Perturbation analysis of a class of conic programming problems under Jacobian uniqueness conditions[1]

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Abstract. We consider the stability of a class of parameterized conic programming problems which are more general than the C^2-smooth parameterization. We show that when the Karush-Kuhn-Tucker (KKT) condition, the constraint nondegeneracy condition, the strict complementary condition and the second order sufficient condition (named as Jacobian uniqueness conditions here) are satisfied at a feasible point of the original problem, the Jacobian uniqueness conditions of the perturbed problem also hold at some feasible point.

Key words. C^2-cone reducible sets; Jacobian uniqueness conditions; KKT system; implicit function theorem.

AMS Subject Classifications.49K40,90C31,49J53

1 Introduction

Consider the optimization problem

$$(\mathrm{P}): \begin{cases} \min\limits_{x\in X} & f(x) \\ \text{s.t.} & G(x) \in K, \end{cases} \tag{1.1}$$

where $f : X \to \Re$ and $G : X \to Y$ are twice continuously differentiable functions, X and Y are two finite dimensional real vector spaces, and K is a closed convex set in Y. Denote the Lagrangian function $L : X \times Y \to \Re$ by:

$$L(x, \lambda) := f(x) + \langle \lambda, G(x)\rangle, (x, \lambda) \in X \times Y, \tag{1.2}$$

and the first order optimality condition, i.e., the KKT condition for problem (P) takes the following form:

$$D_x L(x, \lambda) = 0, \lambda \in \mathcal{N}_K(G(x)), \tag{1.3}$$

where $D_x L(x, \lambda)$ denotes the derivative of $L(x, \lambda)$ at (x, λ) with respect to $x \in X$, and $\mathcal{N}_K(y)$ is the normal cone of K at y in the sense of convex analysis (Rockafellar[19]) :

$$\mathcal{N}_K(y) = \begin{cases} \{d \in Y : \langle d, z - y\rangle \le 0,\ \forall z \in K\} & , \text{ if } y \in K, \\ \emptyset & , \text{ if } y \notin K. \end{cases}$$

For any (x, λ) satisfying (1.3), we call x a stationary point and (x, λ) a KKT point of (P), respectively.

In the last few decades, tremendous progress has been achieved towards stability analysis of solutions to the optimization problem (P) (Bonnans and Shapiro [6], Rockafellar and Wets [20], etc),

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especially when K is $\mathcal{C}^2$-cone reducible in the sense of Bonnans and Shapiro[6, Definition 3.135](Bonnans et al. [2, 3], Bonnans and Shapiro[5, 6]). It is worth mentioning that the class of $\mathcal{C}^2$-cone reducible sets is rich. It includes notably all the polyhedral convex sets and many nonpolyhedral sets such as the cone of symmetric positive semidefinite matrices [6, 21], the second order cone (SOC), and the epigraph cone of the Ky Fan matrix k-norm [9]. Moreover, the Cartesian product of $\mathcal{C}^2$-cone reducible sets is also $\mathcal{C}^2$-cone reducible (see, e.g., [6, 21]). The literature on perturbation analysis of optimization problems is enormous, and even a short summary about the most important results achieved would be far beyond our reach. Here, we only touch those results that are mostly relevant to the research conducted in this paper.

When K is a polyhedral set, the problem (P) is exactly the conventional nonlinear programming (NLP) problem, the stability theory is quite complete. The important concept of strong regularity for generalized equations has been introduced by Robinson [18], and he defined a strong second order sufficient condition for (NLP). He proved that the strong second order sufficient condition and the linear independence constraint qualification imply the strong regularity of the solution to the KKT system. Interestingly, the converse is also true, see Jongen et al. [13], Bonnans and Sulem [7], Dontchev and Rockafellar [8] and Bonnans and Shapiro [6, Proposition 5.38].

When K is a cone of symmetric positive semidefinite matrices, specially for nonlinear semidefinite programming problem with equality constraints, Sun [22] showed that for a locally optimal solution, the following conditions are all equivalent under the Robinson constraint qualification: the strong second order sufficient condition and constraint nondegeneracy, the nonsingularity of Clarke's Jacobian of the KKT system and the strong regularity.

When K is a second order cone, Bonnans and Ramírez C. [4] showed that for a locally optimal solution of the nonlinear SOC programming problem, under the Robinson constraint qualification, the KKT solution mapping is strongly regular if and only if the strong second order sufficient condition and the constraint nondegeneracy hold.

When K is $\mathcal{C}^2$-cone reducible, Ding, Sun and Zhang [11] considered the canonically perturbed optimization problem of (P) recently, they showed that under the Robinson constraint qualification, the KKT solution mapping is robustly isolated calm if and only if both the strict Robinson constraint qualification and the second order sufficient condition hold.

Note that Robinson [18], Sun [22], Bonnans and Ramírez C. [4], Ding, Sun and Zhang [11] only analysed the stability of the canonical perturbed problem which is a special case of $\mathcal{C}^2$-smooth parameterization, and Bonnans and Shapiro [6] studied the stability of general $\mathcal{C}^2$-smooth parameterization. How to get the stability conclusions of the parameterized problems which are more general than $\mathcal{C}^2$-smooth parameterization is worth thinking about. In this aspect, Robinson [17] introduced a general perturbed nonlinear program under pertubations in the objective function and constraint mapping, he proved that when a solution of problem (P) satisfies Jacobian uniqueness conditions and the objective function and the constraint mapping of perturbed problem all have second partial derivatives with respect to the decision variable which are jointly continuous, the Jacobian uniqueness conditions also hold at some solution of perturbed problem. Yin and Zhang [23] showed that the similar conclusions are also true for nonlinear semidefinite programming problems. Inspired by [11], a natural question is asked, is this result still valid when K belongs to the class of $\mathcal{C}^2$-cone reducible sets which are more general? We answer this question in this paper.

In this paper, we assuming that the set K is $\mathcal{C}^2$-cone reducible (see Definition 2.1). The organization of this paper is as follows. In Section 2, we introduce the notion of Jacobian uniqueness conditions for (P) which is equivalent to that of the reduced problem ($\mathcal{P}$) (see (2.4)), it is shown that the Jacobian uniqueness conditions of a solution of (P) imply the nonsingularity of Jacobian of the KKT system. In Section 3, by using the implicit-function theorem, we show that when the Jacobian uniqueness conditions are satisfied at a solution of (P), the perturbed problem which is more general than $\mathcal{C}^2$-smooth parameterization also satisfies the Jacobian uniqueness conditions at some solution. We discussed the differentiability of projection mappings over five types of $\mathcal{C}^2$-cone reducible sets in Section 4, which means that the conclusion we obtained in Section 3 is not only suitable for nonlinear programming problem and semidefinite programming problem, but also for second order cone programming problem, matrix cone programming problems induced by the spectral norm and nuclear norm. We conclude this paper in Section 5.

2 Jacobian uniqueness conditions for problem (P)

For any mapping $G : X \to Y$, we denote the derivative of G at $x \in X$ by $DG(x)$. Let x_0 be a feasible point of (P), i.e., $y_0 := G(x_0) \in K$.

