncertainty leads to robust counterparts as easy as the underlying problem: it can be easily verified that Problem (RP) with $U = [l, u]$ is equivalent to Problem (P) with objective vector u. This approach often leads to very conservative solutions, since all uncertain parameters are allowed to attain their worst case values at the same time. To tackle this problem, Bertsimas and Sim introduced budgeted uncertainty sets [35]. They propose to add a budget constraint to the interval uncertainty set, which limits the number of variables which may differ from their mean value at the same time; see (1). They prove that the corresponding robust counterpart (RP) can be reduced to solving $n+1$ deterministic problems [34]. Therefore, the problem can be solved in polynomial time as soon as the underlying problem can be solved in polynomial time. In a similar way, one can prove that (RP) can be reduced to the solution of only two deterministic problems when considering

$$U = \Big\{ c_0 + \sum_{i=1}^{n} \delta_i e_i \mid \sum_{i=1}^{n} \delta_i \leq \Gamma, \delta_i \in [0, d_i] \Big\} \tag{9}$$

i.e. the variant where the absolute deviation is bounded by Γ. Furthermore, Bertsimas and Sim prove that, if an $(1+\varepsilon)$-approximation algorithm for the deterministic problem exists, then we can approximate the robust min-max version with budgeted uncertainty set with the same accuracy by solving $n+1$ deterministic problems by the approximation algorithm [34]. The results in [34] were later improved in [10, 97]. In [84] the authors prove that it is sufficient to solve $\lceil \frac{n-\Gamma}{2} \rceil + 1$ deterministic problems to calculate an optimal solution of (RP) with budgeted uncertainty.

Similar results as in [34] hold for variable budgeted uncertainty, introduced in [99, 105]. Here, instead of a fixed parameter Γ, we are given a function $\gamma : X \to \mathbb{N}$ and define the uncertainty set as

$$U(x) = \Big\{ c_0 + \sum_{i=1}^{n} \delta_i d_i e_i \mid \sum_{i=1}^{n} \delta_i \leq \gamma(x), \delta_i \in [0, 1] \Big\} ,$$

which requires to extend the definition of the robust counterpart (RP) due to the dependence of U from x. In [99], the author proves that the resulting problem can be solved by solving at most $n(n+1)$ deterministic problems if we assume that γ is an affine function in x. Furthermore, given a dynamic programming algorithm for a combinatorial problem that satisfies certain assumptions, the author derives

a dynamic programming algorithm for Problem (RP) with budgeted and variable budgeted uncertainty. Besides other problems, the latter construction is applicable for the shortest path problem, the traveling salesman problem, and the scheduling problem under budgeted uncertainty sets.

An extension to uncertainty sets modeled by multiple knapsack constraints was studied in [100]. Here the author extends the results in [34] to derive exact and approximation algorithms for the robust counterpart. Furthermore besides other results the *NP*-hardness of the robust counterpart is proved for the case that the number of knapsack constraints is part of the input.

3.2.2 Uncorrelated Ellipsoidal Uncertainty

Another subclass which may lead to tractable robust counterparts is the class of uncorrelated ellipsoidal uncertainty sets, i.e., we have that U is an axis-parallel ellipsoid. In this case the correlation matrix Σ in (3) is a diagonal matrix and the corresponding robust counterpart is given by Problem (5). It has been shown that the latter problem can be solved in polynomial time if X is a matroid [95, 94]. In particular, it can be solved efficiently if $X = \{0,1\}^n$ or if X models the feasible set of the spanning-tree problem. This essentially follows from the submodularity of the objective function in (5).

The latter result can also be derived by interpreting Problem (5) as a bicriteria problem with objective functions $(c_0^\top x, \sigma^\top x)$. Nikolova [94] proved that any optimal solution of Problem (5) is an extreme efficient solution of the bicriteria problem, i.e., it is an extreme point of the pareto frontier. Therefore, all optimal solutions can be obtained by solving the linear problem

$$\min_{x \in X} \; (1-\lambda) c_0^\top x + \lambda \sigma^\top x$$

for appropriate $\lambda \in [0,1]$. For matroids, an optimal solution of the latter problem depends only on the ordering of the coefficients in the objective function, so that we only have to consider the values of λ for which the ordering changes. However, for a given pair of variables the order can change at most once, which shows that the robust problem can be solved by reduction to at most $\binom{n}{2}$ certain problems.

