

Optimal cutting planes from the group relaxations

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Abstract

We study quantitative criteria for evaluating the strength of valid inequalities for Gomory and Johnson’s finite and infinite group models and we describe the valid inequalities that are optimal for these criteria. We justify and focus on the criterion of maximizing the volume of the nonnegative orthant cut off by a valid inequality.

For the finite group model of prime order, we show that the unique maximizer is an automorphism of the *Gomory Mixed-Integer (GMI) cut* for a possibly *different* finite group problem of the same order.

We extend the notion of volume of a simplex to the infinite dimensional case. This is used to show that in the infinite group model, the GMI cut maximizes the volume of the nonnegative orthant cut off by an inequality.

1 Introduction

Cutting planes are important tools to solve integer programming (IP) models. While this technology has seen several revivals and intense research activity since their introduction by Gomory [9–12], some basic aspects are not well understood. In particular, deciding *which* cutting plane family will be most effective in a particular IP instance has been a thorny problem for the community. Some families, like the *Gomory Mixed-Integer (GMI) cuts*, have been enormously useful across all kinds of IP instances [2, 3, 7, 8]. But such empirical observations have never been explained rigorously, to the best of our knowledge.

This problem, known as *cut selection*, has become even more confounded by the advent of the so-called *cut generating functions* which is a modern perspective on the theories of Gomory and Johnson [14, 15], and Balas [1] from the 1970s. This is because recent work in this area has opened the doors to infinitely many distinct families of cutting planes that are computationally accessible. In a sense, this is great news because we now have not only many more choices, but potentially more powerful cutting planes than used before. But this makes the problem of cut selection even more difficult than what it was.

The goal of this paper is to make progress towards establishing rigorous criteria for evaluating the efficacy of cutting planes, and understanding the structure of the optimal cutting

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planes under these criteria. For this purpose, we restrict ourselves to the family of cutting planes studied under the various group relaxations. The group relaxations were first introduced by Gomory [13], and then generalized and investigated by Gomory and Johnson [14–17], as a means to provide a unifying framework for deriving cutting planes for mixed-integer optimization problems.

Our investigations seem to provide analytical evidence supporting the computational success of the Gomory Mixed-Integer (GMI) cuts (see Theorems 2.2 and 2.3).

Our presentation uses notions and terminology that may be unfamiliar in the integer programming community. However, our approach has a three-fold motivation: 1) this new language enables us to state results in a unified manner and borrow necessary mathematical machinery from the areas of pure algebra and analysis; 2) the abstraction allows us to present the proofs in a cleaner and more elegant fashion; 3) the added generalization could be useful in the future to build further bridges between algebra, analysis and mixed-integer optimization. We next review this material and then introduce Gomory’s finite and infinite group models.

Group theoretic preliminaries. We recall that a *group* is a set G endowed with a binary operation mapping $G \times G$ to G , denoted by $+$: $G \times G \rightarrow G$, which satisfies three properties: 1) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in G$ (associativity of $+$), 2) there exists an element $\mathbf{0} \in G$ such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in G$ (existence of identity element), and 3) for every $\mathbf{x} \in G$, there exists an inverse $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$. A group is said to be *abelian* if $+$ is commutative, i.e., $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in G$. All groups considered in this paper will be abelian, so we drop this qualification in the remainder. We define $\mathbf{x} - \mathbf{y} := \mathbf{x} + (-\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in G$. Moreover, we will use the notation $k\mathbf{x}$ to denote \mathbf{x} added to itself k times, for any $k \in \mathbb{N}$ and $\mathbf{x} \in G$.

A group G is said to be a *topological group* if the set G is also endowed with a topology such that the maps $+$: $G \times G \rightarrow G$ and $\text{inv} : G \rightarrow G$, $\mathbf{x} \mapsto -\mathbf{x}$, are continuous with respect to this topology, where $G \times G$ is endowed with the product topology. A topological group is said to be *compact* if G is a compact space under its given topology. We will assume that all topological spaces in this paper are Hausdorff, i.e., for any two distinct points there exist disjoint neighborhoods containing these two points.¹

For any compact, topological group G , there exists a unique measure μ defined on the Borel sets of G such that $\mu(G) = 1$ and $\mu(\mathbf{x} + A) = \mu(A)$ for every Borel subset $A \subseteq G$ (where $\mathbf{x} + A := \{\mathbf{x} + \mathbf{a} : \mathbf{a} \in A\}$); see [21, Chapter 22]. This measure is called the *Haar probability measure*, or *Haar measure* for short.

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ will denote the n -dimensional torus. We will use bold-faced letters like $\mathbf{x}, \mathbf{y}, \mathbf{b}$ etc., to denote elements of a generic group. When the group is a “one-dimensional” group like \mathbb{T}^1 or a finite cyclic group, we will not use the bold font for elements in the group, i.e., refer to elements using x, y, b etc.

Gomory’s group relaxation. Consider a group G and a fixed $\mathbf{b} \in G \setminus \{\mathbf{0}\}$. Gomory’s group relaxation is defined as the set $I_{\mathbf{b}}(G)$ of *finite support* functions $y : G \rightarrow \mathbb{Z}_+$ (i.e., y

¹For a reference on these terminologies, see [21].

takes value 0 on all but a finite subset of G) such that

$$\sum_{\mathbf{x} \in G} y(\mathbf{x}) \mathbf{x} = \mathbf{b}.$$

The usual cases are with $G = \mathbb{T}^n$ (the n -dimensional torus), where it is called the *n-dimensional infinite group relaxation*, and $G = \mathbb{Z}^n / \Lambda$ where Λ is a sub-lattice of \mathbb{Z}^n , where it is called an *n-dimensional finite group relaxation*. These two cases and their use in deriving cutting planes for integer programming have been studied extensively; we refer the reader to the surveys [4–6].

Note that, as $\mathbf{b} \neq \mathbf{0}$, the function y_0 that takes value 0 on all the elements of G is not in $I_{\mathbf{b}}(G)$. In the context of integer programming, the function $y_0 = 0$ corresponds to the optimal solution of the continuous relaxation of the problem. One is interested in finding a halfspace in the space of finite support functions $y : G \rightarrow \mathbb{R}$ that contains $I_{\mathbf{b}}(G)$ but not y_0 . Any such halfspace is described by an inequality of the form $\sum_{\mathbf{x} \in G} \pi(\mathbf{x}) y(\mathbf{x}) \geq 1$, where the function $\pi : G \rightarrow \mathbb{R}$ (which is not necessarily finite support) gives the coefficients of the inequality. We use the notation H_{π} to denote this halfspace, and call π a *valid function* whenever $I_{\mathbf{b}}(G) \subseteq H_{\pi}$.

Most of the literature has focused on the family of valid functions $\pi \geq 0$ called *minimal functions*. A valid function $\pi : G \rightarrow \mathbb{R}_+$ is *minimal* if every valid function $\tilde{\pi} : G \rightarrow \mathbb{R}_+$ such that $\tilde{\pi} \leq \pi$ satisfies $\tilde{\pi} = \pi$. We denote by $\mathcal{M}_{\mathbf{b}}(G)$ the set of minimal functions. The reason for restricting to minimal functions is that if $\pi : G \rightarrow \mathbb{R}_+$ is a valid function, then there always exists a minimal $\pi' \in \mathcal{M}_{\mathbf{b}}(G)$ such that $I_{\mathbf{b}}(G) \subseteq H_{\pi'} \subseteq H_{\pi}$ (thus π' “dominates” π). For any group, minimal functions are characterized by the following theorem [15].

THEOREM 1.1. *A function $\pi : G \rightarrow \mathbb{R}_+$ is a minimal function if and only if it is subadditive (i.e., $\pi(\mathbf{x}) + \pi(\mathbf{y}) \geq \pi(\mathbf{x} + \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in G$), $\pi(\mathbf{0}) = 0$, and $\pi(\mathbf{x}) + \pi(\mathbf{b} - \mathbf{x}) = 1$ for every $\mathbf{x} \in G$ (this property is known as the symmetry property).*

A valid function $\pi : G \rightarrow \mathbb{R}_+$ is *extreme* if $\pi_1 = \pi_2$ for every pair of valid functions $\pi_1, \pi_2 : G \rightarrow \mathbb{R}_+$ such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. It is well known that the set of extreme functions is a subset of $\mathcal{M}_{\mathbf{b}}(G)$. In fact, if G is a finite group, the set $\mathcal{M}_{\mathbf{b}}(G)$ is a polytope defined as

$$\left\{ \pi \in \mathbb{R}_+^G : \begin{array}{ll} \pi(\mathbf{0}) & = 0 \\ \pi(\mathbf{x}) + \pi(\mathbf{y}) & \geq \pi(\mathbf{x} + \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in G \\ \pi(\mathbf{x}) + \pi(\mathbf{b} - \mathbf{x}) & = 1 \quad \forall \mathbf{x} \in G \end{array} \right\}, \quad (1.1)$$

and the extreme functions are precisely the vertices of this polytope.

EXAMPLE 1.2. *We give some well-known examples of minimal functions.*

1. Let $G = \mathbb{T}^n$ and $\mathbf{b} \in \mathbb{T}^n \setminus \{0\}$, and let $i \in \{1, \dots, n\}$ be any index such that $\mathbf{b}_i \neq 0$. The Gomory function $\text{GMI}_{\mathbf{b}_i}^n : \mathbb{T}^n \rightarrow \mathbb{R}_+$ is defined as $\text{GMI}_{\mathbf{b}_i}^n(\mathbf{x}) := \frac{\mathbf{x}_i}{\mathbf{b}_i}$ if $\mathbf{x}_i \leq \mathbf{b}_i$ and $\text{GMI}_{\mathbf{b}_i}^n(\mathbf{x}) := \frac{1 - \mathbf{x}_i}{1 - \mathbf{b}_i}$ otherwise, where on the right hand sides the coordinates $\mathbf{x}_i, \mathbf{b}_i$ are interpreted as real numbers in the interval $[0, 1)$. It is well-known that $\text{GMI}_{\mathbf{b}_i}^n$ is an extreme function. This function is used to derive the well-known GMI cut in integer programming literature.

2. For $G = \mathbb{Z}/q\mathbb{Z}$ and $b \in G \setminus \{0\}$, the function $\text{GOM}_b^q : G \rightarrow \mathbb{R}_+$ defined as $\text{GOM}_b^q(x) := \frac{x}{b}$ when $x \leq b$ and $\text{GOM}_b^q(x) := \frac{1-x}{1-b}$ when $x > b$ is extreme.
3. Let $G = \mathbb{Z}/q\mathbb{Z}$ where q is prime. Let $b_1, b_2 \in G \setminus \{0\}$ and let π be a valid function for $I_{b_1}(G)$. Consider the automorphism $\phi : G \rightarrow G$ that sends b_2 to b_1 (which exists and is unique because q is prime). Then $\pi \circ \phi$ is a valid function for $I_{b_2}(G)$. Furthermore π is minimal (extreme) if and only if $\pi \circ \phi$ is minimal (extreme).

2 Criteria to evaluate the strength of a valid function

2.1 The distance criterion

Consider the case when G is a finite group. Given a valid function π , the halfspace $H_\pi := \{y \in \mathbb{R}^G : \sum_{\mathbf{x} \in G} \pi(\mathbf{x})y(\mathbf{x}) \geq 1\}$ cuts off a portion of the nonnegative orthant \mathbb{R}_+^G that includes the origin. A possible measure one can choose to evaluate to strength of π is the distance of the origin from the hyperplane $\{y \in \mathbb{R}^G : \sum_{\mathbf{x} \in G} \pi(\mathbf{x})y(\mathbf{x}) = 1\}$: the larger this distance, the better π is. It can be shown that this distance is in fact given by $\frac{1}{|\pi|_2}$, where $|\pi|_2 := \sqrt{\sum_{\mathbf{x} \in G} \pi(\mathbf{x})^2}$ is the standard ℓ_2 norm of the vector of coefficients given by π . Thus, one looks for π that minimizes the ℓ_2 norm. Generalizing, one could also look for a function π that minimizes the ℓ_p norm for $p \geq 1$.

