

Underestimate Sequences via Quadratic Averaging

Chenxin Ma · Naga Venkata C. Gudapati ·
Majid Jahani · Rachael Tappenden ·
Martin Takáč

Received: date / Accepted: date

Abstract In this work we introduce the concept of an Underestimate Sequence (UES), which is a natural extension of Nesterov's estimate sequence [1]. Our definition of a UES utilizes three sequences, one of which is a lower bound (or under-estimator) of the objective function. The question of how to construct an appropriate sequence of lower bounds is also addressed, and we present lower bounds for strongly convex smooth functions and for strongly convex composite functions, which adhere to the UES framework. Further, we propose several first order methods for minimizing strongly convex functions in both the smooth and composite cases. The algorithms, based on efficiently updating lower bounds on the objective functions, have natural stopping conditions, which provides the user with a certificate of optimality. Convergence of all algorithms is guaranteed through the UES framework, and we show that all presented algorithms converge linearly, with the accelerated variants enjoying the optimal linear rate of convergence.

Keywords Underestimate Sequence · Estimate Sequence · Quadratic Averaging · Lower bounds · Strongly convex · Smooth minimization · Composite minimization · Accelerated Algorithms

Mathematics Subject Classification (2010) 90C25 · 90C47 · 68Q25

C. Ma, N. V. C. Gudapati, M. Jahani, M. Takáč
Department of Systems and Industrial Engineering, Lehigh University, H.S. Mohler Laboratory, 200 West Packer Avenue, Bethlehem, PA 18015, USA.
E-mail: chm514@lehigh.edu, nag415@lehigh.edu, maj316@lehigh.edu, Takac.MT@gmail.com

R. Tappenden
School of Mathematics and Statistics, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand. E-mail: rachael.tappenden@canterbury.ac.nz

Martin Takáč was supported by NSF Grants CCF-1618717 and CMMI-1663256.

1 Introduction

In this work we are interested in solving the strongly convex, composite, unconstrained optimization problem,

$$\min_{x \in \mathbb{R}^n} \{F(x) := f(x) + h(x)\}. \quad (1)$$

We use x^* to denote the optimal solution of (1), and $F^* := F(x^*)$ to denote the associated optimal function value. It is assumed that $h(x)$ is a convex and possibly nonsmooth function. Furthermore, throughout the paper we make the following assumption regarding the function f .

Assumption 1 *The function $f(\cdot)$ is μ -strongly convex and L -smooth, i.e., for all $x, y \in \mathbb{R}^n$, it holds that*

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2, \quad (2)$$

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \|x - y\|^2. \quad (3)$$

It is straightforward to show that strong convexity of $f(x)$ implies strong convexity of $F(x)$.

For problems of the form (1), which satisfy Assumption 1, it is well known that Nesterov's methods [2–4] converge linearly, with the accelerated variants converging at the optimal rate of $(1 - \sqrt{\mu/L})$.

Nesterov's acceleration approach, and the idea of adding momentum, has led to the extensive analysis of accelerated first order methods in a variety of settings. This includes a recent surge of interest in investigating stochastic gradient methods [7, 5, 6] and their accelerated variants [8–11]. Coordinate descent methods [12, 13] are another class of algorithms that have proved extremely popular, largely because they can take advantage of modern parallel computing architecture [14, 15], and this has also inspired much research into studying their accelerated versions [16–18]. However, while the theoretical and practical performance of Nesterov's methods is well established, a satisfactory geometric interpretation of these approaches has been elusive.

Recently the authors of [19, 20] proposed algorithms for smooth functions (i.e., $h(x) = 0$ in (1)) that enjoy the same optimal rate of convergence as Nesterov's accelerated method, but also have a novel geometric intuition. Specifically, the geometric descent algorithm [19] achieves the optimal linear convergence rate, and shares a geometric intuition similar to that of ellipsoidal methods. The authors illustrate that the optimal rate is achieved by appropriately shrinking two balls that contain x^* (the minimizer of $f(x)$) at each iteration.

Motivated by [19], the paper [20] proposed the Optimal Quadratic Averaging (OQA) algorithm. This algorithm maintains a sequence of quadratic lower bounds of the objective function, and at each iteration the new quadratic lower bound is formed as the optimal average of the current lower bound and the

lower bound from the previous iteration. The gap between the function value $f(x_k)$ and the minimum value of lower bound, ϕ_k^* say, converges to zero at the optimal rate. Importantly, the lower bound also acts a natural stopping criterion for the algorithm, and when $f(x_k) - \phi_k^* \leq \epsilon$, where $\epsilon > 0$ is some stopping tolerance, then the user has a certificate of ϵ -optimality, i.e., it is guaranteed that $f(x_k) - f^* \leq \epsilon$. In practice, the OQA algorithm can be equipped with historical information to achieve further speed up. However, the OQA algorithm and its history based variant need at least two calls of a line search process at every iteration, which can pose a heavy computational burden in terms of function evaluations. The authors in [20] also briefly describe how their *unaccelerated OQA algorithm* can be extended to composite functions, and left as an open problem the possibility of deriving *accelerated proximal* variants.

Very recently, the authors of [21] successfully addressed the open problem in [20] and presented an accelerated algorithm for composite problems of the form (1), that achieves the optimal linear rate of convergence. Their algorithm, called the geometric proximal gradient (GeoPG) method also has a satisfying geometrical interpretation similar to that in [19]. Unfortunately, a major drawback of GeoPG in [21] is that the algorithm is rather complicated, and requires a couple of inner loops to determine necessary algorithm parameters. For example, for GeoPG one must find the root of a specific function and one is also required to compute a minimum enclosing ball via some iterative process; both of these steps must be carried out at every iteration, which is expensive.

In this paper we propose several new algorithms to solve problem (1) that are motivated by, and extend, the previously mentioned works. In particular, we present four algorithms: a Gradient Descent (GD) type algorithm for smooth problems, an accelerated GD type algorithm for smooth problems, a proximal GD type algorithm for composite problems, and an accelerated proximal GD type algorithm for composite problems. Our algorithms all converge linearly, and the accelerated variants converge at the optimal linear rate. These algorithms blend the positive features of Nesterov's methods [2–4] and the OQA algorithm [20], and thus enjoy the advantages of both approaches. First, similarly to Nesterov's methods, no line search is needed by any of our algorithms as long as we make the standard assumption that the Lipschitz constant L is known or is easily computable. Hence, there are no 'inner-loops' in any of our algorithm variants, which ensures that the computational cost is low and is fixed at every iteration. Secondly, our algorithms incorporate quadratic lower bounds so they have natural stopping conditions; a feature that is similar to OQA. However, our algorithms update the quadratic lower bound at each iteration by taking a convex combination of the previous two lower bounds, which is different from OQA.

Another contribution of this work is that we also propose the concept of an UnderEstimate Sequence (UES), which is a natural extension of Nesterov's Esti-

mate Sequence [1]. Perhaps surprisingly, estimate sequences initially appeared to be largely overlooked, but since Nesterov’s work on smoothing techniques in the early 2000s [22], they have seen a significant revival in popularity. For example, the work of Baes in [23], the development of a randomized estimate sequence in [24] and an approximate estimate sequence in [25]. To the best of our knowledge, this is the first work which proposes estimate sequences that form *lower bounds* on the objective function. The UES framework is the powerhouse of our convergence analysis; we prove that each of our proposed algorithms generates a UES, and subsequently the algorithms converge (linearly) to the optimal solution of problem (1). While we describe 4 new algorithms in this work, we stress that the UES framework is general, and it allows a plethora of algorithms to be developed. Moreover, any developed algorithm whose iterates generate a UES is guaranteed to converge linearly to the optimal solution F^* .

1.1 Contributions

In this section we state the main contributions of this paper (listed in no particular order).

- **Underestimate Sequence.** We introduce the concept of an UnderEstimate Sequence (UES), which extends Nesterov’s work on estimate sequences in [1]. The UES consists of three sequences $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$, where for all k , $\phi_k(x)$ is a *global lower bound* on the objective function $F(x)$. While there have been several extensions and variants of Nesterov’s work [1], to the best of our knowledge this is the first time that the estimate sequence framework has been adapted to act as a *lower bound* or *underestimate* of $F(x)$. The UES framework is general, conceptually simple, and it allows the construction of a wide variety of algorithms to solve (1).
- **New algorithms.** We present 4 new algorithms that are computationally efficient and adhere to the UES framework. Crucially, two of our algorithms solve the *composite* problem (1). The algorithms are: (i) SUESA, a GD type algorithm for smooth problems; (ii) ASUESA, an accelerated GD type algorithm for smooth problems; (iii) CUESA, a proximal GD type algorithm for composite problems, and (iv) ACUESA, an accelerated proximal GD type algorithm for composite problems.
- **Algorithms with optimal convergence rate.** Each of the algorithms generate iterates that form a UES, so all four algorithms are guaranteed to converge linearly to the optimal solution of (1). Moreover, the accelerated algorithms (ASUESA and ACUESA) are guaranteed to converge linearly *at the optimal rate*.

- **Algorithms with convergence certificates.** The underestimate sequence builds a global lower bound of $F(x)$ at each iteration, and the gap between the (minimum of the) lower bound and $F(x_k)$ tends to zero. Thus, this difference acts as a kind of surrogate “duality gap”, and once this gap falls below some (user defined) stopping tolerance ϵ , it is guaranteed that the point returned by the algorithm is ϵ -optimal.
- **No line search.** The algorithms developed in this work are computationally efficient and do not involve any ‘inner loops’. In contrast, the methods in [19–21] all involve an exact linesearch or a root finding process to determine necessary algorithmic parameters, which comes with an additional computational cost.

1.2 Paper Outline

The paper is organized as follows. In the next section we introduce the concept of an Underestimate Sequence (UES), and present a proposition which shows that if one has a UES, then it is guaranteed that $F(x_k) - F^* \rightarrow 0$ at a linear rate. Section 3 is dedicated to the discussion of lower bounds for the function $F(x)$ (in both the smooth and composite cases), and these lower bounds are a critical part of the underestimate sequences framework. In Section 4 we propose two algorithms for solving (1) in the smooth case ($h = 0$) and in Section 5 we present two algorithms for solving composite problems of the form (1) ($h \neq 0$). All algorithms in Sections 4 and 5 are supported by convergence theory, which shows that they are guaranteed to converge to the optimal solution of (1) at a linear rate. In Section 6 we present another algorithm which uses an adaptive Lipschitz constant, rather than the true Lipschitz constant. Section 7 presents numerical experiments to demonstrate the practical advantages of our proposed algorithms, and we give concluding remarks in Section 8.

2 Underestimate Sequence

In this section, we present the definition of an Underestimate Sequence (UES) and a proposition showing that if one has a UES then $F(x_k) - F^* \rightarrow 0$.

Definition 1 A series of sequences $\{x_k\}_{k=0}^{\infty}$, $\{\phi_k(x)\}_{k=0}^{\infty}$ and $\{\alpha_k\}_{k=0}^{\infty}$, where $\alpha_k \in (0, 1)$ for all $k \geq 0$, is called an Underestimate Sequence (UES) of the function $F(x)$ if, for all $x \in \mathbb{R}^n$ and for all $k \geq 0$ we have,

$$\phi_k(x) \leq F(x), \tag{4}$$

$$F(x_{k+1}) - \phi_{k+1}^* \leq (1 - \alpha_k)(F(x_k) - \phi_k^*), \tag{5}$$

where $\phi_k^* := \min_x \phi_k(x)$.