For most classical combinatorial optimization problems, it is not known whether Problem (5) is tractable or *NP*-hard. However, again by considering extreme points of the Pareto frontier, Nikolova proves that if the underlying problem admits an FPTAS – which applies in particular if it can be solved in polynomial time – then also Problem (5) admits an FPTAS.

A further oracle based result is proved by Bertsimas and Sim [36]. By replacing the concave square-root function by its subgradients and solving the corresponding linear deterministic problem over x for each value in $\{\sigma^\top x \mid x \in \{0,1\}^n\}$, they obtain an exact algorithm for (5).

In [89] the authors show that the constrained shortest path problem under uncorrelated ellipsoidal uncertainty can be solved in pseudopolynomial time.

3.3 Solution Approaches for *NP*-hard Cases

As discussed in Section 3.1, the robust counterpart (RP) is *NP*-hard for many uncertainty sets U even if the underlying problem (P) is tractable. This implies

that, unless $P{=}NP$, there is no efficient way of reducing (RP) to (P) in general, i.e., no algorithm for solving (RP) by solving a polynomial number of problems of type (P) – as it was the case for interval or budgeted uncertainty; see Section 3.2.1. However, in spite of the NP-hardness, an exact solution of such problems is desirable. Though most of the literature in robust combinatorial optimization concentrates on theoretical complexity issues or the design of new robust optimization paradigms, some papers also discuss exact methods for NP-hard cases. In this section, we describe some approaches that are applicable to general problems of type (RP). These can be divided into two main groups: algorithms relying on an optimization oracle for the underlying problem (RP) and those using integer programming (IP) formulations.

3.3.1 IP-based Approaches

Assuming that a compact integer formulation $X = \{x \in \{0,1\}^n \mid Ax \leq b\}$ is given, we obtain a natural relaxation

$$\min_{Ax \leq b} \max_{c \in U} c^\top x \tag{10}$$

of Problem (RP). A basic, but important observation is that (10) is a convex optimization problem. In fact, the objective function

$$\max_{c \in U} c^\top x \tag{11}$$

is convex, whatever uncertainty set U is considered. This observation gives rise to a straightforward exact solution approach to solve Problem (RP): the solution of the relaxation (10) yields a lower bound for (RP), which can be embedded into a branch-and-bound approach for solving Problem (RP) to optimality.

Even if the convexity of Problem (10) does not depend on the structure of U, different classes of uncertainty sets lead to (practically) more or less efficient algorithms for computing the lower bounds. In case of a discrete set $U = \{c_1, \ldots, c_m\}$, Problem (10) simply reduces to the linear program (LP)

$$\begin{aligned} \min\ & z \\ \text{s.t. } & Ax \leq b \\ & c_i^\top x \leq z \ \forall i = 1, \ldots, m\ . \end{aligned} \tag{12}$$

For polyhedral U (given by an outer description), we can dualize the linear program (11) and again obtain a compact LP formulation for (10). Finally, the ellipsoidal case leads to a second-order cone problem (SOCP) of the form

$$\min_{Ax \leq b} c_0^\top x + \sqrt{x^\top \Sigma^{-1} x}\ . \tag{13}$$

Both LPs and SOCPs can be solved efficiently in theory, e.g., by interior point methods. Moreover, modern IP solvers such as CPLEX [55] or Gurobi [66] can often handle SOCP constraints, so that Problem (RP) can be directly addressed for all mentioned classes of uncertainty sets. These solvers often also allow to include separation algorithms, for cases in which a compact formulation of (P) does not exist, e.g., for the spanning tree problem.

Several authors propose methods to improve the performance of IP-based approaches, either by presenting cutting planes to enforce the model and the resulting dual bounds or by developing alternative formulations. Atamtürk [12] presents extended formulations for mixed-integer 0–1 programs under generalized budgeted uncertainty. Cutting planes exploiting submodularity in the case of uncorrelated ellipsoidal uncertainty are devised in [13]. The authors in [90] develop a dynamic programming algorithm for the robust knapsack problem under budgeted uncertainty and compare it besides others to a compact formulation and the branch-and-cut algorithm developed in [60].