Definition 2.1 *The closed convex set K is said to be $\mathcal{C}^2$-cone reducible at $y_0 \in K$, if there exist an open neighborhood $N \subset Y$ of y_0, a pointed closed convex cone $\mathcal{Q}$ (a cone is said to be pointed if and only if its lineality space is the origin) in a finite dimensional space Z, and a twice continuously differentiable mapping $\Xi : N \to Z$ such that: (i) $\Xi(y_0) = 0 \in Z$, (ii) the derivative mapping $D\Xi(y_0) : Y \to Z$ is onto, and (iii) $K \cap N = \{y \in N \mid \Xi(y) \in \mathcal{Q}\}$. We say that K is $\mathcal{C}^2$-cone reducible if K is $\mathcal{C}^2$-cone reducible at every $y_0 \in K$.*

Condition (iii) of the above definition means that locally the set K can be defined by the constraint $\Xi(y) \in \mathcal{Q}$, and hence locally, near x_0, the feasible set of (P) can be defined by the constraint $\mathcal{G}(x) \in \mathcal{Q}$, where $\mathcal{G}(x) := \Xi(G(x))$. Consequently, in a neighborhood of x_0, the original problem (P) is equivalent to the following, so-called reduced, problem:

$$(\mathcal{P}) : \begin{cases} \min\limits_{x\in X} & f(x) \\ \text{s.t.} & \mathcal{G}(x) \in \mathcal{Q}, \end{cases} \tag{2.4}$$

We have that the feasible sets of (P) and ($\mathcal{P}$) coincide near the point x_0, and hence the sets of optimal solutions of (P) and ($\mathcal{P}$), restricted to a neighborhood of x_0, are the same.

We say that the Robinson constraint qualification (RCQ) for (P) holds at a feasible point x_0 if

$$DG(x_0)X + \mathcal{T}_K(G(x_0)) = Y,$$

where $\mathcal{T}_K(y)$ is denote as the tangent cone of closed convex set K at y:

$$\mathcal{T}_K(y) = \{d \in Y : \text{dist}(y + td, K) = o(t), t \geq 0\}, y \in K, K \subset Y.$$

The strict constraint qualification is said to hold for (P) at x_0 with respect $\lambda_0 \in \Lambda(x_0) \neq \emptyset$ if

$$DG(x_0)X + \mathcal{T}_K(G(x_0)) \cap \lambda_0^{\perp} = Y,$$

Recall that the inner and outer second order tangent sets [6, (3.49) and (3.50)] to the given closed set K in the direction $h \in Y$ can be defined, respectively, by

$$\mathcal{T}_K^{i,2}(y,h) = \left\{ \omega \in Y : \operatorname{dist}(y + th + \frac{1}{2}t^2\omega, K) = o(t^2), t \geq 0 \right\}$$

and

$$\mathcal{T}_K^{2}(y,h) = \left\{ \omega \in Y : \exists t_k \downarrow 0, \text{ s.t. } \operatorname{dist}(y + t_k h + \frac{1}{2}t_k^2\omega, K) = o(t_k^2), \right\},$$

where for any $s \in Y$, $\operatorname{dist}(s,K) := \inf\{\|s-y\|, y \in K\}$. Note that, in general, $\mathcal{T}_K^{i,2}(y,h) \neq \mathcal{T}_K^2(y,h)$ even if K is convex ([6, section 3.3]). However, it follows from [6, Proposition 3.136] that if K is a $\mathcal{C}^2$-cone reducible convex set, then the equality always holds. In this case, $\mathcal{T}_K^2(y,h)$ will be simply called the second order tangent set to K at $y \in K$ in the direction $h \in Y$.

Next, we list some results that are needed for our subsequent discussions from the standard reduction approach. For more details on the reduction approach, one may refer to [6, section 3.4.4]. The following results on the representations of the normal cone and the "sigma term" of the $\mathcal{C}^2$-cone reducible set K are stated in [6, (3.266) and (3.274)].

Lemma 2.1 *Let $y_0 \in K$ be given. Then, there exist an open neighborhood $N \subset Y$ of y_0, a pointed closed convex cone $\mathcal{Q}$ in a finite dimensional space Z, and a twice continuously differentiable function $\Xi : N \to Z$ satisfying conditions (i)-(iii) in Definition 2.1 such that for all $y \in N$ sufficiently close to y_0,*

$$N_K(y) = D\Xi(y)^* N_{\mathcal{Q}}(\Xi(y)), \tag{2.5}$$

where $D\Xi(y)^* : Z \to Y$ is the adjoint of $D\Xi(y)$.

It follows that if $\Lambda(x_0)$ and $\mathcal{M}(x_0)$ are sets of Lagrange multipliers of the problems (P) and ($\mathcal{P}$), respectively, then

$$\Lambda(x_0) = D\Xi(y_0)^* \mathcal{M}(x_0) \tag{2.6}$$

Note that since $D\Xi(y_0)$ is onto, the adjoint mapping $D\Xi(y_0)^*$ is one-to-one. Note also that Robinson constraint qualification for problem (P) holds at x_0 iff it holds for ($\mathcal{P}$) at x_0.

Definition 2.2 *We say that a feasible point x_0 to* (P) *is constraint nondegenerate* [6, (4.172)] *if*

$$DG(x_0)X + \operatorname{lin}\mathcal{T}_K(G(x_0)) = Y. \tag{2.7}$$

Since $D\Xi(y_0)$ is onto, i.e., $D\Xi(y_0)Y = Z$, then by the fact that

$$\begin{aligned} & DG(x_0)X + \operatorname{lin}\mathcal{T}_K(G(x_0)) = Y \\ \Longleftrightarrow\quad & D\Xi(y_0)DG(x_0)X + \operatorname{lin}[D\Xi(y_0)\mathcal{T}_K(G(x_0))] = D\Xi(y_0)Y \\ \Longleftrightarrow\quad & D\mathcal{G}(x_0)X + \operatorname{lin}\mathcal{T}_{\mathcal{Q}}(\mathcal{G}(x_0)) = Z \end{aligned} \tag{2.8}$$

we can say that constraint nondegenerate condition for problem (P) holds at x_0 iff it holds for ($\mathcal{P}$) at x_0.

Definition 2.3 *We say that the strict complementary condition* [6, Definition 4.74] *holds at a feasible point x_0 of the problem* (P) *if there exists a Lagrange multiplier $\lambda \in \Lambda(x_0)$, such that $\lambda \in \operatorname{ri}N_K(G(x_0))$.*

Definition 2.4 *Let x_0 be a feasible point to* (P)*. The critical cone $C(x_0)$ of* (P) *at x_0 is defined by*

$$C(x_0) := \{d \in X : DG(x_0)d \in \mathcal{T}_K(G(x_0)), Df(x_0)d \le 0\}. \tag{2.9}$$

Note that

$$\begin{aligned} C(x_0) &= \{d \in X : DG(x_0)d \in \mathcal{T}_K(G(x_0)), Df(x_0)d \le 0\} \\ &= \{d \in X : D\Xi(y_0)DG(x_0)d \in D\Xi(y_0)\mathcal{T}_K(y_0), Df(x_0)d \le 0\} \\ &= \{d \in X : D\mathcal{G}(x_0)d \in \mathcal{T}_\mathcal{Q}(\mathcal{G}(x_0)), Df(x_0)d \le 0\} \\ &= \{d \in X : D\mathcal{G}(x_0)d \in \mathcal{Q}, Df(x_0)d \le 0\}, \end{aligned} \tag{2.10}$$

hence the critical cones of the problems (P) and ($\mathcal{P}$) are the same.