Proposition 1 *If $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ is an UES of $F(x)$, then*

$$F(x_k) - \phi_k^* \leq \lambda_k(F(x_0) - \phi_0^*), \quad (6)$$

where $\lambda_k = \prod_{i=0}^k (1 - \alpha_i)$. Furthermore, since both $\phi_k^* \leq \phi_k(x^*) \leq F^*$ and $\lambda_k \in (0, 1)$ hold for all $k \geq 0$, the above inequality implies that $\{F(x_k) - F^*\}_{k=0}^\infty$ converges to 0.

Definition 1 is different from Nesterov's Estimate Sequence (ES) in two ways. Firstly, both our UES and Nesterov's ES contain a sequence of estimators $\{\phi_k(x)\}_{k=0}^\infty$ for $F(x)$, and ϕ_k^* converges to F^* as k increases. However, in Definition 1 $\phi_k(x)$ must be a *lower/under estimator* of $F(x)$ for all $k \geq 0$, while this does not necessarily hold for an ES. Nesterov's proof is based on the fact that $F(x_k) \leq \phi_k^*$, but this *does not hold* in our case. Secondly, the definition of an ES only contains two sequences, while the UES has an extra sequence of points $\{x_k\}_{k=0}^\infty$. This enables us to show that the gap between the function value at x_k and ϕ_k^* decreases in the k th iteration.

Proposition 1 shows that any sequences that form a UES (i.e., any sequences that satisfy Definition 1) are guaranteed to converge to the optimal solution of problem (1) and the estimate of the duality gap $F(x_k) - \phi_k^*$ is also guaranteed to converge at a linear rate. Thus, the UES construction provides a general framework for determining whether an optimization algorithm for problem (1) will converge (linearly). In particular, if the iterates generated by an optimization algorithm satisfy Definition 1, then that algorithm is not only convergent, but also achieves a linear rate of convergence.

The UES framework is not only interesting from a theoretical perspective, but it also provides a major practical advantage. In particular, $F(x_k) - \phi_k^*$ provides a natural stopping criterion when designing algorithms, due to the fact that $F(x_k)$ and ϕ_k^* are upper and lower bounds for F^* , respectively. This difference is a kind of surrogate for the duality gap, and subsequently, algorithms that adhere to the UES framework are provided with a certificate of optimality, which is a highly desirable attribute.

3 Lower Bounds via Quadratic Averaging

The purpose of this section is to introduce (global) lower bounds for the function $F(x)$ defined in (1), in both the smooth ($h(x) = 0$) and nonsmooth cases. Lower bounds are the cornerstone of the UES set up, as seen in (4) in Definition 1. Being able to efficiently construct global lower bounds for $F(x)$ will allow the development of practical algorithms whose convergence is guaranteed via the UES framework.

Before stating the lower bounds, several technical results are presented that will be used throughout this paper.

3.1 Preliminary Technical Results

The proximal map is defined as

$$\text{prox}_{h,\gamma}(x) := \arg \min_u \{h(u) + \frac{\gamma}{2} \|x - u\|^2\}, \quad (7)$$

and the proximal gradient is

$$G_\gamma(x) := \gamma \left(x - \text{prox}_{h,\gamma} \left(x - \frac{1}{\gamma} \nabla f(x) \right) \right). \quad (8)$$

Definitions (7) and (8) will be used with $\gamma \equiv L$. Given some point $x \in \mathbb{R}^n$, a short step and a long step are denoted by

$$x^+ := x - \frac{1}{L} G_L(x), \quad (9)$$

$$x^{++} := x - \frac{1}{\mu} G_L(x). \quad (10)$$

In the smooth case ($h \equiv 0$), the proximal gradient is simply the gradient $\nabla f(\cdot)$, so the short and long steps ((9) and (10)) simplify as

$$x^+ = x - \frac{1}{L} \nabla f(x) \quad (11)$$

$$x^{++} = x - \frac{1}{\mu} \nabla f(x). \quad (12)$$

The following Lemma characterizes elements of the subdifferential of $h(x^+)$.

Lemma 1 *Let $G_L(x)$ and x^+ be defined in (8) and (9), respectively. Then, for all $x \in \mathbb{R}^n$, $G_L(x) - \nabla f(x) \in \partial h(x^+)$.*

Proof For a given point $x \in \mathbb{R}^n$,

$$\begin{aligned} x^+ &\stackrel{(9)}{=} x - \frac{1}{L} G_L(x) \\ &\stackrel{(8)}{=} x - \frac{1}{L} L \left(x - \text{prox}_{h,L} \left(x - \frac{1}{L} \nabla f(x) \right) \right) \\ &= \text{prox}_{h,L} \left(x - \frac{1}{L} \nabla f(x) \right) \\ &\stackrel{(7)}{=} \arg \min_u \left\{ h(u) + \frac{L}{2} \|u - \left(x - \frac{1}{L} \nabla f(x) \right)\|^2 \right\}. \end{aligned}$$

This gives

$$0 \in \frac{1}{L} \partial h(x^+) + x^+ - \left(x - \frac{1}{L} \nabla f(x) \right) \stackrel{(9)}{=} \frac{1}{L} \partial h(x^+) - \frac{1}{L} (G_L(x) - \nabla f(x)).$$

Multiplying through by L , and rearranging, gives the result. \square

3.2 A Lower Bound for Smooth Functions

For any point $y \in \mathbb{R}^n$, one can define a lower bound

$$\phi(x; y) := f(y) - \frac{1}{2\mu} \|\nabla f(y)\|^2 + \frac{\mu}{2} \|x - y^{++}\|^2 \leq f(x), \quad (13)$$

which holds with equality $\phi(x; y) = f(x)$ if and only if $x = y$. The lower bound in (13) is a consequence of the assumption in (2) and the equivalence

$$\frac{\mu}{2} \|x - y^{++}\|^2 \stackrel{(12)}{=} \frac{\mu}{2} \|x - y\|^2 + \langle x - y, \nabla f(y) \rangle + \frac{1}{2\mu} \|\nabla f(y)\|^2. \quad (14)$$

Now, a sequence of lower bounds $\{\phi_k(x)\}_{k=0}^\infty$ can be defined in the following way. Using (13) and a given initial point x_0 , define the function

$$\phi_0(x) := \phi(x; x_0) = \phi_0^* + \frac{\mu}{2} \|x - v_0\|^2, \quad (15)$$

where

$$\phi_0^* = f(x_0) - \frac{1}{2\mu} \|\nabla f(x_0)\|^2 \quad \text{and} \quad v_0 = x_0^{++}. \quad (16)$$

Differentiating the expression in (15) w.r.t. x shows that ϕ_0^* and v_0 in (16) are the minimum value and minimizer of $\phi_0(x)$, respectively. This motivates the following construction:

1. $\phi_0(x) := \phi(x; x_0) = \phi_0^* + \frac{\mu}{2} \|x - v_0\|^2$
2. For $k \geq 0$, $\alpha_k \in (0, 1)$, and some point y_k between x_k and v_k , recursively define

$$\phi_{k+1}(x) := (1 - \alpha_k)\phi_k(x) + \alpha_k\phi(x; y_k). \quad (17)$$

Lemma 2 For all $k \geq 0$, ϕ_{k+1} can be written in the canonical form

$$\phi_{k+1}(x) = \phi_{k+1}^* + \frac{\mu}{2} \|x - v_{k+1}\|^2, \quad (18)$$

where $\alpha_k \in (0, 1)$ and

$$v_{k+1} := (1 - \alpha_k)v_k + \alpha_k y_k^{++} \quad (19)$$

$$\begin{aligned} \phi_{k+1}^* := & (1 - \alpha_k)\left(\phi_k^* + \frac{\mu}{2} \|v_{k+1} - v_k\|^2\right) \\ & + \alpha_k \left(f(y_k) - \frac{1}{2\mu} \|\nabla f(y_k)\|^2 + \frac{\mu}{2} \|v_{k+1} - y_k^{++}\|^2 \right). \end{aligned} \quad (20)$$

Proof Using the definitions (19) and (20), for all $k \geq 0$ and all $x \in \mathbb{R}^n$, $\phi_{k+1}(x)$ can be expressed in the form

$$\begin{aligned} \phi_{k+1}(x) &= (1 - \alpha_k) \left(\phi_k^* + \frac{\mu}{2} \|x - v_k\|^2 \right) \\ &\quad + \alpha_k \left(f(y_k) - \frac{1}{2\mu} \|\nabla f(y_k)\|^2 + \frac{\mu}{2} \|x - y_k^{++}\|^2 \right) \\ &= \phi_{k+1}^* + \frac{\mu}{2} \|x - v_{k+1}\|^2. \end{aligned}$$

By taking the derivative of (18) w.r.t. x , we see that the minimizer of $\phi_k(x)$ is $x = v_{k+1}$, where v_{k+1} is defined in (19). Substituting this minimizer into (18) gives the minimum value ϕ_{k+1}^* as in (20). \square

Lemma 3 *An equivalent expression for ϕ_{k+1}^* in (20) is*

$$\phi_{k+1}^* = (1 - \alpha_k) \left(\phi_k^* + \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2 \right) + \alpha_k \left(f(y_k) - \frac{1}{2\mu} \|\nabla f(y_k)\|^2 \right). \quad (21)$$

Proof Using (19) gives the equivalences

$$\|v_{k+1} - v_k\|^2 = \|(1 - \alpha_k)v_k + \alpha_k y_k^{++} - v_k\|^2 = \alpha_k^2 \|v_k - y_k^{++}\|^2 \quad (22)$$

and

$$\begin{aligned} \|v_{k+1} - y_k^{++}\|^2 &= \|(1 - \alpha_k)v_k + \alpha_k y_k^{++} - y_k^{++}\|^2 \\ &= (1 - \alpha_k)^2 \|v_k - y_k^{++}\|^2. \end{aligned} \quad (23)$$

Combining (22) and (23) gives

$$\begin{aligned} &(1 - \alpha_k) \|v_{k+1} - v_k\|_2^2 + \alpha_k \|v_{k+1} - y_k^{++}\|^2 \\ &= \alpha_k^2 (1 - \alpha_k) \|v_k - y_k^{++}\|_2^2 + \alpha_k (1 - \alpha_k)^2 \|v_k - y_k^{++}\|^2 \\ &= \alpha_k (1 - \alpha_k) (\alpha_k + (1 - \alpha_k)) \|v_k - y_k^{++}\|_2^2 \\ &= \alpha_k (1 - \alpha_k) \|v_k - y_k^{++}\|_2^2. \end{aligned} \quad (24)$$

Substituting (24) into (20) gives the result. \square

The following Lemma shows that $\phi_k(x)$ is a (global) lower bound for $f(x)$.

Lemma 4 *For all $k \geq 0$, let $\alpha_k \in (0, 1)$. Then, for all $x \in \mathbb{R}^n$, $\phi_k(x) \leq f(x)$.*

Proof We proceed by induction. When $k = 0$, the result holds trivially. Now, assume that $\phi_k(x) \leq f(x)$. Then

$$\begin{aligned} \phi_{k+1}(x) &\stackrel{(17)}{=} (1 - \alpha_k) \phi_k(x) + \alpha_k \phi(x; y_k), \\ &\stackrel{(13)}{\leq} (1 - \alpha_k) f(x) + \alpha_k f(x) = f(x). \end{aligned}$$

\square

3.3 A Lower Bound for Composite Functions

Here, the previous results are extended from the smooth to the composite setting, so it is assumed that $h(x)$ is not equivalent to the zero function.

The following Lemma defines a lower bound for $F(x)$ in (1). The lower bound is the same as that presented in [20] and [21], with the roles of x and y reversed here; the proof is included for completeness.