A different approach for convex uncertainty sets is based on scenario generation; see, e.g., [91] for the general setting and a convergence analysis. The basic idea is to produce scenarios $c_1, \ldots, c_m$ iteratively and to solve the LP-relaxation (12) in every iteration, yielding an optimal solution (x^*, z^*). Next, a worst case scenario c_{m+1} for x^* is computed, which in our situation can be done by maximizing the linear function $(x^*)^\top c$ over $c \in U$. Now if $c_{m+1}^\top x^* \leq z^*$, then x^* is an optimal solution for the relaxation (10), otherwise c_{m+1} is added to the set of scenarios and Problem (12) is solved again. This approach is compared experimentally to the standard reformulation approach in [60] for budgeted uncertainty and in [32] for budgeted and ellipsoidal uncertainty. The authors of the latter paper discuss many variants and implementation details and conclude that none of the two approaches dominates the other. In [98] telecommunication network design problems under two sources of uncertainty with quadratic dependencies are considered. Besides other results a scenario generation approach is developed and tested for polyhedral and ellipsoidal uncertainty.

Finally, various approaches based on Benders decomposition have been devised for robust mixed-integer programming; see, e.g., [102] for the ellipsoidal uncertainty case or [92] for a problem-specific approach that however allows to address very general uncertainty sets.

3.3.2 Oracle-based Approaches

In many situations, it is preferable to have an algorithm for Problem (RP) that is purely based on an optimization oracle for the underlying problem (P). This may be the case because the underlying problem is well-studied, so that fast solution algorithms exist, or because the underlying problem is so complex that it is not desirable to re-investigate it from a robust optimization point of view. Moreover, there may not be any compact IP formulation at hand.

In the case of interval or budgeted uncertainty, the robust counterpart can be reduced to the solution of at most linearly many instances of the underlying problem, as discussed in Section 3.2.1. For uncorrelated ellipsoidal uncertainty sets, the basic idea of the FPTAS mentioned in Section 3.2.2 can be extended to obtain an exact oracle-based algorithm for (RP); see [95, 94]. The number of oracle calls in the latter approach is exponential in general, however, it is linear in the number of breakpoints of the bicriteria optimization problem $\min_{x \in X}(c_0^\top x, \sigma^\top x)$ derived from the reformulated problem (5).

Few approaches have been presented in the literature that can be applied, in principle, to general sets U. An algorithm based on Lagrangian decomposition has been presented in [18] for the case of uncorrelated ellipsoidal uncertainty. The approach decouples the non-linear objective function from the underlying linear prob-

lem; it requires optimization oracles for both the underlying linear problem (P) and for the unconstrained non-linear problem

$$\min_{x\in\{0,1\}^n} \max_{c\in U} c^\top x \,. \tag{14}$$

The Lagrangian bound can be computed by a subgradient method and then be integrated into a branch-and-bound-scheme. In the uncorrelated ellipsoidal uncertainty case, Problem (14) can be solved in $O(n \log n)$ time [18]. However, for most other types of uncertainty, Problem (14) remains *NP*-hard; see Theorems 1, 3, 4, and 5. Nevertheless, it can be solved in pseudopolynomial time for a fixed finite number of scenarios; see Theorem 2. Moreover, within a branch-and-bound-scheme, it is enough to compute lower bounds for (14). Such bounds can be obtained by relaxing $\{0,1\}^n$ to $[0,1]^n$, leading to a convex problem again, or by considering underestimators in the ellipsoidal case [71].

Lower bounds in a branch-and-bound-scheme can also be obtained by solving the relaxed problem

$$\min_{x\in\mathrm{conv}(X)} \max_{c\in U} c^\top x \,. \tag{15}$$

In case no compact description of $\mathrm{conv}\,(X)$ as in (10) is given, several approaches have been studied which make use of a linear optimization oracle for X. One general approach to solve Problem (15), originally proposed to solve min-max-min-robust counterparts (see Section 4.4), uses column generation to produce new worst case scenarios iteratively. It can be considered a dual approach to the iterative scenario generation method described above, based on the problem

$$\max_{c\in U} \min_{x\in\mathrm{conv}(X)} c^\top x$$

which is equivalent to (15). More precisely, the algorithm starts with a set of feasible solutions $X_0 \subseteq X$ and then alternates between computing a worst-case scenario c^* for Problem (15) over $\mathrm{conv}\,(X_0)$ and computing an optimal solution for (P) with objective c^*, to be added to X_0. The algorithm stops when no feasible solution exists that can improve the worst-case solution. In the former step, the problem reduces to a linear optimization problem over U, with additional linear constraints. In the discrete and polyhedral case, one again obtains an LP, while the ellipsoidal case leads to a quadratic problem. In both cases, the subproblem can thus be solved efficiently. In the latter step, one can use the optimization oracle to check whether a new feasible solution has to be added to X_0. For details see [41].