If G is a group of infinite order, it is important to have a measure on the group G , with respect to which one can integrate so that expressions involving sums in the finite group case can be replaced by integrals. This is why we will consider compact, topological groups with the Haar measure. Since we would like to integrate functions defined in a topological group, we will have to restrict our attention to measurable functions (with respect to the Haar measure). Without further comment, we will assume that all functions π that we consider are measurable; in particular, $\mathcal{M}_{\mathbf{b}}(G)$ is to be thought of as the subset of *measurable* minimal functions from this point on. (For groups of finite order this is no restriction because all functions are measurable.) Under this assumption, for every $p \geq 1$ the ℓ_p norm can be generalized to the standard L_p norm $|\pi|_p := (\int |\pi|^p d\mu)^{1/p}$, where μ is the corresponding Haar measure.

The following result shows that the above optimization problem, i.e., maximizing the distance of the origin from the hyperplane defined by π , has a trivial solution: the L_p norms are minimized by the function which takes value $1/2$ almost everywhere. The result holds for any group – finite or infinite. Note that in the following statement we restrict to functions $\pi \in \mathcal{M}_{G, \mathbf{b}}$, as it is easy to verify that every optimal solution to this problem is a minimal function.

THEOREM 2.1. *Let G be any (finite or infinite) group and $\mathbf{b} \in G \setminus \{0\}$. Let $p \geq 1$. Then $\inf\{|\pi|_p : \pi \in \mathcal{M}_{\mathbf{b}}(G)\} = 1/2$ and is attained by the function which takes value $1/2$ everywhere except for $\mathbf{0}$ and \mathbf{b} , where it takes value 0 and 1 respectively.*

Proof. The proof of this theorem follows directly from the symmetry of π (see Theorem 1.1). Indeed, since π is symmetric, $D := \pi^{-1}([0, 1/2))$ and $D' := \pi^{-1}((1/2, 1])$ have equal measure. Moreover,

$$|\pi|_p^p = \int_D |\pi|^p d\mu + \int_{D'} |\pi|^p d\mu + \int_{\pi^{-1}(\{1/2\})} (1/2)^p d\mu.$$

Since $\mu(D) = \mu(D')$ and π is symmetric, we obtain that

$$\begin{aligned} |\pi|_p^p &= \int_D |\pi|^p d\mu + \int_D |1 - \pi|^p d\mu + \int_{\pi^{-1}(\{1/2\})} (1/2)^p d\mu \\ &= \int_D (|\pi|^p + |1 - \pi|^p) d\mu + \int_{\pi^{-1}(\{1/2\})} (1/2)^p d\mu. \end{aligned}$$

For any real number $0 \leq a \leq 1$, we have that $a^p + (1 - a)^p \geq 2(1/2)^p$ by convexity of the function $x \mapsto x^p$. Since π is bounded between 0 and 1 by Theorem 1.1, we obtain $|\pi|_p \geq 1/2$. \square

It is interesting to observe that the optimal function is not an extreme function. This may be surprising, as extreme functions are usually interpreted as the “best” valid functions. Below we suggest another criterion to evaluate the strength of π , and we will see that it is optimized by an extreme function.

2.2 The volume criterion

Consider first the case when G is a finite cyclic group of prime order, i.e., $G = \mathbb{Z}/q\mathbb{Z}$ with q prime. Take $\pi \in \mathcal{M}_b(G)$. We can easily verify that $\pi(x) > 0$ for every $x \in G \setminus \{0\}$. To see this, assume by contradiction that $\pi(x) = 0$ for some $x \in G \setminus \{0\}$. Since q is prime, there exists $k \in \mathbb{N}$ such that $kx = b$. Since π is subadditive by Theorem 1.1, we have $\pi(b) = \pi(kx) \leq k\pi(x) = 0$, contradicting the fact that $\pi(b) = 1$ (again by Theorem 1.1). This shows that $\pi(x) > 0$ for every $x \in G \setminus \{0\}$, while $\pi(0) = 0$ by Theorem 1.1.

Denote by e^x , $x \in G$, the unit vectors in \mathbb{R}^G . The above observation implies that the halfspace H_π is parallel to e^0 and cuts off a simplex from the $(q - 1)$ -dimensional nonnegative orthant $\mathbb{R}_+^{G \setminus \{0\}}$. This simplex is given by $\text{conv} \left(\{0\} \cup \left\{ \frac{1}{\pi(x)} e^x \right\}_{x \in G \setminus \{0\}} \right)$. Consequently, its $(q - 1)$ -dimensional volume is given by $\frac{1}{(q-1)!} \prod_{x \in G \setminus \{0\}} \frac{1}{\pi(x)}$. We consider this volume as a measure of π : the higher this volume, the “better” the halfspace given by π is. Thus, one looks for $\pi \in \mathcal{M}_b(G)$ that minimizes $\prod_{x \in G \setminus \{0\}} \pi(x)$. Since valid functions cut off the function y defined by $y(x) = 0$ for all $x \in G$, and $I_b(G)$ is contained in the nonnegative orthant, this provides a justification of the volume criterion as a measure of the strength of π .

We remark that, unlike the distance criterion, the function $\pi \mapsto \prod_{x \in G \setminus \{0\}} \pi(x)$ is log-concave, and therefore it has a minimizer π that is a vertex of the polytope in (1.1), i.e., π is an extreme function.

The above definitions were made for a finite group G . We now show how to extend the volume measure defined above to infinite groups.

Let G be a compact topological group with Haar measure μ . Take $\pi \in \mathcal{M}_b(G)$. Let us initially assume that $\pi(\mathbf{x}) > 0$ for every $\mathbf{x} \in G \setminus \{0\}$. We denote by $\mathbb{R}^{(G)}$ the space of finite support functions $y : G \rightarrow \mathbb{R}$. Note that a basis of the vector space $\mathbb{R}^{(G)}$ is given by the family of functions $e^{\mathbf{x}}$, $\mathbf{x} \in G$, where $e^{\mathbf{x}}$ is the function which takes value 1 on \mathbf{x} and 0 elsewhere. Similar to the finite cyclic group case, the halfspace H_π is parallel to e^0 and cuts off a convex set (let us called it a “simplex”) from the set $\mathbb{R}_+^{(G)} \cap \{y : y(0) = 0\}$. This “simplex” is given by $\text{conv} \left(\{0\} \cup \left\{ \frac{1}{\pi(\mathbf{x})} e^{\mathbf{x}} \right\}_{\mathbf{x} \in G \setminus \{0\}} \right)$. However, we cannot compute the volume of this set, as it is

an infinite dimensional object. To overcome this difficulty, we observe that in the finite cyclic group case maximizing the volume of the simplex is equivalent to maximizing its average side length (geometric mean). Therefore, in the infinite group case, we look at the geometric mean of the sides of the “simplex”, i.e., the geometric mean of the function $1/\pi$, which is defined as

$$\exp\left(\frac{1}{\mu(G \setminus \{\mathbf{0}\})} \int_{G \setminus \{\mathbf{0}\}} \ln(1/\pi) d\mu\right) = \exp\left(\int_G \ln(1/\pi) d\mu\right).$$

(The equality holds because the Haar measure satisfies the properties $\mu(G) = 1$ and $\mu(\{\mathbf{x}\}) = \mu(\{\mathbf{y}\})$ for every $\mathbf{x}, \mathbf{y} \in G$, thus $\mu(\{\mathbf{x}\}) = 0$ for every $\mathbf{x} \in G$ because G is infinite; in particular, $\mu(\{\mathbf{0}\}) = 0$ and $\mu(G \setminus \{\mathbf{0}\}) = 1$.) Equivalently, we will minimize $\int_G \ln(\pi) d\mu$.

While we have motivated this formula for functions that are strictly positive everywhere, for minimal functions, this restriction is not necessary for the integral to make sense. Indeed, we will be concerned with integrals of functions of the form $\ln(\pi)$, where π is bounded between 0 and 1. This means that $-\ln(\pi)$ is a nonnegative function taking values in the extended reals, i.e., it could take the value $+\infty$ at some points where π equals 0. For nonnegative, extended real valued functions, integrals are always defined but may equal $+\infty$. Below, we will say a nonnegative extended real valued function is *integrable* if the integral is finite.

2.3 Our results

With the above setup, we state our main results. The first result is about the volume measure for infinite, connected groups, and shows that the average side length of the “simplex” cut off by an optimal inequality is e (the base of the natural logarithm). Moreover, in the usual case $G = \mathbb{T}^n$, the Gomory function $\text{GMI}_{\mathbf{b}_i}^n$ defined in Example 1.2 is an optimal inequality.

THEOREM 2.2. *Let G be any compact, connected topological group with Haar probability measure μ , and $\mathbf{b} \in G \setminus \{\mathbf{0}\}$. Then for every $\pi \in \mathcal{M}_{\mathbf{b}}(G)$, $-\ln(\pi)$ is integrable and*

$$\inf \left\{ \int_G \ln(\pi) d\mu : \pi \in \mathcal{M}_{\mathbf{b}}(G) \right\} = -1.$$

If $G = \mathbb{T}^n$, the infimum is attained by $\text{GMI}_{\mathbf{b}_i}^n$ for any index i such that $\mathbf{b}_i \neq 0$.

For finite cyclic groups of prime order q , we show that the maximum volume is cut off by appropriate automorphisms of the functions GOM_{q-1}^q and GOM_1^q (see part 3. of Example 1.2 for these functions).

THEOREM 2.3. *Let $G = \mathbb{Z}/q\mathbb{Z}$ where q is prime, and let $b \in G \setminus \{0\}$. Then*

$$\inf \left\{ \prod_{x \in G \setminus \{0\}} \pi(x) : \pi \in \mathcal{M}_b(G) \right\} = \frac{(q-1)!}{(q-1)^{q-1}}.$$

Moreover, this infimum is attained by the function $\text{GOM}_{q-1}^q \circ \phi$, where $\phi : G \rightarrow G$ is the automorphism that sends b to $q-1$.

The proofs of Theorems 2.2 and 2.3 form the content of the rest of the paper.

3 Infinite, connected groups: Proof of Theorem 2.2

Theorem 2.2 is proved at the end of this section. The main tool behind this theorem is a rearrangement idea which preserves the properties of subadditivity, symmetry and the values of integrals, and makes the “rearranged” function nondecreasing. The contents of this idea is summarized in Theorem 3.11. One then shows that within the family of nondecreasing, subadditive and symmetric functions, a minimizer of the integral of the logarithm is the identity function; this is shown in Theorem 3.13. Putting these two theorems together gives us Theorem 2.2 (see the final paragraph of this section).

We now begin with the appropriate definitions and results needed to make the rearrangement idea concrete. In the following, given $x \in \mathbb{T}^1$, when we say “ x viewed as a real number” we refer to the unique representative of x in the interval $[0, 1)$.

DEFINITION 3.1. *Let G be any compact, connected topological group with Haar probability measure μ on it. For any nonnegative function $\pi : G \rightarrow \mathbb{R}_+$, define $\hat{\pi} : \mathbb{T}^1 \rightarrow \mathbb{R}_+$ as follows:*

$$\hat{\pi}(x) := \inf\{\alpha \geq 0 : \mu(\pi^{-1}([0, \alpha])) \geq x\} \quad \forall x \in \mathbb{T}^1, \quad (3.1)$$

where the right hand side of the inequality $\mu(\pi^{-1}([0, \alpha])) \geq x$ is viewed as a real number.

