Novel Radar Waveform Optimization for a Cooperative Radar-Communications System

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***Abstract*—We develop and present the novel *minimum estimation error variance* waveform design method, that optimizes the spectral shape of a unimodular radar waveform such that the performance of a joint radar-communications system that shares spectrum is maximized. We also develop the novel spectral water-filling SIC data rate which employs the continuous spectral water-filling algorithm to obtain the optimal communications power spectrum. We also perform a numerical study to compare the performance of the new technique with the previously derived spectral mask shaping method. The global estimation rate and the data rate capture the radar and the communications performance respectively.**

***Index Terms*—Joint Radar-Communications, Radar Waveform Design, Successive Interference Cancellation, Continuous Spectral Water-Filling, Non-convex Optimization**

I. Introduction

Spectral congestion is quickly becoming a problem for the telecommunications sector [1] and cooperative spectrum sharing between radar and communications systems such that both systems mutually benefit from the presence of each other has been proposed as a potential solution [2, 3]. In order to determine how to efficiently share spectral resources and achieve RF convergence [4], a through understanding of the fundamental performance limits of cooperative spectrum sharing is needed. References [3, 5] investigated the fundamental limits of a in-band cooperative radar and communications system and developed inner bounds on performance for such a system. However, these bounds were specifically developed by considering only local estimation errors and with a radar waveform that is suboptimal for joint performance. Generalizing these performance bounds can help to establish limits for cooperative spectrum sharing.

Furthermore, these joint radar-communications performance bounds were found to depend on the shape of the radar waveform spectrum [3, 5]. For a given bandwidth, an impulse-like radar spectral shape (small root mean square (RMS) bandwidth) was found to be more favorable for communications performance, whereas a radar waveform spectrum with more energy at the edges of the bandwidth allocation (large RMS bandwidth) were found to be more favorable for estimation performance. However, the latter waveform also has higher autocorrelation side-lobes or ambiguity which negatively impact the global (local and non-local regime) estimation performance by increasing the radar threshold signal-to-noise ratio (SNR) at which non-local estimation errors occur. Thus, the shape of the radar spectrum poses a trade-off both in terms of radar performance vs. communications performance and in terms of improved estimation performance vs. an increased radar threshold SNR.

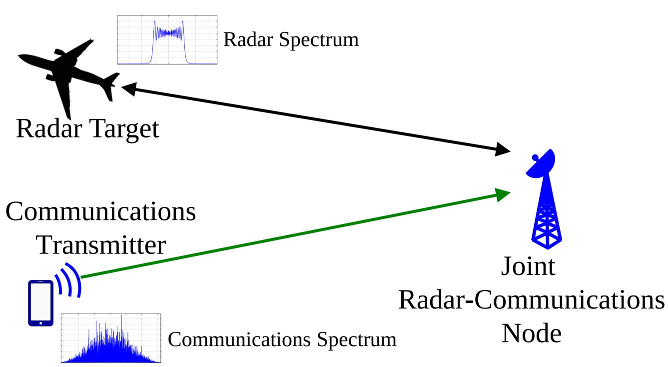

Fig. 1. The joint radar-communications system simulation scenario for radar waveform design. In this scenario, a radar and communications user attempt to use the same spectrum-space-time. This scenario is instructional, and can easily be scaled to more complicated scenarios by using it as a building block to construct real world examples.

Reference [6] generalized the performance bounds developed in References [3, 5] by taking non-local estimation errors into account and tuning the shape of the radar waveform spectrum to maximize joint radar-communications performance. The results presented in this paper is an extension of the work presented in Reference [6]. In this paper, we present a new radar waveform design method for a joint radar-communications system that optimizes the radar waveform spectrum to maximize radar performance or minimize estimation error variance in the non-local (or low-SNR) regime and optimizes the communications power spectrum to maximize communications performance by employing the continuous spectral water-filling algorithm [7]. This novel method designs a jointly optimal radar waveform that is constant modulus, unlike the method presented in Reference [6]. The global estimation rate, introduced in [6], and data rate capture radar

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and communications performance respectively. In order to place emphasis on waveform design approaches and their performance, we assume a simple scenario with a single target and no clutter. The problem scenario considered in this paper is given by Figure 1.

A. Contributions

The main contributions of this paper are summarized below.

- Develop novel *minimum estimation error variance* waveform design method to design a constant modulus radar waveform that maximizes radar performance of a joint radar-communications system
- Develop novel spectral water-filling SIC data rate that maximizes communications performance of a joint radar-communications system
- Perform numerical study on the effects of radar threshold SNR and the order of non-linear chirp phase on waveform design performance
- Compare performance of new waveform design algorithm with previously derived *spectral-mask shaping* waveform design method

B. Background

The performance bounds presented in References [3–5, 8], which only considered local estimation errors, were shown to be dependent primarily on the RMS bandwidth of the radar waveform. Reference [6] extended the estimation rate to consider non-local or global estimation errors and employed a spectral mask to shape the radar waveform spectrum so as to maximize the performance of a joint radar-communications system. An evolutionary optimization algorithm was applied to find the optimal spectral mask that maximizes radar and communications performance (estimation and data rate respectively) and new performance bounds were developed. Performance bounds comparing communications performance versus radar detection performance were derived for a joint radar-communications system in Reference [9].

Modern approaches to the RF convergence problem have looked at waveform design in the context of a single, unified waveform for radar and communications. For example, orthogonal frequency-division multiplexing (OFDM) is commonly chosen for this dual waveform [10–14], where a single transmission is used for communications and monostatic radar. Most results using OFDM waveforms revealed data-dependent ambiguities, opposing cyclic prefix requirements, and demanding peak-to-average power ratio (PAPR) requirements. Spread spectrum waveforms have also been proposed for their autocorrelation properties [15–17], and MIMO radar techniques have been suggested, given that the independent transmitted waveforms allow more degrees of freedom for joint radar-communications co-design [18–20]. Multiple orthogonal linear frequency modulated chirps have also been proposed to accomplish both radar detection and communications transmissions in a MIMO system [21]. Both systems have fundamentally different waveform requirements and that is why, contrary to the aforementioned approaches, the waveform design method proposed in this paper assumes that radar and communications systems transmit separate waveforms.

Researchers have also looked at optimization theory based radar waveform design methods that look to optimize radar performance while the communications system is constrained to reduce interference. Optimization theory is used to maximize some radar performance metrics (detection probability, ambiguity function features etc.) and keep interference to other in-band systems at a minimum [22–24] or impose constraints on the communications rate of other in-band systems [25].

Researchers have searched several other research areas for potential solutions to the spectral congestion problem. Some researchers looked at spatial mitigation as a means to improve spectral interoperability [26–28]. Joint coding techniques, such as robust codes for communications that have desirable radar ambiguity properties, as well as codes that trade data rate and channel estimation error have been investigated as co-design solutions [29–32].

C. Problem Set-up

We consider the scenario shown in Figure 1, which involves a radar and communications user attempting to use the same spectrum-space-time. We consider the joint radar-communications receiver to be a radar transmitter/receiver that can act as a communications receiver. The key assumptions made in this paper for the scenario described in Figure 1 are as follows

- Joint radar-communications receiver is capable of simultaneously decoding a communications signal and estimating a target parameter
- Radar detection and track acquisition have already taken place
- Radar system is an active, single-input single-output (SISO), mono-static, and pulsed system
- Inter-pulse ambiguities between radar pulses not considered
- A single SISO communications transmitter is present
- Only one radar target is present
- Target range or delay is the only parameter of interest
- Target cross-section is well estimated

It should be noted that the performance bounds and results presented in this paper are very closely dependent on the employed receiver model. By employing a mitigation technique called successive interference cancellation (SIC) (discussed later in the paper) at the receiver, the communications data rate at the receiver becomes dependent on the radar waveform spectrum [5]. Employing different mitigation techniques and changing the receiver model will result in a set of performance bounds that are different from the ones presented in this paper.