A related approach to Problem (15) using a Frank-Wolfe type strategy has been devised in [39], where it is applied to general ellipsoidal uncertainty. The algorithm is again of an iterative nature. In each iteration, a set X' of feasible solutions is considered, as well as a point x' in their convex hull. Then the gradient of the objective function (11) in x' is computed and minimized over the set X, using the linear optimization oracle. An exact line search is performed between x' and the resulting linear minimizer, and the set X' is updated. A Frank-Wolfe-approach for the case of uncorrelated ellipsoidal uncertainty has been presented in [36].

4 Robust Two-Stage Problems

A general robust two-stage problem can be formulated as

$$\min_{x \in X} \max_{\xi \in U} \min_{\substack{y \in Y \\ (x,y) \in Z_\xi}} f_\xi(x, y) \tag{2SRP}$$

where $x \in X$ are the first-stage decisions which have to be taken before the scenario is known. After the scenario $\xi \in U$ materializes, we choose the best possible second stage decisions $y \in Y$, such that the pair (x, y) is feasible for the actual scenario, i.e., $(x, y) \in Z_\xi$. As common in robust optimization, we optimize the worst case objective value of $f_\xi(x, y)$ over all scenarios $\xi \in U$. As before, we will concentrate on combinatorial optimization under cost uncertainty and thus assume in the following that $X \subseteq \{0, 1\}^{n_1}$ and $f_\xi(x, y) = \xi^\top(x, y)$. For the second-stage we assume the general case $Y \subseteq \mathbb{R}^{n_2}$ and $Z_\xi \subseteq \mathbb{R}^{n_1+n_2}$ and study the cases of real and integer recourse separately in the following sections. Moreover, we focus on the case where Z_ξ does not depend on ξ.

The two-stage approach is tailored for problems for which a subset of the decisions can be implemented after the scenario is known. Applications occur, e.g., in the field of network design problems where in the first stage a capacity on an edge must be bought such that, after the real costs on each edge are known, a minimum cost flow is sent from a source to a sink [30]. Further applications can be problem formulations involving slack variables or, more generally, auxiliary variables depending on the real decision variables and the realized scenario. In a strictly robust setting, such variables must be determined independently of the scenario, which is not possible in general (and not necessary in practice).

Clearly, one can generalize this approach to more than two stages. A multi-stage approach is applicable when the decisions are made in several steps, assuming that the uncertainty vanishes gradually.

4.1 Adjustable Robustness

Adjustable robustness was first introduced in [21] and can be seen as the beginning of two-stage models in robust optimization. In fact, this approach is often just called Two-Stage Robustness. While the adjustable robust approach was originally introduced for general linear formulations with uncertainty in the constraints, later the approach was applied to combinatorial problems. Considering combinatorial problems with uncertainty only occuring in the cost function, the general linear adjustable robust counterpart is of the form

$$\min_{x \in X} \max_{(c,d) \in U} \min_{\substack{y \in Y \\ (x,y) \in Z}} c^\top x + d^\top y \tag{ARP}$$

for feasible sets $X \subseteq \{0, 1\}^{n_1}$, $Y \subseteq \mathbb{R}^{n_2}$, and $Z \subseteq \mathbb{R}^{n_1+n_2}$. A recent survey on this topic can be found in [109].

Formally, by setting $n_2 = 0$, the classical robust min-max problem is a special case of (ARP) and therefore all *NP*-hardness results from Section 3 carry over to Problem (ARP). Furthermore for interval uncertainty $U = [l, u]$, as for the classical min-max problem, it can be easily verified that Problem (ARP) is equivalent to

the deterministic problem with objective vector u if $Y \subseteq \mathbb{R}^{n_2}_+$. However, for a closer investigation of complexity and solution approaches, an important question is whether we assume Y (or Z) to be a discrete set or not, i.e., whether we consider real or integer recourse.