REMARK 3.2. *We note that for any $x \in [0, 1)$, the set $\{\alpha \geq 0 : \mu(\pi^{-1}([0, \alpha])) \geq x\}$ is nonempty, and so the infimum in (3.1) is a well-defined real number. Indeed, since π is nonnegative, we have $\bigcup_{n \in \mathbb{N}} \pi^{-1}([0, n]) = \pi^{-1}([0, \infty)) = G$. Then, by continuity of measure, $\lim_{n \rightarrow \infty} \mu(\pi^{-1}([0, n])) = \mu(G) = 1$. Thus, for any $x \in [0, 1)$, there must exist some $n \in \mathbb{N}$ such that $\mu(\pi^{-1}([0, n])) \geq x$.*

LEMMA 3.3. *The following are all true.*

1. *The function $\alpha \mapsto \mu(\pi^{-1}([0, \alpha]))$ is nondecreasing and right continuous.*
2. *$\hat{\pi}$ is nondecreasing, i.e., $x \leq y$ implies $\hat{\pi}(x) \leq \hat{\pi}(y)$, where $x, y \in \mathbb{T}^1$ are viewed as real numbers with the standard ordering.*
3. *Let $\bar{x} \in \mathbb{T}^1$ and let $\bar{\alpha} = \hat{\pi}(\bar{x})$. Then $\mu(\pi^{-1}([0, \bar{\alpha}])) = \bar{x} \geq \mu(\pi^{-1}([0, \bar{\alpha})))$, where \bar{x} is viewed as a real number.*
4. *$\hat{\pi}$ is left continuous, i.e., for all $x \in \mathbb{T}^1 \setminus \{0\}$, $\lim_{\varepsilon \rightarrow 0^+} \hat{\pi}(x - \varepsilon) = \hat{\pi}(x)$.*

Proof. For the first property, we observe that $[0, \alpha] = \bigcap_{\varepsilon > 0} [0, \alpha + \varepsilon]$, and therefore $\pi^{-1}([0, \alpha]) = \bigcap_{\varepsilon > 0} \pi^{-1}([0, \alpha + \varepsilon])$. By continuity of measure, $\lim_{\varepsilon \rightarrow 0^+} \mu(\pi^{-1}([0, \alpha + \varepsilon])) = \mu(\pi^{-1}([0, \alpha]))$, establishing property 1. The fact that the function is nondecreasing is clear from its definition.

The second property is clear from the definition of $\hat{\pi}$.

For the third property, since $\mu(\pi^{-1}([0, \bar{\alpha}])) \geq \mu(\pi^{-1}([0, \bar{\alpha})))$, it suffices to show that $\mu(\pi^{-1}([0, \bar{\alpha}])) = \bar{x}$. By property 1, the function $\alpha \mapsto \mu(\pi^{-1}([0, \alpha]))$ is right continuous, and therefore, the infimum in the definition $\hat{\pi}(\bar{x}) = \inf\{\alpha \geq 0 : \mu(\pi^{-1}([0, \alpha])) \geq \bar{x}\}$ is attained at $\alpha = \bar{\alpha}$. Thus $\mu(\pi^{-1}([0, \bar{\alpha}])) = \bar{x}$.

For the fourth property, consider $x \in \mathbb{T}^1 \setminus \{0\}$. For every $0 < \varepsilon < x$, define $t_\varepsilon := \hat{\pi}(x - \varepsilon)$. Since $\hat{\pi}$ is nondecreasing (property 2 above), the function $\varepsilon \mapsto t_\varepsilon$ is nonincreasing and $t_\varepsilon \leq t_0$ for all $0 < \varepsilon < x$. Therefore (see, e.g., [22, Theorem 4.29]), the limit $\bar{t} := \lim_{\varepsilon \rightarrow 0^+} t_\varepsilon$ exists

and $\bar{t} \leq t_0$. We need to show that $\bar{t} \geq t_0$. It suffices to prove that $\mu(\pi^{-1}([0, \bar{t}])) \geq x$, since $t_0 = \hat{\pi}(x) = \inf\{\alpha \geq 0 : \mu(\pi^{-1}([0, \alpha])) \geq x\}$. We observe that $[0, \bar{t}] = \bigcup_{0 < \varepsilon < x} [0, t_\varepsilon]$ since $\bar{t} = \lim_{\varepsilon \rightarrow 0^+} t_\varepsilon$. By continuity of measure, $\mu(\pi^{-1}([0, \bar{t}])) = \lim_{\varepsilon \rightarrow 0^+} \mu(\pi^{-1}([0, t_\varepsilon])) = \lim_{\varepsilon \rightarrow 0^+} (x - \varepsilon) = x$, where the second equation follows from property 3. Therefore, $\mu(\pi^{-1}([0, \bar{t}])) \geq \mu(\pi^{-1}([0, t])) = x$. \square

The next result makes use of Kemperman's theorem [18, Theorem 1.1], which, in our context, states that if A and B are nonempty subsets of G , then $\mu(A + B) \geq \min\{1, \mu(A) + \mu(B)\}$ (where $A + B := \{\mathbf{a} + \mathbf{b} : a \in A, b \in B\}$).

THEOREM 3.4. *Let G be any compact, connected topological group with Haar probability measure μ on it, and let $\pi : G \rightarrow \mathbb{R}_+$ be subadditive on G , with $\pi(0) = 0$. Then $\hat{\pi}$ is subadditive on \mathbb{T}^1 .*

Proof. Consider $x_1, x_2 \in \mathbb{T}^1$. Let $\hat{\pi}(x_i) = \alpha_i$, $i = 1, 2$. It suffices to show that $\mu(\pi^{-1}([0, \alpha_1 + \alpha_2])) \geq x_1 + x_2$, where the right hand side is a sum in \mathbb{T}^1 and its value is viewed as real number.

Subadditivity and nonnegativity of π imply that $\pi^{-1}([0, \alpha_1 + \alpha_2]) \supseteq \pi^{-1}([0, \alpha_1]) + \pi^{-1}([0, \alpha_2])$. This implies:

$$\begin{aligned} \mu(\pi^{-1}([0, \alpha_1 + \alpha_2])) &\geq \mu(\pi^{-1}([0, \alpha_1]) + \pi^{-1}([0, \alpha_2])) \\ &\geq \min\{1, \mu(\pi^{-1}([0, \alpha_1])) + \mu(\pi^{-1}([0, \alpha_2]))\} \\ &\geq x_1 + x_2, \end{aligned}$$

where the second inequality follows from Kemperman's theorem (which can be applied because $\pi^{-1}([0, \alpha_i]) \neq \emptyset$ for $i = 1, 2$, as $\pi(0) = 0$), and the last inequality follows from the fact that $\mu(\pi^{-1}([0, \alpha_i])) = x_i$ because $\hat{\pi}(x_i) = \alpha_i$, for $i = 1, 2$ (property 3 in Lemma 3.3). \square

LEMMA 3.5. *Let G be any compact, connected topological group with Haar probability measure μ on it. Let $\mathbf{b} \in G \setminus \{0\}$. Consider any $\pi : G \rightarrow \mathbb{R}_+$ such that $\pi(\mathbf{x}) + \pi(\mathbf{b} - \mathbf{x}) = 1$ for every $\mathbf{x} \in G$. Then for any $x \in \mathbb{T}^1 \setminus \{0\}$, $\hat{\pi}(x) + \hat{\pi}(-x) \leq 1$.*

Proof. Let $\hat{\pi}(x) = \alpha$. Since $-x$, viewed as a real number, is $1 - x$, we need to show that $\hat{\pi}(1 - x) \leq 1 - \alpha$. For this, it suffices to prove that $\mu(\pi^{-1}([0, 1 - \alpha])) \geq 1 - x$. Define $S := \pi^{-1}([0, \alpha])$ and $S' := \{\mathbf{b} - \mathbf{x} : \mathbf{x} \in S\}$. Since μ is the Haar measure on G , $\mu(S) = \mu(S')$. Moreover, as π satisfies $\pi(\mathbf{x}) + \pi(\mathbf{b} - \mathbf{x}) = 1$ for every $\mathbf{x} \in G$, we must have that $S' = \mathbb{T}^1 \setminus \pi^{-1}([0, 1 - \alpha])$. Since $\hat{\pi}(x) = \alpha$, by property 3 of Lemma 3.3 we obtain $x \geq \mu(S) = \mu(S') = 1 - \mu(\pi^{-1}([0, 1 - \alpha]))$. \square

LEMMA 3.6. *Let G be any compact, connected topological group with Haar probability measure μ on it. For any nonnegative function $\pi : G \rightarrow \mathbb{R}_+$ and any $\beta \geq 0$, we have*

$$\ell(\hat{\pi}^{-1}([0, \beta])) = \mu(\pi^{-1}([0, \beta])),$$

where ℓ denotes the Haar probability measure on \mathbb{T}^1 .

Proof. Property 3 in Lemma 3.3 implies that for all $t \in \mathbb{T}^1$,

$$\hat{\pi}(t) \leq \beta \Leftrightarrow \inf\{\alpha \geq 0 : \mu(\pi^{-1}([0, \alpha])) \geq t\} \leq \beta \Leftrightarrow \mu(\pi^{-1}([0, \beta])) \geq t.$$

Therefore, since $\hat{\pi}$ is nondecreasing,

$$\ell(\hat{\pi}^{-1}([0, \beta])) = \sup\{t \geq 0 : \hat{\pi}(t) \leq \beta\} = \sup\{t \geq 0 : \mu(\pi^{-1}([0, \beta])) \geq t\} = \mu(\pi^{-1}([0, \beta])),$$

where the first supremum is taken over $t \in \mathbb{T}^1$ with the standard order on \mathbb{T}^1 , and the second one is taken over $t \in \mathbb{T}^1$ viewed as a real number. Note that the first supremum is a well-defined real number because $\hat{\pi}(0) = 0$. \square

PROPOSITION 3.7. *Let G be any compact, connected topological group with Haar probability measure μ on it, and let $\pi : G \rightarrow [0, 1]$ be subadditive on G . Let $\hat{\pi}$ be as defined in (3.1). Then $\int_G \ln(\pi) d\mu$ is finite if and only if $\int_{\mathbb{T}^1} \ln(\hat{\pi}) d\ell$ is finite, and in this case*

$$\int_G \ln(\pi) d\mu = \int_{\mathbb{T}^1} \ln(\hat{\pi}) d\ell,$$

where ℓ denotes the Haar probability measure on \mathbb{T}^1 .

Proof. We use the so-called ‘‘layer-cake representation’’ of a nonnegative function (see, e.g., [19, Chapter 1]), which states that for any nonnegative function F defined on a measure space (Ω, ν) ,

$$\int_{\Omega} F d\nu = \int_0^{\infty} \nu(\{\omega \in \Omega : F(\omega) \geq t\}) dt.$$

Therefore, since π takes values in $[0, 1]$, we have that $-\ln(\pi) \geq 0$ and we can write

$$\begin{aligned} \int_G -\ln(\pi) d\mu &= \int_0^{\infty} \mu(\{\mathbf{x} \in G : -\ln(\pi(\mathbf{x})) \geq t\}) dt \\ &= \int_0^{\infty} \mu(\{\mathbf{x} \in G : \pi(\mathbf{x}) \leq e^{-t}\}) dt \\ &= \int_0^1 \frac{1}{s} \mu(\{\mathbf{x} \in G : \pi(\mathbf{x}) \leq s\}) ds, \end{aligned} \tag{3.2}$$

where we use the change of variable $s = e^{-t}$. Since π takes values in $[0, 1]$, so does $\hat{\pi}$ and similarly we get

$$\int_{\mathbb{T}^1} -\ln(\hat{\pi}) d\ell = \int_0^1 \frac{1}{s} \ell(\{x \in \mathbb{T}^1 : \hat{\pi}(x) \leq s\}) ds, \tag{3.3}$$

Applying Lemma 3.6 to (3.2) and (3.3) gives the desired result. \square

We now consider another function derived from $\hat{\pi}$. Let G be any compact, connected topological group with Haar probability measure μ on it. For any nonnegative function $\pi : G \rightarrow \mathbb{R}_+$, let $\hat{\pi} : \mathbb{T}^1 \rightarrow \mathbb{R}_+$ be defined as in (3.1). Define

$$\bar{\pi}(x) := \lim_{\varepsilon \rightarrow 0^+} \hat{\pi}(x + \varepsilon), \tag{3.4}$$

which is well-defined because $\hat{\pi}$ is bounded from below and nondecreasing by property 2 in Lemma 3.3, and so the limit in the definition of $\bar{\pi}$ exists and is finite (see, e.g., [22, Theorem 4.29]).

