Inner Conditions for Error Bounds and Metric Subregularity of Multifunctions

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Abstract

We introduce a new class of sets, functions and multifunctions which is shown to be large and to enjoy some nice common properties with the convex setting. Error bounds for objects attached to this class are characterized in terms of inner conditions of Abadie's type, that is conditions bearing on normal cones and coderivatives at points of the solution set. Application are given to the characterization of metric subregularity of multifunctions and error bounds for functions generalizing the results of [30].

AMS classification numbers: 90C25, 90C31, 49J52, 26E25

Key-words: Error bounds, Abadie's qualification condition, metric subregularity, coderivatives, composite analysis, optimality conditions.

1 Introduction and notations

In this paper, in which we develop some ideas from [6], we are dealing with the equivalent problems of error bounds for functions and metric subregularity of multifunction. The problem of error bounds is the following. Given a function f defined on a metric space with extended real values and given a point $\bar{x} \in [f \leq a] := f^{-1}((-\infty, a])$, we say that the function f has a local error bound at $\bar{x}$ if there exist $\tau > 0$ and a neighborhood U of $\bar{x}$ such that: $\tau\, d(x, [f \leq a]) \leq (f(x) - a)^+$ for every $x \in U$. This problem, which has many important applications in optimization dates back to [17, 27] in which the function f was a finite supremum of affine functions. For general functions existence results can be found in the seminal papers [18, 9]. More recently, existence and characterization of such error bound were given (see e.g [5, 28]). In the quoted references, the existence and characteristic condition are given in terms of the complement $[f > a])$ of $[f \leq a])$. Such conditions can be called "outer conditions". Recently, some authors used successfully "inner conditions" bearing on points of $[f \leq a])$ close to $\bar{x}$. Let us mention [23, 29, 19, 7, 11] mainly devoted to the convex case and [30] which covers the convex composite case.

Let us observe that the quoted references mix error bound results for functions and metric subregularity results for multifunctions which is not surprising since these two problems are equivalent. The present paper is an extension of [30] to a more general class of sets, functions and multifunctions namely the class of Fréchet regular sets, functions and multifunctions introduced in Definition 3.1. This class enjoys many interesting properties of convex sets and is stable by preimages by a smooth mapping under a quite mild condition. It allows to provide inner characterizations of error bounds of functions and metric subregularity of multifunctions extending widely those obtained in [30]. It also yields in Theorem 4.1 a characterization for the existence of multipliers for a large class of constrained optimization problems.

A multifunction (identified with its graph) from a set X into a set Y is a subset of $X \times Y$. Given such a multifunction $F \subset X \times Y$, we define the inverse multifunction $F^{-1} \subset Y \times X$ by $F^{-1} = \{(y,x) : (x,y) \in F$. For $x \in X$, we set $F(x) = \{y \in Y : (x,y) \in F\}$, so that, for $y \in Y$, we have $F^{-1}(y) = \{x \in X : y \in F(x)\}$. We shall denote by $B_r(x)$ (resp. $B_r[x]$) the open (resp. closed) ball with center x and radius r in a metric space X. Given $A \subset X$, the distance function d_A is defined by $d_A(x) = \inf_{a \in A} d(x,a)$. When X is a normed space, we shall denote by B the closed unit ball of X and by B_* the closed unit ball of X^* for the dual norm. Let us recall that the Clarke's penalization principle ([12, Proposition 2.4.3]) says that if a function $h : U \to \mathbb{R}$ has a minimum on a subset C of a metric space U at $\bar{x} \in C$, and if h is Lipschitzian with rank κ on U, then the function $h + \kappa\, d_C$ attains its minimum on U at $\bar{x}$. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, its domain $\operatorname{dom} f$ is the set $f^{-1}(\mathbb{R})$ and the function is said to be proper whenever $\operatorname{dom} f \neq \emptyset$. The Ekeland's variational principle will be used under the following form: for any lower semicontinuous and bounded from below function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a complete metric space (X,d), and for any $x \in \operatorname{dom} f$, we can find $z \in X$ such that $f(z) < f(y) + d(y,z)$ for all $y \in X \setminus \{z\}$ and $f(z) + d(z,x) \leq f(x)$. The indicator function ι_S of a subset $S \subset X$ will be defined as $\iota_S(x) = 0$ if $x \in S$ and $\iota_S(x) = +\infty$ if $x \in X \setminus S$.

Let us recall some different notions of subdifferential and normal and tangent cones that will be needed in the sequel. We refer to [22] and [26] for the details. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function defined on a Banach space X.

- Given $\varepsilon > 0$ and $\bar{x} \in \operatorname{dom} f$, the ε-Fréchet subdifferential $\partial^F_\varepsilon f(\bar{x})$ of the function f at $\bar{x}$ will be the set of those $\xi \in X^*$ for which there exists $r > 0$ such that

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\| \text{ for all } x \in B_r(\bar{x}).$$

- We shall also use the Fréchet subdifferential $\partial^F f(\bar{x})$ defined by

$$\partial^F f(x) = \bigcap_{\varepsilon > 0} \partial^F_\varepsilon f(\bar{x}).$$

In other words, one has $\xi \in \partial^F f(x)$ if and only if, for all $\varepsilon > 0$, there exists $r > 0$ such that $f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\|$ for all $x \in B_r(\bar{x})$.

- The limiting Fréchet (or Kruger-Mordukhovich) subdifferential of f at $x \in \operatorname{dom} f$ is the set $\partial_L f(x)$ of $\xi \in X^*$ such that there exist sequences $((x_n, f(x_n), \varepsilon_n))_{n\in\mathbb{N}} \subset X \times \mathbb{R} \times (0, +\infty)$ converging to $(x, f(x), 0)$ and $(\xi_n)_{n\in\mathbb{N}} \subset X^*$ converging $*$-weakly to ξ such that $\xi_n \in \partial^F_{\varepsilon_n} f(x_n)$ eventually.

- The ε-Fréchet normal cone ε-$N^F_C(x)$ to a closed set $C \subset X$ at $\bar{x} \in C$ is the set $\partial^F_\varepsilon(i_C)(\bar{x})$, that is the set of $\xi \in X^*$ such that there exists $r > 0$ such that $\langle \xi, x - \bar{x}\rangle \le \varepsilon\|x - \bar{x}\|$ for all $x \in B_r(\bar{x})$.

- As usual, the Fréchet normal cone $N^F_C(\bar{x})$ of C to $\bar{x}$ is the set

$$N^F_C(\bar{x}) = \bigcap_{\varepsilon>0} \varepsilon\text{-}N^F_C(\bar{x}) = \partial^F i_C(\bar{x}).$$

- The Limiting Fréchet (or Mordukhovich) normal cone $N_C(x)$ to a closed set $C \subset X$ at $\bar{x} \in C$ is defined as $\partial\iota_C(x)$ where ∂ is the Mordukhovich subdifferential. Equivalently, it is the set of $\xi \in X^*$ for which there exist sequence $(x_n)_{n\in\mathbb{N}} \subset C$ converging to $\bar{x}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ converging to 0, and a sequence $(\xi_n)_{n\in\mathbb{N}}$ in X^* converging $*$-weakly to ξ such that $\xi_n \in \varepsilon_n$-$N^F_C(x_n)$ eventually.

- Given a closed subset $A \ni a$ of a normed space X we classically denote by $T_A(a)$ the contingent (or Bouligand cone) cone at $a \in A$ that is the set of those $u \in X$ such that $\liminf_{t\downarrow 0} t^{-1}d(a + tu, A) = 0$ and by $T^C_A(a)$ the Clarke tangent cone which is the (closed convex) set of vectors $u \in X$ such that

$$\lim_{\begin{cases}(t,x)\to(0^+,a)\\ x\in A\end{cases}} t^{-1}d(x + tu, A) = 0.$$

  The Clarke normal cone $N^C_A(a)$ is the polar of the Clarke tangent cone $T^C_A(a)$, that is $N^C_A(a) = \{\xi \in X^* : \langle \xi, u\rangle \le 0 \text{ for every } u \in T^C_A(a)\}$. We further denote by $f'(x; u)$ the directional derivative,

$$f'(x, u) = \liminf_{(t,v)\to(0^+,u)} \frac{f(x + tv) - f(x)}{t}.$$

2 Inner necessary condition for an error bound

In the following proposition we point out the fact that the existence of a local error bound relative to a sublevel set allows to compute or estimate the approximate and exact normal cone to this sublevel set.

Proposition 2.1 Inner dual necessary condition for an error bound. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a closed proper function defined on a Banach space X. Let $\bar{x} \in [f\leq 0] := f^{-1}(]-\infty, 0])$ be such that there exists $\tau > 0$ and $\rho \in (0, +\infty]$ such that*

$$\tau\, d(x, [f\leq 0]) \leq f^+(x) \text{ for all } x \in B_\rho(\bar{x}).$$

For $x \in B_\rho(\bar{x}) \cap [f\leq 0]$, then for all $\varepsilon > 0$, denoting by B_* the closed unit ball of X^*,

(a)

$$\varepsilon\text{-}N^F_{[f\leq 0]}(x) \cap B_* \subset (1+\varepsilon)\tau^{-1}\partial^F_\varepsilon f^+(x).$$

(b) For every $\sigma \in (0, \tau)$, we have:

$$N^F_{[f\leq 0]}(x) \cap \sigma B_* \subset \partial^F f^+(x) \text{ and } N^F_{[f\leq 0]}(x) = \mathbb{R}_+\partial^F f^+(x).$$

(c)

$$N_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial f^+(x).$$

Proof. (a) Let $\zeta \in \varepsilon\text{-}N^F_{[f\leq 0]}(x) \cap B_*$, so that the function $z \mapsto -\langle \zeta, z\rangle + \varepsilon\|z - x\|$ admits a local minimum on $S = [f\leq 0]$ at x. Thus, by the Clarke's penalization principle (Proposition 2.4.3 in [12]), the function $z \mapsto -\langle \zeta, z\rangle + \varepsilon\|z - x\| + (1+\varepsilon)d(z, S)$ has a local minimum at x. From the local error bound estimate, it follows that the function $z \mapsto -\langle \zeta, z\rangle + \varepsilon\|z - x\| + (1+\varepsilon)\tau^{-1}f^+(z)$ also has a local minimum at x, leading to $\zeta \in (1+\varepsilon)\tau^{-1}\partial^F_\varepsilon f^+(x)$.