4.1.1 Real Recourse

We first discuss the case where no additional integrality constraints occur in the second stage, which has been investigated intensively in the literature.

The adjustable robust counterpart was originally introduced for problems with uncertain constraints and certain objective. In our setting, we can use a straightforward level set transformation to shift the uncertainty from the objective function into the constraints, and hence also apply methods designed for problems with uncertain constraints. Even if the feasible sets Y and Z are given by linear uncertain constraints, the adjustable robust counterpart is NP-hard [88]. Ben-Tal et al. [21] propose to approximate the problem by assuming that the optimal values of the wait and see variables y are affine functions of the uncertainty parameters ξ, i.e., $y = y_0 + W\xi$ for a vector y_0 and a matrix W of appropriate size. The second-stage decisions y are then replaced by the choice of y_0 and W. The authors in [21] prove that in the case of fixed recourse, i.e., if the constraint parameters of the second-stage decisions are not uncertain, the problem is equivalent to a classical min-max problem with uncertainty in the constraints and therefore computationally tractable if we have a separation oracle for U. Note that in the case of these so-called affine recourse decisions U can be replaced by its convex hull if Y and Z can be described by inequalities given by quasiconvex functions in ξ. Affine recourse decisions and related and extended ideas have been studied intensively in the literature [14, 20, 48, 63, 29, 52, 28, 69, 81, 104, 108]. Moreover, it has been proved that under certain assumptions affine decision rules are optimal for the adjustable robust counterpart [31, 70].

However, if we consider the general Problem (ARP) with real recourse for the combinatorial version with cost-uncertainty, we obtain a classical robust min-max problem if U and the feasible sets Y and Z are convex. Indeed applying a classical minimax theorem to the inner problem

$$\max_{(c,d)\in U} \min_{\substack{y\in Y\\ (x,y)\in Z}} c^\top x + d^\top y,$$

we can swap the maximum and the minimum. Therefore in this case (ARP) is equivalent to the classical min-max problem

$$\begin{aligned} \min \max_{(c,d)\in U} \; & c^\top x + d^\top y \\ \text{s.t. } & x \in X, y \in Y, (x,y) \in Z \; . \end{aligned}$$

Note that the latter result even holds without assuming affine decision rules.

For the case of real but not necessarily affine recourse, decomposition methods have been proposed for budget [38] and polyhedral [17] uncertainty.

4.1.2 Integer Recourse

We now focus on the case of integer recourse i.e. $Y \subseteq \mathbb{Z}^{n_2}$, which so far has been investigated much less intensively in the literature. Kasperski and Zieliński [77] consider combinatorial problems, where a subset of the variables of the solution are determined in the second stage. They prove that Problem (ARP) is *NP*-hard for the shortest-path problem, the minimum s-t-cut problem and the minimum assignment problem, even if U contains only two scenarios and $n_1, n_2 \geq 1$. They prove for the same underlying problems that the adjustable counterpart is even strongly *NP*-hard if the number of scenarios is unbounded. Furthermore it has been proved that Problem (ARP) is *NP*-hard for the selection problem even if U only contains two scenarios and that it is strongly *NP*-hard for the spanning-tree problem and the selection problem with unbounded number of scenarios [74, 75].

However, the positive complexity results of the min-max problem for budgeted uncertainty are not transferable to the adjustable robust counterpart: considering budgeted uncertainty for adjustable robustness, note that in general the variants of Problem (ARP) with budgeted uncertainty sets (1) and (2) are not equivalent, different from the classical min-max problem.

In [77], the authors prove that Problem (ARP) is *NP*-hard for the shortest-path problem and the spanning-tree problem with budgeted uncertainty defined as in (1) if the budget parameter Γ is part of the input. It is an open question whether the problem remains *NP*-hard for a constant parameter Γ. In [51], besides other problems, the adjustable robust counterpart of the selection problem is studied under budgeted uncertainty. The authors derive a mixed-integer formulation for the budgeted uncertainty sets (1) and (9) and prove that the problem can be solved in polynomial time for the latter variant.

Approximation algorithms have been developed for the two-stage variant of uncertain network problems and general LP-formulations for an exponential number of scenarios [58, 79]. Furthermore, an exact column-and-constraint generation algorithm has been devised in [110].