LEMMA 3.8. *Let G be any compact, connected topological group with Haar probability measure μ on it and let $\mathbf{b} \in G \setminus \{0\}$. For any $\pi \in \mathcal{M}_{\mathbf{b}}(G)$, let $\hat{\pi}, \bar{\pi}$ be defined as in (3.1) and (3.4). Then $\bar{\pi}$ is nondecreasing and subadditive. Moreover, for any $x \in \mathbb{T}^1 \setminus \{0\}$, $\hat{\pi}(x) + \bar{\pi}(-x) = 1$.*

Proof. We first note that since $\pi \in \mathcal{M}_{\mathbf{b}}(G)$, π satisfies the properties listed in Theorem 1.1. Now, $\bar{\pi}$ is nondecreasing because $\hat{\pi}$ is nondecreasing by property 2 in Lemma 3.3. We check that $\bar{\pi}$ is subadditive. Consider $a, b \in \mathbb{T}^1$ and let $x = a + b$. Then

$$\begin{aligned} \bar{\pi}(x) &= \lim_{\varepsilon \rightarrow 0^+} \hat{\pi}(x + \varepsilon) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} (\hat{\pi}(a + \varepsilon/2) + \hat{\pi}(b + \varepsilon/2)) \quad \text{by subadditivity of } \hat{\pi} \text{ (Theorem 3.4)} \\ &= \lim_{\varepsilon \rightarrow 0^+} \hat{\pi}(a + \varepsilon/2) + \lim_{\varepsilon \rightarrow 0^+} \hat{\pi}(b + \varepsilon/2) \\ &= \bar{\pi}(a) + \bar{\pi}(b). \end{aligned}$$

We now check that for any $x \in \mathbb{T}^1 \setminus \{0\}$, $\hat{\pi}(x) + \bar{\pi}(-x) = 1$. For every $0 < \varepsilon < x$ we have that $\hat{\pi}(x - \varepsilon) + \hat{\pi}(-x + \varepsilon) \leq 1$ by Lemma 3.5. Taking the limit $\varepsilon \rightarrow 0^+$, and using the fact that $\hat{\pi}$ is left continuous by property 4 in Lemma 3.3, we obtain that $\hat{\pi}(x) + \bar{\pi}(-x) \leq 1$. We show that the reverse inequality holds after establishing the following claim.

CLAIM 3.9. *For all $z \in \mathbb{T}^1$, $\bar{\pi}(z) = \sup\{t : \mu(\pi^{-1}([0, t])) \leq z\}$, where z inside the supremum is viewed as a real number.*

Proof of Claim. For every $\varepsilon > 0$ define $t_\varepsilon := \hat{\pi}(z + \varepsilon)$, and therefore $\bar{\pi}(z) = \lim_{\varepsilon \rightarrow 0^+} t_\varepsilon$. By property 3 in Lemma 3.3, $\mu(\pi^{-1}([0, t_\varepsilon])) = z + \varepsilon$. By property 1 in Lemma 3.3 and the continuity of measure, $\mu(\pi^{-1}([0, \bar{\pi}(z)])) = \lim_{\varepsilon \rightarrow 0^+} (z + \varepsilon) = z$. Therefore, $\mu(\pi^{-1}([0, \bar{\pi}(z)])) \leq \mu(\pi^{-1}([0, t_\varepsilon])) = z + \varepsilon$. Thus, $\bar{\pi}(z) \leq \sup\{t : \mu(\pi^{-1}([0, t])) \leq z\}$.

For the reverse inequality, observe that for all $\varepsilon > 0$, $z + \varepsilon/2 = \mu(\pi^{-1}([0, t_{\varepsilon/2}])) \leq \mu(\pi^{-1}([0, t_\varepsilon]))$. This implies that $\sup\{t : \mu(\pi^{-1}([0, t])) \leq z\} \leq t_\varepsilon$ for all $\varepsilon > 0$. Therefore, $\sup\{t : \mu(\pi^{-1}([0, t])) \leq z\} \leq \lim_{\varepsilon \rightarrow 0^+} t_\varepsilon = \bar{\pi}(z)$. \diamond

To complete the proof we need to establish that $\hat{\pi}(x) + \bar{\pi}(-x) \geq 1$. Let $\alpha = \hat{\pi}(x)$. By Claim 3.9 and the fact that $-x$ viewed as a real number is $1 - x$, it suffices to show that $\sup\{t : \mu(\pi^{-1}([0, t])) \leq 1 - x\} \geq 1 - \alpha$. By property 3 in Lemma 3.3, $\mu(\pi^{-1}([0, \alpha])) = x$. By symmetry of π , this implies that $\mu(\pi^{-1}([1 - \alpha, 1])) = x$, and therefore $1 - \mu(\pi^{-1}([0, 1 - \alpha])) = x$. Thus, $\sup\{t : \mu(\pi^{-1}([0, t])) \leq 1 - x\} \geq 1 - \alpha$. \square

We now state our main rearrangement theorem. In its proof, we shall need the following technical lemma about monotone, i.e., nondecreasing or nonincreasing, functions; see [22, Theorem 4.30].

LEMMA 3.10. *Any monotone real valued function defined on any real interval has countably many discontinuities.*

THEOREM 3.11. *Let G be any compact, connected topological group with Haar probability measure μ , and $\mathbf{b} \in G \setminus \{0\}$. Let $\pi \in \mathcal{M}_{\mathbf{b}}(G)$. Define $\hat{\pi}$ as in (3.1) and $\bar{\pi}$ as in (3.4). Then the function*

$$\tilde{\pi} := \frac{\hat{\pi} + \bar{\pi}}{2} \tag{3.5}$$

defined on \mathbb{T}^1 is nondecreasing, subadditive and symmetric in the sense that $\tilde{\pi}(x) + \tilde{\pi}(-x) = 1$ for all $x \in \mathbb{T}^1 \setminus \{0\}$. Further, $\int_G \ln(\pi) d\mu$ exists and is finite if and only if $\int_{\mathbb{T}^1} \ln(\tilde{\pi}) d\ell$ exists and is finite, and in this case

$$\int_G \ln(\pi) d\mu = \int_{\mathbb{T}^1} \ln(\tilde{\pi}) d\ell,$$

where ℓ denotes the Haar probability measure on \mathbb{T}^1 .

Proof. Since $\hat{\pi}$ is nondecreasing by property 2 in Lemma 3.3, and $\bar{\pi}$ is also nondecreasing by Lemma 3.8, so is $\tilde{\pi}$. Since $\hat{\pi}$ and $\bar{\pi}$ are both subadditive by Theorem 3.4 and Lemma 3.8, and subadditivity is preserved by convex combinations, $\tilde{\pi}$ is subadditive. We now check symmetry of $\tilde{\pi}$:

$$\begin{aligned} \tilde{\pi}(x) + \tilde{\pi}(-x) &= \frac{\hat{\pi}(x) + \bar{\pi}(x) + \hat{\pi}(-x) + \bar{\pi}(-x)}{2} \\ &= \frac{(\hat{\pi}(x) + \bar{\pi}(-x)) + (\hat{\pi}(-x) + \bar{\pi}(x))}{2} \\ &= \frac{1+1}{2} && \text{by Lemma 3.8} \\ &= 1 \end{aligned}$$

Since $\hat{\pi}$ is nondecreasing by property 2 in Lemma 3.3, it has countably many discontinuities by Lemma 3.10. Therefore, $\bar{\pi}$ differs from $\hat{\pi}$ only on a countable set, and the same is true for $\tilde{\pi}$. Thus, all three functions have the same value of the integral on the torus, if the integral exists. Proposition 3.7 then gives the final conclusion. \square

We shall also employ the well-known Fatou's lemma from integration theory [19, 21].

THEOREM 3.12. [*Fatou's Lemma*] Let (Ω, ν) be a measure space and let $f_n : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ for $n \in \mathbb{N}$ be a sequence of nonnegative, extended real valued functions that converges pointwise to f almost everywhere. Then $\int_{\Omega} f d\nu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\nu$.

THEOREM 3.13. Let \mathcal{G} be the set of functions $h : \mathbb{T}^1 \rightarrow \mathbb{R}_+$ that are nondecreasing, subadditive, and symmetric in the sense that $h(x) + h(-x) = 1$ for all $x \in \mathbb{T}^1 \setminus \{0\}$. Then $-\ln(h)$ is integrable for all $h \in \mathcal{G}$ and

$$\inf \left\{ \int_{\mathbb{T}^1} \ln(h) d\ell : h \in \mathcal{G} \right\} = -1.$$

Moreover, the infimum is attained by the function $g(x) = x$, where the right hand side is interpreted as a real number.

Proof. For any function h on the torus, we will consider the associated function on $[0, 1]$ which is identical to h on $[0, 1)$ and takes value 1 at 1. Now the integral of such a function on $[0, 1]$ is equal to the integral of h on the torus. Thus, without further comment, below we will consider all functions on the torus as functions defined on $[0, 1]$. With a slight abuse of notation, if h is subadditive on the torus, we will also say the corresponding function on $[0, 1]$ is subadditive.

For any function h on $[0, 1]$, we define $\mathcal{I}(h) := \int_0^1 \ln(h(x)) dx$ (which may be $-\infty$ if $-\ln(h)$ is not integrable). Note that the function $g(x) = x$ satisfies $\mathcal{I}(g) = -1$. We will first assume that h is piecewise linear, i.e., there exist numbers $0 = a_0 < a_1 < \dots < a_m = 1$ (called *breakpoints*) such that h is affine over the interval $[a_{i-1}, a_i]$ for every $i \in \{1, \dots, m\}$. Note

that if h is piecewise linear, then it is also continuous. We will show that if $h : [0, 1] \rightarrow \mathbb{R}_+$ is nondecreasing, subadditive, symmetric, piecewise linear, and satisfies $h(0) = 0$ and $h(1) = 1$, then $\mathcal{I}(h) \geq \mathcal{I}(g)$. Once we establish that this holds for piecewise linear functions, we will then apply Fatou's lemma to conclude that every nondecreasing, subadditive, symmetric h satisfies $\mathcal{I}(h) \geq -1$.

Let $h : [0, 1] \rightarrow \mathbb{R}_+$ be a nondecreasing, subadditive, symmetric, piecewise linear function satisfying $h(0) = 0$ and $h(1) = 1$. We will use the fact that

$$h(1/2^k) \geq 1/2^k \quad \text{for every integer } k \geq 1. \quad (3.6)$$

This is because, by subadditivity and symmetry, $2^{k-1}h(1/2^k) \geq h(1/2) = 1/2$.

For an integer $k \geq 1$, we denote by x^k the minimum x such that $h(x) = 1/2^k$. Note that x^k is well defined as h is continuous and satisfies $h(0) = 0$ and $h(1) = 1$. Moreover, $x^k \leq 1/2^k$ because h is nondecreasing and $h(1/2^k) \geq 1/2^k$ by (3.6). Also, we define $a_t^k = \frac{t}{2^{k-1}}$ for every $t \in \{0, \dots, 2^{k-1} - 1\}$. We define the function $h^k : [0, 1] \rightarrow \mathbb{R}_+$ as follows:

$$h^k(x) := \begin{cases} h(x) & x \in [0, x^k] \\ \frac{1}{2^k} & x \in [x^k, \frac{1}{2^k}] \\ \frac{1}{2^{k-1}} - h^k\left(\frac{1}{2^{k-1}} - x\right) & x \in [\frac{1}{2^k}, \frac{1}{2^{k-1}}] \\ a_t^k + h^k(x - a_t^k) & x \in [a_t^k, a_{t+1}^k], t \in \{1, \dots, 2^{k-1} - 1\}. \end{cases}$$

Note that $h^1 = h$. This is because of the following facts: by definition, $h^1(x) = h(x)$ for every $x \in [0, x^1]$; since h is nondecreasing and $h(1/2) = 1/2$ by symmetry, $h^1(x) = 1/2 = h(x)$ for every $x \in [x^1, 1/2]$; by symmetry of h^1 and h with respect to $1/2$, the two functions coincide also for $x \in [1/2, 1]$.

We remark that in the special case $h = g$ (the identity function) we have $g^k = g$ for every k .