(b) Let $\zeta \in N^F_{[f\leq 0]}(x) \cap \sigma B_*$, so that, for all ε small enough, we get by using (a),

$$(1+\varepsilon)\tau^{-1}\zeta \in N^F_{[f\leq 0]}(x) \cap B_* \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x) \cap B_* \subset (1+\varepsilon)\tau^{-1}\partial^F_\varepsilon f^+(x),$$

and then $\zeta \in \partial^F_\varepsilon f^+(x)$. It follows that $N^F_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial^F f^+(x)$. For the second part of (b), let us simply observe that $\partial^F_\varepsilon f^+(x) \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x)$, thus

$$\mathbb{R}_+\partial^F f^+(x) \subset \mathbb{R}_+\partial^F_\varepsilon f^+(x) \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x) \text{ for all } \varepsilon > 0,$$

and then $N^F_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial^F f^+(x) \subset N^F_{[f\leq 0]}(x)$.

(c) Let $\zeta \in N_{[f\leq 0]}(x)$, so that there exist sequences $(x_n)_{n\in\mathbb{N}}$ converging to x in $[f\leq 0]$, $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ converging to 0 and $(\zeta_n)_{n\in\mathbb{N}} \subset X^*$ converging $*$-weakly to ζ and such that $\zeta_n \in \varepsilon_n\text{-}N^F_{[f\leq 0]}(x_n)$. Let $c > 1$ be such that $\|\zeta_n\|_* < c$ and let $\eta_n = c^{-1}\zeta_n$ so that $\|\eta_n\|_* < 1$ and $\eta_n \in \varepsilon_n\text{-}N^F_{[f\leq 0]}(x_n)$ due to $c > 1$. Thus we get from (a) that $\tau(1+\varepsilon_n)^{-1}\eta_n \in \partial^F_{\varepsilon_n} f^+(x_n)$ hence $\tau c^{-1}\zeta \in \partial f^+(x)$, leading to $N_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial f^+(x)$. ■

Remark 2.1 We derive from (b) that

$$N^F_{[f\leq 0]}(x) \cap \sigma B_* = \partial^F f^+(x) \cap \sigma B_*, \tag{2.1}$$

since it is easily seen that $\partial^F f^+(x) \subset N^F_{[f\leq 0]}(x)$ for every $x \in [f\leq 0]$.

Moreover, as a consequence of Proposition 2.1, one recovers some well-known properties of the subdifferential of the distance function d_C. Indeed the function $f(x) = d_C(x)$ admits a global error bound for the sublevel $[f{\le}0] = C$ with $\tau = 1$. It follows from part (b) of the previous proposition that for all $\sigma \in (0,1)$ and $x \in C$, we have

$$N_C^F(x) \cap \sigma B_* \subset \partial^F d_C(x) \subset N_C^F(x) \cap B_*. \tag{2.2}$$

In fact [26, Lemma 4.21] gives the following better estimate,

$$\partial^F d_C(x) = N_C^F(x) \cap B_*, \tag{2.3}$$

In the sequel, we shall make use of a *subdifferential operator*, that is, an operator associating to any lower semicontinuous $f : X \to \mathbb{R} \cup \{+\infty\}$, and to any $x \in \operatorname{dom} f$, a subset $\partial f(x)$ of X^* which coincides with the the convex subdifferential when f is convex, and which satisfies the following fuzzy Fermat rule on some suitable class of Banach spaces: if $h : X \to \mathbb{R}$ is convex and Lipschitz continuous, and if $\bar{x} \in \operatorname{dom} f$ is a local minimum point of $f + h$, then for every $\delta > 0$ there exist x, $y \in X$, $\xi \in \partial f(x)$, and $\zeta \in \partial h(y)$ such that,

$$\|x - \bar{x}\| \le \delta\,, \quad \|y - \bar{x}\| \le \delta\,, \quad f(x) \le f(\bar{x}) + \delta\,, \quad \text{and} \quad \|\xi + \zeta\|_* \le \delta.$$

The exact Fermat rule corresponds to the cas $\delta = 0$ that is $0 \in \partial f(\bar{x}) + \partial h(\bar{x})$. For example the Fréchet subdifferential in Asplund spaces satisfies the fuzzy Fermat rule (see [15]); and the Clarke-Rockafellar subdifferential in general Banach spaces along with the limiting Fréchet (or Mordukhovich) subdifferential in Asplund spaces satisfies the exact Fermat rule. The interested reader may found in [4, Example 2.1] a list of subdifferential operators satisfying the fuzzy Fermat rule. Given such a subdifferential, the normal cone to a closed subset $S \subset X$ at the point $\bar{x} \in S$ is defined by $N_S(\bar{x}) = \partial \iota_S(\bar{x})$.

3 Inner characterization of error bounds for a class of functions

At this stage, we need to introduce a new class of closed subsets of a Banach space based on the following result.

Lemma 3.1 *Let C be a closed subset of a normed space X and let $\bar{x} \in C$. Then the two following properties are equivalent:*

(a) for all $\varepsilon > 0$, we can find $\rho > 0$ such that,

$$\langle \xi, x - y \rangle \le \varepsilon \|x - y\| \tag{3.1}$$

for every x, $y \in B_\rho(\bar{x}) \cap C$ and every $\xi \in N_C^F(y) \cap B_*$;

(b) for all $\varepsilon > 0$, we can find $\rho > 0$ such that,

$$d_C(x) \geq \langle \xi, x - y \rangle - \varepsilon \|x - y\| \tag{3.2}$$

for every $x \in B_\rho(\bar{x})$, $y \in B_\rho(\bar{x}) \cap C$ and every $\xi \in \partial^F d_C(y)$.

Moreover $\partial^F d_C(\bar{x})$ and $N_C^F(\bar{x})$ are closed whenever these equivalent properties are in force.

Proof. (b) $\Longrightarrow$ (a). Let $\sigma \in (0, 1)$, we observe that (2.2) and (3.2) imply that

$$\langle \xi, x - y \rangle \leq \varepsilon \|x - y\|$$

for every x, $y \in B_\rho(\bar{x}) \cap C$ and every $\xi \in N_C^F(y) \cap \sigma B_*$, thus $\langle \xi, x - y \rangle \leq \sigma^{-1}\varepsilon \|x - y\|$ for every x, $y \in B_\rho(\bar{x}) \cap C$ and every $\xi \in N_C^F(y) \cap B_*$, yielding (a).

(a) $\Longrightarrow$ (b). Let $\varepsilon > 0$, let $\rho > 0$ be such that $\langle \xi, x - y \rangle \leq \varepsilon \|x - y\|$ for every x, $y \in B_{2\rho}(\bar{x}) \cap C$ and every $\xi \in N_C^F(y) \cap B_*$. Let $x \in B_\rho(\bar{x}) \setminus C$, let $y \in B_\rho(\bar{x}) \cap C$, let (z_n) be a sequence of elements of C such that $\lim_{n \to \infty} \|x - z_n\| = d_C(x)$, and let $\xi \in \partial^F d_C(y) \subset N_C^F(y) \cap B_*$. We have

$$\|x - z_n\| \geq \langle \xi, x - z_n \rangle = \langle \xi, x - y \rangle + \langle \xi, y - z_n \rangle.$$

As $z_n \in B_{2\rho}(\bar{x})$ eventually and $\xi \in N_C^F(y) \cap B_*$, we have

$$\langle \xi, y - z_n \rangle = -\langle \xi, z_n - y \rangle \geq -\varepsilon \|z_n - y\| \geq -\varepsilon \|z_n - x\| - \varepsilon \|x - y\|$$

yielding

$$(1 + \varepsilon)\|x - z_n\| \geq \langle \xi, x - y \rangle - \varepsilon \|x - y\|$$

and then, taking the limit as $n \to \infty$,

$$d_C(x) + \varepsilon \|x - y\| \geq (1 + \varepsilon) d_C(x) \geq \langle \xi, x - y \rangle - \varepsilon \|x - y\|,$$

thus $d_C(x) \geq \langle \xi, x - y \rangle - 2\varepsilon \|x - y\|$.

At last, let us prove that $N_C^F(\bar{x})$ is closed under assumption (a). Let $(\xi_n)_{n \in \mathbb{N}} \subset N_C^F(\bar{x})$ be a sequence converging to some $\xi \in X^*$, let $\varepsilon > 0$ and let $\rho > 0$ be such that (3.1) holds true for every x, $y \in B_\rho(\bar{x}) \cap C$ and every $\xi \in N_C^F(y) \cap B_*$. Then $\langle \|\xi_n\|_*^{-1} \xi_n, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|$ for every $x \in B_\rho(\bar{x}) \cap C$ and every $n \in \mathbb{N}$. Taking the limit over $n \to \infty$ yields $\xi \in N_C^F(\bar{x})$. The proof of closedness of $\partial^F d_C(\bar{x})$ is analogous using (b) instead of (a). ■

Definition 3.1

(a) We say that a closed subset $C \subset X$ is a Fréchet set at $\bar{x} \in C$ whenever it satisfies the equivalent properties in Lemma 3.1.