More recently, general approaches for solving Problem (ARP) in the case of integer recourse, at least approximately, have been developed. They use non-linear decision rules [28, 29] or partitionings of the uncertainty sets [27, 101, 107, 67].

4.2 Recoverable Robustness

Recoverable robust optimization problems have been introduced in [86]. The main idea is to calculate a solution which works well in an average scenario and then, after the upcoming scenario is known, can be turned into a feasible solution which is optimal for the given costs in the scenario. As an application example, the approach has been studied for timetabling problems in [86]. Here, we aim at computing a good timetable for the case where no disturbances occur. If a disturbance occurs, we want to slightly change the pre-calculated solution to make it feasible and tractable for the actual situation. Formally, the main idea behind this approach is that a set $\mathcal{A}$ of recovery algorithms is given such that each of them can be applied to a solution x and a scenario ξ to construct a feasible solution for the scenario.

When applied to problems with uncertainty only in the costs, the recoverable robust approach can be interpreted as follows: we aim at computing a solution x,

such that the best feasible solution we can compute from x by one of our algorithms is worst-case optimal. A special case of this variant is the so-called robust optimization with incremental recourse, introduced in [93]. Here, instead of arbitrary algorithms in the second stage, we allow to change the solution x up to a certain distance. In the combinatorial setting, this leads to the problem

$$\min_{x \in X} \; c^\top x + \max_{d \in U} \; \min_{\substack{y \in X \\ \delta(x,y) \leq k}} \; d^\top y \tag{IRRP}$$

where δ is the given distance measure and $k \in \mathbb{N}$. Typical distance measures investigated in the literature are $\delta_1(x, y) = |y \setminus x|$ or $\delta_2(x, y) = |x \setminus y|$ or the symmetric difference.

In the following, we list results for the distance measure δ_1. An overview about further results for different measures can be found in [76]. For discrete uncertainty sets it has been proven that Problem (IRRP) is *NP*-hard for the minimum spanning-tree problem and the selection problem even if U only contains two scenarios and that both problems are strongly *NP*-hard if the number of scenarios is unbounded [75, 73]. The recoverable robust counterpart is strongly *NP*-hard for the shortest-path problem even for two scenarios. The recoverable knapsack problem is weakly *NP*-hard for a fixed number of scenarios and strongly *NP*-hard for an unbounded number of scenarios [46]. For interval uncertainty, the selection problem and the spanning-tree problem are solvable in polynomial time [75, 68]. The shortest path problem is strongly *NP*-hard in this case [42]. For budgeted uncertainty, in [45] a linear integer formulation of quadratic size is derived for the recoverable knapsack problem. For the budgeted uncertainty variants (1) and (9) the recoverable shortest path problem is strongly *NP*-hard [93]. Further results on recoverable robustness for combinatorial problems can be found in [47].

4.3 Bulk Robustness

The concept of *Bulk Robustness* was presented in [3] and studied for the shortest path problem and the minimum matroid basis problem. In contrast to the previous models, this approach considers a set of failure scenarios, where each failure scenario is a set of edges which can break down simultaneously. The aim is to calculate a set of edges such that, if we remove the edges of any failure scenario from this set, it still contains the edge set of a feasible solution of the combinatorial problem. In particular, a solution of a bulk robust counterpart is a superset of a feasible solution, but not a feasible solution itself in general.

As an application, consider a railway system for which a shortest path has to be calculated. Because of possible constructions or accidents it can happen that a section of the railway system is not passable anymore. In this case we can use the bulk robust idea to calculate a set of sections which always contain a feasible path no matter which of the failure scenarios occurs.

The bulk-robust approach is studied for several combinatorial optimization problems in [3, 1, 2]. An extension, which is at the same time related to the recoverable robust idea, was presented in [4] for the shortest-path problem: in a second stage, after the failure scenario is known, it is allowed to add r edges to the precalculated edge-set to connect the given nodes s and t. Note that both approaches consider only constraint uncertainty given by the failure scenarios.