Now let our piecewise linear function h be different from g (so there is at least one breakpoint different from 0 and 1). Let \bar{x} be the smallest nonzero breakpoint of h . The slope of h over the interval $[0, \bar{x}]$ cannot be smaller than 1, otherwise by subadditivity we would obtain $h(1) < 1$, a contradiction. Thus $h(x) \geq x$ for every $x \in [0, \bar{x}]$. Let K be a positive integer such that $1/2^K \leq \bar{x}$.

We will prove that $\mathcal{I}(h^k) \geq \mathcal{I}(g)$ for every $k \in \{1, \dots, K\}$. The proof is by reverse induction on k , where the base case is $k = K$ and our theorem corresponds to $k = 1$.

Fix any integer $k \geq 1$. Then for every integer $\ell \geq k$ we have

$$\begin{aligned} \mathcal{I}(h^\ell) &= \int_0^1 \ln(h^\ell(x)) dx \\ &= \sum_{t=0}^{2^{k-1}-1} \int_{a_t^k}^{a_{t+1}^k} \ln(h^\ell(x)) dx \\ &= \sum_{t=0}^{2^{k-1}-1} \int_{a_t^k}^{a_{t+1}^k} \ln(a_t^k + h^\ell(x - a_t^k)) dx \\ &= \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^{k-1}}} \ln(a_t^k + h^\ell(x)) dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=0}^{2^{k-1}-1} \left(\int_0^{\frac{1}{2^k}} \ln(a_t^k + h^\ell(x)) dx + \int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} \ln \left(a_t^k + \frac{1}{2^{k-1}} - h^\ell \left(\frac{1}{2^{k-1}} - x \right) \right) dx \right) \\
&= \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^k}} (\ln(a_t^k + h^\ell(x)) + \ln(a_{t+1}^k - h^\ell(x))) dx \\
&= \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^k}} \ln((a_t^k + h^\ell(x))(a_{t+1}^k - h^\ell(x))) dx.
\end{aligned}$$

The first equality is the definition of $\mathcal{I}(h^\ell)$; the second equality follows from the additivity of the integral; the third equality is due to the definition of h^ℓ ; the fourth equality follows from the change of variable $y = x - a_t^k$ (but the new variable y is still called x in the integral); the fifth equality uses again the additivity of the integral together with the fact that $h^\ell(x) = \frac{1}{2^{k-1}} - h^\ell(\frac{1}{2^{k-1}} - x)$ for every $x \in [\frac{1}{2^k}, \frac{1}{2^{k-1}}]$; the sixth equality follows from the change of variable $y = \frac{1}{2^{k-1}} - x$ in the second integral.

Note that the above chain of equations also holds for the function g (the identity function), in which case we obtain

$$\mathcal{I}(g) = \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^k}} \ln((a_t^k + x)(a_{t+1}^k - x)) dx.$$

Now assume that $k = \ell = K$ (base case of induction). Fix $x \in [0, \frac{1}{2^k}]$. Note that $h^\ell(x) = h^K(x) \geq h(x) \geq x$, as $0 \leq x \leq \frac{1}{2^k} \leq \bar{x}$ (recall the definitions of \bar{x} and K). Furthermore, $h^\ell(x) \leq \frac{1}{2^\ell} = \frac{1}{2^k}$. Then, for every $t \in \{0, \dots, 2^{k-1} - 1\}$, since the function $\phi(y) := (a_t^k + y)(a_{t+1}^k - y)$ is concave and has its unique maximum point at $y = \frac{a_{t+1}^k - a_t^k}{2} = \frac{1}{2^k}$, we have that $\phi(h^\ell(x)) \geq \phi(x)$, i.e.,

$$(a_t^k + h^\ell(x))(a_{t+1}^k - h^\ell(x)) \geq (a_t^k + x)(a_{t+1}^k - x).$$

This implies

$$\mathcal{I}(h^K) = \mathcal{I}(h^\ell) \geq \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^k}} \ln((a_t^k + x)(a_{t+1}^k - x)) dx = \mathcal{I}(g).$$

This concludes the analysis of the base case.

Now assume that $1 \leq k \leq K - 1$. We claim that $h^k(x) \geq h^{k+1}(x)$ for every $x \in [0, \frac{1}{2^k}]$. This can be verified by observing the following facts:

- (i) for $x \in [0, x^{k+1}]$, we have $h^k(x) = h^{k+1}(x) = h(x)$;
- (ii) for $x \in [x^{k+1}, \frac{1}{2^k} - x^{k+1}]$, we have $h^k(x) \geq \frac{1}{2^{k+1}} = h^{k+1}(x)$;
- (iii) for $x \in [\frac{1}{2^k} - x^{k+1}, x^k]$ (this interval may be empty), we have

$$h^k(x) = h(x) \geq h\left(\frac{1}{2^k}\right) - h\left(\frac{1}{2^k} - x\right) \geq \frac{1}{2^k} - h^{k+1}\left(\frac{1}{2^k} - x\right) = h^{k+1}(x),$$

where the first inequality follows by subadditivity, the second inequality follows by (3.6) and the fact that since $\frac{1}{2^k} - x \leq x^{k+1}$ we have $h(\frac{1}{2^k} - x) = h^{k+1}(\frac{1}{2^k} - x)$, and the last equality follows from the definition of h^{k+1} and the fact that $x \in [\frac{1}{2^k} - x^{k+1}, x^k] \subseteq [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$;

(iv) for $x \in [x^k, \frac{1}{2^k}]$, we have $h^k(x) = \frac{1}{2^k} \geq h^{k+1}(x)$.

Therefore $h^k(x) \geq h^{k+1}(x)$ for $x \in [0, \frac{1}{2^k}]$. Furthermore, $h^k(x) \leq \frac{1}{2^k}$ for $x \in [0, \frac{1}{2^k}]$. Then, for every $t \in \{0, \dots, 2^{k-1} - 1\}$, since the function $\phi(y) := (a_t^k + y)(a_{t+1}^k - y)$ is concave and has its unique maximum point at $y = \frac{1}{2^k}$, we have that $\phi(h^k(x)) \geq \phi(h^{k+1}(x))$ for every $x \in [0, \frac{1}{2^k}]$, i.e.,

$$(a_t^k + h^k(x))(a_{t+1}^k - h^k(x)) \geq (a_t^k + h^{k+1}(x))(a_{t+1}^k - h^{k+1}(x)).$$

This implies

$$\mathcal{I}(h^k) \geq \sum_{t=0}^{2^{k-1}-1} \int_0^{\frac{1}{2^k}} \ln((a_t^k + h^{k+1}(x))(a_{t+1}^k - h^{k+1}(x))) dx = \mathcal{I}(h^{k+1}) \geq \mathcal{I}(g),$$

where the last inequality follows by induction. This concludes the proof for a piecewise linear function h .

Finally, we consider any $h : [0, 1] \rightarrow \mathbb{R}_+$ that is nondecreasing, subadditive and symmetric. For every $n \in \mathbb{N}$, consider the piecewise linear approximation h_n of h obtained by linearly interpolating the values of h at $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$. It is well-known that each h_n is subadditive and symmetric (see, e.g., [5]), and clearly h_n is nondecreasing. By Lemma 3.10, the set of discontinuities of h is countable, and hence of measure zero. Thus, $h_n \rightarrow h$ on all other points, i.e., almost everywhere. By the argument above, $-\mathcal{I}(h_n) = \int_0^1 -\ln(h_n) \leq -\mathcal{I}(g) = 1$. Applying Fatou's Lemma (Theorem 3.12) to the sequence $-\ln(h_n) \rightarrow -\ln(h)$, we obtain that $-\mathcal{I}(h) \leq \liminf_{n \rightarrow \infty} -\mathcal{I}(h_n) \leq 1$. Therefore, $-\ln(h)$ is integrable and $\mathcal{I}(h) \geq -1$. \square

Proof of Theorem 2.2. For any $\pi \in \mathcal{M}_{\mathbf{b}}(G)$, we consider $\tilde{\pi} : \mathbb{T}^1 \rightarrow \mathbb{R}_+$ as defined in (3.5) which, by Theorem 3.11, is nondecreasing, subadditive, symmetric and satisfies

$$\int_G \ln(\pi) d\mu = \int_{\mathbb{T}^1} \ln(\tilde{\pi}) d\ell,$$

where ℓ denotes the Haar probability measure on \mathbb{T}^1 , if either integral exists and is finite. By Theorem 3.13, $-\ln(\tilde{\pi})$ is integrable and $\int_{\mathbb{T}^1} \ln(\tilde{\pi}) d\ell \geq -1$. As a consequence, $-\ln(\pi)$ is integrable and $\int_G \ln(\pi) d\mu \geq -1$ for all $\pi \in \mathcal{M}_{\mathbf{b}}(G)$. The statement about $\text{GMI}_{\mathbf{b}_i}^n$ follows from the fact that the integral of the logarithm of $\text{GMI}_{\mathbf{b}_i}^n$ is equal to the integral of the logarithm of $\text{GMI}_{\mathbf{b}_i}^1$, which is easily verified to be -1 by elementary calculus. \square

We close this section by observing that the Gomory function $\text{GMI}_{\mathbf{b}_i}^n$ is not the unique minimizer of $\int_{\mathbb{T}^1} \ln(\pi) d\mu$. Indeed, already for $n = 1$, the function defined by $\pi(x) := \text{GMI}_{\mathbf{b}}^1(kx)$ for $x \in \mathbb{T}^1$ achieves value of the integral equal to -1 , for every $k \in \mathbb{N}$.

4 Finite cyclic groups of prime order: Proof of Theorem 2.3

It turns out that there is an analogue of Theorem 3.4 for $G = \mathbb{Z}/q\mathbb{Z}$ with q prime. We again consider the natural total order on G .

THEOREM 4.1. *Let $G = \mathbb{Z}/q\mathbb{Z}$ with q prime, and let $\pi : G \rightarrow \mathbb{R}_+$ be a subadditive function on G such that $\pi(0) = 0$ and π is not identically 0. Then $\hat{\pi} : G \rightarrow \mathbb{R}_+$ defined as*

$$\hat{\pi}(x) := \min\{\alpha \geq 0 : |\pi^{-1}((0, \alpha])| \geq x\} \quad \forall x \in G \quad (4.1)$$

is finite-valued, subadditive on G and nondecreasing. (On the right-hand side, x is viewed as a number in $\{0, \dots, q-1\}$.) Moreover, $|\pi^{-1}(\{\beta\})| = |\hat{\pi}^{-1}(\{\beta\})|$ for all $\beta > 0$. Finally, if there exists some $b \in G$ such that $\pi(x) + \pi(b-x) = 1$ for all $x \in G$, then $\hat{\pi}(x) + \hat{\pi}(q-1-x) = 1$ for all $x \in G$.

Proof. Subadditivity of π implies that $\pi(x) > 0$ for all $x \neq 0$ because q is prime and π is not identically 0. Thus, for every $x \in \{0, \dots, q-1\}$, there exists $\alpha \geq 0$ such that $|\pi^{-1}((0, \alpha])| \geq x$. So the minimum in (4.1) is taken over a nonempty set. Since G is finite, this set has a minimum, which is attained at some $\alpha \in \pi(G)$. This shows that the minimum is well-defined. The nondecreasing property follows from definition.

The proof of subadditivity is along the same lines as for a compact, connected group; instead of Kemperman's theorem (which holds only when the topological group is connected, and $G = \mathbb{Z}/q\mathbb{Z}$ is not connected) one needs to use the Cauchy–Davenport theorem [20] which states that for any nonempty subsets $A, B \subseteq G$, we have the inequality $|A + B| \geq \min\{q, |A| + |B| - 1\}$. Consequently, any nonempty $A, B \subseteq G$ such that $0 \notin B$ satisfy $|(A + B) \cup A| \geq \min\{q, |A| + |B|\}$: apply Cauchy–Davenport theorem to $A, B \cup \{0\}$.

Consider $x_1, x_2 \in G$ and let $\alpha_i = \hat{\pi}(x_i)$, $i = 1, 2$. Therefore, $|\pi^{-1}((0, \alpha_i])| \geq x_i$, $i = 1, 2$. Moreover, by subadditivity of π ,

$$(\pi^{-1}((0, \alpha_1]) + \pi^{-1}((0, \alpha_2])) \cup \pi^{-1}((0, \alpha_1]) \subseteq \pi^{-1}((0, \alpha_1 + \alpha_2]).$$

Therefore, we obtain

$$\begin{aligned} |\pi^{-1}([0, \alpha_1 + \alpha_2])| &\geq |(\pi^{-1}((0, \alpha_1]) + \pi^{-1}((0, \alpha_2])) \cup \pi^{-1}((0, \alpha_1])| \\ &\geq \min\{q, |\pi^{-1}((0, \alpha_1])| + |\pi^{-1}((0, \alpha_2])|\} \\ &\geq x_1 + x_2 \end{aligned}$$

where we use the fact that $\pi(0) = 0$, and so $0 \notin \pi^{-1}((0, \alpha_2])$. Note that the last term $x_1 + x_2$ is a sum in G and is viewed as a real number, thus $x_1 + x_2 < q$.

We next confirm that $|\pi^{-1}(\{\beta\})| = |\hat{\pi}^{-1}(\{\beta\})|$ for all $\beta > 0$. To show this, it suffices to prove that $|\pi^{-1}((0, \beta])| = |\hat{\pi}^{-1}((0, \beta])|$ for all $\beta > 0$. Observe that for any $t \in G$,

$$\hat{\pi}(t) \leq \beta \Leftrightarrow \min\{\alpha \geq 0 : |\pi^{-1}((0, \alpha])| \geq t\} \leq \beta \Leftrightarrow |\pi^{-1}((0, \beta])| \geq t.$$

Therefore, since $\hat{\pi}$ is nondecreasing and $\hat{\pi}(0) = 0$,

$$|\hat{\pi}^{-1}((0, \beta])| = \max\{t \in G : \hat{\pi}(t) \leq \beta\} = \max\{t \in G : |\pi^{-1}((0, \beta])| \geq t\} = |\pi^{-1}((0, \beta])|.$$

We finally verify that the symmetry condition is preserved. Consider any $x \in G$ and let $\alpha = \hat{\pi}(x)$. By definition, $|\pi^{-1}((0, \alpha])| \geq x$. By symmetry of π , $t \in \pi^{-1}((0, \alpha))$ if and only if $b - t \in \pi^{-1}((1 - \alpha, 1))$. We now observe that $\alpha = \hat{\pi}(x)$ implies that

$$\begin{aligned} & x \geq |\pi^{-1}((0, \alpha])| \\ \Rightarrow & x \geq |\pi^{-1}((1 - \alpha, 1))| \\ \Rightarrow & x \geq q - 1 - |\pi^{-1}((0, 1 - \alpha])| \\ \Rightarrow & |\pi^{-1}((0, 1 - \alpha])| \geq q - 1 - x. \end{aligned}$$

Thus, by definition of $\hat{\pi}$, $\hat{\pi}(q - 1 - x) \leq 1 - \alpha$. Therefore, we have $\hat{\pi}(x) + \hat{\pi}(q - 1 - x) \leq 1$. By subadditivity of $\hat{\pi}$, $\hat{\pi}(x) + \hat{\pi}(q - 1 - x) \geq \hat{\pi}(q - 1)$. We finally observe that $\hat{\pi}(q - 1) = 1$ because $\pi(b) = 1$. by symmetry of π , and $\pi(x) \leq 1$ for all $x \in G$. Therefore, $\hat{\pi}(x) + \hat{\pi}(q - 1 - x) = 1$. \square

LEMMA 4.2. *Let $G = \mathbb{Z}/q\mathbb{Z}$ where q is prime, and let $b \in G \setminus \{0\}$. Then the function $h = \text{GOM}_{q-1}^q \circ \phi$, where $\phi : G \rightarrow G$ is the automorphism that sends b to $q - 1$, satisfies $\hat{h}(x) = \frac{x}{q-1}$ for every $x \in G$, where on the right hand side x is viewed as a real number.*

Proof. Since GOM_{q-1}^q takes all the distinct values in $\left\{0, \frac{1}{q-1}, \frac{2}{q-1}, \dots, \frac{q-2}{q-1}, 1\right\}$, so does the function h . Since the rearrangement \hat{h} is nondecreasing, we obtain the result. \square

Proof of Theorem 2.3. Let $G = \mathbb{Z}/q\mathbb{Z}$ where q is a prime number, and let $b \in G \setminus \{0\}$. By virtue of Theorem 4.1 and Lemma 4.2, we may assume that $b = q - 1$ and restrict attention to functions $\pi : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{R}_+$ such that $\pi(0) = 0$, π is subadditive, nondecreasing, and $\pi(x) + \pi(q - 1 - x) = 1$ for every $x \in \mathbb{Z}/q\mathbb{Z}$. To simplify the notation, we scale the function values by a factor of $q - 1$. Then $\pi(x) + \pi(b - x) = q - 1 = b$ for every $x \in G$, and the statement of Theorem 2.3 reads as follows:

$$\inf \left\{ \prod_{x \in G \setminus \{0\}} \pi(x) : \pi \in \mathcal{M}_b(G) \right\} = (q - 1)!. \quad (4.2)$$

Since we will not use the group structure of $\mathbb{Z}/q\mathbb{Z}$, we will see the domain of π simply as the set $I := \{0, \dots, q - 1\} = \{0, \dots, b\}$. The Gomory function (scaled by a factor of $q - 1$) is the function g defined by $g(x) := x$ for every $x \in I$. The left hand side of (4.2), when $\pi = g$, is immediately verified to be equal to $(q - 1)!$. We will prove that this function minimizes the left hand side of (4.2). For this purpose, for a function π as above we define $\mathcal{P}(\pi) := \pi(1)\pi(2) \cdots \pi(b)$. Therefore our theorem is: $\mathcal{P}(\pi) \geq \mathcal{P}(g)$ for every $\pi : I \rightarrow \mathbb{R}_+$ such that $\pi(0) = 0$, π is subadditive, nondecreasing, and $\pi(x) + \pi(q - 1 - x) = b$ for every $x \in I$. We will assume $q \geq 3$ (i.e., $b \geq 2$), as for $q = 2$ there is only one minimal function and Theorem 2.3 is immediately verified in this case. In particular, b is an even number.

For every $k \in \{1, \dots, K\}$ (where K will be determined later), we partition the discrete interval I into 2^k discrete sub-intervals $I_1^k, \dots, I_{2^k}^k$ of the same cardinality plus a set A^k of additional points that are not covered by these sub-intervals.

For $k = 1$, define $s^1 := \frac{b}{2}$ (which is an integer), $I_1^1 := [1, s^1] \cap \mathbb{Z}$, $I_2^1 := (s^1, b - 1] \cap \mathbb{Z}$. Note that $|I_1^1| = |I_2^1|$ and I_2^1 is the symmetric of I_1^1 with respect to s^1 , i.e., $I_2^1 = 2s_1 - I_1^1$. The set of additional points of the partition is $A^1 := \{0, s^1, b\}$.

For $k \geq 2$, the partition is constructed starting from the partition of level $k - 1$: every interval of level $k - 1$ is split into two intervals, plus a (possibly empty) set of additional points. The construction depends on which of the following two cases holds:

Case (a): $\pi(\lfloor s^{k-1} \rfloor) \geq \lfloor s^{k-1} \rfloor$;

Case (b): $\pi(\lfloor s^{k-1} \rfloor) < \lfloor s^{k-1} \rfloor$.

Define

$$s^k := \begin{cases} \frac{\lfloor s^{k-1} \rfloor}{2} & \text{if Case (a) holds,} \\ \frac{\lceil s^{k-1} \rceil}{2} & \text{if Case (b) holds.} \end{cases}$$

Also, define $I_1^k := [1, s^k) \cap \mathbb{Z}$ and $I_2^k := 2s^k - I_1^k = (s^k, 2s^k - 1] \cap \mathbb{Z}$ (i.e., I_2^k is the symmetric of I_1^k with respect to s^k). The intervals I_3^k and I_4^k are defined as the symmetric of I_2^k and I_1^k (respectively) with respect to s^{k-1} . The intervals $I_5^k, I_6^k, I_7^k, I_8^k$ (assuming $k \geq 3$) are obtained by symmetrizing $I_4^k, I_3^k, I_2^k, I_1^k$ (respectively) with respect to s^{k-2} , and so forth. The formal definition is by induction as follows: $I_1^k := [1, s^k) \cap \mathbb{Z}$; for $h \in \{1, \dots, k\}$, assuming that the intervals $I_1^k, \dots, I_{2^{h-1}}^k$ have been defined, we define

$$I_t^k := 2s^{k-h+1} - I_{2^h-t+1}^k \text{ for every } t \in \{2^{h-1} + 1, \dots, 2^h\}. \quad (4.3)$$

Finally, we define $A^k := I \setminus \bigcup_{t=1}^{2^k} I_t^k$.

The above construction is repeated until we reach an index K such that $s^K \in \{\frac{1}{2}, 1\}$; note that such an index K necessarily exists.

CLAIM 4.3. *The following properties hold for every $k, \ell \in \{1, \dots, K\}$ with $\ell \geq k$:*

- (i) $I = \bigcup_{t=1}^{2^k} I_t^k \cup A^k$, and the union is disjoint;
- (ii) $|I_t^k| = |I_{t'}^k|$ for every $t, t' \in \{1, \dots, 2^k\}$;
- (iii) $I_t^k \setminus A^\ell = \bigcup_{j \in J_t^{k,\ell}} I_j^\ell$ for every $t \in \{1, \dots, 2^k\}$, where $J_t^{k,\ell} = \{2^{\ell-k}(t-1) + 1, \dots, 2^{\ell-k}t\}$;
- (iv) $I_t^k \cap A^\ell = 2s^{k-h+1} - (I_{2^h-t+1}^k \cap A^\ell)$ for every $t \in \{1, \dots, 2^k\}$, where h is the integer such that $2^{h-1} < t \leq 2^h$;
- (v) $\pi(\lfloor s^k \rfloor) \geq \lfloor s^k \rfloor$ or $\pi(\lceil s^k \rceil) \geq \lceil s^k \rceil$;²
- (vi) $\pi(2s^k) \geq 2s^k$.

Furthermore, for $k = K$ the following holds:

- (vii) $I_t^K = \emptyset$ for every $t \in \{1, \dots, 2^K\}$, and $I = A^K$.

²When $k = K$ and $s^K = \frac{1}{2}$, the symbol $\pi(\lfloor s^k \rfloor)$ has no meaning, as $\lfloor s^k \rfloor = 0 \notin I$; in this case we mean that the alternative $\pi(\lceil s^k \rceil) \geq \lceil s^k \rceil$ holds.

Proof of Claim. Properties (i) and (ii) follow directly from the construction of the partition.

We prove (iii). Note that $I_j^\ell \cap A^\ell = \emptyset$ for every j and ℓ by property (i). Then it suffices to show that

$$I_j^\ell \cap I_t^k \neq \emptyset \text{ if and only if } j \in J_t^{k,\ell}, \text{ and in this case } I_j^\ell \subseteq I_t^k. \quad (4.4)$$

We proceed by induction on ℓ .

If $\ell = k$ then (4.4) is immediately verified, as $J_t^{k,\ell} = \{t\}$ for every $t \in \{1, \dots, 2^K\}$ and thanks to property (i).