If x_0 is a stationary point of (P), namely $\Lambda(x_0) \neq \emptyset$, then

$$\begin{aligned} C(x_0) &= \{d \in X : DG(x_0)d \in \mathcal{T}_K(G(x_0)), Df(x_0)d = 0\} \\ &= \{d \in X : DG(x_0)d \in C_K(G(x_0), y_0)\}, \end{aligned}$$

where for any $y \in K$, $C_K(y, \lambda)$ is the critical cone of K at y with respect to $\lambda \in N_K(y)$ defined by

$$C_K(y, \lambda) := \mathcal{T}_K(y) \cap \lambda^\perp, \tag{2.11}$$

and for any $s \in Y$, $s^\perp := \{z \in Y | \langle z, s\rangle = 0\}$.

Recall that for any set $D \subseteq Y$, the support function of the set D is defined as

$$\sigma(y, D) := \sup_{z\in D} \langle z, y\rangle, \ y \in Y,$$

we can state in the following definition the second order sufficient condition for (P).

Definition 2.5 *Let x_0 be a stationary point to* (P)*, we say that the second order sufficient condition holds at x_0 if*

$$\sup_{\lambda\in\Lambda(x_0)} \left\{d^T \nabla^2_{xx} L(x_0, \lambda)d - \sigma(\lambda, \mathcal{T}^2_K(G(x_0), DG(x_0)d)\right\} > 0, \forall d \in C(x_0)\setminus \{0\}. \tag{2.12}$$

Combining this and [6, P.242], we can claim that the second order sufficient condition for problem (P) holds at x_0 iff it holds for ($\mathcal{P}$) at x_0, namely,

$$\sup_{\mu\in\mathcal{M}(x_0)} d^T \nabla^2_{xx} \mathcal{L}(x_0, \mu)d > 0, \forall d \in C(x_0)\setminus \{0\}. \tag{2.13}$$

where $\mathcal{L} : X \times Z \to \Re$ is the Lagrangian function of the reduced problem ($\mathcal{P}$), defined by $\mathcal{L}(x, \mu) := f(x) + \langle \mu, \mathcal{G}(x)\rangle$.

Finally, we give the definition of Jacobian uniqueness conditions for (P):

Definition 2.6 *Let x_0 be a feasible point to* (P)*, we say that the Jacobian uniqueness conditions hold at x_0 if*

i)the point x_0 is a stationary point of (P)*;*

ii)the constraint nondegenerate condition holds at x_0;

iii)the strict complementary condition holds at x_0;

iv)the second order sufficient condition holds at x_0.

Lemma 2.2 *The Jacobian uniqueness conditions for problem* (P) *hold at* x_0 *iff they hold for problem* $(\mathcal{P})$ *at* x_0.

The KKT system (1.3) is equivalent to the following system of nonsmooth equations:

$$F(x,\lambda) = 0, \tag{2.14}$$

where $F : X \times Y \to X \times Y$ is the natural mapping defined by

$$F(x,\lambda) := \begin{bmatrix} \nabla_x L(x,\lambda) \\ G(x) - \Pi_K(G(x)+\lambda) \end{bmatrix} = 0. \tag{2.15}$$

Since K is $\mathcal{C}^2$-cone reducible, we know from [2, Theorem 7.2] that Π_K is directionally differentiable at C, where $\Pi_K : Y \to Y$ is the metric projection operator over K, i.e., for any $C \in Y$,

$$\Pi_K(C) := \text{argmin}\left\{\frac{1}{2}\|y - C\|^2, y \in K\right\}.$$

Lemma 10 of [11] is very useful to our research, we restate it as the following Lemma:

Lemma 2.3 *Let* $C \in Y$, $\overline{A} = \Pi_K(C)$, *and* $\overline{B} = C - \overline{A}$.
(i) Let $\Delta A, \Delta B \in Y$. $\Delta A - \Pi_K'(C; \Delta A + \Delta B) = 0$ *if and only if*

$$\begin{cases} \Delta A \in C_K(\overline{A}, \overline{B}), \\ \Delta B - \frac{1}{2}\nabla\Upsilon(\Delta A) \in [C_K(\overline{A}, \overline{B})]^\circ, \\ \langle \Delta A, \Delta B\rangle = -\sigma(\overline{B}, \mathcal{T}_K^2(\overline{A}, \Delta A)), \end{cases} \tag{2.16}$$

where $\Upsilon(\cdot)$ *is the quadratic function defined by*

$$\Upsilon(D) := -\sigma(\overline{B}, \mathcal{T}_K^2(\overline{A}, D)) \geq 0, \forall D \in C_K(\overline{A}, \overline{B}).$$

(ii) Let $\mathcal{A} : X \to Y$ *be a linear operator. Then, the following two statements are equivalent:*

(a) $\Delta B \in Y$ *is a solution to the following system of equations*

$$\begin{cases} \mathcal{A}^*\Delta B = 0, \\ \Pi_K'(C; \Delta B) = 0; \end{cases}$$

(b) $\Delta B \in \left[\mathcal{A}X + \mathcal{T}_K(\overline{A}) \cap \overline{B}^\perp\right]^\circ$.

In order to complete the proof of the following Lemma, we shall make an assumption first as follows which will be verified to be correct when K belongs to the five specific $\mathcal{C}^2$-cone reducible sets in Section 4.

Assumption 2.1 *If the strict complementary condition holds at a feasible point* x_0 *of the problem* (P), $\lambda \in \Lambda(x_0)$ *is the corresponding Lagrange multiplier, then* $\Pi_K(\cdot)$ *is differentiable (in the sense of Fréchet) at* $\lambda + G(x_0)$.

Now, we are ready to introduce the result which plays an important role in our subsequent analysis.

Lemma 2.4 *Let (x_0, λ_0) be a feasible point to* (P) *at which the Jacobian uniqueness conditions are satisfied, then the Jacobian of F is nonsingular at (x_0, λ_0).*

Proof. The constraint nondegeneracy condition is assumed to hold at x_0, which imply $\Lambda(x_0)$ is a singleton [6, Proposition 4.75], we denote it by λ_0 which is unique. By Assumption 2.1, we know that $\Pi_K(\lambda_0 + G(x_0))$ is differentiable, i.e., $D\Pi_K(\lambda_0 + G(x_0))H = \Pi'_K(\lambda_0 + G(x_0); H)$, where $\Pi'_K(\cdot; H)$ is the directional derivative of $\Pi_K(\cdot)$ for any direction $H \in Y$.