(a) *We say that C is Fréchet regular at $\bar{x} \in C$ whenever it is Fréchet at $\bar{x}$ and $N^F_C(\bar{x}) = N^C_C(\bar{x})$.*

One easily observe that if C is a Fréchet set at $\bar{x} \in C$ and if X is Asplund, then $N^F_C(\bar{x}) = N^L_C(\bar{x})$. Closed convex sets are clearly Fréchet regular at each of their points. Fréchet regular sets have some elementary properties listed in the following:

Proposition 3.1 *If C is Fréchet regular at x, then it is tangencially regular at x, that is $T^C_C(x) = T_C(x)$. Moreover, one has $\partial^F d_C(x) = \partial^C d_C(x)$, and the function d_C is regular at x that is d_C has a one-sided directional derivative at x which coincides with the Clarke directional $d^{\uparrow}_C(x,\cdot)$.*

Proof. We have $N^F_C(x) = N^C_C(x)$, thus $N^F_C(x)$ is closed convex and $N^F_C(x) \subset [T_C(x)]^-$, hence,

$$T_C(x) \subset [N^F_C(x)]^- = [N^C_C(x)]^- = T^C_C(x) \subset T_C(x),$$

that is $T^C_C(x) = T_C(x)$. Using (2.3), we get

$$N^C_C(x) \cap B_* = N^F_C(x) \cap B_* = \partial^F d_C(x), \tag{3.3}$$

and then, using the fact that $\partial^C d_C(x) \subset N^C_C(x) \cap B_*$ (see [12, Proposition 2.4.2]),

$$\partial^C d_C(x) \subset N^C_C(x) \cap B_* = N^F_C(x) \cap B_* = \partial^F d_C(x) \subset \partial^C d_C(x).$$

At last, let us prove that d_C is regular at x. One easily checks that $d'_C(x,u) \geq \langle \xi, u\rangle$, for any $u \in X$ and $\xi \in \partial^F d_C(x) = \partial^C d_C(x)$ which yields

$$d'_C(x,u) \geq \sup_{\xi \in \partial^C d_C(x)} \langle \xi, u\rangle = d^{\uparrow}_C(x,u).$$

Then we get

$$\liminf_{t\to 0^+} \frac{d_C(x+tu) - d_C(x)}{t} \geq d'_C(x,u) \geq d^{\uparrow}_C(x,u) \geq \limsup_{t\to 0^+} \frac{d_C(x+tu) - d_C(x)}{t},$$

hence the result. ■

Notation 3.1 *If C is Fréchet regular at x, we then set*

$$\begin{cases} N_C(x) := N^F_C(x) = N^C_C(x) \\ \\ \partial d_C(x) := \partial^F d_C(x) = \partial^C d_C(x). \end{cases} \tag{3.4}$$

Observe that if C is Fréchet regular at x, we get from (2.3) that,

$$\partial d_C(x) = N_C(x) \cap B_*. \tag{3.5}$$

An interesting property of Fréchet regularity lies in the fact that the class of Fréchet regular sets is stable under preimages by a smooth mapping under a quite mild condition as shown in next result which provides a large class of Fréchet regular sets.

Theorem 3.1 *Let X, Y be Banach spaces, let $K \subset Y$ be closed, let $\phi : X \to Y$ be a $\mathcal{C}^1$ mapping defined on an open subset of X and let $\bar{x} \in S := \phi^{-1}(K)$. Assume that K is Fréchet regular at $\phi(x)$ for every $x \in S$ near $\bar{x}$, and that we can find $\tau > 0$ and an open neighborhood V of $\bar{x}$ such that,*

$$\tau\, d_S(x) \leq d_K(\phi(x)) \text{ for all } x \in V. \tag{3.6}$$

Then, S is Fréchet regular on $S \cap V$ and, for every $x \in S \cap V$, we have

$$N_S(x) \cap \tau\, B_* \subset \phi'(x)^*\big(\partial d_K(\phi(x))\big) = \phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big), \tag{3.7}$$

and

$$N_S(x) = \phi'(x)^*\big(N_K(\phi(x))\big). \tag{3.8}$$

Proof. Let us begin by showing that $N_S^C(x) = N_S^F(x)$ for every $x \in S \cap V$. We may assume that K is Fréchet regular at $\phi(x)$ for $x \in S \cap V$. Let $f(x) = d_K(\phi(x))$, and let $x \in S \cap V$. As the function d_K is regular at $\phi(x)$ (see Proposition 3.1), we know from [12, Theorem 2.3.10] that

$$\partial^C f(x) = \phi'(x)^*\big(\partial d_K(\phi(x))\big).$$

As easily seen, we have $\phi'(x)^*\big(\partial d_K(\phi(x))\big) = \phi'(x)^*\big(\partial^F d_K(\phi(x))\big) \subset \partial^F f(x)$ and then $\partial^C f(x) = \partial^F f(x)$. As K is tangencially regular at $\phi(x)$ and (3.6) holds true, we know from [9, Corollary 5.1] that,

$$T_S(x) = T_S^C(x) = \{u \in X : \phi'(x)(u) \in T_K(\phi(x))\}.$$

Setting $\ell = \phi'(x)$, we derive that $[\ell^*(N_K(\phi(x)))]^- = T_S^C(x)$, so that

$$N_S^C(x) = \mathrm{cl}\,(\ell^*(N_K(\phi(x)))). \tag{3.9}$$

Now we get from (2.1) that, for every $\sigma \in (0, \tau)$,

$$N_S^F(x) \cap \sigma B_* = \partial^F f(x) \cap \sigma B_*. \tag{3.10}$$

As $\partial^F f(x)$ is closed (it is equal to $\partial^C f(x)$), we deduce that $N_S^F(x) \cap \sigma B_*$ and then $N_S^F(x)$ is closed. Using Proposition 2.1 (b) and (3.5), we obtain

$$N_S^F(x) = \mathbb{R}_+ \partial^F f(x) = \mathbb{R}_+ \ell^*(N_K(\phi(x)) \cap B_*) = \ell^*(N_K(\phi(x))) \tag{3.11}$$

thus, by (3.9), $N_S^C(x) = \mathrm{cl}\,\big(N_S^F(x)\big) = N_S^F(x)$. Then, letting σ increase to τ in (3.10) and taking into account the closedness of $N_S^F(x)$ and of $\partial^F f(x)$, we obtain

$$N_S^F(x) \cap \tau\, B_* \subset \phi'(x)^*\big(\partial d_K(\phi(x))\big). \tag{3.12}$$

At last it remains to prove that S is a Fréchet set on $S \cap V$. Let $x \in S \cap V$, let $\varepsilon > 0$, let $\eta > 0$ be such that $\langle \zeta, y_2 - y_1 \rangle \leq \varepsilon \|y_2 - y_1\|$ for all y_2, $y_1 \in B_\eta(\phi(x)) \cap K$ and all $\zeta \in N_K(y_1) \cap B_*$, and let $\delta > 0$ be such that $B_\delta(x) \subset V$, $\phi(B_\delta(x)) \subset B_\eta(\phi(x))$ and $\|\phi(v) - \phi(u) - \phi'(u)(v-u)\| \leq \varepsilon \|v - u\|$ for all v, $u \in B_\delta(x)$. Given v, $u \in B_\delta(x) \cap S$, let $\xi \in N_S(u) \cap \tau B_* \subset \phi'(u)^*\big(\partial d_K(\phi(u))\big)$ and let $\zeta \in \partial d_K(\phi(u)) = N_K(\phi(u)) \cap B_*$ be such that $\xi = \phi'(u)^*(\zeta)$. We get,

$$\langle \xi, v - u \rangle = \langle \zeta, \phi'(u)(v-u) \rangle \leq \langle \zeta, \phi(v) - \phi(u) \rangle + \varepsilon \|v - u\| \leq (c+1)\varepsilon \|v - u\|,$$

where c is a lipschitz rate of ϕ on $B_\delta(x)$, which shows that S is Fréchet regular at x. Finally (3.8) follows from (3.11) and (3.7) from (3.12). ■

We derive from Theorem 3.1 examples of Fréchet regular sets in the following:

Example 3.1

(a) If $\phi : U \to Y$ is a $\mathcal{C}^1$ mapping defined on an open subset U of a Banach space X with values in a Banach space Y and if $\phi(\bar{x}) = 0$ and $\phi'(\bar{x})$ is surjective at some $\bar{x} \in U$, then we know from the Lyusternik-Graves Theorem that we can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that $\tau\, d_S(x) \le \|\phi(x)\|$ for every $x \in V$, where $S = g^{-1}(0)$. We obtain from Theorem 3.1 that S is Fréchet regular at $S \cap V$. Then a submanifold $M \subset \mathbb{R}^d$ is Fréchet regular at every $x \in M$.

(b) Let us consider the classical framework in constrained optimization. Let $\phi : U \to Y$ be a $\mathcal{C}^1$ mapping $\phi : U \to Y$ where U is an open subset of a Banach space X and Y is a Banach space. Given a closed convex cone $K \subset Y$ we consider the constraint set S defined by

$$S = \{x \in U : \phi(x) \in K\} = f^{-1}(]-\infty, 0]) = f^{-1}(0), \tag{3.13}$$

where the function $f : U \to \mathbb{R}$ is defined by $f(x) = d_K(\phi(x))$. Assume that for some $\bar{x} \in S$, we can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that

$$\tau\, d_S(x) \le f(x) \ \text{ for every } \ x \in V. \tag{3.14}$$

Then Theorem 3.1 tells us that the constraint set S is Fréchet regular at every $x \in S \cap V$ since closed convex sets are clearly Fréchet regular at each of their points. Several condition are known ensuring (3.14). For example condition:

$$\liminf_{\begin{cases} x \to \bar{x} \\ \phi(x) \notin K \end{cases}} \quad d_*\big(0, \phi'(x)^*(\partial d_K(\phi(x)))\big) > 0. \tag{3.15}$$

This follows from the fact that $\phi'(x)^*(\partial d_K(\phi(x))) = \partial f(x)$ where ∂ is, for example the Clarke's subdifferential and from [5, Proposition 4.1 and Theorem 5.1] or [28, Theorem 2.2]. Observe that (3.15) is in force whenever the following stronger condition holds:

$$d_*(0, \phi'(\bar{x})^*(K^- \cap S_*)) > 0, \tag{3.16}$$

where S_* is the closed unit sphere of Y^*, due to the continuity of ϕ' and to the fact that $\partial d_K(\phi(x)) \subset K^- \cap S_*$ whenever $\phi(x) \notin K$. If Y is finite dimensional, then compactness of the unit sphere leads to the fact that (3.15) is satisfied whenever

$$\ker(\phi'(\bar{x})^*) \cap K^- \cap (\phi(\bar{x}))^\perp = \{0\}. \tag{3.17}$$

For example if $Y = \mathbb{R}^p \times \mathbb{R}^m$ with $K = \{0_p\} \times \mathbb{R}^m_-$, condition (3.17) becomes

$$\begin{cases} \mu \ge 0 \text{ and } \sum_{i=1}^{p} \lambda_i \phi_i'(\bar{x}) + \sum_{j=p+1}^{p+m} \mu_j \phi_j'(\bar{x}) = 0 \\ \\ \sum_{j=p+1}^{p+m} \mu_j \phi_j(\bar{x}) = 0 \end{cases} \quad \text{implies } (\lambda, \mu) = (0_p, 0_m),$$

that is the classical Mangasarian-Fromowitz condition.

(b) Condition

$$(3.18) \qquad N_S(x) \cap \tau B_* \subset \phi'(x)^*(\partial d_K(\phi(x))) = \partial f(x) \text{ for every } x \in S \cap V,$$

used in Theorem 3.1 can be viewed as a general Abadie's condition. In the case where $\phi = (\phi_1, \cdots, \phi_m)$ with $\phi_1, \cdots, \phi_m$ convex differentiable and $K = \mathbb{R}^m_-$, and setting $f(x) = d_K(\phi(x)) = \sum_{k=1}^m \phi_k^+(x)$ whenever $\mathbb{R}^m$ is equiped with the ℓ_1 norm. It follows that, for any $x \in C$, we have

$$\phi'(x)^*\big(\partial d_K(\phi(x))\big) = \sum_{k\in K(x)} [0,1]\phi_k'(x),$$

where $K(x) = \{k \in [1,m] : \phi_k(x) = 0\}$. Thus condition (3.18) implies the dual form of the usual Abadie's qualification condition that is

$$N_S(x) = \sum_{k\in K(x)} \mathbb{R}_+\phi_k'(x) \text{ for every } x \in S \text{ near } \bar{x}$$

Conversely, it is shown in the proof of [23, Theorem 3] that this condition implies (3.18) for some $\tau > 0$. The dual form of Abadie's condition dates back to [16] and is sometimes taken as a definition of Abadie's condition as, for example in [23].

The following lemma is our key tool.

Lemma 3.2 *Let $f : X \to \mathbb{R}_+$ be a lower semicontinuous function, let $\bar{x} \in S = [f{\le}0] = [f{=}0]$ and let ∂ be a subdifferential operator. Assume that we can find a multifunction $\Sigma \subset X \times X^*$ such that for all $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ such that*

$$(3.19) \qquad f(x) - \langle \xi, x-z\rangle \ge -\varepsilon\|x-z\| \text{ for every } x \in U,\ z \in U\cap S \text{ and } \xi \in \Sigma(z).$$

Assume also that, setting $N_S^\partial(z) = \partial\iota_S(z)$, there exists a neighborhood V of $\bar{x}$ such that,

$$(3.20) \qquad N_S^\partial(z) \cap \tau B_* \subset \Sigma(z) \text{ for every } z \in S\cap V.$$

Then, for every $\sigma \in (0,\tau)$, we can find $\rho > 0$ such that,

$$\sigma\, d(x,S) \le f(x) \text{ for every } x \in B_\rho(\bar{x}).$$

Proof. Let $\varepsilon := \tau - \sigma$ and let $\rho > 0$ be such that for every $x \in B_{2\rho}(\bar{x})$, $z \in B_{2\rho}(\bar{x}) \cap S$ and every $\xi \in \Sigma(z)$, we have

$$(3.21) \qquad f(x) - \langle \xi, x-z\rangle \ge -\varepsilon\|x-z\| \text{ and } N_S^\partial(z)\cap\tau B_* \subset \Sigma(z).$$

Let $x \in (X\setminus S)\cap B_\rho(\bar{x})$ and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. For each $n \in \mathbb{N}$, we can find $y_n \in S$ such that $h(y_n) \le d_S(x) + \delta_n$ where $h(z) = \iota_S(z) + \|z-x\|$. From Ekeland's principle, there exists $z_n \in S$ such that $\|z_n - y_n\| \le \sqrt{\delta_n}$, so that $\lim_{n\to\infty}\|z_n - y_n\| = 0$, and that z_n minimizes the function

$h(\cdot) + \sqrt{\delta_n}\|z_n - \cdot\|$ on X. From the fuzzy Fermat rule, for every $n \in \mathbb{N}$, we then can find $x_n \in X$, ζ_n, $\eta_n \in X^*$ such that

$$\|x_n - z_n\| \le \delta_n,\ \|\zeta_n\|_* \le 1,\ \|\eta_n\|_* \le \sqrt{\delta_n},\ \langle \zeta_n, x_n - x\rangle = \|x - x_n\|,$$

and $s_n \in S$ with $\|s_n - z_n\| \le \delta_n$ and $\omega_n \in N_S^{\partial}(s_n)$ such that

$$\|\zeta_n + \eta_n + \omega_n\|_* \le \delta_n.$$

As $\lim_{n\to\infty} \|x - s_n\| = d_S(x)$, we have $s_n \in B_{2\rho}(\bar{x})$ eventually and,

$$\|\omega_n\|_* \le (1 + \sqrt{\delta_n} + \delta_n),$$

thus we get from (3.20) that $\xi_n := \tau(1 + \sqrt{\delta_n} + \delta_n)^{-1}\omega_n \in \Sigma(s_n)$. As $s_n \in S$ we get, returning to 3.21 and setting $k_n = (1 + \sqrt{\delta_n} + \delta_n)^{-1}$,

$$\begin{aligned} f(x) &\ge \langle \xi_n, x - s_n\rangle - \varepsilon\|x - s_n\| \\ &\ge \tau k_n\langle \omega_n, x - x_n\rangle - 2\tau\delta_n - \varepsilon\|x - s_n\| \\ &\ge \tau k_n\langle -\zeta_n, x - x_n\rangle - \tau k_n(\sqrt{\delta_n} + \delta_n)\|x - x_n\| - 2\tau\delta_n - \varepsilon\|x - s_n\|. \end{aligned}$$

As $\langle -\zeta_n, x - x_n\rangle = \|x - x_n\|$ and $\lim_{n\to\infty} \|x - s_n\| = \lim_{n\to\infty} \|x - x_n\| = d_S(x)$, we obtain by letting n go to $+\infty$,

$$f(x) \ge (\tau - \varepsilon)\, d_S(x) = \sigma\, d_S(x) \ \text{ for every } \ x \in B_\rho(\bar{x}).$$

■

The next theorem is an error bound characterization from which we shall derive metric subregularity characterizations in section 5.

Theorem 3.2 *Let X, Y be Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $K \subset Y$ be closed, let $\phi : X \to Y$ be a $\mathcal{C}^1$ mapping defined on an open subset of X and let $\bar{x} \in S := \phi^{-1}(K)$. Assume that K is Fréchet regular at every $\phi(x)$ for $x \in S$ close to $\bar{x}$. Then, the two following properties are equivalent.*

(a) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that

$$N_S^{\partial}(x) \cap \tau B_* \subset \phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big) \ \text{ for every } \ x \in S \cap V. \qquad (3.22)$$

(b) For every $\sigma \in (0, \tau)$, we can find a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_S(x) \le d_K(\phi(x)) \ \text{ for every } \ x \in U.$$

Moreover S is Fréchet regular on $S \cap U$.

Proof. The fact that (b) $\Longrightarrow$ (a) follows from Theorem 3.1 observing that $N_S^{\partial}(x) = N_S(x)$ for every $x \in S$ close to $\bar{x}$ since S is then Fréchet regular near $\bar{x}$ and using the fact that $\phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big)$ is closed for any $x \in S$ close to $\bar{x}$.

Let us prove that (a) $\Longrightarrow$ (b). We may assume that V is open and that K is Fréchet regular at $\phi(x)$ for every $x \in S \cap V$. Setting $f(x) = d_K(\phi(x))$, let $x \in S \cap V$ and let $\sigma \in (0, \tau)$. As K is Fréchet regular at $\phi(x)$, we obtain from Proposition 3.1 that $N_K(\phi(x)) \cap B_* = \partial^F d_K(\phi(x))$, thus

$$N_S^{\delta}(x) \cap \tau B_* \subset \phi'(x)^*\big(\partial^F d_K(\phi(x))\big).$$

Let us set $\Sigma(x) = \phi'(x)^*\big(\partial^F d_K(\phi(x))\big)$, we claim that f satisfies (3.19). Indeed, given $\varepsilon > 0$, let $\rho > 0$ be such that

$$d_K(u) - \langle \zeta, u - v \rangle \geq -\varepsilon \|u - v\|$$

for all $u \in B_\rho(\phi(\bar{x}))$, $v \in B_\rho(\phi(\bar{x})) \cap K$ for all $\zeta \in \partial^F d_K(v)$. Let c be a lipschitz rate of ϕ near $\bar{x}$, let $\eta \in (0, \rho)$ be such that $\phi(B_\eta(\bar{x})) \subset B_\rho(\phi(\bar{x}))$ and that

$$\|\phi(x) - \phi(z) - \phi'(z)(x - z)\| \leq \varepsilon \|x - z\| \text{ on } B_\eta(\bar{x})$$

Now, let $x \in B_\eta(\bar{x})$, $z \in B_\eta(\bar{x}) \cap S$, let $\xi = \phi'(z)^*(\zeta) \in \Sigma(z)$ where $\zeta \in \partial^F d_K(\phi(z))$, so that,

$$\begin{aligned} f(x) &\geq \langle \zeta, \phi(x) - \phi(z) \rangle - \varepsilon \|\phi(x) - \phi(z)\| \\ &\geq \langle \zeta, \phi'(z)(x - z) \rangle - \varepsilon \|x - z\| \|\zeta\|_* - c\varepsilon \|x - z\| \\ &\geq \langle \xi, x - z \rangle - (c + 1)\varepsilon \|x - z\|. \end{aligned}$$

Thus the conclusion follows from Lemma 3.2. ■

Remark 3.1 It is noteworthy that Theorem 3.2 and all results in the sequel of the present paper apply with $\partial = \partial^F$ if X is Asplund and with $\partial = \partial^C$ in a general Banach space X.

Theorem 3.2 is quite versatile as shown by the following:

Theorem 3.3 *Let X, Y, Z be Banach spaces let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $K \subset Y$, $L \subset Z$ be closed, and let $\phi : X \to Y$, $\psi : X \to Z$ be $\mathcal{C}^1$ mappings defined on an open subset of X, and let $\bar{x} \in S := \phi^{-1}(K) \cap \psi^{-1}(L)$. Assume that K (resp. L) is Fréchet regular at every $\phi(x)$ (resp. $\psi(x)$) for $x \in S$ close to $\bar{x}$, then the two following properties are equivalent.*

(a) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that

(3.23) $$N_S^{\partial}(x) \cap \tau B_* \subset \phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big) + \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big)$$

for every $x \in S \cap V$.

(b) *For every $\sigma \in (0, \tau)$, we can find a neighborhood U of $\bar{x}$ such that*

$$\sigma\, d_S(x) \leq d_K(\phi(x)) + d_L(\psi(x)) \ \textit{ for every } \ x \in U.$$

Moreover S is Fréchet regular on $S \cap U$.

Proof. It is straightforward to check that $K \times L$ is Fréchet regular at $x \in K \cap L$ whenever K and L are Fréchet regular at x. Let us endow $Y \times Z$ with the norm $\|(y, z)\| = \|y\| + \|z\|$, so that the unit ball of the dual norm is $B_* \times B_*$. Now we have $d_{K\times L}(y, z) = d_K(y) + d_L(z)$, $(\phi, \psi)^{-1}(K \times L) = \phi^{-1}(K) \cap \psi^{-1}(L)$ and $(\phi, \psi)'(x)^*(\xi, \zeta) = \phi'(x)^*(\xi) + \psi'(x)^*(\zeta)$ for $(\xi, \zeta) \in Y^* \times Z^*$. Applying Theorem 3.2 in that case leads to the result. ∎

The previous theorem has the following consequence.

Corollary 3.1 *Let X be a Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $K \subset X$, $L \subset X$ be closed subsets which are Fréchet regular near $\bar{x} \in K \cap L$. Then the two following properties are equivalent.*

(a) *We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that*

$$N^{\partial}_{K\cap L}(x) \cap \tau B_* \subset N_K(x) \cap B_* + N_L(x) \cap B_* \ \textit{ for every } \ x \in K \cap L. \tag{3.24}$$

(b) *For every $\sigma \in (0, \tau)$, we can find a neighborhood U of $\bar{x}$ such that*

$$\sigma\, d_{K\cap L}(x) \leq d_K(x) + d_L(x) \ \textit{ for every } \ x \in U.$$

Proof. Apply Theorem 3.3 with $Y = Z = X$ and $\phi = \psi = I_X$. ∎

4 The Lagrange-Karush-Kuhn-Tucker property

In this section, we apply results of the preceding section to optimality conditions. It is well known that existence of multipliers in mathematical programming is obtained in presence of some qualification condition. Then, it seems interesting to identify some characteristic condition guaranteeing existence of multipliers like in [20]. To this end, let us begin by the definition of the Lagrange-Karush-Kuhn-Tucker property. We consider a closed subset $K \subset Y$ of a Banach space Y and a $\mathcal{C}^1$ mapping $\phi : U \to Y$ defined on an open subset of a Banach space X and we set $S = \phi^{-1}(K)$. Given $\bar{x} \in S$, we assume that K is Fréchet regular at $\phi(x)$ for every $x \in S$ close to $\bar{x}$.

Definition 4.1 *We say that the pair (ϕ, K) has the Lagrange-Karush-Kuhn-Tucker property near $\bar{x} \in S$ whenever there exists $\theta > 0$ and $\rho > 0$ such that for any function f which have a local minimum on S at some $x \in S \cap B_\rho(\bar{x})$ and is κ-Lipschitzian near x, then we can find $\xi \in X^*$ such that,*

$$\begin{cases} 0 \in \partial^C f(x) + \phi'(x)^*(\xi) \\ \\ \xi \in N_K(\phi(x) \cap \theta\kappa B_*. \end{cases} \tag{4.1}$$

In the Lagrange case, corresponding to $K = \{0\}$, and if f is differentiable at x, then property (4.1) reduces to $f'(x) + \phi'(x)^*(\xi) = 0$, that is ξ is a classical Lagrange multiplier. If K is a closed convex cone then ξ is a Karush-Kuhn-Tucker multiplier. It is pleasant that these two cases may be treated in a unified way. Our result is the following:

Theorem 4.1 *Let $K \subset Y$ be a closed subset of a Banach space Y, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $\phi : U \to Y$ be a $\mathcal{C}^1$ mapping defined on an open subset of a Banach space X and let $S = \phi^{-1}(K)$. Given $\bar{x} \in S$, we assume that K is Fréchet regular at $\phi(x)$ for every $x \in S$ close to $\bar{x}$. Then the following properties are equivalent.*

(a) The set S has the following error bound property near $\bar{x}$: there exist $\tau > 0$ such that for every $\sigma \in (0, \tau)$ we can find $\rho > 0$ such that

$$\sigma\, d_S(x) \leq d_K(\phi(x)) \;\text{ for every }\; x \in B_\rho(\bar{x}). \tag{4.2}$$

(b) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that

$$N_S^\partial(x) \cap \tau B_* \subset \phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big) \;\text{ for every }\; x \in S \cap V. \tag{4.3}$$

(c) The pair (ϕ, K) has the Lagrange-Karush-Kuhn-Tucker property near $\bar{x} \in S$.

Proof. We know from Theorem 3.3 that (a) and (b) are equivalent.

(a) $\Longrightarrow$ (c) Let $\sigma \in (0, \tau)$ and let $\rho > 0$ satisfying (4.2). We may assume that K is Fréchet regular at $\phi(x)$ for every $x \in S \cap B_\rho(\bar{x})$. Let f be κ-Lipschitzian near x having a local minimum on S at $x \in S \cap B_\rho(\bar{x})$. We can find $r > 0$ such that f is κ-Lipschitzian on $B_{2r}(x) \subset B_\rho(\bar{x})$ and attains its minimum on $S \cap B_{2r}(x)$ at x. Applying the Clarke's penalization principle, we obtain that for every $z \in B_r(x)$,

$$f(z) + \sigma^{-1}\kappa\, d_K(\phi(z)) \geq f(z) + \kappa\, d_S(z) = f(z) + \kappa\, d_{S \cap B_{2r}(x)}(z) \geq f(x).$$

It follows that the function $f + \gamma\,(d_K \circ \phi)$ with $\gamma = \sigma^{-1}\kappa$ has a local minimum at x, thus we get, using the facts that the function d_K is Fréchet regular at $\phi(x)$ and that K is Fréchet regular at $\phi(x)$,

$$0 \in \partial^C f(x) + \gamma\, \partial^C (d_K \circ \phi)(x) = \partial^C f(x) + \gamma\, \phi'(x)^*\big(N_K(\phi(x)) \cap B_*\big),$$

leading to (4.1) for some $\xi \in N_K(\phi(x))$ satisfying $\|\xi\|_* \leq \theta\kappa$ with $\theta = \sigma^{-1}$.

(c) $\Longrightarrow$ (a) Let $\rho > 0$ be such that the Lagrange-Karush-Kuhn-Tucker property is satisfied by the pair (ϕ, K) on $B_{2\rho}(\bar{x})$ and let c be a lipschitz rate of ϕ on $B_{2\rho}(\bar{x})$. Let $\tau = \theta^{-1}$, let $\sigma \in (0, \tau)$ and let $0 < \varepsilon < (1 + c)^{-1}(\tau - \sigma)$. We may assume that

$$\|\phi(x) - \phi(z) - \phi'(z)(x - z)\| \leq \varepsilon \|x - z\|,$$

for every x, $z \in B_{2\rho}(\bar{x})$. Given $x \in (X \setminus S) \cap B_\rho(\bar{x})$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0, let $x_n \in S$ be such that $h(x_n) \le d_S(x) + \delta_n$ where

$$h(z) = \iota_S(z) + \|z - x\|,$$

so that $\lim_{n\to\infty} \|x - x_n\| = d_S(x)$. From Ekeland's principle applied with the metric $d(x, y) = \sqrt{\delta_n}\|x - y\|$, we can find $z_n \in S$ such that

$$h(z_n) + \sqrt{\delta_n}\|x_n - z_n\| \le h(x_n), \tag{4.4}$$

and that z_n minimizes the function $h(\cdot) + \sqrt{\delta_n}\|z_n - \cdot\|$ on X. Observe that (4.4) yields $\|x_n - z_n\| \le \sqrt{\delta_n}$ thus $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Then, we get

$$\|x - z_n\| \le \|x - x_n\| + \|x_n - z_n\| \le d_S(x) + \delta_n + \sqrt{\delta_n} \le \|x - \bar{x}\| + \delta_n + \sqrt{\delta_n},$$

so that $z_n \in B_{2\rho}(\bar{x})$ eventually. Setting

$$f(z) = \|z - x\| + \sqrt{\delta_n}\|z - z_n\|,$$

this function is convex Lipschitzian with rank $\kappa := 1 + \sqrt{\delta_n}$ and has a global minimum on S at $z_n \in B_{2\rho}(\bar{x})$. Thus we can find, from (b) vectors $\zeta_n \in \partial\|\cdot\|(z_n - x)$, $\eta_n \in \sqrt{\delta_n}\,\partial\|\cdot\|(0)$, $\xi_n \in N_K(\phi(z_n))$ such that $\|\xi_n\|_* \le \theta(1 + \sqrt{\delta_n})$ and