Assume now $\ell = k + 1$. We prove (4.4) by induction on h , where h is the integer such that $2^{h-1} < t \leq 2^h$. If $h = 0$, then $t = 1$ and $J_t^{k,k+1} = \{1, 2\}$. Since $I_1^k = [1, s^k) \cap \mathbb{Z}$, $I_1^{k+1} = [1, s^{k+1}) \cap \mathbb{Z}$, $I_2^{k+1} = (s^{k+1}, 2s^{k+1} - 1] \cap \mathbb{Z}$ and $2s^{k+1} - 1 < s^k$, we have $I_1^{k+1} \cup I_2^{k+1} \subseteq I_1^k$. Furthermore, one verifies that every $x \in I_j^{k+1}$ with $j \geq 3$ satisfies $x > s^k$, and therefore (4.4) is satisfied. If $h \geq 1$, by (4.3) we have $I_t^k = 2s^{k-h+1} - I_{2^h-t+1}^k$. Furthermore, since $2^{h-1} < t \leq 2^h$, every $j \in J_t^{k,k+1}$ satisfies $2^h < j \leq 2^{h+1}$. Then, again by (4.3), we have $I_j^{k+1} = 2s^{k-h+1} - I_{2^{h+1}-j+1}^{k+1}$ for every $j \in J_t^{k,k+1}$. This implies that $I_j^{k+1} \cap I_t^k \neq \emptyset$ if and only if $I_{2^{h+1}-j+1}^{k+1} \cap I_{2^h-t+1}^k \neq \emptyset$, and $I_j^{k+1} \subseteq I_t^k$ if and only if $I_{2^{h+1}-j+1}^{k+1} \subseteq I_{2^h-t+1}^k$. By induction, each of these two properties holds if and only if $2^{h+1} - j + 1 \in J_{2^h-t+1}^{k,k+1}$. It can be checked that this condition is equivalent to $j \in J_t^{k,k+1}$.

We finally assume $\ell \geq k + 2$. As shown above, $I_j^{k+1} \cap I_t^k \neq \emptyset$ if and only if $j \in J_t^{k,k+1}$, and in this case $I_j^{k+1} \subseteq I_t^k$. By induction, $I_{j'}^\ell \cap I_j^{k+1} \neq \emptyset$ if and only if $j' \in J_j^{k+1,\ell}$, and in this case $I_{j'}^\ell \subseteq I_j^{k+1}$. This implies that $I_{j'}^\ell \cap I_t^k \neq \emptyset$ if and only if $j' \in J_j^{k+1,\ell}$ for some $j \in J_t^{k,k+1}$, and in this case $I_{j'}^\ell \subseteq I_t^k$. It can be checked that the condition $j' \in J_j^{k+1,\ell}$ for some $j \in J_t^{k,k+1}$ is equivalent to $j' \in J_t^{k,\ell}$.

To show (iv), we use notation $J_t^{k,\ell}$ as in part (iii) and define $t' := 2^h - t + 1$. Observe that

$$\begin{aligned} 2s^{k-h+1} - (I_{t'}^k \cap A^\ell) &= 2s^{k-h+1} - (I_{t'}^k \setminus (I_{t'}^k \setminus A^\ell)) \\ &= 2s^{k-h+1} - I_{t'}^k \setminus \bigcup_{j' \in J_{t'}^{k,\ell}} I_{j'}^\ell \\ &= (2s^{k-h+1} - I_{t'}^k) \setminus \bigcup_{j' \in J_{t'}^{k,\ell}} (2s^{k-h+1} - I_{j'}^\ell) \\ &= I_t^k \setminus \bigcup_{j \in J_t^{k,\ell}} I_j^\ell \\ &= I_t^k \cap A^\ell, \end{aligned}$$

where the second and the last equalities hold by property (iii), and the fourth equality can be justified as follows. First, $2s^{k-h+1} - I_{t'}^k = I_t^k$ by (4.3). Moreover, if we define $h' := \ell - k + h$, since $2^{h'-1} < j' \leq 2^{h'}$ for every $j' \in J_{t'}^{k,\ell}$, (4.3) also implies $I_{j'}^\ell = 2s^{\ell-h'+1} - I_{2^{h'}-j'+1}^\ell = 2s^{k-h+1} - I_{2^{h'}-j'+1}^\ell$ for every $j' \in J_{t'}^{k,\ell}$. It can be checked that $\{2^{h'} - j' + 1 : j' \in J_{t'}^{k,\ell}\} = J_t^{k,\ell}$, proving the correctness of the fourth equality and thus showing (iv).

We prove (v) by induction on k . For $k = 1$, recall that $s^1 = \frac{b}{2}$ is an integer. Suppose by contradiction that $\pi(s^1) < s^1$. Then, by subadditivity, $\pi(b) = \pi(2s^1) \leq 2\pi(s^1) < 2s^1 = b$, a contradiction to the property $\pi(b) = b$.

Now take $k \geq 2$ and assume that (v) holds for the index $k - 1$. Consider first Case (a), that is $\pi(\lfloor s^{k-1} \rfloor) \geq \lfloor s^{k-1} \rfloor$. Suppose by contradiction that $\pi(\lfloor s^k \rfloor) < \lfloor s^k \rfloor$ and $\pi(\lceil s^k \rceil) < \lceil s^k \rceil$. Since $\lfloor s^{k-1} \rfloor = \lfloor s^k \rfloor + \lceil s^k \rceil$, by subadditivity we have $\pi(\lfloor s^{k-1} \rfloor) = \pi(\lfloor s^k \rfloor + \lceil s^k \rceil) \leq \pi(\lfloor s^k \rfloor) + \pi(\lceil s^k \rceil) < \lfloor s^k \rfloor + \lceil s^k \rceil = \lfloor s^{k-1} \rfloor$, a contradiction to the condition of Case (a).

We now consider Case (b), that is $\pi(\lfloor s^{k-1} \rfloor) < \lfloor s^{k-1} \rfloor$. Assume $\pi(\lfloor s^k \rfloor) < \lfloor s^k \rfloor$ and $\pi(\lceil s^k \rceil) < \lceil s^k \rceil$. Since $\lceil s^{k-1} \rceil = \lfloor s^k \rfloor + \lceil s^k \rceil$, by subadditivity we have $\pi(\lceil s^{k-1} \rceil) = \pi(\lfloor s^k \rfloor + \lceil s^k \rceil) \leq \pi(\lfloor s^k \rfloor) + \pi(\lceil s^k \rceil) < \lfloor s^k \rfloor + \lceil s^k \rceil = \lceil s^{k-1} \rceil$. Therefore $\pi(\lfloor s^{k-1} \rfloor) < \lfloor s^{k-1} \rfloor$ and $\pi(\lceil s^{k-1} \rceil) < \lceil s^{k-1} \rceil$, a contradiction to the induction hypothesis.

We now prove (vi). The statement is clearly true for $k = 1$, as $2s^1 = b$ and $\pi(b) = b$. Assume now $k \geq 2$. If $\pi(\lfloor s^{k-1} \rfloor) \geq \lfloor s^{k-1} \rfloor$, then $s^k = \frac{\lfloor s^{k-1} \rfloor}{2}$ and $\pi(2s^k) = \pi(\lfloor s^{k-1} \rfloor) \geq \lfloor s^{k-1} \rfloor = 2s^k$. If $\pi(\lfloor s^{k-1} \rfloor) < \lfloor s^{k-1} \rfloor$, then by (v) we have $\pi(\lceil s^{k-1} \rceil) \geq \lceil s^{k-1} \rceil$. Since $s^k = \frac{\lceil s^{k-1} \rceil}{2}$, we have $\pi(2s^k) = \pi(\lceil s^{k-1} \rceil) \geq \lceil s^{k-1} \rceil = 2s^k$.

We finally prove (vii). Since $s^K \in \{\frac{1}{2}, 1\}$, we have $I_1^K = [1, s^K) \cap \mathbb{Z} = \emptyset$. Then, by property (ii), $I_t^K = \emptyset$ for every $t \in \{1, \dots, 2^K\}$. Property (i) then implies that $I = A^K$. \diamond

For every $k \in \{1, \dots, K\}$, define

$$x^k := \max\{x \in I : \pi(x) < s^k\}, \text{ with } x^k = 0 \text{ if } \pi(x) \geq s^k \text{ for every } x \in I.$$

Note that $x^k < s^k$: this is because of Claim 4.3(v) and the fact that π is nondecreasing. We define the following function $\pi^k : I \rightarrow \mathbb{R}_+$:

$$\pi^k(x) := \begin{cases} \pi(x) & \text{if } x \in I_1^k \text{ and } x \leq x^k \\ s^k & \text{if } x \in I_1^k \text{ and } x > x^k \\ 2s^{k-h+1} - \pi^k(2s^{k-h+1} - x) & \text{if } x \in I_t^k, t \in \{2^{h-1} + 1, \dots, 2^h\}, h \in \{1, \dots, k\} \\ x & \text{otherwise (i.e., } x \in A^k) \end{cases}$$

Note that the third line of the above formula allows us to define π^k on I_2^k (for $h = 1$), then on $I_3^k \cup I_4^k$ (for $h = 2$), then on $I_5^k \cup I_6^k \cup I_7^k \cup I_8^k$ (for $h = 3$), and so forth.

CLAIM 4.4. *We have $\pi^1 = \pi$ and $\pi^K = g$.*

Proof of Claim. To see that $\pi^1 = \pi$, recall that $s^1 = \frac{b}{2}$ and note the following facts: $\pi^1(x) = \pi(x)$ for every integer $1 \leq x \leq x^1$. Since π is nondecreasing and $\pi(s^1) = s^1$ by symmetry of π , by definition of x^1 we have $\pi^1(x) = s^1 = \pi(x)$ for every integer $x^1 < x < s^1$. Since both π^1 and π are symmetric with respect to $s^1 = \frac{b}{2}$, the two functions coincide also for $x > s^1$. Finally, since $s^1 \in A^1$ we have $\pi^1(s^1) = \pi(s^1) = s^1$.

To see that $\pi^K = g$, recall that $I = A^K$ by Claim 4.3(vii), and therefore $\pi^K(x) = x$ for every $x \in I$, i.e., $\pi^K = g$. \diamond

By Claim 4.4, it suffices to show that $\mathcal{P}(\pi^k) \geq \mathcal{P}(\pi^{k+1})$ for every $k \in \{1, \dots, K - 1\}$.

It is convenient to use the following notation: for $k \in \{1, \dots, K\}$ and $t \in \{1, \dots, 2^k\}$, we define Δ_t^k as an integer number such that $I_t^k = \Delta_t^k + I_1^k$ if t is odd and $I_t^k = \Delta_t^k + I_2^k$ if t

is even. The existence of Δ_t^k is guaranteed by the fact that the sets $I_1^k, \dots, I_{2^k}^k$ are discrete intervals of the same cardinality. When $k < K$, Δ_t^k is uniquely defined for every t ; when $k = K$, all the intervals are empty (see Claim 4.3(vii)) and we can define $\Delta_t^k = 0$.

CLAIM 4.5. *If $k \in \{1, \dots, K\}$ and $t \in \{1, \dots, 2^k\}$ with t odd, then $\Delta_t^k = \Delta_{t+1}^k$.*

Proof of Claim. The proof is by induction on h , where h is the integer such that $2^{h-1} < t+1 \leq 2^h$. If $h = 1$ then $t = 1$ and the statement is true, as $\Delta_1^k = \Delta_2^k = 0$ by definition. Thus we assume $h \geq 2$. We have

$$I_t^k = 2s^{k-h+1} - I_{2^h-t+1}^k = 2s^{k-h+1} - \Delta_{2^h-t+1}^k - I_2^k,$$

where we used the definitions of I_t^k and $\Delta_{2^h-t+1}^k$, and fact that $2^h - t + 1$ is an even number. Similarly,

$$I_{t+1}^k = 2s^{k-h+1} - I_{2^h-t}^k = 2s^{k-h+1} - \Delta_{2^h-t}^k - I_1^k.$$

Since $2^h - t$ and $2^h - t + 1$ are both $\leq 2^{h-1}$, by induction $\Delta_{2^h-t}^k = \Delta_{2^h-t+1}^k$. Since $I_2^k = 2s^k - I_1^k$, this shows that $I_t^k = c + I_1^k$ and $I_{t+1}^k = c + I_2^k$, where $c := 2s^{k-h+1} - \Delta_{2^h-t+1}^k - 2s^k$, which implies that $\Delta_t^k = \Delta_{t+1}^k$. \diamond

CLAIM 4.6. *Fix $k, \ell \in \{1, \dots, K\}$ with $\ell \geq k$, and $t \in \{1, \dots, 2^k\}$. Then $\pi^\ell(x) = \Delta_t^k + \pi^\ell(x - \Delta_t^k)$ for every $x \in I_t^k$.*

Proof of Claim. We assume $k < K$, otherwise there is nothing to prove, as $I_t^K = \emptyset$ for every t by Claim 4.3(vii).