Lemma 5 (Lemma 6.1 in [20]; Lemma 3.1 in [21]) *Given a point $y \in \mathbb{R}^n$, let $G_L(y)$ and y^+ be defined in (8) and (9), respectively. Then for all $x \in \mathbb{R}^n$*

$$\varphi(x; y) := F(y^+) + \langle G_L(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2 + \frac{1}{2L} \|G_L(y)\|^2 \leq F(x). \quad (25)$$

Proof By Assumption 1 (μ -strongly convex)

$$f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2 \leq f(x), \quad \forall x, y \in \mathbb{R}^n, \quad (26)$$

and (L -smooth)

$$\begin{aligned} f(y^+) &\leq f(y) + \langle \nabla f(y), y^+ - y \rangle + \frac{L}{2} \|y^+ - y\|_2^2 \\ &\stackrel{(9)}{=} f(y) - \frac{1}{L} \langle \nabla f(y), G_L(y) \rangle + \frac{1}{2L} \|G_L(y)\|_2^2. \end{aligned} \quad (27)$$

Combining (26) and (27) gives

$$\begin{aligned} F(y^+) &\leq F(x) - \langle \nabla f(y), x - y \rangle - \frac{\mu}{2} \|x - y\|^2 - \frac{1}{L} \langle \nabla f(y), G_L(y) \rangle \\ &\quad + \frac{1}{2L} \|G_L(y)\|_2^2 + (h(y^+) - h(x)) \\ &= F(x) - \langle \nabla f(y), x - y^+ \rangle - \frac{\mu}{2} \|x - y\|^2 \\ &\quad + \frac{1}{2L} \|G_L(y)\|_2^2 + (h(y^+) - h(x)) \\ &= F(x) - \langle \nabla f(y) - G_L(y), x - y^+ \rangle - \frac{\mu}{2} \|x - y\|^2 \\ &\quad + \frac{1}{2L} \|G_L(y)\|_2^2 + (h(y^+) - h(x)) - \langle G_L(y), x - y^+ \rangle \\ &\leq F(x) - \frac{\mu}{2} \|x - y\|^2 + \frac{1}{2L} \|G_L(y)\|_2^2 - \langle G_L(y), x - y^+ \rangle \\ &= F(x) - \frac{\mu}{2} \|x - y\|^2 - \frac{1}{2L} \|G_L(y)\|_2^2 - \langle G_L(y), x - y \rangle. \end{aligned}$$

Rearranging gives the result. \square

Before stating the next result, which shows that $\varphi(x; y)$ is a quadratic lower bound, we give the following equivalence, which is the composite version of (14),

$$\frac{\mu}{2} \|x - y^{++}\|^2 \stackrel{(10)}{=} \frac{\mu}{2} \|x - y\|^2 + \langle x - y, G_L(y) \rangle + \frac{1}{2\mu} \|\nabla G_L(y)\|^2. \quad (28)$$

Lemma 6 *For all $x, y \in \mathbb{R}^n$, the lower bound (25) has the canonical form*

$$\varphi(x; y) = \varphi^* + \frac{\mu}{2} \|x - y^{++}\|^2, \quad (29)$$

where

$$\varphi^* = F(y^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y)\|^2. \quad (30)$$

Proof Minimizing $\varphi(x; y)$ in (25) w.r.t. x , and using the definition in (7), yields the minimizer

$$y^{++} = \arg \min_x \varphi(x; y). \quad (31)$$

The corresponding minimal value is

$$\begin{aligned}
\varphi^* &:= \min_x \varphi(x; y) = \varphi(y^{++}; y) \\
&\stackrel{(25)}{=} F(y^+) + \langle G_L(y), y^{++} - y \rangle + \frac{\mu}{2} \|y^{++} - y\|^2 + \frac{1}{2L} \|G_L(y)\|^2 \\
&\stackrel{(10)}{=} F(y^+) - \frac{1}{\mu} \langle G_L(y), G_L(y) \rangle + \frac{\mu}{2} \|\frac{1}{\mu} G_L(y)\|^2 + \frac{1}{2L} \|G_L(y)\|^2 \\
&= F(y^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y)\|^2,
\end{aligned}$$

which is equivalent to (31). (Note also that (31) and (30) are the minimizer and minimum value of (29), respectively.) Furthermore,

$$\begin{aligned}
\varphi(x; y) &\stackrel{(29)}{=} \varphi^* + \frac{\mu}{2} \|x - y^{++}\|^2 \\
&= F(y^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y)\|^2 + \frac{\mu}{2} \|x - y^{++}\|^2 \\
&\stackrel{(28)}{=} F(y^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y)\|^2 \\
&\quad + \frac{\mu}{2} \|x - y\|^2 + \frac{1}{2\mu} \|G_L(y)\|^2 + \langle G_L(y), x - y \rangle \\
&= F(y^+) + \frac{1}{2L} \|G_L(y)\|^2 + \frac{\mu}{2} \|x - y\|^2 + \langle G_L(y), x - y \rangle,
\end{aligned}$$

which confirms that (29) is equivalent to (25). \square

Remark 1 Lemma 6 shows that the lower bound (25) (equivalently (29)) is a quadratic lower bound for $F(x)$.

Now, a sequence of lower bounds $\{\varphi_k(x)\}_{k=0}^\infty$ can be defined in the following way. Using (25) and a given initial point x_0 , define the function

$$\varphi_0(x) := \varphi(x; x_0) = \varphi_0^* + \frac{\mu}{2} \|x - v_0\|_2^2, \quad (32)$$

where

$$\varphi_0^* := F(x_0^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(x_0)\|^2, \quad v_0 = x_0^{++}. \quad (33)$$

Differentiating (32) w.r.t. x shows that the minimum value and minimizer of $\varphi_0(x)$ are given by (33). This motivates the following construction

1. $\varphi_0(x) := \varphi(x; x_0) = \varphi_0^* + \frac{\mu}{2} \|x - v_0\|_2^2$,
2. For $k \geq 0$ and some point y_k between x_k and v_k we recursively define

$$\varphi_{k+1}(x) := (1 - \alpha_k) \varphi_k(x) + \alpha_k \varphi(x; y_k). \quad (34)$$

Lemma 7 For all $k \geq 0$, φ_{k+1} can be written in the canonical form

$$\varphi_{k+1}(x) := \varphi_{k+1}^* + \frac{\mu}{2} \|x - v_{k+1}\|_2^2, \quad (35)$$

where

$$v_{k+1} := (1 - \alpha_k)v_k + \alpha_k y_k^{++} \quad (36)$$

$$\begin{aligned} \varphi_{k+1}^* &:= (1 - \alpha_k)\left(\varphi_k^* + \frac{\mu}{2}\|v_{k+1} - v_k\|_2^2\right) \\ &+ \alpha_k \left(F(y_k^+) + \left(\frac{1}{2L} - \frac{1}{2\mu}\right)\|G_L(y_k)\|^2 + \frac{\mu}{2}\|v_{k+1} - y_k^{++}\|^2\right). \end{aligned} \quad (37)$$

Proof Using (34) and (35), for all $k \geq 0$ and all $x \in \mathbb{R}^n$, $\varphi_{k+1}(x)$ can be expressed in the form

$$\begin{aligned} \varphi_{k+1}(x) &= (1 - \alpha_k)\left(\varphi_k^* + \frac{\mu}{2}\|x - v_k\|_2^2\right) \\ &+ \alpha_k \left(F(y_k^+) + \left(\frac{1}{2L} - \frac{1}{2\mu}\right)\|G_L(y_k)\|^2 + \frac{\mu}{2}\|x - y_k^{++}\|^2\right) \\ &= \varphi_{k+1}^* + \frac{\mu}{2}\|x - v_{k+1}\|_2^2. \end{aligned}$$

Taking the derivative of (35) w.r.t. x shows that the minimizer of $\varphi_k(x)$ is $x = v_{k+1}$. Substituting $x = v_{k+1}$ into the above gives (37). \square

Lemma 8 *An equivalent expression for φ_{k+1}^* in (37) is*

$$\begin{aligned} \varphi_{k+1}^* &= (1 - \alpha_k)\left(\varphi_k^* + \alpha_k \frac{\mu}{2}\|v_k - y_k^{++}\|_2^2\right) \\ &+ \alpha_k \left(F(y_k^+) + \left(\frac{1}{2L} - \frac{1}{2\mu}\right)\|G_L(y_k)\|^2\right). \end{aligned} \quad (38)$$

Proof The proof follows the same arguments as for Lemma 3; noting that (19) and (36) are equivalent, and then combining (24) and (37) gives the result. \square

Lemma 9 *For all $k \geq 0$ and $\forall x \in \mathbb{R}^n$, $\varphi_k(x) \leq F(x)$.*

Proof When $k = 0$, the result holds trivially. Now assume that $\varphi_k(x) \leq F(x)$. Then

$$\begin{aligned} \varphi_{k+1}(x) &= (1 - \alpha_k)\varphi_k(x) + \alpha_k\varphi(x; y_k) \\ &\leq (1 - \alpha_k)F(x) + \alpha_k F(x) = F(x). \end{aligned}$$

This completes the proof. \square

4 Algorithms and Convergence Guarantees for Smooth Functions

The purpose of this section is to demonstrate that the UES framework, and the previously presented lower bounds, are *useable* definitions that give rise to *efficient implementable algorithms*. Throughout this section we consider smooth optimization problems (problems of the form (1) with $h \equiv 0$) and, as for all results in this work, we suppose that Assumption 1 holds.

We present two algorithms whose iterates fit the Underestimate Sequence framework described in Section 2, and use the lower bounds developed in Section 3.2. The first algorithm is a gradient descent type method, while the second algorithm is a gradient descent type method that incorporates an acceleration strategy. As will be shown, both algorithms are supported by convergence guarantees, which are established via the UES framework.

4.1 An Underestimate Sequence Algorithm for Smooth Functions

We are now ready to present an algorithm that fits our UES framework; a brief description follows.

Algorithm 1 Smooth Underestimate Sequence Algorithm (SUESA)

- 1: Initialization: Set $k = 0$, $\epsilon > 0$, initial point $x_0 \in \mathbb{R}^n$ and compute μ , L .
 - 2: Set $\phi_0(x)$ as in (15), with v_0 and ϕ_0^* as in (16), and let $\alpha_k = \frac{\mu}{L}$.
 - 3: **while** $f(x_k) - \phi_k^* > \epsilon$ **do**
 - 4: Set $y_k = x_k$ and $y_k^{++} = x_k^{++}$.
 - 5: Set $x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$.
 - 6: Update v_{k+1} and ϕ_{k+1}^* as in (19) and (20), respectively.
 - 7: $k = k + 1$.
 - 8: **end while**
-

The Smooth (functions) UnderEstimate Sequence Algorithm (SUESA) presented in Algorithm 1 solves the problem (1) in the smooth case, i.e., when $h = 0$. The algorithm proceeds as follows. First, an initial point $x_0 \in \mathbb{R}^n$ is chosen, as well as some stopping tolerance $\epsilon > 0$. Secondly, the point $v_0 = x_0^{++}$ (i.e., v_0 is the long step from x_0) is constructed, as well as the lower bound $\phi_0(x)$ with minimum value ϕ_0^* . The algorithm uses a fixed step size of $\alpha = \mu/L$ at every iteration. Next, the main loop commences and an iteration proceeds as follows. One sets $y_k = x_k$ (i.e., y_k is not explicitly used in SUESA); x_k is updated by taking a gradient descent step with the step size $\frac{1}{L}$, resulting in the new point x_{k+1} ; the point $v_{k+1} = x_{k+1}^{++}$ is constructed and the lower bound $\phi_{k+1}(x)$ is updated.

The algorithm constructs two points at every iteration, namely x_k and v_k , and the values $\phi_k(x)$ and ϕ_k^* . The point v_k and the value ϕ_k^* are used for the lower bound, which is essential for the stopping criterion. The stopping condition $f(x_k) - \phi_k^* \leq \epsilon$ provides a certificate of optimality; once the stopping condition is satisfied, it is guaranteed that x_k gives a function value $f(x_k)$ that is at most ϵ from the true solution f^* .