$$0 = \zeta_n + \eta_n + \phi'(z_n)^*(\xi_n).$$

Thus we get $\|\zeta_n\|_* = 1$, $\|\eta_n\|_* \le \sqrt{\delta_n}$, $\langle \zeta_n, z_n - x\rangle = \|x - z_n\|$, and

$$\begin{cases} -\zeta_n - \eta_n = \phi'(z_n)^*(\xi_n) \\ \\ \xi_n \in N_K(\phi(z_n)) \end{cases},$$

and then $\tau_n\xi_n \in \partial d_K(\phi(z_n))$ with $\tau_n = \left(\theta(1 + \sqrt{\delta_n})\right)^{-1}$. Now, as K is Fréchet regular at $\phi(\bar{x})$, we may assume that for every $x \in B_{2\rho}(\bar{x})$, $z \in B_{2\rho}(\bar{x}) \cap S$ and for every $\chi \in \partial d_K(\phi(z))$, we have $\langle \chi, \phi(x) - \phi(z)\rangle \ge -\varepsilon\|\phi(x) - \phi(z)\|$. We have

$$\lim_{n\to\infty} \|x - z_n\| = \lim_{n\to\infty} \|x - x_n\| = d_S(x),$$

Observing that $d_K(\phi(z_n)) = 0$ and that $\tau_n\xi_n \in \partial d_K(\phi(z_n))$ we get,

$$\begin{aligned} d_K(\phi(x)) &\ge \langle \tau_n\xi_n, \phi(x) - \phi(z_n)\rangle - \varepsilon\|\phi(x) - \phi(z_n)\| \\ &\ge \tau_n\langle \phi'(z_n)^*(\xi_n), x - z_n\rangle - \varepsilon\,\|x - z_n\| - \varepsilon c\|x - z_n\| \\ &\ge \tau_n\langle -\zeta_n, x - z_n\rangle - (\varepsilon\,(1 + c) + \tau_n\sqrt{\delta_n})\|x - z_n\| \\ &\ge (\tau_n - \varepsilon\,(1 + c) - \tau_n\sqrt{\delta_n})\|x - z_n\|. \end{aligned}$$

Letting n go to $+\infty$, we get the desired conclusion:

$$d_K(\phi(x)) \geq (\tau - (1+c)\varepsilon)\, d_S(x) \geq \sigma\, d_S(x) \text{ for every } x \in B_\rho(\bar{x}).$$

■

It follows from Theorem 4.1 that the error bound property (4.2) is the ultimate qualification condition in order to derive existence of multipliers.

Remark 4.1 Theorem 4.1 extends [20, Theorem 2] in which the same conclusion was obtained in the particular case of finite dimensions with $K = \{0\}^p \times \mathbb{R}^q_-$.

5 Applications to metric subregularity of multifunctions

Given metrics denoted by d on X and Y, we assume that $X \times Y$ is endowed with a metric also denoted by d inducing the product topology and satisfying the condition $d((x_1, y_1), (x_1, y_2)) \leq d(y_1, y_2)$, and let $f : X \to \mathbb{R}$ be defined by

$$f(x) = d_F(x, \bar{y}),$$

so that $[f{\leq}0] = F^{-1}(\bar{y})$ whenever F is closed. Given $\tau > 0$, let us define on $X \times Y$ a norm $\|(\cdot,\cdot)\|^\tau$ whose associate metric is denoted by d^τ by,

$$\|(u,v)\|^\tau = \tau\, \|u\| + \|v\|. \tag{5.1}$$

The following proposition which is inspired from a part of the proof of [29, Theorem 3.1] establishes the equivalence between a local error bound for the function f and metric subregularity of the multifunction F.

Proposition 5.1 *Let $F \subset X \times Y$ be a closed multifunction, let $(\bar{x}, \bar{y}) \in F$ and let $\tau > 0$. Let us consider the two following properties:*

(a) we can find a neighborhood V of $\bar{x}$ such that, setting $S = F^{-1}(\bar{y})$, we have

$$\tau\, d_S(x) \leq d_{F(x)}(\bar{y}) \text{ for all } x \in V; \tag{5.2}$$

(b) we can find a neighborhood W of $\bar{x}$ such that

$$\tau\, d_S(x) \leq d_F(x, \bar{y}).$$

Then (b) $\Longrightarrow$ (a) and (a) $\Longrightarrow$ (b) whenever $d = d^\tau$ on $X \times Y$. We say that F is metrically subregular at $(\bar{x}, \bar{y}) \in F$ whenever (a) is satisfied.

Proof. (b) $\Longrightarrow$ (a). Let $\rho > 0$, be such that $\tau\, d_S(x) \le d_F(x, \bar y)$ for all $x \in B_\rho(\bar x)$. Then, for every $x \in B_\rho(\bar x)$ and $y \in F(x)$, we have,

$$\tau\, d_S(x) \le d_F(x, \bar y) \le d((x, y), (x, \bar y)) \le d(y, \bar y)$$

thus $\tau\, d_S(x) \le d_{F(x)}(\bar y)$.

(a) $\Longrightarrow$ (b). Assume that

$$\tau\, d_S(x) \le d_{F(x)}(\bar y) \;\text{ for every }\; x \in B_{2\rho}(\bar x).$$

We claim that $\tau\, d_S(x) \le d_F^\tau(x, \bar y)$ for every $x \in B_\rho(\bar x)$. Indeed, let us assume by contradiction that $\tau\, d_S(x_0) > d_F^\tau(x_0, \bar y)$ for some $x_0 \in B_\rho(\bar x)$. Then we can find $(u, v) \in F$ such that $\tau\, d_S(x_0) > \tau\, d(x_0, u) + d(\bar y, v) \ge \tau\, d(x_0, u) + d_{F(u)}(\bar y)$. Now we have $\tau\, d(x_0, u) < \tau\, d_S(x_0) \le \tau d(x_0, \bar x)$, so that $d(\bar x, u) < 2\rho$ and then,

$$\tau\, d_S(x_0) > \tau\, d(x_0, u) + d_{F(u)}(\bar y) > \tau\, d(x_0, u) + \tau\, d_S(u),$$

leading to the contradiction $d_S(x_0) > d(x_0, u) + d_S(u)$. ∎

Remark 5.1 One may introduce a closed constraint $A \subset X$ in the definition of metric subregularity. The multifunction F is metrically regular at $(\bar x, \bar y) \in F \cap (A \times Y)$ with respect to A if we can find $\tau > 0$ and a neighborhood V of $\bar x$ such that we can find a neighborhood V of $\bar x$ such that, setting $S = F^{-1}(\bar y) \cap A$, we have

$$\tau\, d_S(x) \le d_{F(x)}(\bar y) + d_A(x) \;\text{ for all }\; x \in V. \tag{5.3}$$

With a proof analogous to the one of Proposition 5.1 one easily see that (5.3) implies that there exists a neigbhborhood U of $\bar x$ such that,

$$\tau\, d_S(x) \le \hat d_F(x, \bar y) + d_A(x) \;\text{ for all }\; x \in U, \tag{5.4}$$

where $\hat d_\tau((x_1, y_1), (x_2, y_2)) = (\tau + 1) d(x_1, x_2) + d(y_1, y_2)$.

Given a multifunction $F \subset X \times Y$ and a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, the ∂-coderivative at $(\bar x, \bar y) \in F$ is the multifunction $D^{*\partial} F(\bar x, \bar y) \subset Y^* \times X^*$ defined by

$$D^{*\partial} F(\bar x, \bar y) = \{(\zeta, \xi) \in Y^* \times X^* : (\xi, -\zeta) \in N^\partial F(\bar x, \bar y)\}.$$

If F is Fréchet regular at $(\bar x, \bar y)$, we simply write $D^* F(\bar x, \bar y)$.

Theorem 5.1 Characterization of metric subregularity of Fréchet regular multifunctions. *Let X, Y be Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $G \ni (\bar x, \bar y)$ be a closed multifunction which is Fréchet regular at $(x, \bar y)$ for every $x \in S := G^{-1}(\bar y)$ close to $\bar x$. Then the multifunction G is metrically* *subregular at $(\bar{x}, \bar{y})$ if and only if we can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that,*

$$N_S^{\partial}(x) \cap \tau B_* \subset D^*G(x, \bar{y})(B_*) \text{ for every } x \in S \cap V. \tag{5.5}$$

More precisely, if (5.5) is satisfied, then for all $\sigma \in (0, \tau)$ we can find a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_{G^{-1}(\bar{y})}(x) \le d_{G(x)}(\bar{y}) \text{ for every } x \in U.$$

.

Proof. We know from Proposition 5.1 that G is metrically subregular at $(\bar{x}, \bar{y})$ if and only if we can find $\tau > 0$ and a neighborhood W of $\bar{x}$ such that $\tau\, d_S(x) \le d_G^{\tau}(\Phi(x))$ for every $x \in W$ where $S = G^{-1}(\bar{y})$ and $\Phi(x) = (x, \bar{y})$. As G is Fréchet regular at $(x, \bar{y})$ for every $x \in S := G^{-1}(\bar{y})$ close to $\bar{x}$, this is equivalent, by Theorem 3.2, to the existence of a neighborhood U of $\bar{x}$ such that

$$N_S^{\partial}(x) \cap \tau B_* \subset \Phi'(x)^*\big(N_G(\Phi(x)) \cap B_*\big) \text{ for every } x \in S \cap U.$$

Now we have $\Phi'(x)^*(\xi, \zeta) = \xi$, the dual norm of $\|\cdot\|^{\tau}$ is $\|(\xi, \zeta)\|_*^{\tau} = \max(\tau^{-1}\|\xi\|_*, \|\zeta\|_*))$, thus $N_G(\Phi(x)) \cap B_* = N_G(x, \bar{y}) \cap (\tau\, B_* \times B_*)$, yielding the conclusion of the theorem. ∎

Analogously to the convex composite multifunctions introduced in [30], we shall use Fréchet regular composite multifunctions. Let us begin by a characterization.

Theorem 5.2 *Let X, Y, Z, W be Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $F \subset Z \times Y$ be a multifunction, let $L \subset W$ be closed let ϕ (resp. ψ) be a $\mathcal{C}^1$ mapping defined near $\bar{x} \in X$ with values in Z (resp. W) and let $G = F \circ \phi$. Assume that*

- $(\phi(\bar{x}), \bar{y}) \in F$, $\psi(\bar{x}) \in L$,
- *for every $x \in S := G^{-1}(\bar{y}) \cap \psi^{-1}(L)$ close to $\bar{x}$, F is Fréchet regular at $(\phi(x), \bar{y})$ and L is Fréchet regular at $\psi(x)$.*

Then the two following properties are equivalent.

(a) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that, for every $x \in S \cap V$,

$$N_S^{\partial}(x) \cap \tau B_* \subset \phi'(x)^*\big(D^*F(\phi(x), \bar{y})(B_*) \cap B_*\big) + \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big). \tag{5.6}$$

(b) For all $\sigma \in (0, \tau)$ we can find a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_S(x) \le d_F(\phi(x), \bar{y}) + d_L(\psi(x))) \text{ for every } x \in U.$$

Proof. Let us endow $Z \times Y$ with the norm $\|(u,v)\| = \|u\| + \|v\|$ so that $B_* = B_* \times B_*$. Let us set $\Phi(x) = (\phi(x), \bar{y})$, thus $S = \Phi^{-1}(F) \cap \psi^{-1}(L)$, $\Phi'(x)(u) = (\phi'(x)(u), 0)$ for all $u \in X$, and $\Phi'(x)^*(\xi, \zeta) = \phi'(x)^*(\xi)$ for all $(\xi, \zeta) \in Y^* \times Y^*$. Observing that (5.6) is equivalent to $N_S^\partial(x) \cap \tau B_* \subset \Phi'(x)^*\big(N_F(\Phi(x), \bar{y}) \cap (B_* \times B_*)\big) + \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big)$, the conclusion of the theorem follows from Theorem 3.3. ∎

We give now a characterization of metric subregularity with respect to a constraint for Fréchet regular composite multifunctions.

Theorem 5.3 Characterization of metric subregularity with respect to a constraint of Fréchet regular composite multifunctions. *Let X, Y, Z, W be Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $F \subset Z \times Y$ be a multifunction, let $L \subset W$ be closed let ϕ (resp. ψ) be a $\mathcal{C}^1$ mapping defined near $\bar{x} \in X$ with values in Z (resp. W) and let $G = F \circ \phi$. Assume that*

- *$(\phi(\bar{x}), \bar{y}) \in F$, $\psi(\bar{x}) \in L$,*
- *for every $x \in S := G^{-1}(\bar{y}) \cap \psi^{-1}(L)$ close to $\bar{x}$, F is Fréchet regular at $(\phi(x), \bar{y})$ and L is Fréchet regular at $\psi(x)$.*
- *Assume also that, for some $t > 0$,*

  (5.7) $$t\, d_G(x,y) \le d_F(\phi(x), y) \ \text{ for } \ (x,y) \ \text{ close to } (\bar{x}, \bar{y}),$$

  and that, setting $A = \psi^{-1}(L)$, for some $s > 0$,

  (5.8) $$s\, d(x, A) \le d_L(\psi(x)) \ \text{ for } \ x \ \text{ close to } \bar{x}.$$

Then the two following properties are equivalent.

(a) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that, for every $x \in S \cap V$,

(5.9) $$N_S^\partial(x) \cap \tau B_* \subset \phi'(x)^*\big(D^* F(\phi(x), \bar{y})(B_*)\big) + \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big).$$

(b) We can find $\sigma > 0$ and a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_S(x) \le d_{F(\phi(x))}(\bar{y}) + d_A(x) \ \text{ for every } \ x \in U.$$

Proof. Let us prove that (a) $\Longrightarrow$ (b). We may assume that (5.7) is satisfied for the ℓ^1 norms on $X \times Y$ and $Z \times Y$, so that the dual closed unit balls are the product of the coresponding dual closed unit balls. Let $\Phi(x,y) = (\phi(x), y)$, so that $G = \Phi^{-1}(F)$. Then we derive from Theorem 3.1 that we can find neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that G is Fréchet regular on $U \times V$, A is Fréchet regular on U and,

(5.10) $$\begin{cases} N_A(x) \cap s\, B_* \subset \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big), \\ \\ N_G(x,y) \cap t\,(B_* \times B_*) \subset \Phi'(x,y)^*\big(N_F(\phi(x), y) \cap (B_* \times B_*)\big), \end{cases}$$

for every $(x,y) \in G \cap (U \times V)$ and $x \in A \cap U$. We have $\Phi'(x,y)(u,v) = (\phi'(x)(u),v)$ for all $u \in X$ and $v \in Y$, and $\Phi'(x,y)^*(\xi,\zeta) = (\phi'(x)^*(\xi),\zeta)$ for all $(\xi,\zeta) \in Z^* \times Y^*$. Now, let $(\xi,\zeta) \in N_G(x,y)$, it follows from (5.10) that $(\xi,\zeta) = (\phi'(x)^*(\chi),\zeta)$ with $(\chi,\zeta) \in N_F(\phi(x),y)$ and $\|\chi\|_* \le t^{-1}(\|\xi\|_* + \|\zeta\|_*)$. Thus, using the fact that $N_G(x,y) = \Phi'(x,y)^*(N_F(\phi(x),y)))$ (see (3.8]), if $\xi \in \phi'(x)^*\big(D^*F(\phi(x),\bar{y})(B_*)\big)$, we have

$$\xi \in \phi'(x)^*\big(D^*F(\phi(x),\bar{y})(B_*) \cap t^{-1}(\|\xi\|_* + 1)B_*\big).$$

It follows that (5.9) takes the form,

$$N_S^{\partial}(x) \cap \tau B_* \subset \phi'(x)^*\big(D^*F(\phi(x),\bar{y})(B_*) \cap (\tau + c + 1)t^{-1}B_*\big) + \psi'(x)^*\big(N_L(\psi(x)) \cap B_*\big)$$

where c is a lipschitz rate of ψ on U. From (5.8) and Theorem 3.1, we know that $A = \psi^{-1}(L)$ is Fréchet regular and that $N_A(x) = \psi'(x)^*(N_L(\psi(x)))$ near $\bar{x}$. Then we have,

$$N_S^{\partial}(x) \cap \tau B_* \subset \phi'(x)^*\big(D^*F(\phi(x),\bar{y})(B_*) \cap (\tau + c + 1)t^{-1}B_*\big) + N_A(x) \cap cB_*,$$

yielding:

$$N_S^{\partial}(x) \cap \hat{\tau} B_* \subset \phi'(x)^*\big(D^*F(\phi(x),\bar{y})(B_*) \cap B_*\big) + N_A(x) \cap B_*,$$

with $\hat{\tau} = \tau(1 + c + (\tau + c + 1)t^{-1})^{-1}$. Applying Theorem 5.2 with $\psi = Id$, for any $\sigma \in (0,\hat{\tau})$, we can find a neighborhood U of $\bar{x}$ such that, for every $x \in U$,

$$\sigma\, d_S(x) \le d_F(\phi(x),\bar{y}) + d_A(x) \le d_{F(\phi(x))}(\bar{y}) + d_A(x),$$

leading to (b).

Let us prove the reverse implication. Relying on (5.4), we can find a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_S(x) \le \hat{d}_G(x,\bar{y}) + d_A(x) \ \text{ for every } \ x \in U, \tag{5.11}$$

where $G = F \circ \phi$ and $\hat{d}$ is the distance associated with the norm $\|(u,v)\| = (\sigma+1)\|u\| + \|v\|$ on $X \times Y$ whose dual closed unit ball is $(\sigma + 1)B_* \times B_*$. Applying Theorem 3.3, we derive that S is Fréchet regular on U, and that for every $x \in S \cap U$,

$$N_S^{\partial}(x) \cap \sigma B_* = N_S(x) \cap \sigma B_* \subset \Omega'(x)^*\big(N_G(x,\bar{y}) \cap ((\sigma+1)B_* \times B_*)\big) + N_A(x) \cap B_*,$$

where $\Omega(x) = (x,\bar{y})$. As $\Omega'(x)^*(\xi,\zeta) = \xi$, we get

$$N_S(x) \cap \sigma B_* \subset D^*G(x,\bar{y})(B_*) \cap (\sigma+1)B_* + N_A(x) \cap B_*,$$

and then, setting $\hat{\sigma} = \sigma(\sigma+1)^{-1}$ and using (5.10),

$$\begin{aligned} N_S(x) \cap \hat{\sigma} B_* &\subset D^*G(x,\bar{y})(B_*) + N_A(x) \cap B_* \\ &\subset D^*G(x,\bar{y})(B_*) + \psi'(x)^*\big(N_L(\psi(x)) \cap s^{-1}B_*\big). \end{aligned}$$

From (5.7) and Theorem 3.1, we get $N_G(x,\bar{y}) = \Phi'(x,y)\big(N_F(\phi(x),\bar{y})\big)$, thus

$$D^*G(x,\bar{y}) = \phi'(x)^* \circ D^*F(\phi(x),\bar{y}),$$

yielding (5.9) with $\tau = \frac{\hat{\sigma}}{1+s^{-1}}$. ■

Remark 5.2 In the convex composite setting (F and L closed convex), assumption (5.7) is fulfilled if the Robinson qualification assumption :

$$\mathbb{R}_+\big(\mathrm{dom}\, F - \phi(\bar{x}) - \phi'(\bar{x})(X)\big) = Z$$

holds true. Indeed this condition yields $\mathbb{R}_+\big(F - \Phi(\bar{x},\bar{y}) - \Phi'(\bar{x},\bar{y})(X \times Y)\big) = Z \times Y$, where $\Phi(x,y) = (\phi(x), y)$ which in turn implies (5.7) by [9, Theorem 4.2]. As well, the same theorem shows that the qualification assumption

$$\mathbb{R}_+(L - \psi(\bar{x}) - \psi'(\bar{x})X)) = W$$

leads to (5.8). Better than a sufficient condition such as the Robinson's one, (5.7) and (5.8) are characterized in our more general setting by (5.10).