We first show that $\pi^\ell(x) = \Delta_t^k + \pi^\ell(x - \Delta_t^k)$ for every $x \in I_t^k \setminus A^\ell$. The proof is by induction on h , where h is the integer number such that $2^{h-1} < t \leq 2^h$; note that $h \leq k$. The statement is trivially true for $h = 0$, as in this case $t = 1$ and $\Delta_1^k = 0$.

Suppose now $h \geq 1$. We assume t odd (the other case is similar). By Claim 4.3(iii), we have $I_t^k \setminus A^\ell = \bigcup_{j \in J_t^{k,\ell}} I_j^\ell$, where $J_t^{k,\ell} = \{2^{\ell-k}(t-1)+1, \dots, 2^{\ell-k}t\}$. If we define $h' = \ell - k + h$, since $2^{h-1} + 1 \leq t \leq 2^h$ we have that $J_t^{k,\ell} \subseteq \{2^{h'-1} + 1, \dots, 2^{h'}\}$. Then, for every $x \in I_t^k \setminus A^\ell$,

$$\begin{aligned} \pi^\ell(x) &= 2s^{\ell-h'+1} - \pi^\ell(2s^{\ell-h'+1} - x) \\ &= 2s^{k-h+1} - \pi^\ell(2s^{k-h+1} - x) \\ &= 2s^{k-h+1} - \Delta_{2^h-t+1}^k - \pi^\ell(2s^{k-h+1} - x - \Delta_{2^h-t+1}^k) \\ &= 2s^{k-h+1} - \Delta_{2^h-t+1}^k - 2s^\ell + \pi^\ell(2s^\ell - 2s^{k-h+1} + x + \Delta_{2^h-t+1}^k) \\ &= \Delta_t^k + \pi^\ell(x - \Delta_t^k). \end{aligned}$$

The first equality is by definition of π^ℓ ; the second equality holds because $h' = \ell - k + h$; the third equality follows by induction after observing that $2s^{k-h+1} - x \in I_{2^h-t+1}^k$ by (4.3) and $2^h - t + 1 \leq 2^{h-1}$; the fourth equality is due to the definition of π^ℓ together with the fact that $2^h - t + 1$ is even and therefore $2s^{k-h+1} - x - \Delta_{2^h-t+1}^k \in I_2^k$; the last equality holds because $2s^\ell - 2s^{k-h+1} + x + \Delta_{2^h-t+1}^k \in I_1^k$, and therefore

$$\Delta_t^k = -2s^\ell + 2s^{k-h+1} - \Delta_{2^h-t+1}^k. \quad (4.5)$$

We now assume $x \in I_t^k \cap A^\ell$ (with t odd again). We show by induction on h that $x - \Delta_t^k \in A^\ell$ (where h is defined as above). To see this, note that by Claim 4.3(iv) we have $2s^{k-h+1} - x \in I_{2^{h-t+1}}^k \cap A^\ell$. Then, by induction and the fact that $2^h - t + 1$ is even, $2s^{k-h+1} - x - \Delta_{2^{h-t+1}}^k \in I_2^k \cap A^\ell$, which (again by Claim 4.3(iv)) implies that $2s^\ell - (2s^{k-h+1} - x - \Delta_{2^{h-t+1}}^k) \in I_1^k \cap A^\ell$. Equation (4.5) then implies $x - \Delta_t^k \in A^\ell$.

It follows that if $x \in I_t^k \cap A^\ell$ then

$$\pi^\ell(x) = x = \Delta_t^k + (x - \Delta_t^k) = \Delta_t^k + \pi^\ell(x - \Delta_t^k),$$

where the first equality holds because $x \in A^\ell$ and the last equality because $x - \Delta_t^k \in A^\ell$. \diamond

We now fix $k \in \{1, \dots, K-1\}$ and show that $\mathcal{P}(\pi^k) \geq \mathcal{P}(\pi^{k+1})$. Since $A^k \subseteq A^{k+1}$ by construction of the partition, we have $\pi^k(x) = \pi^{k+1}(x) = x$ for every $x \in A^k$. Therefore, it suffices to prove that $\frac{\mathcal{P}(\pi^k)}{\prod_{x \in A^k \setminus \{0\}} \pi^k(x)} \geq \frac{\mathcal{P}(\pi^{k+1})}{\prod_{x \in A^k \setminus \{0\}} \pi^{k+1}(x)}$.

Fix $\ell \in \{k, k+1\}$. Then (we use the standard convention that the product of no elements is equal to 1)

$$\begin{aligned} \frac{\mathcal{P}(\pi^\ell)}{\prod_{x \in A^k \setminus \{0\}} \pi^\ell(x)} &= \prod_{x \in I \setminus A^k} \pi^\ell(x) \\ &= \prod_{t=1}^{2^{k-1}} \prod_{x \in I_{2^{t-1}}^k \cup I_{2^t}^k} \pi^\ell(x) \\ &= \prod_{t=1}^{2^{k-1}} \prod_{x \in I_{2^{t-1}}^k \cup I_{2^t}^k} \left(\Delta_{2^t}^k + \pi^\ell(x - \Delta_{2^t}^k) \right) \\ &= \prod_{t=1}^{2^{k-1}} \prod_{x \in I_1^k \cup I_2^k} \left(\Delta_{2^t}^k + \pi^\ell(x) \right) \\ &= \prod_{t=1}^{2^{k-1}} \left(\prod_{x \in I_1^k} \left(\Delta_{2^t}^k + \pi^\ell(x) \right) \prod_{x \in I_2^k} \left(\Delta_{2^t}^k + 2s^k - \pi^\ell(2s^k - x) \right) \right) \\ &= \prod_{t=1}^{2^{k-1}} \prod_{x \in I_1^k} \left(\Delta_{2^t}^k + \pi^\ell(x) \right) \left(\Delta_{2^t}^k + 2s^k - \pi^\ell(x) \right). \end{aligned}$$

The first equality follows from the fact that $\pi^\ell(x) = x$ for every $x \in A^k$. The second equality hold because of Claim 4.3(i). The third equality is due to Claim 4.6 and Claim 4.5, which ensures that $\Delta_{2^{t-1}}^k = \Delta_{2^t}^k$. The fourth equality holds by definition of $\Delta_{2^t}^k$ and by Claim 4.5 again. In the fifth equality we have used the fact that $\pi^\ell(x) = 2s^k - \pi^\ell(2s^k - x)$ for every $x \in I_2^k$: this follows directly from the definition of π^ℓ when $\ell = k$ and also when $\ell = k+1$ and $x \in I_2^k \setminus A^{k+1} = I_3^{k+1} \cup I_4^{k+1}$; when $\ell = k+1$ and $x \in I_2^k \cap A^{k+1}$, we know by Claim 4.3(iv) that $2s^k - x \in I_1^k \cap A^{k+1}$ and therefore $\pi^{k+1}(x) = x = 2s^k - (2s^k - x) = 2s^k - \pi^{k+1}(2s^k - x)$. The sixth equality follows from the fact that $I_k^2 = 2s^k - I_1^k$ by (4.3).

We claim that $\pi^k(x) \geq \pi^{k+1}(x)$ for every $x \in I_1^k$. This can be verified by observing the following facts:

- (i) for $x \leq x^{k+1}$, note that $x^{k+1} \leq x^k$ because $s^{k+1} \leq s^k$ and π is nondecreasing. So we have $\pi^k(x) = \pi^{k+1}(x) = \pi(x)$;
- (ii) for $x^{k+1} < x < \min\{2s^{k+1} - x^{k+1}, x^k + 1\}$, we have $\pi^k(x) = \pi(x) \geq s^{k+1} = \pi^{k+1}(x)$;
- (iii) for $2s^{k+1} - x^{k+1} \leq x \leq x^k$ (this interval may be empty), we have

$$\pi^k(x) = \pi(x) \geq \pi(2s^{k+1}) - \pi(2s^{k+1} - x) \geq 2s^{k+1} - \pi^{k+1}(2s^{k+1} - x) = \pi^{k+1}(x),$$

where the first inequality holds by subadditivity, and the second inequality follows from Claim 4.3(vi) and the fact that since $2s^{k+1} - x \leq x^{k+1}$ we have $\pi^{k+1}(2s^{k+1} - x) = \pi(2s^{k+1} - x)$; for the last equation, note that since $x \geq 2s^{k+1} - x^{k+1} > s^{k+1}$, there are two alternatives: either $x \in I_2^{k+1}$ and thus the equation holds by definition of π^{k+1} , or $x \in A^{k+1}$, in which case $2s^{k+1} - x \in A^{k+1}$ by Claim 4.3(iv) and thus $2s^{k+1} - \pi^{k+1}(2s^{k+1} - x) = 2s^{k+1} - (2s^{k+1} - x) = x = \pi^{k+1}(x)$;

- (iv) for $x^k < x < s^k$, we have two cases: if $x \leq s^{k+1}$, then $\pi^k(x) = s^k \geq s^{k+1} \geq \pi^{k+1}(x)$; otherwise, if $x > s^{k+1}$, then

$$\pi^k(x) = s^k \geq 2s^{k+1} - \frac{1}{2} \geq 2s^{k+1} - \pi^{k+1}(1) \geq 2s^{k+1} - \pi^{k+1}(2s^{k+1} - x) = \pi^{k+1}(x).$$

The first inequality follows from the definition of s^k . The second inequality holds because $\pi^{k+1}(1) \in \{\pi(1), s^{k+1}, 1\}$, and these values are all $\geq \frac{1}{2}$ (note that $bf(1) \geq \pi(b \cdot 1) = b$ thus $\pi(1) \geq 1$). For the third inequality, note that since $x < s^k$, we have $x \leq 2s^{k+1} - 1$ and thus $2s^{k+1} - x \geq 1$; also, since $x > s^{k+1}$, we have $2s^{k+1} - x < s^{k+1}$; this implies that $2s^{k+1} - x \in I_1^{k+1}$ and thus $\pi^{k+1}(2s^{k+1} - x) \geq \pi^{k+1}(1)$, as π^{k+1} is easily verified to be nondecreasing on I_1^{k+1} . Finally, the last equality holds by definition of π^{k+1} , using again the fact that $2s^{k+1} - x \in I_1^{k+1}$ and thus $x \in I_2^{k+1}$.

Therefore $\pi^k(x) \geq \pi^{k+1}(x)$ for every $x \in I_1^k$. Furthermore, $\pi^k(x) \leq s^k$ for $x \in I_1^k$. Now fix $t \in \{1, \dots, 2^{k-1}\}$. Since the function $\phi(y) := (\Delta_{2t}^k + y)(\Delta_{2t}^k + 2s^k - y)$ is concave and has its unique maximum point at $y = s^k$, we have that $\phi(\pi^k(x)) \geq \phi(\pi^{k+1}(x))$ for every $x \in I_1^k$, i.e.,

$$(\Delta_{2t}^k + \pi^k(x))(\Delta_{2t}^k + 2s^k - \pi^k(x)) \geq (\Delta_{2t}^k + \pi^{k+1}(x))(\Delta_{2t}^k + 2s^k - \pi^{k+1}(x)). \quad (4.6)$$

This implies $\mathcal{P}(\pi^k) \geq \mathcal{P}(\pi^{k+1})$. □

We observe that the above proof easily implies that g is the unique minimizer of the functional $\pi \mapsto \mathcal{P}(\pi)$. Indeed, if $\pi \neq g$, then by Claim 4.4 there exists $k \in \{1, \dots, K-1\}$ such that $\pi^k \neq \pi^{k+1}$. As these two functions coincide on A^k , this means that there exists $x \in I \setminus A^k$ such that $\pi^k(x) \neq \pi^{k+1}(x)$. By Claim 4.6, x can be assumed to be in $I_1^k \cup I_2^k$. Thus exactly one of x and $2s^k - x$ is in I_1^k , and the other is in I_2^k . We assume $x \in I_1^k$ and $2s^k - x \in I_2^k$ without loss of generality. Then (4.6) is a strict inequality for this particular x . This implies that $\mathcal{P}(\pi^k) > \mathcal{P}(\pi^{k+1})$.

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