If Step 5 is considered in isolation, then one sees that at every iteration of SUESA, the point x_k is updated via a standard gradient descent step. That

is, a step of size $1/L$ in the direction of the negative gradient is taken from the current point x_k , resulting in the new point x_{k+1} . However, SUESA is different from the standard gradient descent method, because SUESA also involves several other ingredients, including the points v_k and lower bound values ϕ_k^* .

The following result provides a convergence guarantee for SUESA. In particular, Theorem 1 shows that the iterates generated by Algorithm 1 form an underestimate sequence (i.e., they satisfy Definition 1) and therefore, Algorithm 1 is guaranteed to converge (linearly) to the solution of problem (1).

Theorem 1 *Let Assumption 1 hold. The sequences $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by SUESA (Algorithm 1) form a UES.*

Proof We must show that the iterates generated by Algorithm 1 satisfy the conditions of Definition 1. Note that, $\alpha_k = \mu/L \in (0, 1)$ for all $k \geq 0$, so by Lemma 4, (4) holds. Thus, it remains to prove (5). Combining the definition of x_{k+1} (Step 5 in Algorithm 1) with (3), gives

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), -\frac{1}{L} \nabla f(x_k) \rangle + \frac{L}{2} \|\frac{1}{L} \nabla f(x_k)\|^2 \\ &= f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2. \end{aligned} \quad (39)$$

Subtracting ϕ_{k+1}^* in (21) from both sides of the above gives,

$$\begin{aligned} f(x_{k+1}) - \phi_{k+1}^* &\leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 - (1 - \alpha_k) (\phi_k^* + \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2) \\ &\quad - \alpha_k (f(y_k) - \frac{1}{2\mu} \|\nabla f(y_k)\|^2) \\ &\leq (1 - \alpha_k) (f(x_k) - \phi_k^*) - (\frac{\alpha_k}{2\mu} - \frac{1}{2L}) \|\nabla f(x_k)\|^2 \\ &\quad - (1 - \alpha_k) \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2 \\ &\leq (1 - \alpha_k) (f(x_k) - \phi_k^*), \end{aligned}$$

where the last step follows because $\alpha_k = \frac{\mu}{L}$, so $\frac{\alpha_k}{2\mu} - \frac{1}{2L} = \frac{\mu}{2\mu L} - \frac{1}{2L} = 0$, and $(1 - \alpha_k) \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2 \geq 0$. Therefore, the sequences $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by Algorithm 1 form a UES. \square

Corollary 1 *Let Assumption 1 hold. Then the sequences $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by SUESA (Algorithm 1) form a UES, so SUESA converges at a linear rate*

$$f(x_k) - \phi_k^* \leq (1 - \frac{\mu}{L})^k (f(x_0) - \phi_0^*). \quad (40)$$

Corollary 1 is simply a consequence Proposition 1, which states that if $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ form an underestimate sequence, then (6) holds (i.e., linear convergence). Theorem 1 shows that SUESA (Algorithm 1) generates iterates forming a UES, implying that SUESA converges linearly to the optimal solution. Moreover, $\alpha_k = \frac{\mu}{L}$ for all k in SUESA, so recalling the definition of λ_k in Proposition 1, confirms the rate $(1 - \frac{\mu}{L})$ in Corollary 1.

We remark that there are other ways to prove convergence of Algorithm 1. For example, one can proceed by proving that the distance between x_k and the minimizer of the lower bound in k th iteration shrinks at a fixed rate. That is, since $\alpha_k = \frac{\mu}{L}$, we have the following equality,

$$\begin{aligned} x_{k+1} - v_{k+1} &= \left(x_k - \frac{1}{L}\nabla f(x_k)\right) - \left((1 - \alpha_k)v_k - \alpha_k\left(x_k - \frac{1}{\mu}\nabla f(x_k)\right)\right) \\ &= \left(1 - \frac{\mu}{L}\right)(x_k - v_k). \end{aligned} \quad (41)$$

Equation (41) illustrates that, after each iteration of Algorithm 1, the line joining x_{k+1} and v_{k+1} is parallel to the line joining x_k and v_k from the previous iteration (see the blue lines in Figure 1). Moreover, the distance between the two points is reduced by precisely $(1 - \frac{\mu}{L})$ at every iteration. Intuitively, the solution x_k and the minimizer v_k are becoming ever closer, and eventually they both converge to the optimal solution x^* .

One can visualize the fact above using the following toy example. Consider the (smooth) regularized logistic regression problem, i.e., problem (1) with $h = 0$ and

$$f(x) = \sum_{i=1}^m \log(1 + \exp(-y_i \langle x, a_i \rangle)) + \frac{\lambda}{2} \|x\|^2,$$

where $a_i \in \mathbb{R}^n$ is the i th feature vector with corresponding (binary) label $y_i \in \mathbb{R}$. For this example we randomly generate 100 two dimensional data points with binary labels $\{a_i, y_i\}$ (so $m = 100$ and $n = 2$) as shown in the left hand plot in Figure 1. (Each point a_i is plotted on a 2D grid, and the point is colored green or red to highlight its label y_i). Parameter $\lambda = 0.01$, so the strong convexity constant is $\mu = 0.01$. Algorithm 1 is used to solve this problem, starting from the point $x_0 = (-20, 10)^T$, and the iterates are shown in the right hand plot in Figure 1.

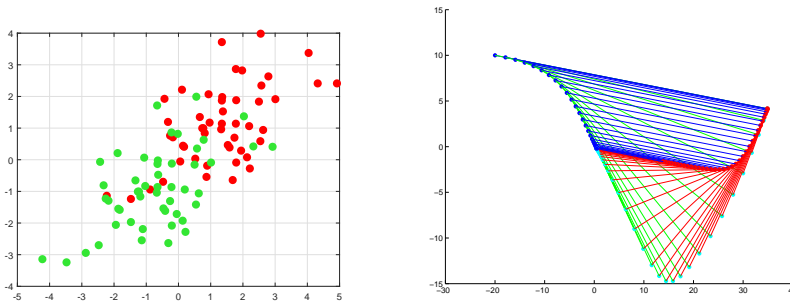


Fig. 1: Left: Randomly generated two classes of 2D data. Right: A 2D illustration for Option 1. The blue, red and green points represent $\{x_k\}$, $\{v_k\}$, $\{x_k^{++}\}$ respectively.

4.2 An Accelerated Underestimate Sequence Algorithm for Smooth Functions

We now present an accelerated first order algorithm for solving problems of the form (1) when $h = 0$; a description will follow.

Algorithm 2 Accelerated Smooth Underestimate Sequence Algorithm (ASUESA)

- 1: Initialization: Set $k = 0$, $\epsilon > 0$, initial point $x_0 \in \mathbb{R}^n$ and compute μ , L .
 - 2: Set $\phi_0(x)$ as in (15), with v_0 and ϕ_0^* as in (16). Let $\alpha_k = \sqrt{\frac{\mu}{L}}$, $\beta_k = \frac{1}{1+\alpha_k}$.
 - 3: **while** $f(x_k) - \phi_k^* > \epsilon$ **do**
 - 4: Set $y_k = \beta_k x_k + (1 - \beta_k)v_k$.
 - 5: Set $x_{k+1} = y_k - \frac{1}{L}\nabla f(y_k)$.
 - 6: Update v_{k+1} and ϕ_{k+1}^* as in (19) and (20), respectively.
 - 7: $k = k + 1$.
 - 8: **end while**
-

The Accelerated Smooth UnderEstimate Sequence Algorithm (ASUESA) presented in Algorithm 2 solves (1) in the smooth case, i.e., when $h = 0$, and can be described as follows. Algorithm initialization is similar to that of SUESA (Algorithm 1), where an initial point $x_0 \in \mathbb{R}^n$ and some stopping tolerance $\epsilon > 0$ are chosen, the point $v_0 = x_0^{++}$ is constructed and the lower bound $\phi_0(x)$ and minimum value ϕ_0^* are evaluated. For ASUESA one sets $\alpha_k = \sqrt{\mu/L}$ and the parameter $\beta_k = \frac{1}{1+\alpha_k}$ is also used. Parameter α_k is fixed for all iterations, and subsequently so too is β_k . The main loop proceeds as follows. At every iteration one sets y_k to be a convex combination of the points x_k and v_k ; a gradient descent step is taken from y_k , resulting in the new point x_{k+1} ; the point v_{k+1} is constructed using (19) and the lower bound $\phi_{k+1}(x)$ is updated via (20).

Notice that Algorithm 2 can be viewed as an accelerated version of Algorithm 1. In contrast to Algorithm 1, ASUESA constructs *three* points at every iteration, namely x_k , v_k and y_k , where the intermediate vector y_k is a convex combination of the points x_k and v_k (i.e., for ASUESA $x_k \neq y_k$.) Notice also that x_{k+1} is the result of a gradient descent step taken from the point y_k . The variable ϕ_k^* is also maintained and is used in the stopping condition.

The following result provides a convergence guarantee for ASUESA. Theorem 2 shows that the iterates generated by Algorithm 2 fit the UES framework (i.e., they satisfy Definition 1) and therefore, Algorithm 2 is guaranteed to converge (linearly at the optimal rate) to the solution of problem (1) (see Corollary 2).

Theorem 2 *Let Assumption 1 hold. The series of sequences $\{x_k\}_{k=0}^\infty$, $\{\phi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by ASUESA in Algorithm 2 form an UES.*

Proof At every iteration of ASUESA the function value is reduced as follows

$$\begin{aligned} f(y_k^+) &= f(y_k - \frac{1}{L}\nabla f(y_k)) \leq f(y_k) + \langle \nabla f(y_k), -\frac{\nabla f(y_k)}{L} \rangle + \frac{L}{2} \left\| \frac{\nabla f(y_k)}{L} \right\|^2 \\ &= f(y_k) - \frac{1}{2L} \|\nabla f(y_k)\|^2. \end{aligned} \quad (42)$$

Moreover,

$$\begin{aligned} f(x_{k+1}) - \phi_{k+1}^* &\stackrel{(21)}{=} f(y_k^+) - (1 - \alpha_k)\phi_k^* \\ &\quad - \alpha_k \left(f(y_k) - \frac{\|\nabla f(y_k)\|^2}{2\mu} + \frac{\mu}{2}(1 - \alpha_k)\|v_k - y_k^{++}\|^2 \right) \\ &\stackrel{(42)}{\leq} f(y_k) - \frac{1}{2L}\|\nabla f(y_k)\|^2 - (1 - \alpha_k)\phi_k^* \\ &\quad - \alpha_k \left(f(y_k) - \frac{\|\nabla f(y_k)\|^2}{2\mu} + \frac{\mu}{2}(1 - \alpha_k)\|v_k - y_k^{++}\|^2 \right) \\ &= (1 - \alpha_k)(f(y_k) - \phi_k^*) - \left(\frac{1}{2L} - \frac{\alpha_k}{2\mu} \right) \|\nabla f(y_k)\|^2 \\ &\quad - \alpha_k \frac{\mu}{2}(1 - \alpha_k)\|v_k - y_k^{++}\|^2. \end{aligned}$$

By completing the square term one obtains

$$\begin{aligned} f(x_{k+1}) - \phi_{k+1}^* &= (1 - \alpha_k)(f(y_k) - \phi_k^*) - \left(\frac{1}{2L} - \frac{\alpha_k}{2\mu} \right) \|\nabla f(y_k)\|^2 \\ &\quad - \frac{\mu}{2}\alpha_k(1 - \alpha_k) \left(\|v_k - y_k\|^2 + \left\| \frac{1}{\mu}\nabla f(y_k) \right\|^2 + \frac{2}{\mu}\langle \nabla f(y_k), v_k - y_k \rangle \right) \\ &= (1 - \alpha_k)(f(y_k) - \phi_k^*) - \left(\frac{1}{2L} - \frac{\alpha_k}{2\mu} + \frac{\alpha_k(1 - \alpha_k)}{2\mu} \right) \|\nabla f(y_k)\|^2 \\ &\quad - \alpha_k(1 - \alpha_k)\langle \nabla f(y_k), v_k - y_k \rangle \\ &\leq (1 - \alpha_k)(f(y_k) - \phi_k^*) - \left(\frac{1}{2L} - \frac{\alpha_k^2}{2\mu} \right) \|\nabla f(y_k)\|^2 \\ &\quad - \alpha_k(1 - \alpha_k)\langle \nabla f(y_k), v_k - y_k \rangle \\ &= (1 - \alpha_k)(f(y_k) - \phi_k^*) - \alpha_k(1 - \alpha_k)\langle \nabla f(y_k), v_k - y_k \rangle, \end{aligned} \quad (43)$$

where the last step follows because $\alpha_k = \sqrt{\frac{\mu}{L}}$ in ASUESA, so $\frac{1}{2L} - \frac{\alpha_k^2}{2\mu} = 0$.