II. The Joint Radar-Communications System - Receiver Model and Performance Metrics

In this section, we present both the model used to represent the joint radar-communications receiver and the performance metrics used in this paper to characterize radar and communications performance for the joint radar-communications system. A discussion of the SIC mitigation techniques employed at the receiver is also provided. We present a table of significant notation that is employed in this paper in Table I.

TABLE I
SURVEY OF NOTATION

Variable	Description
$\langle\cdot\rangle$	Expectation
$\lVert\cdot\rVert$	L2-norm or absolute value
$Q_M(.)$	Marcum Q-function
$\delta(.)$	Dirac-delta function
f	Frequency
t	Time
B	Full bandwidth of the system
B_{rms}	Root-mean-squared radar bandwidth
$x(t)$	Unit-variance transmitted radar signal
$X(f)$	Radar signal frequency response
P_{rad}	Radar power
τ	Time delay to target
a	Target complex combined antenna, cross-section, and propagation gain
T	Radar pulse duration
δ	Radar duty factor
P_{com}	Total communications power
b	Complex combined antenna gain and communications propagation loss
$n(t)$	Receiver thermal noise
$n_{\mathrm{resi}}(t)$	Post-SIC radar residual
$\sigma^2_{\mathrm{noise}}$	Thermal noise power
k_B	Boltzmann constant
T_{temp}	Absolute temperature
$\sigma^2_{\tau,\mathrm{proc}}$	Variance of range fluctuation process
σ^2_{CRLB}	Cramèr-Rao lower bound or estimation error variance
ISNR	Integrated radar SNR
$p_1,\ldots,p_N$	Phase parameters of polynomial chirp

A. Successive Interference Cancellation Receiver Model

We present the joint radar-communications receiver model that employs SIC, an interference mitigation technique. It is this receiver model that causes communications performance to be closely tied to the spectrum of the radar waveform. This receiver model was first developed in Reference [3].

We assume we have some knowledge of the radar target range (or time-delay), based on prior observations, up to some random fluctuation or process noise which is modeled as a zero-mean random variable $n_{\tau,\mathrm{proc}}(t)$. Using this information, we can generate a predicted radar return and subtract it from the joint radar-communications received signal. Since there is some error in the predicted and actual target locations, this predicted radar signal suppression leaves behind a residual contribution, $n_{\mathrm{resi}}(t)$, to the joint received signal. By lowering the communications rate, the receiver can perfectly decode the communications message from the radar-suppressed joint received signal (which consists of the communications signal, thermal noise and radar residual). The joint radar-communications receiver uses the decoded communications message to reconstruct and remove the communications waveform from the received signal to obtain a radar return signal free of communications interference. This method of interference cancellation is called SIC. SIC is the same optimal multiuser detection technique used for a two user multiple-access communications channel [7, 33], except it is now reformulated for a communications and radar user instead of two communications users. The block diagram of the joint radar-communications system considered in this scenario is shown in Figure 2. For a joint radar-communications received

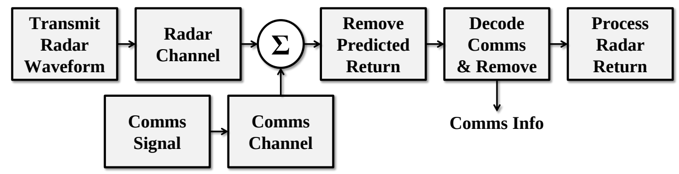

Fig. 2. Joint radar-communications system block diagram for SIC scenario. The radar and communications signals have two effective channels, but arrive converged at the joint receiver. The radar signal is predicted and removed, allowing a reduced rate communications user to operate. Assuming near perfect decoding of the communications user, the ideal signal can be reconstructed and subtracted from the original waveform, allowing for unimpeded radar access.

signal, $z(t)$, given by

$$z(t) = b\,\sqrt{P_{\mathrm{com}}}\,r(t) + n(t) + \sqrt{P_{\mathrm{rad}}}\,a\,x(t-\tau)\,,$$

the received signal at the communications receiver with the predicted radar return suppressed, $\tilde{z}(t)$, is given by [3, 5]

$$\begin{aligned}\tilde{z}(t) &= b\,\sqrt{P_{\mathrm{com}}}\,r(t) + n(t) \\ &\quad + \sqrt{P_{\mathrm{rad}}}\,a[x(t-\tau) - x(t-\tau_{\mathrm{pre}})]\,,\end{aligned} \tag{1}$$

where $x(t–\tau_{\mathrm{pre}})$ is the predicted radar return signal, and τ_{pre} is the predicted target delay. When applying SIC, the interference residual plus noise signal $n_{\mathrm{int+n}}(t)$, from the communications receiver's perspective, is given by [3, 5]

$$\begin{aligned}n_{\mathrm{int+n}}(t) &= n(t) + n_{\mathrm{resi}}(t) \\ &= n(t) + \sqrt{\lVert a\rVert^2\,P_{\mathrm{rad}}}\,n_{\tau,\mathrm{proc}}(t)\,\frac{\partial x(t-\tau)}{\partial t},\end{aligned} \tag{2}$$

where $n_{\tau,\mathrm{proc}}(t)$ is the process noise with variance $\sigma^2_{\tau,\mathrm{proc}}$.

It should be noted that SIC performance is highly sensitive to model mismatch errors since they introduce larger residuals in the SIC process, negatively impacting interference cancellation performance. Potential sources for model mismatch include dynamic range constraints on the receiver or transmitter, phase noise etc. Insufficient transmitter or receiver dynamic range implies that if one received signal is stronger than the other signal, mitigating the stronger signal through SIC will be incredibly difficult, resulting in high residual values being present in the weaker signal after SIC. Communications signal mitigation by 40-50 dB has been demonstrated experimentally [34, 35]. As a result, a dynamic range of 50 dB is sufficient to avoid model mismatch errors for the joint radar-communications receiver. Additionally, it should be noted that the performance of the SIC receiver has a complex dependency on receiver phase noise. Large phase noise can introduce larger post-radar suppression residual values, negatively impacting joint radar-communications performance. A better analysis of the relationship between the SIC receiver and phase noise can be found in [36].

B. Spectral Water-filling SIC Data Rate

We develop and present the novel spectral water-filling SIC data rate, which utilizes the continuous spectral water-filling algorithm [7, 37] to determine the optimal communications power distribution over frequency. The continuous spectral water-filling algorithm optimizes the data rate for a given noise power spectral density [7, 37]. Once the receiver model is known, the communications transmitter can easily determine the noise spectral density at the receiver, $N_{\text{int+n}}(f)$, and apply the continuous spectral water-filling algorithm to determine the optimal communications transmit power distribution, $P(f)$. This communications power distribution, $P(f)$, maximizes the communications data rate at which the joint radar-communications receiver decodes the communications message. We define this maximized communications rate as the spectral water-filling SIC data rate. The spectral water-filling SIC data rate is the data rate that will be used to measure communications performance. The continuous spectral water-filling algorithm is a continuous form extension of the water-filling algorithm employed in References [3, 5]. Figure 3 highlights how the continuous spectral water-filling algorithm selects the optimal power distribution.

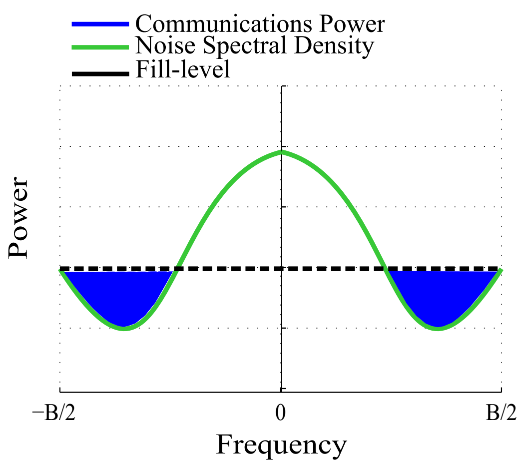

Fig. 3. A notional example of the continuous spectral water-filling algorithm. The black, dashed line indicates the fill level (maximum amount of communications power that can be allocated at any frequency), the green curve represents the noise power spectral density $N_{\text{int+n}}(f)$, and the optimal communications power spectral distribution is shown in blue.

As mentioned earlier, since we employ the SIC model at the joint radar-communications receiver, the receiver will decode the communications message after the predicted radar signal has been mitigated from the received signal. As a result, from the communications receiver's perspective, the channel will be corrupted by noise given by Equation (2). In order to find the noise spectral density, $N_{\text{int+n}}(f)$, we first calculate the autocorrelation function of the time- and band-limited noise signal, $n(t)$ (since the received signal is also time- and band-limited),

$$\begin{aligned}
\gamma(\alpha) &= \langle n_{\text{int+n}}(t)\, n^*_{\text{int+n}}(t-\alpha)\rangle \\
&= \langle n(t)\, n^*(t-\alpha)\rangle + \langle n_{\text{resi}}(t)\, n_{\text{resi}}(t-\alpha)\rangle \\
&= k_B\, T_{\text{temp}}\, B\, \text{sinc}(\pi\, B\, \alpha) \\
&\qquad + \|a\|^2\, P_{\text{rad}}\, \sigma^2_{\tau,\text{proc}}\, \frac{\partial x(t-\tau)}{\partial t}\, \frac{\partial x^*(t-\tau-\alpha)}{\partial t} \\
&= k_B\, T_{\text{temp}}\, B\, \text{sinc}(\pi\, B\, \alpha) + (4\pi^2)\, \|a\|^2\, P_{\text{rad}}\, \sigma^2_{\tau,\text{proc}} \\
&\qquad \cdot \int_{-\infty}^{\infty} df\, f^2 X(f) X^*(f) \mathrm{e}^{i 2\pi f \alpha} \\
&= k_B\, T_{\text{temp}}\, B\, \text{sinc}(\pi\, B\, \alpha) + (4\pi^2)\, \|a\|^2\, P_{\text{rad}}\, \sigma^2_{\tau,\text{proc}}\, g(\alpha)\,,
\end{aligned} \tag{3}$$

where Parseval's theorem and the time-shift and time derivative properties of the Fourier transform are used between the second and third steps, $\text{sinc}(x) = \frac{\sin(x)}{x}$, and $g(\alpha)$ is the inverse Fourier transform with respect to α of $G(f) = \|X(f)\|^2 f^2$. Since the noise power spectral density and autocorrelation are Fourier transform pairs, the noise power spectral density is given by

$$\begin{aligned}
N_{\text{int+n}}(f) &= N(f) + N_{\text{resi}}(f) \\
&= k_B\, T_{\text{temp}}\, \Pi_B(f) \\
&\quad + (4\pi^2)\, \|a\|^2\, P_{\text{rad}}\, \sigma^2_{\tau,\text{proc}}\, \|X(f)\|^2\, f^2\,,
\end{aligned} \tag{4}$$

where $N(f)$ and $N_{\text{resi}}(f)$ are the Fourier transforms of $n(t)$ and $n_{\text{resi}}(t)$ respectively, and $\Pi_B(f)$ is a top-hat or rectangular function from $\frac{-B}{2}$ to $\frac{B}{2}$. The optimal communications power spectrum determined by the continuous spectral water-filling algorithm is given by

$$P(f) = \left(\mu - \frac{N_{\text{int+n}}(f)}{b^2}\right)^+\,, \tag{5}$$

where $(x)^+ = x$ if $x \geq 0$; otherwise $(x)^+ = 0$ and μ is a constant that is determined from the power constraint

$$P_{\text{com}} = \int_{\frac{-B}{2}}^{\frac{B}{2}} df\, P(f) = \int_{\frac{-B}{2}}^{\frac{B}{2}} df\, \left(\mu - \frac{N_{\text{int+n}}(f)}{b^2}\right)^+\,. \tag{6}$$

The spectral water-filling SIC data rate (the corresponding data rate for the channel with noise spectral density $N_{\text{int+n}}(f)$) is given by [7, 37]

$$R_{\text{com}} = \int_{\frac{-B}{2}}^{\frac{B}{2}} df\, \log\left(1 + \frac{b^2\, P(f)}{N_{\text{int+n}}(f)}\right)\,. \tag{7}$$

It should be noted that due to the complexity involved in determining analytical solutions for the integrals shown in Equations (6) and (7), these integrals are evaluated numerically to determine the optimal value for μ and the communications data rate.

C. Global Estimation Rate

Here, we provide a brief discussion of the global estimation rate which was first developed in Reference [6]. We measure radar performance by the estimation rate [3, 5] which measures the amount of information contained in radar returns. The estimation rate is upper bounded as follows:

$$R_{\text{est}} \leq \frac{\delta}{2T} \log_2 \left[1 + \frac{\sigma^2_{\tau,\text{proc}}}{\sigma^2_{\text{est}}} \right] , \tag{8}$$

where σ^2_{est} is the range estimation noise variance, which is bounded locally by the Cramér-Rao lower bound (CRLB) [38]. The estimation rate was extended in Reference [6] to account for global estimation errors. The method of interval errors [33, 39, 40] is employed to calculate the effect of non-local errors on time-delay estimation performance. A closed-form solution of the probability of side-lobe confusion, $P_{s.l.}$ is obtained in terms of the values and locations of the side-lobe peaks, integrated radar SNR, and the Marcum Q-function Q_M [33]. The method of intervals time-delay estimation variance is then given by

$$\sigma^2_{\text{est}} = [1 - P_{s.l.}(\text{ISNR})]\, \sigma^2_{\text{CRLB}}(\text{ISNR}) \,+\, P_{s.l.}(\text{ISNR})\, \phi^2_{s.l.} \,, \tag{9}$$

where $\phi_{s.l.}$ is the offset in time (seconds) between the autocorrelation peak side-lobe and main-lobe [5]. The probability of side-lobe confusion, $P_{s.l.}$, is given by [33]

$$\begin{aligned} P_{s.l.}(\text{ISNR}) = 1 - Q_M &\left(\sqrt{\frac{\text{ISNR}}{2}\left(1 + \sqrt{1 - \|\rho\|^2}\right)}, \right. \\ &\left. \sqrt{\frac{\text{ISNR}}{2}\left(1 - \sqrt{1 - \|\rho\|^2}\right)} \right) \\ +Q_M &\left(\sqrt{\frac{\text{ISNR}}{2}\left(1 - \sqrt{1 - \|\rho\|^2}\right)}, \right. \\ &\left. \sqrt{\frac{\text{ISNR}}{2}\left(1 + \sqrt{1 - \|\rho\|^2}\right)} \right), \end{aligned} \tag{10}$$

where ρ is the ratio of the main-lobe to the peak side-lobe of the autocorrelation function. For a radar system performing time-delay estimation, the CRLB for time delay estimation is given by [41]

$$\sigma^2_{\text{CRLB}} = (8\pi^2 B^2_{\text{rms}} \text{ISNR})^{-1} \,, \tag{11}$$

where the RMS bandwidth is given by

$$B^2_{\text{rms}} = \frac{\int f^2 \, \|X(f)\|^2 \, df}{\int \|X(f)\|^2 \, df} \,. \tag{12}$$

A more intuitive understanding of how the estimation rate metric captures target parameter estimation performance and the implications of altering the estimation rate can be found in [4]. The estimation rate is extended to account for Doppler measurement and continuous signaling radars in Reference [42].

III. Non-linear Chirp with Parametric Polynomial Phase

In this section, we briefly introduce the novel parameterized non-linear chirp that will be used to design the optimal radar waveform in the minimum estimation error variance waveform design method. We also derive an approximate closed-form solution for the spectrum for a special case of this non-linear chirp waveform.

One desirable property for radar waveforms is to have a peak-to-average power ratio as close as possible to 1 (the smallest possible value). Thus, most radar systems now require the signal to be constant modulus or unimodular, which keeps the peak and the average power the same over any time period, granting the signal the smallest possible peak-to-average power ratio of 1. To ensure that the optimized radar waveform is unimodular, we begin by considering the following unimodular non-linear chirp signal with a polynomial phase

$$x(t) = \mathrm{e}^{i\pi(\sum_{m=1}^{N} p_m t^{2m})}, \tag{13}$$

where N is a positive integer and $p_m \in \mathbb{R}, \forall m$ are phase coefficients. We let the polynomial phase to have only even terms to ensure symmetry in the frequency domain. The shape of the waveform spectrum is determined by the phase coefficients. The minimum estimation error variance method selects the appropriate phase coefficient values so as to optimize the shape of the radar spectrum to maximize joint radar-communications performance.

In the following discussion, we derive an approximate expression for the spectrum of the non-linear chirp waveform shown in Equation (13) for the case that $N = 2$.

A. Spectrum of Non-linear Chirp with Parametric Polynomial Phase

Due to the increased complexity involved in evaluating the spectrum for higher values of N, we consider the simple case of $N = 2$. The spectrum of the band-limited non-linear chirp with bandwidth B and time-duration T is given by

$$\begin{aligned} X(f) &= \int_{\frac{-T}{2}}^{\frac{T}{2}} dt \, \mathrm{e}^{i\pi(p_1\, B^2\, t^2 \,+\, p_2\, B^4\, t^4)} \, \mathrm{e}^{-i2\,\pi\, f\, t} \\ &= \int_{\frac{-T}{2}}^{\frac{T}{2}} dt \, \mathrm{e}^{i\pi(p_1\, B^2\, t^2 \,+\, p_2\, B^4\, t^4 \,-\, 2\, f\, t)} \\ &= \int_{\frac{-T}{2}}^{\frac{T}{2}} dt \, \mathrm{e}^{i\phi(t,f)} \,. \end{aligned} \tag{14}$$

In order to obtain a closed form solution for the above integral, we employ the principle of stationary phase [43]. We first find the points in time, t_0, where the phase, $\phi(t, f)$, is stationary i.e. when

$$\begin{aligned} \frac{\partial \phi(t,f)}{\partial t}\Big|_{t=t_0} &= 0 \\ \Rightarrow \pi(2\, p_1\, B^2\, t_0 \,+\, 4\, p_2\, B^4\, t_0^3 \,-\, 2\, f) &= 0 \\ \Rightarrow 2\, p_1\, B^2\, t_0 \,+\, 4\, p_2\, B^4\, t_0^3 \,-\, 2\, f &= 0 \end{aligned} \tag{15}$$

Solving for t_0, we get

$$\begin{aligned} t_0 = &\frac{-6^{\frac{2}{3}}\, B^6\, p_1\, p_2}{Q} \\ &+ \frac{6^{\frac{1}{3}}\, (9\, B^8\, p_2^2\, f \,+\, \sqrt{3\, B^{16}\, p_2^3\, (2\, B^2\, p_1^3 \,+\, 27\, p_2\, f^2)})^{\frac{2}{3}}}{Q} \end{aligned} \tag{16}$$

where

$$Q = 6\,B^4\,p_2\,(9\,B^8\,p_2^2\,f + \sqrt{3\,B^{16}\,p_2^3\,(2\,B^2\,p_1^3\,+\,27\,p_2\,f^2)})^{\frac{1}{3}}\,.$$

Using the principle of stationary phase, the expression for an approximation of the spectrum is given by [43]

$$\begin{aligned} X(f) &\approx 2\,\sqrt{\frac{-\pi}{2\,\phi''(t_0,f)}}\,\mathrm{e}^{-i\frac{\pi}{4}}\,x(t_0)\,\mathrm{e}^{i\phi(t_0,f)} \\ &= 2\,\sqrt{\frac{-1}{4\,p_1\,B^2\,+\,24\,p_2\,B^4\,t_0^2}}\mathrm{e}^{-i\frac{\pi}{4}}\,\mathrm{e}^{i\pi(p_1\,B^2\,t_0^2 + p_2\,B^4\,t_0^4)} \\ &\quad\cdot \mathrm{e}^{i\pi(p_1\,B^2\,t_0^2 + p_2\,B^4\,t_0^4 - 2\,f\,t_0)} \end{aligned} \tag{17}$$

where $\phi''(t,f) = \frac{\partial^2\phi(t,f)}{\partial t^2} = \pi(2\,p_1\,B^2 + 12\,p_2\,B^4\,t^2)$. Although we do not use the above-discussed expression for the radar spectrum in our numerical study, nevertheless, the above result may be useful in other studies.

IV. Radar Waveform Design Methods

In this section, we present the two radar waveform design algorithms discussed in this paper. We first briefly discuss the spectral mask shaping method that was first introduced in Reference [6]. A novel radar waveform design method, the minimum estimation error variance method, is also presented in this section. The spectral mask shaping method will be used as a baseline to compare the performance of the minimum estimation error variance method presented in this paper.

A. Spectral Mask Shaping Method

We present the radar waveform design method presented in Reference [6]. This method will be used as a performance baseline to compare the performance of the novel radar waveform design method presented in this paper. The radar waveform can be designed to maximize radar estimation rate, communications rate, or some weighting therein. Without consideration of global error, waveform design can be simplified to tuning B_{rms} [3]. A closed-form, parameterized spectral mask is used to tune B_{rms} to jointly maximize both the radar and communications users' information rate.

We assume we have a linear frequency-modulated (FM) chirp which spans from $-B/2$ to $B/2$ in time T. We then apply a frequency domain spectral mask weighting to the chirp, $W(f) = x + z\,f^2\,, |f| \leq \frac{B}{2}$. The RMS bandwidth of the resulting weighted chirp is found by assuming the chirp spectrum is approximately flat using the principle of stationary phase (PSP) [43]. As a result, the RMS bandwidth is easily calculable in closed-form for the polynomial [6]:

$$B_{\mathrm{rms}} = \sqrt{\frac{\frac{x^2\,B^3}{12} + \frac{x\,z\,B^5}{40} + \frac{z^2\,B^7}{448}}{x^2\,B + \frac{x\,z\,B^3}{6} + \frac{z^2\,B^5}{80}}}. \tag{18}$$

Using differential evolution (DE) [44] to tune B_{rms}, the following objective function (or cost function) is optimized to maximize joint performance (radar and communications users' information rate)

$$R_{\mathrm{total}} = R_{\mathrm{est}}(B_{\mathrm{rms}})^{\alpha}\,\tilde{R}_{\mathrm{com}}(B_{\mathrm{rms}})^{(1-\alpha)}, \tag{19}$$

where $\tilde{R}_{\mathrm{com}}(B_{\mathrm{rms}})$ is the SIC communications data rate defined in Reference [3] (not to be confused with the spectral water-filling SIC data rate defined in this paper) and α is a blending parameter that is varied from 0 to 1. When $\alpha = 0$, only communications rate is considered. When $\alpha = 1$, only the radar estimation rate is considered. In between, the product is jointly maximized. Note that for $\alpha = 0.5$, R_{total} represents the geometric mean of the two rates. This provides a more numerically stable error term, even when $\tilde{R}_{\mathrm{com}} \gg R_{\mathrm{est}}$.

B. Minimum Estimation Error Variance Method

The waveform design algorithm that we propose in this section designs an optimal non-linear chirp radar waveform (as modeled in Section III) from a global estimation rate perspective. In other words, we first design the waveform to minimize the global estimation error variance (estimation error variance taking into account both non-local and local estimation errors), given by Equation (9). This minimization of the global estimation error variance is accomplished by minimizing the estimation error variance at the radar threshold SNR of the radar estimator. The threshold point of an estimator is the estimator (or radar) SNR value at which the estimator's performance deviates from the CRLB [38] due to error contributions from non-local estimation errors. At SNR values lower than the threshold point, due to autocorrelation main-lobe - side-lobe confusion, non-local estimation errors begin to contribute to estimator's error variance which causes the estimation performance to degrade and deviate from the CRLB [33]. Since the threshold point is the SNR point at which an estimator's performance deviates from the CRLB and also the SNR point at which non-local estimation errors contribute to estimation performance, minimizing the CRLB at the threshold point gives the lowest possible global estimation error variance or highest possible global estimation rate.

For a given SNR, we have to design a radar waveform that has a threshold point at that SNR and has the best (or smallest) estimation error variance. We first eliminate all radar waveforms that have a threshold point higher than the current SNR and then, from the remaining feasible solution set, we find the radar waveform that minimizes the CRLB given by Equation (11). We perform the first elimination step by imposing the following constraint on the ratio of the global estimation error variance (given by Equation (9)) and the CRLB (given by Equation (11))

$$\frac{\sigma_{\mathrm{est}}^2}{\sigma_{\mathrm{CRLB}}^2} \leq \delta_{\mathrm{constraint}}, \tag{20}$$

where $\delta_{\mathrm{constraint}}$ is a parameter whose value determines the size of the feasible solution set. We discuss how to tune this parameter in Section V. By ensuring the above ratio stays below $\delta_{\mathrm{constraint}}$, any radar waveforms with higher threshold points (SNR values) are eliminated. Figure 4 depicts how this constraint works on eliminating radar waveforms with higher threshold points.

We also introduce an additional constraint on spectral leakage (constraint $\boldsymbol{C_2}$) to the waveform optimization problem in order to obtain optimal radar waveforms that not only

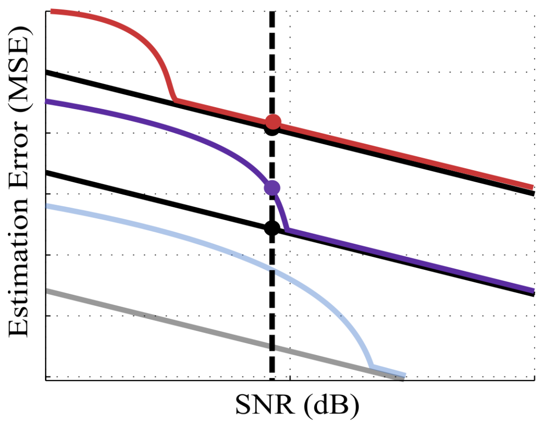

Fig. 4. A notional example depicting the impact of the constraint given by Equation (20) on the feasible set for optimization. The dashed vertical line indicates the given SNR. The red, purple and blue solid curves indicate the estimator performance for different radar waveforms and the black solid lines indicate the CRLB for each radar waveform. The black dots indicate the CRLB values for various feasible radar waveforms at the given SNR. The red and purple dots indicate the actual estimation error variance (estimation performance) for various feasible radar waveforms at the given SNR. The grayed out curves indicate estimation performance for unfeasible radar waveforms at the given SNR. Minimizing the CRLB over the feasible set ensures that the optimal radar waveform will have the lowest estimation threshold point (or best estimation performance, taking both local and non-local estimation errors).

ensure optimal joint radar-communications performance, but also satisfy additional real-world properties that a traditional radar waveform would. Since the system can only receive signals whose spectrum lies within the system's bandwidth, any electromagnetic radio frequency (RF) energy that leaks outside of the bandwidth will be lost. To minimize this loss of RF energy, we introduce a constraint on the amount of energy present in the radar spectrum at frequencies out of the system bandwidth range. We enforce this spectral leakage constraint by having the radar spectrum be below a thresholding spectral mask such as the one seen in Figure 5.

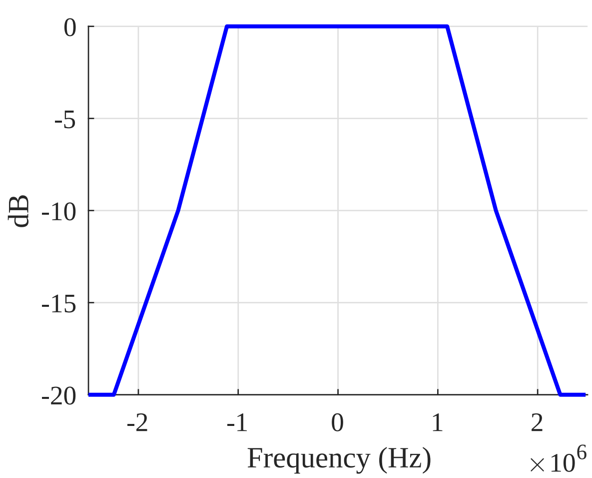

Fig. 5. Spectral Leakage Mask used constrain the amount of energy in the radar spectrum leaking out at frequencies out of the system bandwidth range. The spectral leakage constraint is enforced by having the radar spectrum be below this thresholding spectral leakage mask.

We consider the non-linear chirp waveform given by Equation (13). The spectral shape of the waveform is determined by the parameters $p_m, m = 1, \ldots, N$. In order to design the radar waveform spectrum that minimizes the global estimation performance, we solve the following optimization problem:

$$\begin{aligned} \underset{\bar{p}}{\text{minimize}} \quad & \frac{1}{8\pi^2 B_{\text{rms}}(\bar{p})^2 TB(\text{SNR})}, \\ \text{subject to} \quad & p_m \in [0, 10] \ \forall m \\ & \frac{\sigma^2_{\text{est}}}{\sigma^2_{\text{CRLB}}} \leq \delta_{\text{constraint}} \\ & \mathbb{1}_A(\bar{p}) = 1 \qquad (\boldsymbol{C_2}) \end{aligned} \tag{21}$$

where $\bar{p} = (p_1, \ldots, p_N)$, and $p_1, \ldots, p_N$ are the coefficients of the polynomial phase for the unimodular waveform in Equation (13), and $B_{\text{rms}}(\bar{p})$ is given by Equation (12). The constraint $\boldsymbol{C_2}$ constrains the coefficients $\bar{p}$ such that the resulting spectrum of the waveform stays below a certain masking threshold, which is represented by an indicator function, where A is the set of all phase coefficients that let the resulting masked spectrum stay below the masking threshold as shown in Figure 5.

Once the optimal radar waveform that maximizes the radar performance of a joint radar-communications system is designed, the continuous spectral water-filling algorithm described in Section II-B is employed to determine the spectral water-filling SIC data rate that maximizes the communications performance of a joint radar-communications system. This optimization process is called the minimum estimation error variance method. It should be noted that the optimization problem described in Equation (21) is a non-convex optimization problem.

C. Impact of Threshold Point SNR

As mentioned in Section I, we saw from References [3, 5] that the spectral shape of the radar waveform (the radar RMS bandwidth) impacts the the performance of a joint radar-communications system. Shaping the radar spectrum imposes a trade-off both in terms of radar performance vs. communications performance and in terms of improved estimation performance vs. an increased radar threshold SNR. In this subsection, we briefly discuss how the choice of the threshold SNR impacts both the shape of the radar waveform spectrum and the performance of the joint radar-communications system.

Selecting a low value for the threshold SNR implies that even for small radar SNR values, the probability of side-lobe confusion for the radar waveform autocorrelation function (which causes the estimator performance to deviate from the CRLB) is small. Radar waveforms with more energy at frequencies closer to center of the bandwidth allocation can have such autocorrelation functions. However, such a radar waveform has a smaller RMS bandwidth which degrades the overall estimation performance as seen in Equation (11). Furthermore, as we observe from Equation (4), radar waveforms with more spectral energy at the bandwidth center will reduce the noise spectral density, $N_{\text{int+n}}(f)$, due to minimal radar residual values ($N_{\text{resi}}(f)$), thereby maximizing the data rate.

Conversely, selecting a larger value for the threshold point implies there is more ambiguity in the radar waveform autocorrelation function (higher side-lobes), which occurs for radar waveforms with more energy at frequencies closer to the edges of the bandwidth allocation. Such waveforms also have larger RMS bandwidth values and a better estimation performance. Finally, radar waveforms with more spectral energy at the bandwidth edges have larger $N_{\mathrm{resi}}(f)$ values and consequently, larger $N_{\mathrm{int+n}}(f)$ values which degrade the communications data rate.

Thus, we see that selecting a low radar SNR threshold point increases the communications performance and decreases the radar performance but also results in a radar waveform with low side-lobes in the autocorrelation. Similarly, selecting a high threshold point increases the radar performance and decreases the communications performance but also results in a radar waveform with large autocorrelation side-lobes. The objective is to select a threshold point that optimizes the spectral shape of the radar waveform such that the performance with respect to radar and communications is jointly maximized.

The results from the numerical study of the above optimization problem are discussed in Section V.

V. Simulation Results

In this section, we present an example of the waveform design technique discussed in this paper, the minimum estimation error variance method, for an example parameter set. The parameters used in the example are shown in Table II. Additionally, a performance comparison of the minimum estimation error variance method with the previously derived spectral mask shaping method is also provided. We also study the effect of the order of the non-linear chirp phase on joint radar-communications performance. In order to better solve the non-convex optimization problem described in Section IV-B, all the results presented below were obtained by solving the optimization problem in Equation (21) using *fmincon* [45] for 100 Monte-Carlo runs with randomized initial solutions and selecting the solution with the highest objective value.

TABLE II
Parameters for Waveform Design Methods

Parameter	Value
Bandwidth (B)	5 MHz
Center Frequency	3 GHz
Effective Temperature (T_{temp})	1000 K
Communications Range	10 km
Communications Power (P_{com})	0.3 W
Communications Antenna Gain	0 dBi
Communications Receiver Side-lobe Gain	10 dBi
Radar Target Range	200 km
Radar Antenna Gain	30 dBi
Target Cross Section	10 m^2
Target Process Standard Deviation ($\sigma_{\tau,\mathrm{proc}}$)	100 m
Time-Bandwidth Product (TB)	128
Radar Duty Factor (δ)	0.01
Threshold Point Constraint ($\delta_{\mathrm{constraint}}$)	$1 + 0.01$

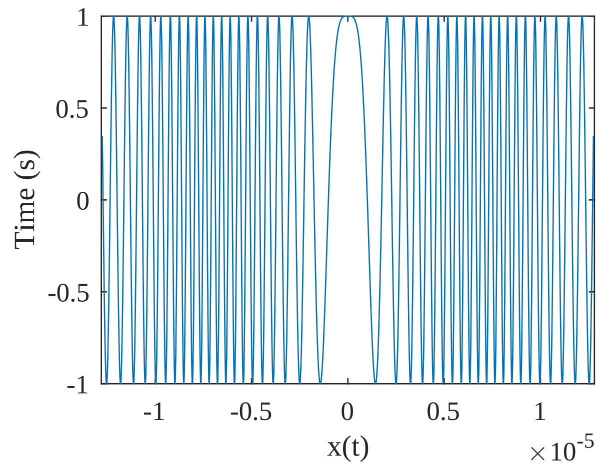

Fig. 6. Real valued amplitude of waveform vs. time (s). We see that the radar waveform has a chip shape similar to a non-linear chirp in the time domain. The constant modulus nature of the radar waveform is also clearly evident.

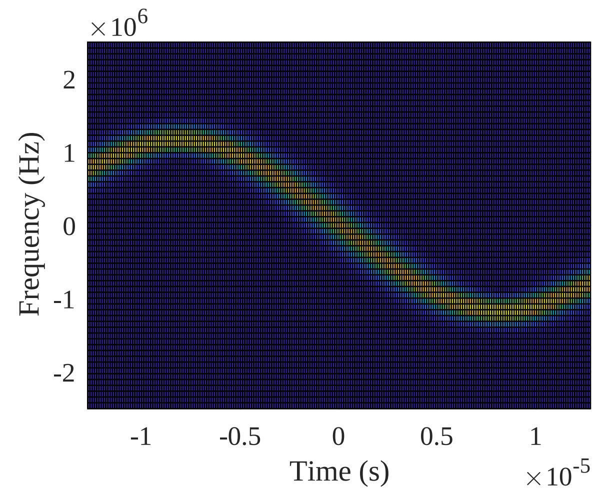

Fig. 7. Short-time Fourier transform spectrogram vs. time (s) and frequency (Hz). We observe that the optimal radar waveform has a non-linear time-frequency representation and is similar to a sum of polynomials.

A. Minimum Estimation Error Variance Method Optimal Waveform Shape

Here we present an example of a joint radar-communications optimal radar waveform designed by the minimum estimation error variance method. Figures 6 and 7 show time domain and time-frequency representations of the non-linear chirp waveform with polynomial phase shown in Equation (13) for the number of polynomial phase coefficients, $N = 6$ and a threshold SNR value of 0dB. Figure 6 shows the real valued amplitude of the waveform as a function of time. We see that the radar waveform has a chip shape similar to a non-linear chirp in the time domain. The constant modulus nature of the radar waveform is also apparent from the figure. Figure 7 shows the short-time Fourier transform spectrogram as a function of time and frequency. From this figure, we observe that the optimal radar waveform has a non-linear time-frequency representation and is similar to a sum of polynomials.

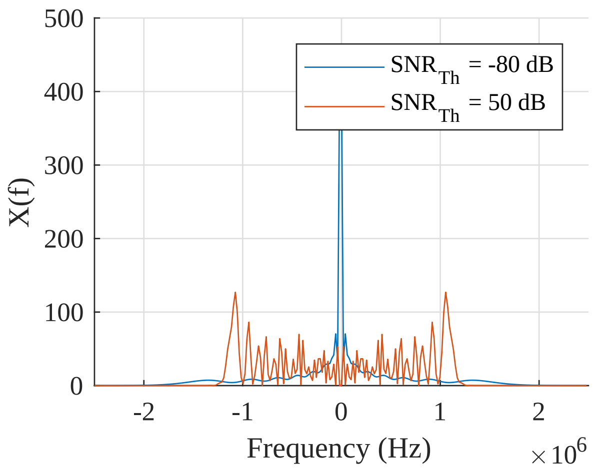

Fig. 8. The minimum estimation error variance optimized radar waveform spectrum for different threshold SNR values. The radar waveform optimization was done for $N = 6$. We see the optimal radar spectrum has more spectral energy at the edges of the bandwidth for high threshold SNR values and has more spectral energy closer to the center for low threshold SNR values.

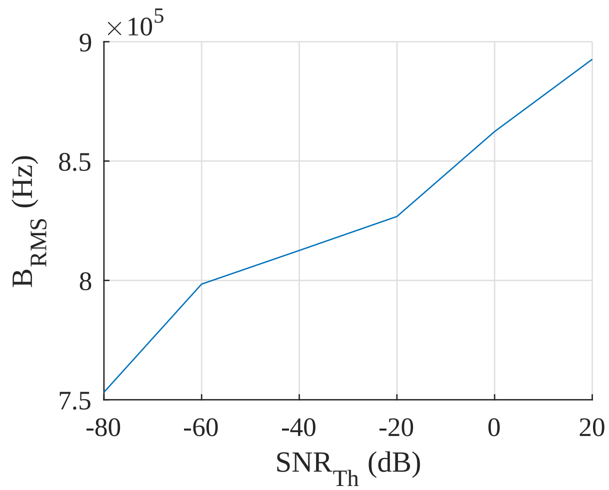

Fig. 9. RMS bandwidth of the optimized radar waveform vs. SNR. As expected, the optimal RMS bandwidth increases as we increase the threshold SNR. From Equation (11), we see that the optimal RMS bandwidth increasing as the threshold SNR increases will thereby reduce the CRLB. As a result, we see that the estimation performance increases with SNR.

B. Impact of Threshold SNR

We now discuss the numerical results from implementing the minimum estimation error variance method in Section IV-B. First, we highlight the impact of the threshold SNR value on the shape of the radar spectrum. We consider two threshold SNR values of -80dB and 50dB and we choose $N = 6$ in Equation (13), i.e., $x(t) = e^{i\pi(\sum_{m=1}^{6} p_m t^{2m})}$. The minimum estimation error variance optimized radar waveform spectrum for this set of parameters is shown in Figure 8. From Figure 8, we see the optimal radar spectrum has more spectral energy at the edges of the bandwidth for high threshold SNR values and has more spectral energy closer to the center for low threshold SNR values, as we stated in Section IV-C.

We also study the impact of the threshold SNR (or radar SNR) on the system performance. For the purpose of this study, we choose $N = 6$. For different values of SNR, we optimize the shape of the waveform, i.e., optimize the coefficients $\bar{p} = (p_1, \ldots, p_6)$, to minimize the CRLB achieved with the waveform. We also impose the constraint $\sigma^2_{\text{est}}/\sigma^2_{\text{CRLB}} \leq \delta_{\text{constraint}}$, which ensures that for the given SNR, our feasible solution set include only waveforms whose threshold SNR is less than or equal to the given SNR (as discussed in Section IV-B). $\delta_{\text{constraint}}$ is tuned so that the ratio between the estimation error variance (which characterizes estimation performance in this paper) and the CRLB remains close to 1. For this simulation, we consider a $\delta_{\text{constraint}}$ value of $1 + \epsilon$, where ϵ introduces some flexibility to the constraint and typically has a value of 0.01.

Figure 9 shows the RMS bandwidth values achieved with each optimized waveform for various values of threshold SNR. As expected, the optimal RMS bandwidth increases as we increase the threshold SNR. From Equation (11) and Section IV-C, we see that the optimal RMS bandwidth increasing as the threshold SNR increases will thereby reduce the CRLB as stated in Section IV-C.

Figure 10 shows the autocorrelation function achieved with each optimized waveform for various values of threshold SNR.

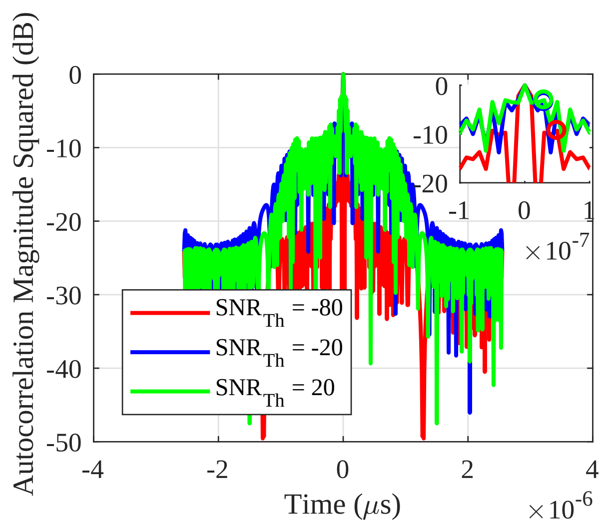

Fig. 10. Autocorrelation function of the optimized radar waveform vs. SNR. As expected, the peak side-lobe of the autocorrelation function increases as we increase the threshold SNR. This trend is observed because a higher threshold SNR implies the optimal waveform has more ambiguity which translates into higher peak autocorrelation side-lobes.

For SNR values -80dB, -20dB and 20dB, we observed that the peak side-lobes in all three cases occur at $\pm 0.2 - 0.5\mu s$ and have values of -10dB, -7dB, and -6dB respectively. As expected, the peak side-lobe of the autocorrelation function increases as we increase the threshold SNR. As mentioned in Section IV-C, a higher threshold SNR implies the optimal waveform has more ambiguity which translates into higher peak autocorrelation side-lobes.

Now, for each threshold SNR value we considered and the optimal waveform shape parameters $p_1, p_2, p_3, p_4, p_5, p_6$ we obtained above, we evaluate the radar estimation rate bound in Equation (8) and the spectral water-filling SIC data rate in Equation (7) corresponding to each of these waveforms.

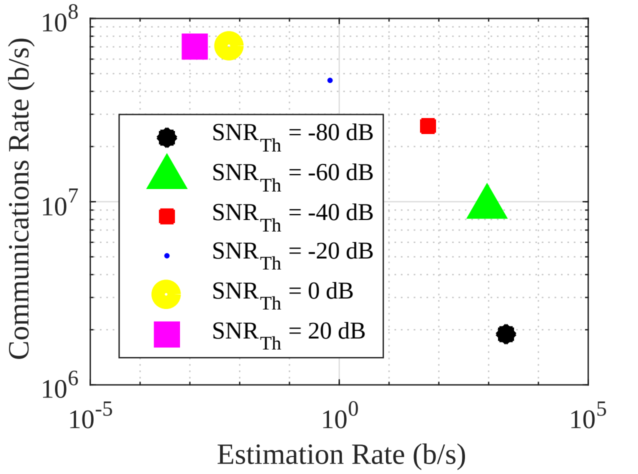

Fig. 11. Estimation and data rates vs. threshold SNR. Clearly, we see the performance of the system improve with respect to the estimation rate and degrade with respect to the spectral water-filling SIC data rate as we increase the threshold SNR.

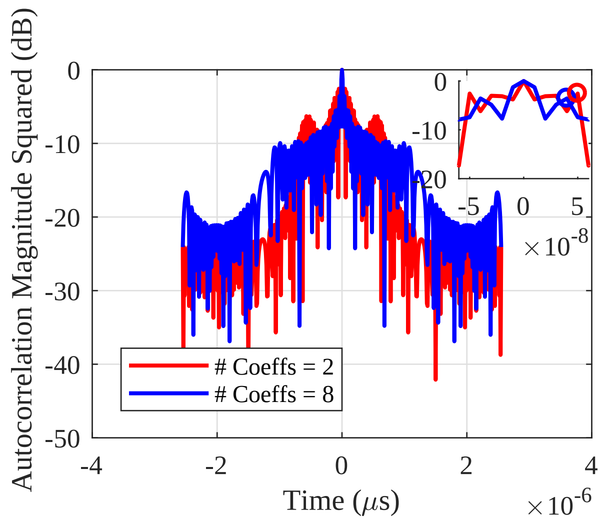

Fig. 12. Autocorrelation function of the optimized radar waveform vs. N. We clearly see that the autocorrelation peak side-lobes decrease as N increases, which causes the estimation performance to improve overall as N increases.

Figure 11 shows the plot of estimation rate and the data rate against the threshold SNR value. Clearly, according to the figure, the performance of the system improves with respect to the estimation rate as we increase the threshold SNR, which is expected as the minimum achievable CRLB decreases with threshold SNR, and the estimation rate increases with decreasing CRLB according to Equation (8) and Equation (9). However, we observe that the spectral water-filling SIC data rate reduces as the threshold SNR increases. This trend occurs because, as we stated in Section IV-C, as the threshold SNR increases, the noise spectral density, $N_{\mathrm{int+n}}(f)$ achieves higher values due to larger radar residual values, which reduces the spectral water-filling SIC data rate.

C. Impact of Order of Chirp Phase

We first investigate the relationship between the autocorrelation peak side-lobe levels and the order of the non-linear chirp waveform's phase, N. Figure 12 shows the autocorrelation function for $N = 2$ and $N = 8$ at a threshold SNR value of 0dB. We clearly see that the autocorrelation peak side-lobes decrease as N increases, which causes the estimation performance to improve overall as N increases.

As the shape of the waveform explicitly depends on the coefficients $p_1, \ldots, p_N$ in Equation (21), we now study the effect of the number of coefficients, N, on both the estimation and the data rates. For this study, we choose a threshold SNR value of 0dB and vary N from 1 to 8. For each N and threshold SNR value, we solve Equation (21) and evaluate the estimation rate from Equation (8) and spectral water-filling SIC data rate from Equation (7). Figure 13 shows plots of these rates against N. From Figure 13, we see that as N increases, the estimation rate increases and the spectral water-filling SIC data rate decreases. The improvement in estimation rates as N increases is because there are more degrees of freedom available to shape the optimal radar waveform spectrum obtained by solving the optimization problem in Equation (21). As a result of these increased degrees of freedom, optimal radar waveforms are obtained that have better or lower autocorrelation side-lobe levels (a trend we observed earlier). Additionally, increasing N means increasing the amount of energy at higher frequencies for the radar waveform spectrum, which results in the B_{rms} value increasing, there by improving local estimation performance given by Equation (11). This increase in local estimation performance, coupled with lower autocorrelation side-lobe levels, results in an overall increase in the estimation rate as N increases. Furthermore, the increase in the radar waveform's spectral content at higher frequencies, due to an increase in N, means that the noise spectral density, $N_{\mathrm{int+n}}(f)$, achieves higher values due to larger radar residual values, which reduces the spectral water-filling SIC data rate.

D. Performance Comparison of Waveform Design Algorithms

Next, we compare the performance of the minimum estimation error variance method against the spectral mask shaping method in [6]. We conduct a Monte Carlo study with 50 runs to compare the performance of these methods. For this study, we choose the SNR of 7.6dB, and set $N = 6$. In each Monte Carlo run, we evaluate the estimation rates and the communications rates (spectral water-filling SIC data rate for the minimum estimation error variance method and the SIC data rate for the spectral mask shaping method) from the two methods. Table III shows the average of estimation rates ($\mathrm{R_{est}}$) and communications rates ($\mathrm{R_{com}}$) from the Monte-Carlo study.

From Table III, we clearly observe that the minimum estimation error variance method outperforms the spectral-mask shaping method in terms of estimation rate and communications rate. Furthermore, a significant increase in the achieved communications rate highlights the impact of the continuous water-filling algorithm. Thus, we see the explicit advantage of the proposed method over the method in [6] in that the minimum estimation error variance method proposed in this paper designs radar waveforms that are constant modulus and

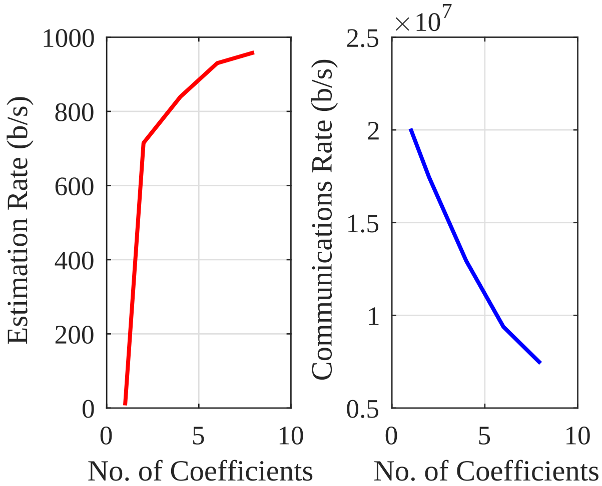

Fig. 13. Estimation and data rates vs. N. We see that as N increases, the estimation rate increases and the spectral water-filling SIC data rate decreases. The improvement in estimation rates as N increases is because there are more degrees of freedom available to shape the optimal radar waveform spectrum, which results in optimal waveforms that have better or lower autocorrelation side-lobe levels. Additionally, increasing N means increasing the amount of energy at higher frequencies for the radar waveform spectrum, which improves local estimation performance. Furthermore, the increase in the radar waveform's spectral content at higher frequencies, due to an increase in N, means that the noise spectral density, $N_{\text{int+n}}(f)$, achieves higher values due to larger radar residual values, which reduces the spectral water-filling SIC data rate.

ensures better estimation performance and better communications performance over the spectral shaping method.

TABLE III
MINIMUM ESTIMATION ERROR VS. SPECTRAL-MASK SHAPING FOR SNR = 7.6DB

Performance metric (average values)	**Min. Est. Error**	**Spectral-Mask Shaping**
R_{est} (b/s)	1.38×10^3	3.05×10^3
R_{com} (b/s)	6.92×10^6	1.38×10^4

VI. CONCLUSIONS

In this paper, we presented a novel radar waveform design technique that maximizes the performance of a spectrum sharing, joint radar-communications system. The global estimation rate, an extension on the estimation rate that takes into account non-local or global estimation errors, and the data rate are used to measure radar and communications performance respectively. We developed the novel minimum estimation error variance radar waveform design method that selects the phase parameters of a nonlinear chirp radar waveform to maximize radar performance. We also developed the spectral water-filling SIC data rate which is the maximized communications data rate for a joint radar-communications receiver employing SIC. This maximized data rate was obtained by employing the continuous spectral water-filling algorithm which determines the optimal communications power spectral distribution for a given noise spectral density. We presented examples of the minimum estimation error variance radar waveform design method for an example parameter set and also compared the method's performance against the performance of the previously derived spectral mask shaping method. We saw that the minimum estimation error variance method is able to achieve higher estimation and communications data rate values than the spectral mask shaping method. We also observed that the optimal estimation rate increases for higher radar SNR values whereas the optimal spectral-water-filling SIC data rate decreases for higher radar SNR values. Finally, we observed that the estimation rate increases and the spectral water-filling SIC data rate decreases as the number of coefficients in the phase of the non-linear chirp increases.

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