A natural way to adapt the bulk-robust concept to problems with only cost-uncertainty is to ask for a set of edges such that the best solution contained in the set is worst-case optimal. However, this approach is only reasonable if the vector x is either restricted or penalized by the objective function, as otherwise the optimal solution is always $\mathbf{1}$. We thus obtain the two problem variants

$$\min_{x\in\{0,1\}^n} \max_{c\in U} \min_{\substack{y\leq x\\ y\in X}} c^\top x + d^\top y$$

and

$$\min_{\substack{x\in\{0,1\}^n\\ |x|\leq k}} \max_{c\in U} \min_{\substack{y\leq x\\ y\in X}} d^\top y\ .$$

A generalization of the former model was mentioned in [77], but to the best of our knowledge, no complexity results have been devised for this problem. Note however that both variants are at least as hard as the original bulk-robust problem: the failure scenarios can be modeled by large enough costs in the corresponding scenarios. This implies that both variants are *NP*-hard for the shortest path problem and the assignment problem [3, 1].

4.4 K-Adaptability

As mentioned in Section 4.1, Problem (2SRP) in general is *NP*-hard. Besides using affine decision rules, another way to approximate Problem (2SRP) is to limit the number of second-stage solutions by a given parameter K and calculate these solutions in the first stage. This idea, called K-adaptability, has been introduced in [26] and has been applied to robust two-stage problems in [67]. For the case that only the objective function is uncertain, the authors of [67] show that this problem is equivalent to the exact problem (2SRP) for large enough K. Furthermore, by dualizing the inner max-min problem, the authors provide a mixed-integer linear programming formulation of polynomial size, which they evaluated for the shortest path problem, besides others. For the general case of constraint uncertainty the authors prove that even evaluating the objective function is *NP*-hard.

As a special case of K-adaptability in two-stage robust optimization, in [41] the authors study problems of the form

$$\min_{x^{(1)},\dots,x^{(k)}\in X} \max_{c\in U} \min_{i=1,\dots,k} c^\top x^{(i)}\ , \tag{KAP}$$

where again $X \subseteq \{0,1\}^n$ and $U \subseteq \mathbb{R}^n$. Problem (KAP) is an extension of the classical min-max problem (RP) and yields better solutions in general if $k > 1$. The idea is to calculate k solutions by solving problem (KAP) once and to quickly choose the best of them after the scenario is known.

It is shown in [41] that Problem (KAP) with an additional uncertain constant is *NP*-hard for $X = \{0,1\}^n$ and any fixed k even if U is a polyhedron given by an inner description. Furthermore, it is shown that Problem (KAP) is equivalent to

$$\min_{x\in\mathrm{conv}(X)} \max_{c\in U} c^\top x$$

for $k \geq n+1$, which is a continuous convex problem. By using results on the equivalence of optimization and separation for convex problems proved in [65],

the authors show that for general bounded convex uncertainty sets over which a linear function can be optimized efficiently, Problem (KAP) for $k \geq n+1$ can be solved in polynomial time as soon as the underlying deterministic problem can be solved in polynomial time. Furthermore, the authors provide an exact oracle-based algorithm to solve the problem for $k \geq n+1$, see Section 3.3.2, as well as a heuristic approach for $k \leq n$. For discrete uncertainty sets, the problem is analyzed in [40]. The authors show that the complexity of Problem (KAP) coincides with the complexity of the corresponding min-max problem (RP) for many classical combinatorial optimization problems.

5 Conclusion

Considering all classical types of uncertainty sets discussed above, the main dividing line between hard and easy cases seems to be the inclusion of correlations: in the case of interval uncertainty, where all cost coefficients can vary independently, the robust counterpart inherits the complexity of the underlying problem. In the case of uncorrelated ellipsoidal uncertainty, it is not known yet whether the same is true, but positive general results exist. On the other hand, uncertainty sets allowing to model correlations, i.e., general discrete, polyhedral, and ellipsoidal sets, usually lead to *NP*-hard counterparts. The budgeted uncertainty case is on the borderline, as it takes correlations into account in a rudimentary way, without increasing the complexity.

Apart from many interesting open complexity theoretic questions, we think that there is a lot of potential for improving exact methods for general classes of uncertainty sets. By a more extensive use of techniques from mathematical programming and non-linear (robust) optimization, we believe that such methods can become more powerful even in the combinatorial optimization setting. In particular, such non-linear methods are relevant when dealing with non-finite and non-polyhedral uncertainty sets. Such sets often yield a more realistic description of the given uncertainty, while at the same time not necessarily leading to harder robust counterparts.

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