Now, rearranging the expression for y_k in Step 6 in Algorithm 2 gives

$$v_k = \frac{1}{1 - \beta_k}(y_k - \beta_k x_k) = y_k + \frac{\beta_k}{1 - \beta_k}(y_k - x_k), \quad (44)$$

and notice also that $\beta_k = \frac{1}{1 + \alpha_k}$ for all k , so

$$1 - \beta_k = 1 - \frac{1}{1 + \alpha_k} = \frac{\alpha_k}{1 + \alpha_k} = \alpha_k \beta_k \quad \Rightarrow \quad \frac{\alpha_k \beta_k}{1 - \beta_k} = 1. \quad (45)$$

Thus, by the convexity of f we have

$$\begin{aligned} -\langle \nabla f(y_k), v_k - y_k \rangle &= -\langle \nabla f(y_k), y_k + \frac{\beta_k}{1 - \beta_k}(y_k - x_k) - y_k \rangle \\ &\leq \frac{\beta_k}{1 - \beta_k}(f(x_k) - f(y_k)). \end{aligned} \quad (46)$$

Using (46) in (43) gives

$$\begin{aligned} f(x_{k+1}) - \phi_{k+1}^* &\leq (1 - \alpha_k)(f(y_k) - \phi_k^*) + \alpha_k(1 - \alpha_k) \frac{\beta_k}{1 - \beta_k} (f(x_k) - f(y_k)) \\ &\stackrel{(45)}{=} (1 - \alpha_k)(f(y_k) - \phi_k^*) + (1 - \alpha_k)(f(x_k) - f(y_k)) \\ &= (1 - \alpha_k)(f(x_k) - \phi_k^*). \end{aligned}$$

□

Corollary 2 *Let Assumption 1 hold. Then, the sequence of iterates $\{x_k\}_{k \geq 0}$ generated by Algorithm 2 exhibits the optimal linear rate of convergence*

$$f(x_k) - \phi_k^* \leq \left(1 - \sqrt{\frac{\mu}{L}}\right)^k (f(x_0) - \phi_0^*).$$

Corollary 2 shows that ASUESA converges linearly at the optimal rate. The difference in convergence rates between Algorithms 1 and 2 is essentially explained by the quadratic term $\|v_k - y_k^{++}\|^2$, which is entirely ignored in the proof of Theorem 1. Thus, in the proof of Theorem 2, one is able to incorporate another term containing $\|\nabla f(y_k)\|^2$, which leads to a larger allowable value of α_k , and ultimately, a tighter bound for Algorithm 2.

5 Algorithms and Convergence Guarantees for Composite Functions

The purpose of this section is to extend the results presented in Section 4 from the smooth to the composite setting, i.e., here we suppose that $h(x) \neq 0$. In particular, we present two algorithms whose iterates fit the Underestimate Sequence framework described in Section 2, and use the lower bounds developed in Section 3.3. Both algorithms appear to fit the composite setting very naturally; the first algorithm is a proximal gradient descent type method, while the second algorithm is an accelerated proximal gradient variant. The algorithms also incorporate stopping conditions that provide a certificate of optimality. We establish convergence guarantees for both algorithms via the UES framework and for all results we suppose that Assumption 1 holds.

5.1 A Composite Underestimate Sequence Algorithm

We now present an algorithm to solve 1, which is based on the UES framework. A brief description will follow.

The Composite (functions) UnderEstimate Sequence Algorithm (CUESA) presented in Algorithm 3 solves problem (1) when $h \neq 0$. The algorithm is described now. First, an initial point $x_0 \in \mathbb{R}^n$ is chosen, as well as some stopping

Algorithm 3 Composite UES Algorithm (CUESA)

-
- 1: Initialization: Set $k = 0$, $\epsilon > 0$, initial point $x_0 \in \mathbb{R}^n$ and compute μ , L .
 - 2: Set $\varphi_0(x)$ as in (32), with v_0 and φ_0^* as in (33). Let $\alpha_k = \frac{\mu}{L}$.
 - 3: **while** $F(x_k) - \varphi_k^* > \epsilon$ **do**
 - 4: Set $x_{k+1} = x_k - \frac{1}{L}G_L(x_k)$,
 - 5: Set $y_k = x_k$, $y_k^+ = x_k^+$, and $y_k^{++} = x_k^{++}$
 - 6: Update v_{k+1} and φ_{k+1}^* as in (36) and (37) respectively.
 - 7: $k = k + 1$.
 - 8: **end while**
-

tolerance $\epsilon > 0$. Secondly, the point $v_0 = x_0^{++}$ is constructed, as well as the lower bound $\varphi_0(x)$ with minimum value φ_0^* . The algorithm uses a fixed step size of $\alpha_k = \frac{\mu}{L}$ at every iteration. Next, the main loop commences and an iteration proceeds as follows. One sets $y_k = x_k$ (so y_k is not explicitly used in CUESA); x_k is updated by taking a *proximal* gradient descent step with the step size $\frac{1}{L}$, resulting in the new point x_{k+1} ; the point $v_{k+1} = x_{k+1}^{++}$ is constructed and the lower bound $\varphi_{k+1}(x)$ is updated.

The algorithm utilizes two points at every iteration, namely x_k and v_k , as well as the values $\varphi_k(x)$ and φ_k^* . The point v_k and the value φ_k^* are used for the lower bound, which is essential for the stopping criterion.

Considering only Step 5, one sees that at every iteration of CUESA the point x_k is updated via a proximal gradient descent step. That is, a step of size $1/L$ in the direction of the negative proximal gradient is taken from the current point x_k , resulting in the new point x_{k+1} . What makes CUESA distinct from a standard proximal gradient method is the inclusion of several other ingredients related to the lower bound $\varphi_k(x)$, which guarantee an ϵ -optimal solution.

Now we present a convergence guarantee for CUESA. Theorem 3 shows that the iterates generated by Algorithm 1 form an underestimate sequence (i.e., they satisfy Definition 1) and therefore, Algorithm 1 is guaranteed to converge (linearly) to the solution of problem (1).

Theorem 3 *Let Assumption 1 hold. The sequences $\{x_k\}_{k=0}^\infty$, $\{\varphi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by CUESA (Algorithm 3) form a UES.*

Proof From Step 5 in CUESA, one sees that $y_k = x_k$ for all k , so it also follows that $y_k^+ = x_{k+1}$ for all k . Now, using $y = x = x_k$ in the lower bound (25) gives

$$F(x_{k+1}) \leq F(x_k) - \frac{1}{2L}\|G_L(x_k)\|_2^2. \quad (47)$$

Thus,

$$\begin{aligned}
& F(x_{k+1}) - \varphi_{k+1}^* \\
&= (1 - \alpha)F(x_{k+1}) + \alpha F(x_{k+1}) - \varphi_{k+1}^* \\
&\stackrel{(21)}{=} (1 - \alpha)F(x_{k+1}) + \alpha F(x_{k+1}) - (1 - \alpha_k) (\varphi_k^* + \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2) \\
&\quad - \alpha_k \left(F(y_k^+) + \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y_k)\|^2 \right) \\
&= (1 - \alpha)F(x_{k+1}) - (1 - \alpha_k) (\varphi_k^* + \alpha_k \frac{\mu}{2} \|v_k - y_k^{++}\|_2^2) \\
&\quad - \alpha_k \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y_k)\|^2 \\
&\leq (1 - \alpha) (F(x_{k+1}) - \varphi_k^*) - \alpha_k \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(y_k)\|^2 \\
&\stackrel{(47)}{\leq} (1 - \alpha) (F(x_k) - \frac{1}{2L} \|G_L(x_k)\|^2 - \phi_k^*) - \alpha \left(\frac{1}{2L} - \frac{1}{2\mu} \right) \|G_L(x_k)\|^2 \\
&\leq (1 - \alpha) (F(x_k) - \phi_k^*) + \left(\frac{\alpha}{2\mu} - \frac{\alpha}{2L} - (1 - \alpha) \frac{1}{2L} \right) \|G_L(x_k)\|^2 \\
&\leq (1 - \alpha_k) (F(x_k) - \varphi_k^*),
\end{aligned}$$

where the last step follows because $\alpha_k = \frac{\mu}{L}$ so $\frac{\alpha_k}{2\mu} - \frac{1}{2L} = \frac{1}{2L} - \frac{1}{2L} = 0$. \square

Corollary 3 *Let Assumption 1 hold. Then, the sequence of iterates $\{x_k\}_{k \geq 0}$ generated by Algorithm 3 exhibits a linear rate of convergence*

$$F(x_k) - \varphi_k^* \leq \left(1 - \frac{\mu}{L}\right)^k (F(x_0) - \varphi_0^*).$$

5.2 An Accelerated Composite UES Algorithm

An accelerated algorithm for convex composite problems is now presented.

Algorithm 4 Accelerated Composite UES Algorithm (ACUESA)

- 1: Initialization: Set $k = 0$, $\epsilon > 0$, initial point $x_0 \in \mathbb{R}^n$ and compute μ , L .
 - 2: Set $\varphi_0(x)$ as in (32), with v_0 and φ_0^* as in (33). Let $\alpha_k = \sqrt{\frac{\mu}{L}}$, $\beta_k = \frac{1}{1 + \alpha_k}$.
 - 3: **while** $F(x_k) - \varphi_k^* > \epsilon$ **do**
 - 4: Set $y_k = \beta_k x_k + (1 - \beta_k) v_k$.
 - 5: Set $x_{k+1} = y_k - \frac{1}{L} G_L(y_k)$.
 - 6: Update v_{k+1} and φ_{k+1}^* as in (36) and (37) respectively.
 - 7: $k = k + 1$.
 - 8: **end while**
-

The Accelerated Composite UnderEstimate Sequence Algorithm (ACUESA) presented in Algorithm 4 solves (1) when $h \neq 0$. The algorithm proceeds as

follows. ACUESA is initialized with a starting point $x_0 \in \mathbb{R}^n$, a stopping tolerance $\epsilon > 0$, the point $v_0 = x_0^{++}$ as well as the construction of the lower bound $\varphi_0(x)$ and minimum value φ_0^* . For ACUESA one sets $\alpha_k = \sqrt{\mu/L}$ and the parameter $\beta_k = \frac{1}{1+\alpha_k}$ is also used. Notice that parameters α_k and β_k are fixed for all iteration. The main loop proceeds as follows. At every iteration one sets y_k to be a convex combination of the points x_k and v_k ; a gradient descent step is taken *from* y_k , resulting in the new point x_{k+1} ; the point v_{k+1} is constructed and the lower bound $\varphi_{k+1}(x)$ is updated.