6 Applications to error bounds of Fréchet regular composite functions

Let us consider Banach spaces X and Y and a function $g : X \to \mathbb{R} \cup \{+\infty\}$ of the type $g = f \circ \phi$ where $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a Fréchet regular function (that is a function whose epigraph is Fréchet regular), and ϕ is a $\mathcal{C}^1$ mapping defined in a neighborhood of $\mathrm{dom}\, f$ with values in Y. The class of such functions extends widely the class of convex composite functions. Given a Fréchet regular function $f : Y \to \mathbb{R} \cup \{+\infty\}$ and $y \in Y$, we have $\partial^F f(y) = \partial^C f(y)$ since $N^F_{\mathrm{epi}\, f}(x, f(x)) = N^C_{\mathrm{epi}\, f}(x, f(x))$, and then we set $\partial f(y) := \partial^F f(y) = \partial^C f(y)$. We shall also use the asymptotic subdifferential $\partial^\infty f(x) = \{\xi \in X^* : (\xi, 0) \in N_{\mathrm{epi}\, f}(x, f(x))\}$. Let us begin by the computation of the subdifferential of $f \circ \phi$.

Proposition 6.1 *Let X, Y be Banach spaces, let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a Fréchet regular function, let ϕ is a $\mathcal{C}^1$ mapping defined on an open set of X with values in Y and let $g = f \circ \phi$. Assume that, for some $t > 0$ we can find open neighborhoods U of $\bar{x} \in \mathrm{dom}\, g$ and V of $g(\bar{x})$ such that*

(6.1) $$t\, d_{\mathrm{epi}\, g}(x, \alpha) \le d_{\mathrm{epi}\, f}(\phi(x), \alpha) \text{ for every } (x, \alpha) \in U \times V.$$

Then $\mathrm{epi}\, g$ is Fréchet regular on $U \times V$ and, reducing U if necessary, for all $(x, \alpha) \in \mathrm{epi}\, g \cap (U \times V)$ we have:

(6.2) $$\partial g(x) = \phi'(x)^*(\partial f(\phi(x))),$$

and, for every $\rho > 0$,

(6.3) $$\partial g(x) \cap \rho B_* \subset \phi'(x)^*(\partial f(\phi(x)) \cap t^{-1}(\rho + 1)B_*).$$

Proof. We may assume that $V = (g(\bar{x}) - \varepsilon, g(\bar{x}) + \varepsilon)$ for some $\varepsilon > 0$ and that $g(x) > g(\bar{x}) - \varepsilon$ for every $x \in U$. We have $\mathrm{epi}\, g = \Phi^{-1}(\mathrm{epi}\, f)$ where $\Phi(x, \alpha) = (\phi(x), \alpha)$. We may also assume that (6.1) holds for the ℓ^1 norms on $X \times \mathbb{R}$ and $Y \times \mathbb{R}$. Then we know from Theorem 3.2 that $\operatorname{epi} g$ is Fréchet regular on $\operatorname{epi} g \cap (U \times V)$ and that, for every $(x, \alpha) \in \operatorname{epi} g \cap (U \times V)$,

$$N_{\operatorname{epi} g}(x, \alpha) = \Phi'(x, \alpha)^*(N_{\operatorname{epi} f}(\phi(x), \alpha)),$$

along with,

$$N_{\operatorname{epi} g}(x, \alpha) \cap t(B_* \times [-1, 1]) \subset \Phi'(x, \alpha)^*(N_{\operatorname{epi} f}(\phi(x), \alpha) \cap (B_* \times [-1, 1])). \tag{6.4}$$

Now, if $(x, \alpha) \in \operatorname{epi} g \cap (U \times V)$, we have $g(\bar{x}) - \varepsilon < g(x) \leq \alpha < g(\bar{x}) + \varepsilon$, so that $(x, g(x)) \in \operatorname{epi} g \cap (U \times V)$ thus $N_{\operatorname{epi} g}(x, g(x)) = \Phi'(x, g(x))^*(N_{\operatorname{epi} f}(\phi(x), f(\phi(x)))$. As $\Phi'(x, \alpha)^*(\xi, s) = (\phi'(x)^*(\xi), s)$, we get

$$N_{\operatorname{epi} g}(x, g(x)) = \big\{(\phi'(x)^*(\xi), s) : (\xi, s) \in N_{\operatorname{epi} f}(\phi(x), f(\phi(x)))\big\},$$

and then $(\zeta, -1) \in N_{\operatorname{epi} g}(x, g(x))$ iff $\zeta = \phi'(x)^*(\xi)$ with $(\xi, -1) \in N_{\operatorname{epi} f}(\phi(x), f(\phi(x)))$ leading to (6.2). At last $(\zeta, -1) \in N_{\operatorname{epi} g}(x, g(x))$ implies by 6.4 that $\zeta = \phi'(x)^*(\xi)$ with $\xi \in \partial f(\phi(x))$ and $\|\xi\|_* \leq t^{-1}\|(\zeta, -1)\|_* \leq t^{-1}(\|\zeta\|_* + 1)$ from which we get (6.3). ■

Applying the results of the preceding section, we can characterize the existence of an error bound for a Fréchet regular composite function.

Theorem 6.1 *Let X, Y be Banach spaces, let ∂ be a subdifferential operator such that $\partial^F \subset \partial \subset \partial^C$, let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a Fréchet regular function, let ϕ be a $\mathcal{C}^1$ mapping defined near $\bar{x} \in X$ with values in Y and let $g = f \circ \phi$. Assume that $g(\bar{x}) \leq 0$ and that for some $t > 0$ and for every $(x, \alpha) \in \operatorname{epi} g$ close to $(\bar{x}, 0)$,*

$$t\, d_{\operatorname{epi} g}(x, \alpha) \leq d_{\operatorname{epi} f}(\phi(x), \alpha). \tag{6.5}$$

Then the two following properties are equivalent:

(a) We can find $\tau > 0$ and a neighborhood V of $\bar{x}$ such that,

$$N_S^{\partial}(x) \cap \tau B_* \subset \begin{cases} \phi'(x)^*\big([0,1]\partial f(\phi(x))\big) & \text{if } f(\phi(x)) = 0 \\ \\ \phi'(x)^*\big(\partial^\infty f(\phi(x))\big) & \text{if } f(\phi(x)) < 0, \end{cases}$$

for every $x \in S \cap V$ where $S = [g \leq 0]$ with the convention $0\, \partial f(\phi(x)) = \partial^\infty f(\phi(x))$.

(b) The function g has an error bound at $\bar{x}$ and for every $\sigma \in (0, \tau)$ we can find a neighborhood U of $\bar{x}$ such that

$$\sigma\, d_{[g \leq 0]}(x) \leq g(x)^+ \text{ for every } x \in U.$$

Proof. We apply Theorem 5.3 with $L = W = \{0\}$ and $\psi = 0$ to the multifunctions $F = \operatorname{epi} f$ and $G = \operatorname{epi} g$ at $(\bar{x}, 0)$ observing that $G^{-1}(0) = [g{\leq}0]$ and that $d_{G(x)}(0) = d_{[g(x),+\infty)}(0) = g(x)^+$. Condition (5.7) takes the form (6.5) and it remains to translate condition (5.9) in our setting. This condition reads,

$$N_S^F(x) \cap \tau B_* \subset \phi'(x)^*\big(D^*(\operatorname{epi} f)(\phi(x), 0)([-1, 1])\big) \text{ for every } x \in S \cap U.$$

where $S = [g{\leq}0]$, for some $\tau > 0$ and some neighborhood U of $\bar{x}$. Let us compute $N_{\operatorname{epi} f}(\phi(x), 0)$ whenever $x \in S \cap U$. If $f(\phi(x)) < 0$, then $-\delta > f(\phi(x))$ for every $\delta \in (0, \delta_0)$ with $\delta_0 = -f(\phi(x)) > 0$, so that $(\phi(x), \pm\delta) \in \operatorname{epi} f$. Now let $(\xi, s) \in N_{\operatorname{epi} f}(\phi(x), 0)$ and $\varepsilon > 0$, so that we can find $0 < \delta < \delta_0$ such that $s(\pm\delta) \leq \varepsilon\delta$ yielding $|s| \leq \varepsilon$ for all $\varepsilon > 0$, that is $s = 0$. If $f(\phi(x)) = 0$, we have $N_{\operatorname{epi} f}(\phi(x), 0) = [T^C_{\operatorname{epi} f}(\phi(x), 0)]^-$ and $T^C_{\operatorname{epi} f}(\phi(x), 0)$ is the epigraph of the Clarke-Rockafellar directional derivative $f^{\uparrow}(\phi(x), \cdot)$ yielding $s \leq 0$. Applying Proposition 6.1 and reducing U if necessary, we get,

- If $f(\phi(x)) < 0$, then

$$\begin{aligned} D^*(\operatorname{epi} f)(\phi(x), 0)([-1, 1]) &= D^*(\operatorname{epi} f)(\phi(x), 0)([0, 1]) \\ &= D^*(\operatorname{epi} f)(\phi(x), 0)(0) \\ &= \partial^\infty f(\phi(x)). \end{aligned}$$

- If $f(\phi(x)) = 0$, we have $D^*(\operatorname{epi} f)(\phi(x), 0)(0) = \partial^\infty f(\phi(x)) = 0\,\partial f(\phi(x))$ and, for $t \in (0, 1]$,

$$D^*(\operatorname{epi} f)(\phi(x), 0)(t) = t\,\partial f(\phi(x)),$$

leading to the conclusion of the theorem. ■

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