Let $d = (d_x, d_\lambda) \in X \times Y$ be arbitrarily chosen such that

$$DF(x_0, \lambda_0)d = \left[\begin{array}{c} \nabla^2_{xx} L(x_0, \lambda_0)d_x + DG(x_0)^* d_\lambda \\ DG(x_0)d_x - D\Pi_K(G(x_0) + \lambda_0)(DG(x_0)d_x + d_\lambda) \end{array} \right] = 0. \tag{2.17}$$

By part (i) of Lemma 2.3, we know from the second equation of (2.17) that

$$DG(x_0)d_x \in C_K(G(x_0, \lambda_0)), \langle DG(x_0)d_x, d_\lambda \rangle = -\sigma(\lambda_0, \mathcal{T}^2_K(G(x_0), DG(x_0)d_x)).$$

Thus, we have $d_x \in C(x_0)$. By taking the inner product between d_x and both sides of the first equation of (2.17), respectively, we obtain that

$$\begin{aligned} 0 &= \langle d_x, \nabla^2_{xx} L(x_0, \lambda_0)d_x \rangle + \langle DG(x_0)d_x, d_\lambda \rangle \\ &= \langle d_x, \nabla^2_{xx} L(x_0, \lambda_0)d_x \rangle - \sigma(\lambda_0, \mathcal{T}^2_K(G(x_0), DG(x_0)d_x). \end{aligned}$$

Hence, we can conclude from $d_x \in C(x_0)$ and the second order sufficient condition (2.12) that

$$d_x = 0.$$

Thus, (2.17) reduces to

$$\left[\begin{array}{c} DG(x_0)^* d_\lambda \\ D\Pi_K(G(x_0) + \lambda_0)d_\lambda \end{array} \right] = 0. \tag{2.18}$$

By part (ii) of Lemma 2.3, we have

$$d_\lambda \in [DG(x_0)X + \mathcal{T}_K(G(x_0)) \cap \lambda_0^\perp]^\circ,$$

where for any given nonempty convex cone $K \in Y$, we use K° to denote the polar of K, i.e., $K^\circ := \{z \in Y | \langle z, s \rangle \leq 0, \forall s \in K\}$. Combining this and the constraint nondegeneracy condition and the strict complementary condition of x_0 and [6, Proposition 4.73], we obtain that $d_\lambda = 0$. Therefore, the Jacobian of F is nonsingular at (x_0, λ_0). □

Remark 2.1 *The above proof is similar to that of Proposition 21 in Ding, Sun and Zhang[11], here we need to consider the derivative of $\Pi_K(\cdot)$ and the fact that when the strict complementary condition holds at x_0, the constraint nondegeneracy condition is equivalent to the strict constraint qualification.*

3 Perturbation analysis of problem (P)

Consider the parameterized optimization problem in the form

$$(\mathrm{P}_u) : \left\{ \begin{array}{ll} \min\limits_{x \in X} & f(x, u) \\ \text{s.t.} & G(x, u) \in K, \end{array} \right. \tag{3.19}$$

depending on the parameter vector u varying in a Banach space $\mathcal{U}$. Here $f : X \times \mathcal{U} \to \Re$, and $G : X \times \mathcal{U} \to Y$ all have second partial derivatives with respect to x which are jointly continuous on $X \times \mathcal{U}$, X and Y are two finite dimensional real vector spaces, and K is a closed convex set in Y. We assume that for a given value u_0 of the parameter vector, problem (P_{u_0}) coincides with the unperturbed problem (P).

The KKT system of (P_u) is equivalent to the following system of nonsmooth equations:

$$F(x, \lambda, u) = 0, \tag{3.20}$$

where $F : X \times Y \times \mathcal{U} \to X \times Y$ is the natural mapping defined by

$$F(x, \lambda, u) := \left[\begin{array}{c} \nabla_x L(x, \lambda, u) \\ G(x, u) - \Pi_K(G(x, u) + \lambda) \end{array} \right] = 0, \tag{3.21}$$

and the Lagrangian function $L : X \times Y \times \mathcal{U} \to \Re$ is defined by:

$$L(x, \lambda, u) := f(x, u) + \langle \lambda, G(x, u) \rangle, (x, \lambda, u) \in X \times Y \times \mathcal{U}. \tag{3.22}$$

Since we assume that K belongs to the class of $\mathcal{C}^2$-cone reducible sets, so in a neighborhood of x_0, the parameterized optimization problem (P_u) is equivalent to the following reduced problem:

$$(\mathcal{P}_u) : \left\{ \begin{array}{ll} \min\limits_{x \in X} & f(x, u) \\ \text{s.t.} & \mathcal{G}(x, u) \in \mathcal{Q}, \end{array} \right. \tag{3.23}$$

where $\mathcal{G}(x, u) := \Xi(G(x, u))$. By property (iii) of Definition 2.1 we have that for u sufficiently close to u_0, the feasible sets of (P_u) and $(\mathcal{P}_u)$ coincide near the point x_0. Therefore, the sets of optimal solutions of (P_u) and $(\mathcal{P}_u)$, restricted to a neighborhood of x_0, are the same for all u sufficiently close to u_0.

Now, we are ready to state the main result of this paper.

Theorem 3.1 *Suppose that $f(x, u)$, $G(x, u)$ all have second partial derivatives with respect to x which are jointly continuous on $X \times \mathcal{U}$. Let $u_0 \in \mathcal{U}$, and let (x_0, λ_0) be a feasible point to the (P_{u_0}) problem at which the Jacobian uniqueness conditions are satisfied.*

Then, there exist open neighborhoods $M(u_0) \subset \mathcal{U}$ and $N(x_0, \lambda_0) \subset X \times Y$, and a continuous function $\mathcal{Z} : M \to N$, such that $\mathcal{Z}(u_0) = (x_0, \lambda_0)$, and for each $u \in M$, $\mathcal{Z}(u)$ is both the unique KKT point of (P_u) in N and the unique zero in N of the function $F(\cdot, \cdot, u)$. Further, if $\mathcal{Z}(u) := (x(u), \lambda(u))$, then for each $u \in M$, $x(u)$ is an isolated local minimizer of (P_u) at which the Jacobian uniqueness conditions are satisfied.

Proof. By Lemma 2.4, the Jacobian of F is nonsingular at (x_0, λ_0), then, by the implicit-function theorem[15, Theorems 1-2(4.XVII)], there exist open neighborhoods $M_0(u_0) \subset \mathcal{U}$ and $N_0(x_0, \lambda_0) \subset X \times Y$, and a continuous function $\mathcal{Z} : M_0 \to N_0$ such that $\mathcal{Z}(u_0) = (x_0, \lambda_0)$, and for each $u \in M_0$, $\mathcal{Z}(u)$ is the unique zero of $F(\cdot, \cdot, u)$ in N_0, i.e., $F(\mathcal{Z}(u), u) = 0$.

In addition, there are some open neighborhoods $M_1(u_0)$ and $N_1(x_0, \lambda_0)$ such that for $(x, \lambda) \in N_1$ and $u \in M_1$, the problem (P_u) is equivalent to $(\mathcal{P}_u)$.

Let $N := N_0 \cap N_1$, then since $\mathcal{Z}$ is continuous and $\mathcal{Z}(u_0) = (x_0, \lambda_0)$, it is possible to find an open neighborhood $M_2(u_0) \subset M_1 \cap M_0$ such that $u \in M_2$ implies $\mathcal{Z}(u) \in N$. Because $F(\mathcal{Z}(u), u) = 0$, then for each $u \in M_2$, $\mathcal{Z}(u)$ satisfies the KKT condition for (P_u). Let $\mathcal{Z}(u) = (\tilde{x}, \tilde{\lambda})$, by the second equation of (3.21), we have $G(\tilde{x}, u) = \Pi_K(G(\tilde{x}, u) + \tilde{\lambda})$, which means that $\tilde{\lambda} \in N_K(G(\tilde{x}, u))$, so $N_K(G(\tilde{x}, u)) \neq \emptyset$, then $G(\tilde{x}, u) \in K$, i.e., $\tilde{x}$ is a feasible point of (P_u), so $\mathcal{Z}(u)$ is the KKT point of (P_u), it is the only such point in N because it is the only zero of $F(\cdot, \cdot, u)$ there.

Further, if $\mathcal{Z}(u) := (x(u), \lambda(u))$, we have $\lambda(u) \in N_K(G(x(u), u))$, by Lemma 2.1, we know that there exists a unique element $\mu(u)$ in $N_{\mathcal{Q}}(\mathcal{G}(x(u), u))$ such that $\lambda(u) = D\Xi(G(x(u), u))^* \mu(u)$. By the strict complementary condition for prblem (P_{u_0}) at (x_0, λ_0), namely, $\lambda(u_0) \in \mathrm{ri} N_k(G(x(u_0))$, there is a unique element $\mu(u_0) \in \mathrm{ri} N_{\mathcal{Q}}(\mathcal{G}(x(u_0))$. Since $\mathcal{Q}$ is a closed convex cone, and $\mathcal{G}(x(u_0)) = \Xi(G(x_0)) = 0$, then $\mu(u_0) \in \mathrm{ri} N_{\mathcal{Q}}(\mathcal{G}(x(u_0))) = \mathrm{ri}\mathcal{Q}^\circ$, so there exists an open neighborhood $M_3(u_0) \subset M_2$ such that for any $u \in M_3$, $\mu(u) \in \mathrm{ri}\mathcal{Q}^\circ$, i.e.,

$$\langle \mu(u), z \rangle < 0, \forall z \in \mathcal{Q} \setminus \{0\}.$$

Note that $\mu(u) \in N_{\mathcal{Q}}(\mathcal{G}(x(u), u))$ implies $\mu(u) \in [|\mathcal{G}(x(u), u)|]^\perp$ and $\mathcal{G}(x(u), u) \in \mathcal{Q}$, namely,

$$\langle \mu(u), \mathcal{G}(x(u), u) \rangle = 0,$$

so we obtain that $\mathcal{G}(x(u), u) = 0$, then $\mu(u) \in \mathrm{ri}\mathcal{Q}^\circ = \mathrm{ri} N_{\mathcal{Q}}(\mathcal{G}(x(u), u))$, which means that the strict complementary condition for $(\mathcal{P}_u)$ holds at $(x(u), \mu(u))$, $\forall u \in M_3$, by Lemma 2.2, the strict complementary condition will hold for $\mathcal{Z}(u)$.

By the constraint nondegeneracy condition of (P_u) at x_0 and Lemma 2.2, we obtain $D\mathcal{G}(x_0)X + \mathrm{lin}\mathcal{T}_{\mathcal{Q}}(\mathcal{G}(x_0)) = Z$, i.e., $D\mathcal{G}(x_0)X = Z$, which implies that $D\mathcal{G}(x_0)X$ is onto, then there exists an open neighborhood $M_4(u_0) \subset M_3$ such that for any $u \in M_4$, $D\mathcal{G}(x(u), u)X = Z$. Thus, since $\mathcal{G}(x(u), u) = 0$, we have $\mathrm{lin}\mathcal{T}_{\mathcal{Q}}(\mathcal{G}(x(u), u)) = 0$, hence, $D\mathcal{G}(x(u), u)X + \mathrm{lin}\mathcal{T}_{\mathcal{Q}}(\mathcal{G}(x(u), u)) = Z$, which means that the constraint nondegeneracy condition of $(\mathcal{P}_u)$ holds at $x(u)$, from Lemma 2.2, we know that the constraint nondegeneracy condition of (P_u) also holds at $x(u)$.

Now we shall show that the second order sufficient condition also holds at $\mathcal{Z}(u)$. We observe first that $\mathcal{Z}(u)$ is KKT point by each $u \in M_4$, consider the set-valued function

$$\Gamma(u) := C(x(u), u),$$

where $C(x(u), u)$ is the critical cone of (P_u). We may assume with no loss of generality that $\Gamma(u) \setminus \{0\}$ is nonempty. For each $u \in M_4$, it is easy to verify that the graph of Γ is closed; thus, if we let B be the unit sphere in X, the function $\Gamma(u) \cap B$ is upper semicontinuous on M_4, and $\Gamma(u) \cap B$ is also nonempty. Then consider the functional

$$\delta(u) := \min_{d \in \Gamma(u) \cap B} \left\{ d^T \nabla^2_{xx} \mathcal{L}(x(u), \mu(u)) d \right\}.$$

Obviously, the quantity being minimized is jointly continuous in u and d, and since the constraint set $\Gamma(u) \cap B$ is upper semicontinuous in u, follows from [1, Theorem 2,p.116] that δ is lower semicontinuous in u for $u \in M_4$. Also since the second order sufficient condition holds for $\mathcal{Z}(u_0)$ and together with (2.12) and (2.13), we have $\delta(u_0) > 0$; hence, there exist an open neighborhood $M(u_0) \subset M_4$ such that for each $u \in M$, we also have $\delta(u) > 0$, combining this and use (2.12) and (2.13) again, we obtain that for each $u \in M$, $\mathcal{Z}(u)$ satisfies the second order sufficient condition. □

4 About five specific $\mathcal{C}^2$-cone reducible sets

It is well known that the class of $\mathcal{C}^2$-cone reducible sets is rich. It includes notably all the polyhedral convex sets and many nonpolyhedral sets such as the cone of symmetric positive semidefinite matrices [6, 21], the second order cone (SOC), and the epigraph cone of the Ky Fan matrix k-norm [9]. Specially, for the matrix space $\Re^{m\times n}$, if $k=1$ then the Ky Fan k-norm is the spectral norm of matrices, and if $k=m$ then the Ky Fan k-norm is just the nuclear norm of matrices. Moreover, the Cartesian product of $\mathcal{C}^2$-cone reducible sets is also $\mathcal{C}^2$-cone reducible (see, e.g., [6, 21]). Whether the conclusion we obtained in Theorem 3.1 is valid for these $\mathcal{C}^2$-cone reducible sets we mentioned above? In this section, we shall check it, in another word, we only need to verify that whether the Assumption 2.1 holds when the set K is one of the $\mathcal{C}^2$-cone reducible sets we listed above.

Nonlinear programming. When $K=\{0\}_q\times\Re^p_-$, the problem (P) is a nonlinear programming problem. Let $G(x)=(h(x),g(x))\in\{0\}_q\times\Re^p_-$. Assume that the strict complementary condition holds at a feasible point x_0 , and $\lambda_0:=(\mu_0,\nu_0)\in\Lambda(x_0)\subset\Re^q\times\Re^p$ is the corresponding Lagrange multiplier, i.e., $\nu_{0i}-g_i(x_0)>0$, $i=1,\cdots,p$. Since $\nu_{0i}+g_i(x_0)\neq 0$, $i=1,\cdots,p$, hence $\Pi_{\Re^p_-}(\cdot)$ is differentiable at $g(x_0)+\nu_0$, further, $\Pi_K(\cdot)$ is differentiable at $G(x_0)+\lambda_0$.

Semidefinite programming. When $K=S^p_+$, where S^p_+ is the cone of all $p\times p$ positive semidefinite matrices, the problem (P) is a semidefinite programming problem. We assume that the strict complementary condition holds at a feasible point x_0 , and $\Omega\in\Lambda(x_0)$ is the corresponding Lagrange multiplier. Let $A:=G(x_0)+\Omega$ have the following spectral decomposition

$$A=P\Lambda P^T,$$

where Λ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors. Then by the strict complementary condition of x_0, we can just assume

$$\Lambda=\begin{bmatrix}\Lambda_\alpha & 0\\ 0 & \Lambda_\gamma\end{bmatrix}, G(x_0)=P\begin{bmatrix}\Lambda_\alpha & \\ & 0\end{bmatrix}P^T, \Omega=P\begin{bmatrix}0 & \\ & \Lambda_\gamma\end{bmatrix}P^T,$$

where α and γ are index sets of positive and negative eigenvalues of A, respectively, as

$$\alpha=\{i:\lambda_i(A)>0\}, \gamma=\{i:\lambda_i(A)<0\}.$$

then A is nonsingular, by [16, Corollary 10], we know that $\Pi_{S^p_+}(\cdot)$ is differentiable at $G(x_0)+\Omega$.

Second order cone programming. When $K=Q_{m+1}:=\{s=(s_0,\bar{s})\in\Re\times\Re^m : s_0\geq\|\bar{s}\|\}$, where $\|\cdot\|$ denotes the Euclidean norm, the problem (P) is a second order cone programming problem. We assume that the strict complementary condition holds at a feasible point x_0 , and $\lambda_0\in\Lambda(x_0)$ is the corresponding Lagrange multiplier, namely, $G(x_0)-\lambda_0\in\mathrm{int}Q_{m+1}$. By [14, Lemma 3.3], one of the following three cases must be holds for each block pair $G(x_0)$ and λ_0:

(i) $G(x_0)\in\mathrm{int}Q_{m+1}$ and $\lambda_0=0$;

(ii) $G(x_0)=0$ and $-\lambda_0\in\mathrm{int}Q_{m+1}$;

(iii) $G(x_0)\in\mathrm{bd}Q_{m+1}\backslash\{0\}$ and $-\lambda_0\in\mathrm{bd}Q_{m+1}\backslash\{0\}$.

Thus, by [14, Lemma 3.4], for each case of $G(x_0)$ and λ_0 mentioned above, $\Pi_{Q_{m+1}}(\cdot)$ is continuously differentiable at $G(x_0)+\lambda_0$.

Matrix cone programming induced by the spectral norm. When $K = \mathcal{K}_2$, where $\mathcal{K}_2$ denotes the epigraph cone of spectral norm, i.e., $\mathcal{K}_2 = \{(t, X) \in \Re \times \Re^{m\times n} | \|X\|_2 \leq t\}$, the problem (P) is a matrix cone programming problem induced by the spectral norm. Note that $\|\cdot\|_2$ denotes the spectral norm, i.e., for each $X \in \Re^{m\times n}$, $\|X\|_2$ is the largest singular value of X; and $\|\cdot\|_*$ denotes the nuclear norm, i.e., for each $X \in \Re^{m\times n}$, $\|X\|_*$ is the sum of singular values of X. We assume that the strict complementary condition holds at a feasible point x_0 , and $\lambda_0 \in \Lambda(x_0)$ is the corresponding Lagrange multiplier, then we have $\lambda_0 \in \mathrm{ri} N_{\mathcal{K}_2}(G(x_0))$.

Let $\Re^{m\times n}$ be the linear space of all $m \times n$ real matrices equipped with the inner product $\langle X, Y\rangle := \mathrm{Tr}(X^T Y)$ for X and Y in $\Re^{m\times n}$, where 'Tr" denotes the trace, i.e., the sum of the diagonal entries of a squared matrix. Denote the Euclidean space $\Re \times \Re^{m\times n}$ by $\mathcal{X}$, and the natural inner product of $\mathcal{X}$ is given by

$$\langle (t, X), (\tau, Y)\rangle_{\mathcal{X}} := t\tau + \langle X, Y\rangle, \ \forall (t, X) \text{ and } (\tau, Y) \in \Re \times \Re^{m\times n}.$$

From [10, Proposition 11], we know that the polar of $\mathcal{K}_2$ is $-\mathcal{K}_*$, where $\mathcal{K}_*$ is the epigraph cone of nuclear norm, denoted by $\mathcal{K}_* = \{(t, X) \in \Re \times \Re^{m\times n} | \|X\|_* \leq t\}$.

From now on, without loss of generality, we assume that $m \leq n$. Let $X \in \Re^{m\times n}$ be given. We use $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_m(X)$ to denote the singular values of X (counting multiplicity) being arranged in non-increasing order. Denote $\sigma(X) := (\sigma_1(X), \sigma_2(X), \cdots, \sigma_m(X))^T \in \Re^m$ and $\Sigma(X) := \mathrm{diag}(\sigma(X))$. Let $X \in \Re^{m\times n}$ admit the following singular value decomposition (SVD):

$$X = U[\Sigma(X)\ 0]V^T = U[\Sigma(X)\ 0][V_1\ V_2]^T = U[\Sigma(X)\ 0]V_1^T, \tag{4.24}$$

where $U \in \mathcal{O}^m$ and $V = [V_1\ V_2] \in \mathcal{O}^n$ are orthogonal matrices, $V_1 \in \Re^{n\times m}$, $V_2 \in \Re^{n\times(n-m)}$. Let $\mu_1 > \mu_2 > \cdots > \mu_r > \mu_{r+1} = 0$ be the distinct singular values of X, and define

$$\alpha_k := \{i \in \{1, \cdots, m\} : \sigma_i(X) = \mu_k\}, k = 1, \cdots, r + 1. \tag{4.25}$$

Denote $G(x_0) = (t_0, X_0)$, and $\lambda_0 = (\zeta_0, Y_0)$, we summarize some properties of $N_{\mathcal{K}_2}(\cdot)$ in the following proposition. For details, see[12, Proposition 3.2, Proposition 4.3].

Proposition 4.1 *Assume that* $(\zeta_0, Y_0) \in \mathrm{ri} N_{\mathcal{K}_2}(t_0, X_0)$, *then* (ζ_0, Y_0) *and* (t_0, X_0) *can be formulated as follows:*

(i) if $(t_0, X_0) \in \mathrm{int}\mathcal{K}_2$, *then* $(\zeta_0, Y_0) = (0, 0)$;

(ii) if $(t_0, X_0) = (0, 0)$, *then* $(\zeta_0, Y_0) \in \mathrm{int} - \mathcal{K}_*$;

(iii) if $(t_0, X_0) \in \mathrm{bd}\mathcal{K}_2 \backslash \{(0, 0)\}$, *then there exists two orthogonal matrices* $\overline{U} \in \Re^{m\times m}$ *and* $\overline{V} \in \Re^{n\times n}$, *such that*

$$X_0 = \overline{U}[\Sigma(X_0)\ 0]\overline{V}^T,\ Y_0 = \overline{U}[\Sigma(Y_0)\ 0]\overline{V}^T, \tag{4.26}$$

and $\|Y_0\|_* = -\zeta_0$. *Moreover, let* $\alpha = \{i \in \{1, \cdots, m\} : \sigma_i(Y_0) > 0\}$, $\alpha_1 = \{i \in \{1, \cdots, m\} : \sigma_i(X_0) = t_0\}$, *then it holds that* $\alpha = \alpha_1$.

Proposition 4.2 *Assume that* $(\zeta_0, Y_0) \in \mathrm{ri} N_{\mathcal{K}_2}(t_0, X_0)$, *then* $\Pi_{\mathcal{K}_2}(\cdot)$ *is differentiable at* $(\eta, \Gamma) := (t_0 + \zeta_0, X_0 + Y_0)$.

Proof. By Proposition 4.1, we know that there are three cases about (ζ_0, Y_0) and (t_0, X_0):

Case 1: $(t_0, X_0) \in \mathrm{int}\mathcal{K}_2$, $(\zeta_0, Y_0) = (0, 0)$. In this case, $(\eta, \Gamma) \in \mathrm{int}\mathcal{K}_2$, which means that $\eta > \|\Gamma\|_2$, we know from condition (i) of [10, Theorem 3], $\Pi_{\mathcal{K}_2}(\cdot)$ is differentiable at (η, Γ).

Case 2: $(t_0, X_0) = (0, 0)$, $(\zeta_0, Y_0) \in \mathrm{int} - \mathcal{K}_*$. In this case, $(\eta, \Gamma) \in \mathrm{int} - \mathcal{K}_*$, which means that $\eta < -\|\Gamma\|_*$, we know from condition (iii) of [10, Theorem 3], $\Pi_{\mathcal{K}_2}(\cdot)$ is differentiable at (η, Γ).

Case 3: $(t_0, X_0) \in \mathrm{bd}\mathcal{K}_2 \backslash \{(0, 0)\}$. In this case, by (iii) of Proposition 4.1, we obtain that $(\zeta_0, Y_0) \in \mathrm{bd} - \mathcal{K}_* \backslash \{(0, 0)\}$, and

$$(\eta, \Gamma) = \overline{U}[\Sigma(X_0) + \Sigma(Y_0)\ 0]\overline{V}^T. \tag{4.27}$$

Thus, we have

$$\|\Gamma\|_2 = \sigma_1(X_0) + \sigma_1(Y_0) > \sigma_1(X_0) = \|X_0\|_2 = t_0 > t_0 + \zeta_0 = \eta, \tag{4.28}$$

and

$$\|\Gamma\|_* = \|X_0\|_* + \|Y_0\|_* = \|X_0\|_* - \zeta_0 > -\|X_0\|_2 - \zeta_0 = -(t_0 + \zeta_0) = -\eta. \tag{4.29}$$

Hence, we obtain that

$$\|\Gamma\|_2 > \eta > -\|\Gamma\|_*. \tag{4.30}$$

For the sake of convenience, let $\sigma_0(\Gamma) = +\infty$ and $\sigma_{m+1}(\Gamma) = -\infty$. Let $s_0 = 0$ and $s_k = \sum_{i=1}^{k} \sigma_i(\Gamma)$, $k = 1, \cdots, m$. Let $\bar{k}$ be the smallest integer $k \in \{0, 1, \cdots, m\}$ such that

$$\sigma_{k+1}(\Gamma) \leq (s_k + \eta)/(k + 1) < \sigma_k(\Gamma). \tag{4.31}$$

Denote $\theta(\eta, \sigma(\Gamma)) \in \Re$ by

$$\theta(\eta, \sigma(\Gamma)) := (s_{\bar{k}} + \eta)/(\bar{k} + 1). \tag{4.32}$$

Without loss of generality, we assume that $1 \leq k \leq m$. We observe that

$$s_{\bar{k}} + \eta = \sum_{i=1}^{\bar{k}} \sigma_i(\Gamma) + \eta = \sum_{i=1}^{\bar{k}} \sigma_i(\Gamma) + \|X_0\|_2 - \|Y_0\|_* = \sum_{i=1}^{\bar{k}} (\sigma_i(X_0) + \sigma_i(Y_0)) + \sigma_1(X_0) - \sum_{i=1}^{m} \sigma_i(Y_0). \tag{4.33}$$

If $\bar{k} + 1 \in \alpha$, then $\sigma_1(X_0) = \cdots = \sigma_{\bar{k}+1}(X_0)$ and $\sigma_1(Y_0) \geq \cdots \geq \sigma_{\bar{k}+1}(Y_0) > 0$, we have

$$\begin{aligned}
(\bar{k} + 1)(\sigma_{\bar{k}+1}(\Gamma)) &= (\bar{k} + 1)(\sigma_{\bar{k}+1}(X_0) + \sigma_{\bar{k}+1}(Y_0)) \\
&> \sum_{i=1}^{\bar{k}} \sigma_i(X_0) + \sigma_1(X_0) - \sum_{i=\bar{k}+1}^{m} \sigma_i(Y_0) \\
&= s_{\bar{k}} + \eta,
\end{aligned}$$

namely, $\sigma_{\bar{k}+1}(\Gamma) > (s_{\bar{k}} + \eta)/(\bar{k} + 1)$, this is conflict with the definition of $\bar{k}$, hence, there must be $\bar{k} + 1 \notin \alpha$, which implies that $\sigma_{\bar{k}+1}(Y_0) = \cdots = \sigma_m(Y_0) = 0$ and $\sigma_{\bar{k}+1}(X_0) < \sigma_1(X_0)$, then we have

$$\begin{aligned}
(\bar{k} + 1)(\sigma_{\bar{k}+1}(\Gamma)) &= (\bar{k} + 1)(\sigma_{\bar{k}+1}(X_0)) \\
&< \sum_{i=1}^{\bar{k}} \sigma_i(X_0) + \sigma_1(X_0) \\
&= \sum_{i=1}^{\bar{k}} \sigma_i(X_0) + \sigma_1(X_0) - \sum_{i=\bar{k}+1}^{m} \sigma_i(Y_0) \\
&= s_{\bar{k}} + \eta,
\end{aligned}$$

i.e., $\sigma_{\bar{k}+1}(\Gamma) < (s_{\bar{k}} + \eta)/(\bar{k}+1)$, together with (4.30), we know from condition (ii) of [10, Theorem 3], $\Pi_{\mathcal{K}_2}(\cdot)$ is differentiable at (η, Γ). □

Matrix cone programming induced by the nuclear norm. When $K = \mathcal{K}_*$, the problem (P) is a matrix cone programming problem induced by the nuclear norm. We also assume that the strict complementary condition holds at a feasible point x_0 , and $\lambda_0 \in \Lambda(x_0)$ is the corresponding Lagrange multiplier, then we have $\lambda_0 \in \mathrm{ri} N_{\mathcal{K}_*}(G(x_0))$.

We also denote $G(x_0) = (t_0, X_0)$, and $\lambda_0 = (\zeta_0, Y_0)$, from Proposition 3.2 of [24], we obtain the following proposition.

Proposition 4.3 *Assume that* $(\zeta_0, Y_0) \in \mathrm{ri} N_{\mathcal{K}_*}(t_0, X_0)$*, then* (ζ_0, Y_0) *and* (t_0, X_0) *can be formulated as follows:*

(i) if $(t_0, X_0) \in \mathrm{int}\mathcal{K}_*$*, then* $(\zeta_0, Y_0) = (0, 0)$*;*

(ii) if $(t_0, X_0) = (0, 0)$*, then* $(\zeta_0, Y_0) \in \mathrm{int} - \mathcal{K}_2$*;*

(iii) if $(t_0, X_0) \in \mathrm{bd}\mathcal{K}_* \backslash \{(0, 0)\}$*, then there exists two orthogonal matrices* $\overline{U} \in \Re^{m\times m}$ *and* $\overline{V} \in \Re^{n\times n}$*, such that*

$$X_0 = \overline{U}[\Sigma(X_0)\ 0]\overline{V}^T,\ Y_0 = \overline{U}[\Sigma(Y_0)\ 0]\overline{V}^T, \tag{4.34}$$

and $\|Y_0\|_2 = -\zeta_0$*. Moreover, let* $\alpha = \{i \in \{1, \cdots, m\} : \sigma_i(X_0) > 0\}$*,* $\alpha_1 = \{i \in \{1, \cdots, m\} : \sigma_i(Y_0) = -\zeta_0\}$*, then it holds that* $\alpha = \alpha_1$*.*

Proof. It is easy to see that case (i) and case (ii) is right, we will mainly focus on the case (iii).

Since $(\zeta_0, Y_0) \in N_{\mathcal{K}_*}(t_0, X_0)$, we have

$$\mathcal{K}_* \ni (t_0, X_0) \perp (\zeta_0, Y_0) \in -\mathcal{K}_2, \tag{4.35}$$

then we obtain that

$$\langle X_0, Y_0\rangle = -t_0\zeta_0,\ -\zeta_0 \geqslant \|Y_0\|_2, \tag{4.36}$$

by employing von Neumanns trace inequality, we have

$$-t_0\zeta_0 = \langle X_0, Y_0\rangle \leqslant \langle \sigma(X_0), \sigma(Y_0)\rangle \leqslant \|Y_0\|_2\|X_0\|_* \leqslant -\zeta_0\|X_0\|_* = -t_0\zeta_0,$$

hence, the inequalities above are equalities in fact, so $\|Y_0\|_2 = -\zeta_0$, and there exists two orthogonal matrices $\overline{U} \in \Re^{m\times m}$ and $\overline{V} \in \Re^{n\times n}$, such that

$$X_0 = \overline{U}[\Sigma(X_0)\ 0]\overline{V}^T,\ Y_0 = \overline{U}[\Sigma(Y_0)\ 0]\overline{V}^T. \tag{4.37}$$

Let $\alpha = \{i \in \{1, \cdots, m\} : \sigma_i(X_0) > 0\}$, $\beta = \{i \in \{1, \cdots, m\} : \sigma_i(X_0) = 0\}$, $\beta_0 = \{m+1, \cdots, n\}$, and $\hat{\beta} = \beta \cup \beta_0$, then

$$\begin{aligned} Y_0 &= \begin{bmatrix} \overline{U}_\alpha & \overline{U}_\beta \end{bmatrix} \begin{bmatrix} \Sigma_\alpha(Y_0) & 0 & 0 \\ 0 & \Sigma_\beta(Y_0) & 0 \end{bmatrix} \begin{bmatrix} \overline{V}_\alpha^T \\ \overline{V}_{\hat{\beta}}^T \end{bmatrix} \\ &= \begin{bmatrix} \overline{U}_\alpha\Sigma_\alpha(Y_0) & \overline{U}_\beta\Sigma_\beta(Y_0) & 0 \end{bmatrix} \begin{bmatrix} \overline{V}_\alpha^T \\ \overline{V}_{\hat{\beta}}^T \end{bmatrix} \\ &= \overline{U}_\alpha\Sigma_\alpha(Y_0)\overline{V}_\alpha^T + \overline{U}_\beta\Sigma_\beta(Y_0)\overline{V}_\beta^T, \end{aligned}$$

which, together with Proposition 3.2 of [24] and $(\zeta_0, Y_0) \in \mathrm{ri} N_{\mathcal{K}_*}(t_0, X_0)$, we have $\Sigma_\alpha(Y_0) = -\zeta_0 I_{|\alpha|}$, and $\sigma_1(\Sigma_\beta(Y_0)) < -\zeta_0$, i.e., $\alpha = \alpha_1$. □

Proposition 4.4 *Assume that* $(\zeta_0, Y_0) \in \mathrm{ri}N_{\mathcal{K}_*}(t_0, X_0)$*, then* $\Pi_{\mathcal{K}_*}(\cdot)$ *is differentiable at* $(\eta, \Gamma) := (t_0 + \zeta_0, X_0 + Y_0)$.

Proof. By Proposition 4.2 and Proposition 4.3, it is easy to see that $\Pi_{\mathcal{K}_2}(\cdot)$ is differentiable at $(-\eta, -\Gamma)$, by Moreau decomposition [9, (3.190)]:

$$\Pi_{\mathcal{K}_*}(t, X) = (t, X) + \Pi_{\mathcal{K}_2}(-t, -X) \ \forall (t, X) \in \Re \times \Re^{m\times n},$$

we obtain that $\Pi_{\mathcal{K}_*}(\cdot)$ is differentiable at (η, Γ). □

Remark 4.1 *When* K *is the Cartesian product of some of the five* $\mathcal{C}^2$*-cone reducible sets we listed above, the Assumption 2.1 also holds obviously.*

5 conclusion

In this paper, we consider the stability of a class of perturbed problems of conic programming problems which are more general than the $\mathcal{C}^2$-smooth parameterization. $\mathcal{C}^2$-smooth parameterization require that the objective function and the constraint mapping of perturbed problem are twice continuously differentiable, here we assume that they all have second partial derivatives only with respect to the decision variable which are jointly continuous. For problem (P) with a $\mathcal{C}^2$-cone reducible convex set, we showed that when a feasible point to the unperturbed problem satisfies the Jacobian uniqueness conditions, the Jacobian of the KKT system is nonsingular at the KKT point, then by using implicit-function theorem, we proved that the locally optimal solution to the perturbed problem is isolated and can be formulated by parameter vector. Further, we showed that the Jacobian uniqueness conditions also hold at some feasible point to the perturbed problem. Finally, we verified that the conclusion we obtained above is suitable for nonlinear programming problem, semidefinite programming problem, second order cone programming problem and matrix cone programming problems induced by the spectral norm and nuclear norm.

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