Algorithm 4 can be viewed as the accelerated version of Algorithm 3. In contrast to Algorithm 3, ACUESA constructs *three* points at every iteration, namely x_k , v_k and y_k , where the intermediate vector y_k is a convex combination of the points x_k and v_k (i.e., for ACUESA $x_k \neq y_k$.) Notice also that x_{k+1} is the result of a gradient descent step taken from the point y_k . The variable φ_k^* are also maintained and is used in the stopping condition.

The following result provides a convergence guarantee for ACUESA. Theorem 4 shows that the iterates generated by Algorithm 4 fit the UES framework (i.e., they satisfy Definition 1), so Algorithm 4 is guaranteed to converge (linearly at the optimal rate) to the solution of problem (1) (see Corollary 4).

Theorem 4 *Let Assumption 1 hold. The sequences $\{x_k\}_{k=0}^\infty$, $\{\varphi_k(x)\}_{k=0}^\infty$ and $\{\alpha_k\}_{k=0}^\infty$ generated by ACUESA (Algorithm 4) form a UES.*

Proof From Step 7 in ACUESA,

$$x_{k+1} = y_k - \frac{1}{L}G_L(y_k) \stackrel{(9)}{=} y_k^+. \quad (48)$$

Hence,

$$\begin{aligned} F(x_{k+1}) - \varphi_{k+1}^* &= (1 - \alpha_k)F(x_{k+1}) + \alpha_k F(x_{k+1}) - \varphi_{k+1}^* \\ &\stackrel{(21)}{=} (1 - \alpha_k)F(x_{k+1}) + \alpha_k F(x_{k+1}) - (1 - \alpha_k)\varphi_k^* - \alpha_k F(y_k^+) \\ &\quad - \alpha(1 - \alpha)\frac{\mu}{2}\|v_k - y_k^{++}\|_2^2 - \alpha_k \left(\frac{1}{2L} - \frac{1}{2\mu}\right) \|G_L(y_k)\|^2 \\ &\stackrel{(48)}{=} (1 - \alpha_k)F(x_{k+1}) - (1 - \alpha_k)\varphi_k^* \\ &\quad - \alpha_k(1 - \alpha_k)\frac{\mu}{2}\|v_k - y_k^{++}\|_2^2 + \left(\frac{\alpha_k}{2\mu} - \frac{\alpha_k}{2L}\right) \|G_L(y_k)\|^2 \\ &= (1 - \alpha_k) \left(F(x_{k+1}) - \varphi_k^* - \alpha_k\frac{\mu}{2}\|v_k - y_k^{++}\|_2^2\right) \\ &\quad + \left(\frac{\alpha_k}{2\mu} - \frac{\alpha_k}{2L}\right) \|G_L(y_k)\|^2 \end{aligned} \quad (49)$$

By considering the expression for y_k in Step 6 in Algorithm 4 and noticing that $\beta_k = \frac{1}{1+\alpha_k}$ for all k , (44) and (45) hold. Thus, combining (44) and (10)

gives

$$\begin{aligned}
& \frac{\alpha_k \mu}{2} \|v_k - y_k^{++}\|^2 \\
&= \frac{\alpha_k \mu}{2} \left\| \frac{\beta_k}{1-\beta_k} (y_k - x_k) + \frac{1}{\mu} G_L(y_k) \right\|^2 \\
&= \frac{\alpha_k \mu}{2} \frac{\beta_k^2}{(1-\beta_k)^2} \|x_k - y_k\|^2 + \frac{\alpha_k}{2\mu} \|G_L(y_k)\|^2 - \frac{\alpha_k \beta_k}{1-\beta_k} \langle G_L(y_k), x_k - y_k \rangle \\
&\stackrel{(45)}{=} \frac{\mu}{2} \frac{\beta_k}{1-\beta_k} \|x_k - y_k\|^2 + \frac{\alpha_k}{2\mu} \|G_L(y_k)\|^2 - \langle G_L(y_k), x_k - y_k \rangle. \tag{50}
\end{aligned}$$

Substituting (50) into (49) results in

$$\begin{aligned}
& F(x_{k+1}) - \varphi_{k+1}^* \\
&= (1 - \alpha_k) \left(F(x_{k+1}) - \varphi_k^* - \frac{\mu}{2} \frac{\beta_k}{1-\beta_k} \|x_k - y_k\|^2 + \langle G_L(y_k), x_k - y_k \rangle \right) \\
&\quad - (1 - \alpha_k) \frac{\alpha_k}{2\mu} \|G_L(y_k)\|^2 + \left(\frac{\alpha_k}{2\mu} - \frac{\alpha_k}{2L} \right) \|G_L(y_k)\|^2 \\
&= (1 - \alpha_k) \left(F(x_{k+1}) - \varphi_k^* - \frac{\mu}{2} \frac{\beta_k}{1-\beta_k} \|x_k - y_k\|^2 + \langle G_L(y_k), x_k - y_k \rangle \right) \\
&\quad + (1 - \alpha_k) \frac{1}{2L} \|G_L(y_k)\|^2.
\end{aligned}$$

Using a rearrangement of the lower bound (25), and (48), gives

$$\begin{aligned}
& F(x_{k+1}) - \varphi_{k+1}^* \\
&\leq (1 - \alpha_k) \left(F(x_k) - \langle G_L(y_k), x_k - y_k \rangle - \frac{\mu}{2} \|x_k - y_k\|^2 - \frac{1}{2L} \|G_L(y_k)\|^2 - \varphi_k^* \right) \\
&\quad + (1 - \alpha_k) \left(-\frac{\mu}{2} \frac{\beta_k}{1-\beta_k} \|x_k - y_k\|^2 + \langle G_L(y_k), x_k - y_k \rangle + \frac{1}{2L} \|G_L(y_k)\|^2 \right) \\
&= (1 - \alpha_k) (F(x_k) - \varphi_k^*) + (1 - \alpha_k) \left(-\frac{\mu}{2} \|x_k - y_k\|^2 - \frac{\mu}{2} \frac{\beta_k}{1-\beta_k} \|x_k - y_k\|^2 \right) \\
&\leq (1 - \alpha_k) (F(x_k) - \varphi_k^*) - \frac{\mu}{2} (1 - \alpha_k) \frac{1}{1-\beta_k} \|x_k - y_k\|^2 \\
&\leq (1 - \alpha_k) (F(x_k) - \varphi_k^*).
\end{aligned}$$

Thus, the iterates generated by ACUESA form a UES. \square

Corollary 4 *Let Assumption 1 hold. Then, the sequence of iterates $\{x_k\}_{k \geq 0}$ generated by Algorithm 4 exhibits the optimal linear rate of convergence*

$$F(x_k) - \varphi_k^* \leq \left(1 - \sqrt{\frac{\mu}{L}}\right)^k (F(x_0) - \varphi_0^*).$$

6 An algorithm with adaptive L

In the algorithms presented so far, the Lipschitz constant L is explicitly used in each algorithm. However, by studying the convergence proofs for Algorithms 1–4 one notices that the role of the Lipschitz constant L is to enforce a reduction in the function value from one iteration to the next (see the first step in the proofs of Theorems 1–4). Thus, it is natural to ask the question, ‘Can an

adaptive Lipschitz constant, L_k say, be used in place of the true Lipschitz constant L ?'. In this section we show that, using a strategy similar to that proposed by Nesterov in [3,4], it is possible to employ an adaptive Lipschitz constant while preserving convergence guarantees.

6.1 The Inequality

When the Lipschitz constant L is unknown, or is expensive to compute, it may be preferable to employ an ‘adaptive’ Lipschitz constant, L_k say, i.e., determine a value L_k that approximates L locally. This approach has been previously studied by Nesterov in [3,4], and it has the added advantage that L_k may be smaller than the true Lipschitz constant L , which can lead to large step sizes. Throughout the algorithm certain inequalities must hold to ensure that convergence guarantees are maintained. The relevant inequalities are as follows.

Smooth case. For smooth functions, (39) and (42) must hold for SUESA and ASUESA, respectively. This means that at every iteration, if L_k satisfies

$$f\left(y_k - \frac{1}{L_k} \nabla f(y_k)\right) \leq f(y_k) - \frac{1}{2L_k} \|\nabla f(y_k)\|^2, \quad (51)$$

then convergence guarantees for SUESA and ASUESA are maintained. If L_k is chosen to satisfy (51) then we will show that we have the improvement $\alpha_k = \mu/L_k$ (or $\alpha_k = \sqrt{\mu/L_k}$ for the accelerated case) at every iteration.

Non-smooth case. For composite functions, (47) and (49) must hold for CUESA and ACUESA, respectively. So, if L_k satisfies

$$F\left(y_k - \frac{1}{L_k} G_{L_k}(y_k)\right) \leq F(y_k) - \frac{1}{2L_k} \|G_{L_k}(y_k)\|_2^2, \quad (52)$$

then the algorithms are still guaranteed to converge. This also implies the improvement $\alpha_k = \mu/L_k$ (or $\alpha_k = \sqrt{\mu/L_k}$ for the accelerated case) at every iteration.

With these two inequalities in mind, the adaptive Lipschitz process can be described briefly as follows. When initializing Algorithms 1–4, choose an initial estimate $L_0 > 0$, and increase and decrease factors $u > 1$ and $d > 1$ respectively. To find the appropriate L_k , at iteration k , one starts with the value L_{k-1} (i.e., the adaptive Lipschitz constant from the previous iteration) and increases it via multiplication with u , or decreases it via division by d , until (51) (or (52)) is satisfied. At iteration k , once an L_k is found such that (51) (or (52) in the composite case) holds, then the iteration proceeds with L_k used in place of L .

Note that, using this process, it is possible that at some iteration, $L_k < L$, i.e., L_k may be smaller than L . In this case, the stepsize $1/L_k$ is used, which is larger than $1/L$.

The pseudocode is presented in Algorithm 5. Note that determining the adaptive Lipschitz constant occurs as an inner loop within one of Algorithms 1–4. Thus, we use the iteration counter s in Algorithm 5 to distinguish it from the outer loop counter k . Note that the strategy above holds for Algorithms 2 and

Algorithm 5 Finding L_k in iteration k of Algorithms 2 and 4.

```

1: Input:  $x_k, v_k, u > 1, d > 1$  and  $L_{k-1}$ .
2: Initialize: If  $k = 0$  let  $L_s = L_0$ , or if  $k \geq 1$  then  $L_s = L_{k-1}/d$ .
3: for  $s = 0, 1, 2, \dots$  do
4:    $\alpha_s = \sqrt{\frac{\mu}{L_s}}, \beta_s = \frac{1}{1+\alpha_s}$ .
5:   Set  $y_s = \beta_s x_k + (1 - \beta_s)v_k$ .
6:   Set  $x_s = y_s - \frac{1}{L_s}G(y_s)$ .
7:   if (51) or (52) holds then
8:     Break.
9:   else
10:     $L_{s+1} = u \cdot L_s$ .
11:   end if
12: end for
13: Output:  $L_k = L_s, \alpha_k = \alpha_s, \beta_k = \beta_s, y_k = y_s$  and  $x_{k+1} = x_s$ .
```

4, but it is straightforward to adapt it to Algorithms 1 and 3 by modifying the variables α_s and β_s .

We now present several theoretical result related to this setup.

Lemma 10 *Let $u, d > 1$. If $L_0 \leq L$ then for all k , $L_k \leq L \cdot u$. If $L_0 \geq L$ then for the first $k = \frac{\log \frac{L}{L_0}}{\log d}$ iterations, $L_k = L_0 \cdot d^k$.*

Proof Note that, if $L_k \geq L$ then L_k is guaranteed to satisfy (51) or (52).

Now suppose that at the start of iteration k , we have a trial value $L_s < L$. If L_s satisfies either (51) or (52) then one sets $L_k = L_s$ and terminates the inner loop, noting that $L_k < L$. If L_s does not satisfy (51) or (52) then it is increase by multiplication with u . Suppose that L_s is the largest possible trial value satisfying $L_s < L$ and $L_s \cdot u \geq L$, but with (51) or (52) not holding. Then multiplying L_s by u once results in $L_s \cdot u \geq L$ and $L_s \cdot u \leq L \cdot u$, and this value is guaranteed to satisfy (51) or (52) so we set $L_k = L_s$.

On the other hand, suppose that at iteration $k = 0$, the initial value happens to satisfy $L_0 > L$. Then this L_0 will satisfy (51) or (52) so it will be accepted with $L_k = L_0$ at iteration $k = 0$. At iteration $k = 1$ the first trial value is

set to $L_s = L_{k-1}/d = L_0/d$. If $L_s \geq L$, then one accepts $L_k = L_1 = L_s$ and the iteration proceeds. At the start of any iteration, if the previous value $L_{k-1} \geq L$, then the first trial value is divided by $d > 1$. Thus, it is guaranteed that $\frac{L_0}{d^K} \geq L$, where

$$K = \left\lceil \frac{\log \frac{L_0}{L}}{\log d} \right\rceil.$$

In other words, $L_k \geq L$ must hold for $k = \{1, 2, \dots, K\}$.

Proposition 2 *The maximum number of times that Lines 4 to 10 are executed during the first K iterations is bounded by*

$$\left\lceil \frac{\max\{\log(Ld/L_0), 0\}}{\log u} \right\rceil + (K - 1) \left\lceil \frac{\log d}{\log u} + 1 \right\rceil. \quad (53)$$

Proof In the first iteration, at most we need to execute the procedure from line 4 to line 10 of Algorithm 5 $\left\lceil \frac{\log(Ld/L_0)}{\log u} \right\rceil$ times, assuming $L_0/d < L$. In the case of $L_0/d \geq L$, this procedure is carried out once. In the k th iteration when $k \geq 2$, since we know that $L_{k-1} \geq L$, the above procedure should be run $\left\lceil \frac{\log d}{\log u} + 1 \right\rceil$ times if $L_{k-1}/d < L$ occurs. Thus, we obtain (53).

7 Numerical Experiments

In this section, we present numerical results to compare our proposed algorithms with several other methods that have an optimal convergence rate. The algorithms are as follows, and are summarized in Table 1.

OQA. The Optimal Quadratic Averaging algorithm (OQA) [20], which builds upon the work in [19], maintains a quadratic lower bound on the objective function value at every iteration. The quadratic lower bound is called ‘optimal’ because it is the ‘best’ lower bound that can be obtained as a convex combination of the previous 2 quadratic lower bounds. In OQA, x_{k+1} is set to be the minimizer of $f(x)$ on the line joining the points x_k^+ and the minimizer of the current quadratic lower bound. In [20] the author suggest a variant of OQA, which we call OQA+ here, that computes x_k^+ via a line search that does not use the true Lipschitz constant L . We compare both OQA and OQA+ in our experiments.

NEST. We use NEST to denote the algorithm described in Chapter 2 of [2]. Further, NEST+ is a variant of NEST in which the Lipschitz constant L is adaptively update via the strategy in [3, 4].

GD. We also implement a Gradient Descent (GD) method which uses a fixed stepsize of $\frac{1}{L}$. Note that this is similar to Algorithm 1, although GD does not maintain any kind of lower bound. As the only non-optimal algorithm,

Gradient Descent provides a benchmark that will enable us to observe any performance advantages of the optimal methods.

Algorithm	Description
OQA	Optimal Quadratic Averaging Algorithm
OQA+	Optimal Quadratic Averaging Algorithm with $x_k^+ = \text{line-search}(x_k, x_k - \nabla f(x_k))$
ASUSEA	Accelerated Smooth Underestimate Sequence Algorithm
ASUSEA+	Accelerated Smooth Underestimate Sequence Algorithm with adaptive Lipschitz constant
SUESA	Smooth Underestimate Sequence Algorithm
NEST	Algorithm described in Chapter 2 of [2]
NEST+	Algorithm described in Chapter 2 of [2] with adaptive Lipschitz constant
CUSEA	Composite Underestimate Sequence Algorithm
CUESA+	Composite Underestimate Sequence Algorithm with adaptive Lipschitz constant
ACUESA	Accelerated Composite Underestimate Sequence Algorithm
ACUESA+	Accelerated Composite Underestimate Sequence Algorithm with adaptive Lipschitz constant
CNEST	Algorithm (4.9) described in [3] with fixed Lipschitz constant
CNEST+	Algorithm (4.9) described in [3] with adaptive Lipschitz constant

Table 1: Description of implemented algorithms

7.1 Empirical Risk Minimization

We consider two Empirical Risk Minimization (ERM) problems, which are popular in the machine learning literature. In particular, we study ERM with a squared hinge loss

$$f(x) = \frac{1}{m} \sum_{i=1}^m (\max\{0, y_i - a_i^T x\})^2 + \frac{\lambda}{2} \|x\|^2, \quad (54)$$

and ERM with a logistic loss (also called logistic regression)

$$f(x) = \frac{1}{m} \sum_{i=1}^m \log(1 + e^{-y_i a_i^T x})^2 + \frac{\lambda}{2} \|x\|^2. \quad (55)$$

In each case $y_i \in \{-1, +1\}$ is the label and $a_i \in \mathbb{R}^n$ represents the training data for $i = \{1, 2, \dots, m\}$. All the datasets in our experiments come from LIBSVM database [26]. Also note that for all experiments we have $\mu = \lambda$.

Comparison on Decreasing Objective Values

In the first experiment we compare the OQA, ASUESA and NEST algorithms (both the standard and adaptive Lipschitz variants) and investigate how the objective function values behave on several test problems. The test problems considered in this experiment are the `a1a` dataset with a squared hinge loss and a value $\lambda = 10^{-4}$, the `rcv1` dataset with a logistic loss and a value $\lambda = 10^{-4}$, and the `covtype` dataset with a squared hinge loss and a value $\lambda = 10^{-5}$.

In Figure 2 we plot the gap $f(x_k) - \phi_k^*$ vs the number of function evaluations and the gap $f(x_k) - \phi_k^*$ vs the cpu time. The figure shows the advantages of

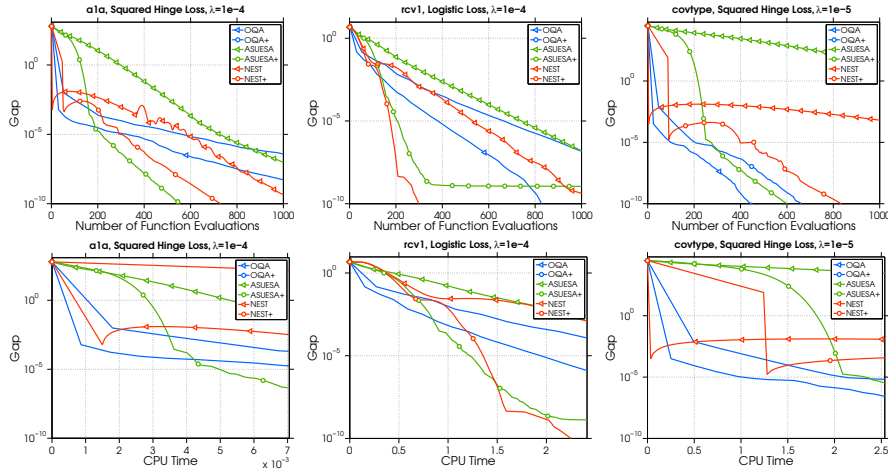


Fig. 2: Evolution of the gap $f(x_k) - \phi_k^*$ for each algorithm compared with the number of function evaluations and cputime.

using an adaptive Lipschitz constant with the adaptive methods performing better than their original versions in most cases. Figure 2 also shows that ASUESA+ performs very well, being the best algorithm on the first dataset, and the second best algorithm on the other two datasets.

Theory and Practice for OQA and ASUESA

In this numerical experiment we study ASUESA and OQA and investigate how their practical performance compares with that predicted by theory. For the OQA algorithm a line search is needed to determine a necessary algorithmic variable, and to ensure that theory for OQA holds, the line search should be exact. In this experiment we will use bisection to compute this variable, but we will restrict the number of bisection steps allowed to $b = 2, 5, 20$. Figure 3 plots the ratio $(f(x_k) - \phi_k^*) / (f(x_{k-1}) - \phi_{k-1}^*)$ for ASUESA and for three instances of OQA, where each instance uses a different number of bisection steps $b = 2, 5, 20$. We also plot $1 - \sqrt{\frac{b}{L}}$ (black dots), which is the amount of decrease in the gap $f(x_k) - \phi_k^*$ at each iteration predicted by the theory. (In theory, we should have $(f(x_k) - \phi_k^*) / (f(x_{k-1}) - \phi_{k-1}^*) = 1 - \sqrt{\frac{b}{L}}$ for all $k \geq 0$.)

From the plots in Figure 3 we see that ASUESA performs very well, and as predicted by the theory, with the ratio $(f(x_k) - \phi_k^*) / (f(x_{k-1}) - \phi_{k-1}^*)$ always strictly below the theoretical bound. On the other hand, the quality of line search affects OQA significantly. The fewer the number of line search (bisection) iterations, the more likely it is for OQA to violate the theoretical results. Note that this is not necessarily surprising because the theory for OQA requires the exact minimizer along a line segment to be found, so 2

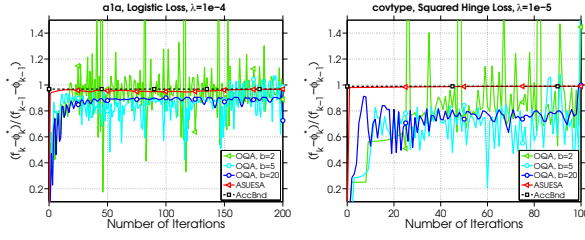


Fig. 3: Comparison of $\frac{f(x_k) - \phi_k^*}{f(x_{k-1}) - \phi_{k-1}^*}$ for ASUESA and for OQA with different numbers of bisection steps ($b = 2, 5, 20$). The black dots are $1 - \sqrt{\frac{\mu}{L}}$.

or 5 iterations of bisection may be simply too few to find it. Notice that when $b = 2$, the green line shows that OQA behaves erratically, with the ratio $(f(x_k) - \phi_k^*) / (f(x_{k-1}) - \phi_{k-1}^*)$ being greater than 1 on many iterations, indicating that the gap is growing on those iterations. When we use OQA with $b = 5$ steps of bisection at each iteration (light blue line), the algorithm performs better, and often, but not always, the ratio is less than 1. Finally, the dark blue line shows the behaviour of OQA when $b = 20$ steps of bisection at each iteration. The dark blue line is always below the theoretical bound of $1 - \sqrt{\frac{\mu}{L}}$, indicating good algorithmic performance (often better than predicted by theory). However, the line search needed by OQA comes at an additional computational cost, which can still mean that the overall runtime is longer for OQA than for ASUESA, as we now show.

Here a similar experiment is performed to compare the theoretical and practical performance of SUESA and ASUESA. We have already seen that the theoretical results for ASUESA give a proportional reduction of $1 - \sqrt{\frac{\mu}{L}}$ in the gap at every iteration. However, for SUESA, the proportional reduction in the gap is $1 - \frac{\mu}{L}$. We investigate how these theoretical bounds compare with the practical performance of each of these algorithms. We use the `ala`, `rcv1` and `covtype` datasets for this experiment, and for each of the three datasets we form both a logistic loss, and a squared hinge loss to create 6 problem instances. The results are shown in Figure 4.

Figure 4 presents the ratio $\frac{f(x_k) - \phi_k^*}{f(x_{k-1}) - \phi_{k-1}^*}$ for SUESA and ASUESA. Also displayed is the theoretical (unaccelerated) rate $1 - \frac{\mu}{L}$ (the green line) and the theoretical (accelerated) rate $1 - \sqrt{\frac{\mu}{L}}$ (the black line). One sees that the practical performance of SUESA is very similar to that predicted by the theory because the blue line matches the green line closely. Another observation is that for the accelerated algorithm (ASUESA), in practice, the reduction in the gap $f(x_k) - \phi_k^*$ is often more optimistic than the theoretical rate.

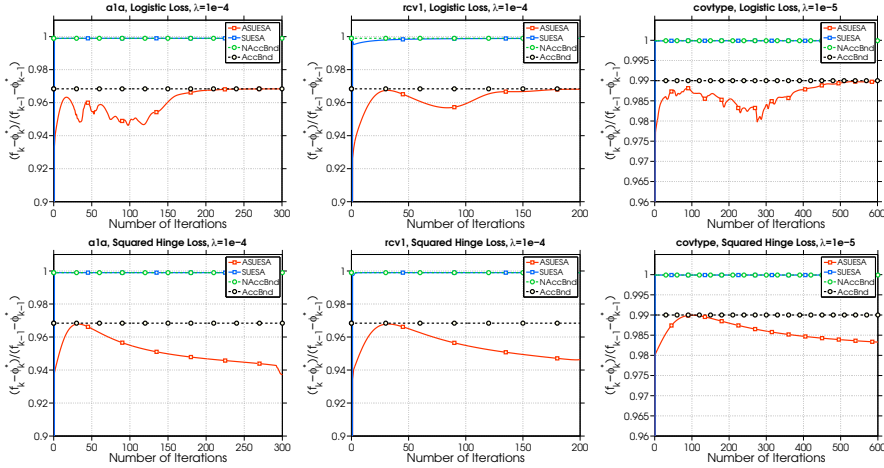


Fig. 4: Comparison of $\frac{f(x_k) - \phi_k^*}{f(x_{k-1}) - \phi_{k-1}^*}$ for SUESA and ASUESA and $1 - \frac{\mu}{L}$ (green line) and $1 - \sqrt{\frac{\mu}{L}}$ (black line).

7.2 Experiments on composite functions

In this section we perform several numerical experiments on problems with a composite objective. Specifically, we consider the elastic net problem, which is problem (1) with

$$F(x) = \frac{1}{n} \sum_{i=1}^n (a_i^T x - y_i)^2 + \frac{\lambda_1}{2} \|x\|_2^2 + \frac{\lambda_2}{2} \|x\|_1. \quad (56)$$

Notice that the first two terms in (56) are smooth, while the ℓ_1 -norm term makes (56) nonsmooth overall. We compare our Algorithm 3 and 4 (CUESA and ACUESA) with the one proposed in [3] (NEST). As stated previously, each of these algorithms can be implemented with either a fixed L or an adaptive L , and we will compare each algorithm under both of these two options.

For these experiments we again use the 3 datasets `ala`, `rcv1` and `covtype`. For the `ala` data the regularization parameters were set to $\lambda_1 = \lambda_2 = 10^{-4}$, for the `rcv1` data the regularization parameters were set to $\lambda_1 = 10^{-4}$ and $\lambda_2 = 10^{-5}$, and for the `covtype` data the regularization parameters were set to $\lambda_1 = 10^{-4}$ and $\lambda_2 = 10^{-6}$.

The results of this experiment are presented in Figure 5, and they show the clear practical advantage of the ACUESA algorithm. The ACUESA algorithm outperforms the CNEST algorithm in all problem instances. Interestingly, on the `rcv1` dataset, the CUESA+ algorithm (CUESA with an adaptive Lipschitz constant) performs better than the accelerated ACUESA algorithm, although

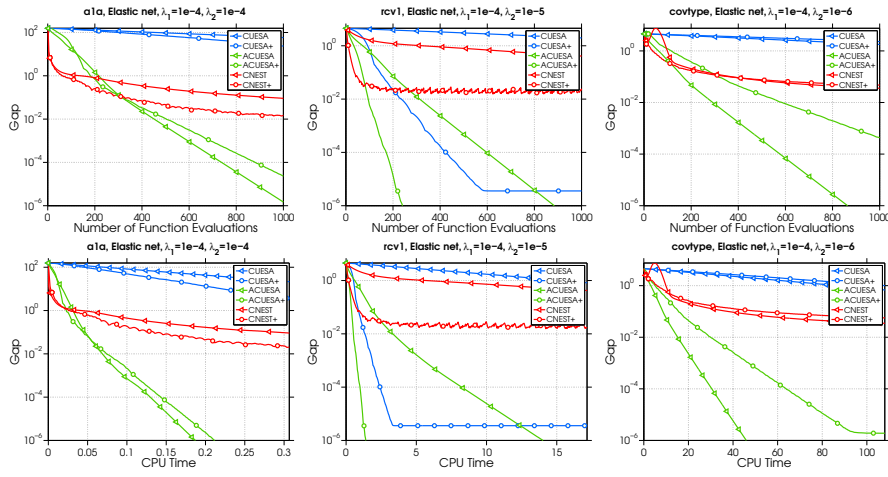


Fig. 5: Comparison of how gaps between the objective values and the minimum amount of the lower bounds decreases for different algorithms. We observe the advantage of ACUESA+ in both number of function evaluations and running time.

the ACUESA+ (accelerated plus adaptive Lipschitz constant) algorithm is still the best overall.

In the final numerical experiment presented here, we investigate the theoretical vs practical performance of CUESA and ACUESA. We set up three problems using each of the 3 datasets already described, and the results are presented in Figure 6.

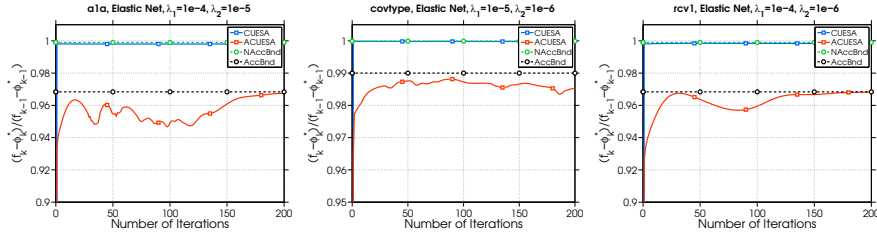


Fig. 6: Comparison of $\frac{f(x_k) - \phi_k^*}{f(x_{k-1}) - \phi_{k-1}^*}$ for CUESA and ACUESA and $1 - \frac{\mu}{L}$ (green line) and $1 - \sqrt{\frac{\mu}{L}}$ (black line). Here we have a similar observation as in Figure 5.

As before, the green line represents the theoretical (unaccelerated) rate $1 - \frac{\mu}{L}$ and the black line represents the theoretical (accelerated) rate $1 - \sqrt{\frac{\mu}{L}}$. Note that the practical performance of CUESA closely matches the theoretical rate.

We also observe that the practical performance of ACUESA is always at least as good as the theoretical rate, and can often get better decrease in the gap per iteration than $1 - \sqrt{\frac{\mu}{L}}$.

All the numerical results presented in this section strongly support the practical success of the SUESA, ASUESA, CUESA and ACUESA algorithms.

8 Conclusion

In this paper we studied efficient algorithms for solving the strongly convex composite problem (1). We proposed four new algorithms — SUESA, ASUESA, CUESA and ACUESA — to solve (1) in both the smooth and composite cases. All of these algorithms maintain a global lower bound on the objective function value, which can be used as an algorithm stopping condition to provide a certificate of convergence. Moreover, we proposed a new underestimate sequence framework that incorporates three sequences, one of which is a global lower bound on the objective function, and this framework was used to establish convergence guarantees for the algorithms proposed here. Our algorithms have a linear rate of convergence, and the two accelerated variants (ASUESA and ACUESA) converge at the optimal linear rate. We also presented a strategy to adaptively select a local Lipschitz constant for the situation when one does not wish to, or cannot, compute the true Lipschitz constant. Numerical experiments show that our algorithms are computationally competitive when compared with other state-of-the-art methods including Nesterov’s accelerated gradient methods and optimal quadratic averaging methods.

References

1. Y. Nesterov, Doklady AN SSSR (translated as Soviet Math. Dokladi) **3**, 54317547 (1983)
2. Y. Nesterov, *Introductory lectures on convex optimization: A basic course* (Kluwer Academic Publishers, Boston, 2004)
3. Y. Nesterov, Gradient methods for minimizing composite objective function. CORE Discussion Paper 2007/76, Université Catholique de Louvain (2007)
4. Y. Nesterov, *Mathematical Programming* **140**(1), 125 (2013)
5. M. Schmidt, N. Le Roux, F. Bach, *Mathematical Programming* **162**(1-2), 83 (2017)
6. R. Johnson, T. Zhang, in *Advances in neural information processing systems* (2013), pp. 315–323
7. H. Robbins, S. Monro, *The annals of mathematical statistics* pp. 400–407 (1951)
8. A. Cotter, O. Shamir, N. Srebro, K. Sridharan, in *Advances in neural information processing systems* (2011), pp. 1647–1655
9. S. Shalev-Shwartz, T. Zhang, in *Advances in Neural Information Processing Systems* (2013), pp. 378–385
10. D. Kingma, J. Ba, arXiv preprint arXiv:1412.6980 (2014)
11. A. Nitanda, in *Advances in Neural Information Processing Systems* (2014), pp. 1574–1582
12. Y. Nesterov, *SIAM Journal on Optimization* **22**(2), 341 (2012)

13. P. Richtárik, M. Takáč, *Mathematical Programming* **144**(1-2), 1 (2014)
14. M. Jaggi, V. Smith, M. Takáč, J. Terhorst, S. Krishnan, T. Hofmann, M.I. Jordan, in *Advances in Neural Information Processing Systems* (2014), pp. 3068–3076
15. C. Ma, V. Smith, M. Jaggi, M.I. Jordan, P. Richtárik, M. Takáč, arXiv preprint arXiv:1502.03508 (2015)
16. O. Fercoq, P. Richtárik, *SIAM Journal on Optimization* **25**(4), 1997 (2015)
17. Z. Allen-Zhu, Z. Qu, P. Richtárik, Y. Yuan, in *International Conference on Machine Learning* (2016), pp. 1110–1119
18. C. Ma, M. Jaggi, F.E. Curtis, N. Srebro, M. Takáč, arXiv (2017)
19. S. Bubeck, Y.T. Lee, M. Singh, A geometric alternative to Nesterov’s accelerated gradient descent. Tech. rep., Microsoft Research (2015)
20. D. Drusvyatskiy, M. Fazel, S. Roy, An optimal first order method based on optimal quadratic averaging. Tech. rep., University of Washington (2016)
21. S. Chen, S. Ma, W. Liu, Geometric descent method for convex composite minimization. Tech. rep., The Chinese University of Hong Kong, Shatin, NT, Hong Kong (2016)
22. Y. Nesterov, *Mathematical Programming* **103**(1), 12717152 (2005)
23. M. Baes, Estimate sequence methods: extensions and approximations. Tech. Rep. Optimization-Online 2372, Université Catholique de Louvain (2009)
24. Z. Lu, L. Xiao, *Mathematical Programming* **152**, 615 (2015)
25. H. Lin, J. Mairal, Z. Harchaoui, A universal catalyst for first-order optimization. Tech. rep., INRIA, France (2015)
26. C.C. Chang, C.J. Lin, *ACM Transactions on Intelligent Systems and Technology* pp. 2:27:1–27:27 (2011). <http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets>