ON LEXICOGRAPHIC APPROXIMATIONS OF INTEGER PROGRAMS

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ABSTRACT. We use the lexicographic order to define a hierarchy of primal and dual bounds on the optimum of a bounded integer program. These bounds are constructed using lex maximal and minimal feasible points taken under different permutations. Their strength is analyzed and it is shown that a family of primal bounds is tight for any 0\1 program with nonnegative linear objective, and a different family of dual bounds is tight for any packing- or covering-type 0\1 program with an arbitrary linear objective. The former result yields a structural characterization for the optimum of 0\1 programs, with connections to matroid optimization, and a heuristic for general integer programs. The latter result implies a stronger polyhedral representation for the integer feasible points and a new approach for deriving strong valid inequalities to the integer hull. Since the construction of our bounds depends on the computation of lex optima, we derive explicit formulae for lex optima of some special polytopes, such as polytopes that are monotone with respect to each variable, and integral polymatroids and their base polytopes. We also classify P and NP-hard cases of computing lex bounds and lex optima.

1. INTRODUCTION

The lexicographic order is a translation invariant total order (also called a term order) on $\mathbb{Z}^n$ and is abbreviated as lex order. For any permutation σ of $\{1,\dots,n\}$ and $x,y \in \mathbb{Z}^n$, we say that x is lexicographically less than equal to y under σ if either $x = y$ or there is some i such that $x_{\sigma(i)} < y_{\sigma(i)}$ and $x_{\sigma(k)} = y_{\sigma(k)}$ for all $k > i$. This is written as $x \preccurlyeq_\sigma y$. Alternatively, by denoting $\sigma \cdot x := (x_{\sigma(1)},\dots,x_{\sigma(n)})$ to be the action of the permutation σ on a point x, we have $x \preccurlyeq_\sigma y$ if and only if $\sigma\cdot x \preccurlyeq \sigma \cdot y$, where $\preccurlyeq$ is the lex order under identity permutation. Clearly, $x \preccurlyeq_\sigma y$ implies $(x_{\sigma(i)},\dots,x_{\sigma(n)}) \preccurlyeq (y_{\sigma(i)},\dots,y_{\sigma(n)})$ for $1 \le i < n$.

Let X be a nonempty compact set and assume $X \subset [0,u]$ for a given integral vector u. The associated discrete set is $X_I := X \cap \mathbb{Z}^n$,[1] and when it is nonempty, the lexmax and lexmin integral points in X are defined for every permutation σ as the optimal solutions to the following discrete optimization problems:

$$\begin{aligned} (\mathsf{LexMax}^\sigma_X) \quad & \theta^\sigma_X := \arg\max_{\preccurlyeq_\sigma} \{x \mid x \in X_I\}, \\ (\mathsf{LexMin}^\sigma_X) \quad & \gamma^\sigma_X := \arg\min_{\preccurlyeq_\sigma} \{x \mid x \in X_I\}. \end{aligned} \tag{1}$$

These points exist because X is compact and $X_I \neq \emptyset$, and are unique because $\preccurlyeq_\sigma$ is a total order. The fact that the lex order reverses after complementing variables, just like the usual partial order $\le$ on $\mathbb{R}^n$, implies that $\gamma^\sigma_X = u - \theta^\sigma_{u-X}$, where $u - X := \{u - x \mid x \in X\}$. The

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[1] Literature typically denotes X_I to be the convex hull of $X \cap \mathbb{Z}^n$, but we deviate from this since we are mostly interested in only the integer points in X.

lexmax and lexmin points are necessarily extreme points of the convex hull of X_I. They may not be distinct for different permutations. A worst-case example of this is $X = [0, u]$, for which we have $\theta^\sigma_X = u$ and $\gamma^\sigma_X = 0$ for all σ.

The lex order is commonly used in computational algebraic geometry for ordering monomials in a polynomial, and hence is indirectly connected to integer programming through the algebraic methods in literature that rely on computing the Gröbner basis of certain ideals [BW05]. A more direct connection is through optimization problems where the lex order is used for selecting the best optimal solution with respect to some criteria. These include, for example, fair allocation problems in decision theory [Fis74; OS09], location problems [Ogr97], bottleneck problems in combinatorial optimization [BR91], scheduling problems [CP79], characterizing optimal bases of a matroid [Fuj80; GGJ16], computing nucleolus of cooperative games [MPS79], and computing optimal flows in a network [Meg74; HT94]. In some of these applications, the feasible set is defined by lex constraints to signify preference with respect to some fixed solutions. The general question is: given a problem $\max\{f(x) \mid x \in X_I\}$ where $f\colon X_I \mapsto \mathbb{Z}$ is bounded (and also typically linear), find the optimal solution x^* that is lex maximal (or minimal) with respect to all the optimal solutions. This can be modeled as a problem in (1) of finding the lex optima of the set $\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x \in X, f(x) - x_{n+1} \geq 0\}$. Another area where lex optimization arises is finding lex best solutions from all the Pareto optima of multiobjective programs [Ehr05; ZÁ17]. A new application for problem (1) is in strengthening disjunctive cutting planes for mixed integer programs as part of a branch-and-cut algorithm that also involves branching lexicographically and thereby introducing lex constraints to the LP relaxation [Gup17].

It is clear from the definition that the lex optima of X can be found by a greedy algorithm that solves integer programs recursively as follows: for $i = n, \ldots, 1$,

$$\begin{aligned}(\theta^\sigma_X)_{\sigma(i)} &= \max\{x_{\sigma(i)} \mid x \in X_I,\, x_{\sigma(k)} = (\theta^\sigma_X)_{\sigma(k)},\ k > i\},\\ (\gamma^\sigma_X)_{\sigma(i)} &= \min\{x_{\sigma(i)} \mid x \in X_I,\, x_{\sigma(k)} = (\gamma^\sigma_X)_{\sigma(k)},\ k > i\}.\end{aligned} \tag{2}$$

This is the classical and most common method in literature. A practical implementation of this recursion has been to solve linear relaxations at each step, round the optimal solution, and backtrack whenever the linear program is infeasible. Another method for finding lex optima is to create a weighted scalar objective $\sum_i \lambda_i x_i$, for some nonnegative vector λ, and solve a single integer program that is equivalent to the lex optimization problem. Two different ways of constructing these λ's have been known in the context of multiobjective integer programs [She82], and these results also apply to the problem of finding lex optima after transforming the multiobjective program. The lesser of these two sequences has also been independently derived while studying superincreasing knapsack polytopes [Gup16], and is given by $\lambda_{\sigma(i)} = 1 + \sum_{j<i} \lambda_{\sigma(j)} u_{\sigma(j)}$. The permuted sequence $\sigma \cdot \lambda$ is referred to as a superincreasing sequence. In particular, we have that for any $x, y \in [0, u] \cap \mathbb{Z}^n$, $x \preccurlyeq_\sigma y$ if and only if $\lambda x \leq \lambda y$. When u is all ones and the permutation is identity, we have λ as the powers of 2. All of the known scalarization sequences suffer from the drawback that $\max_i \lambda_i = \Omega(2^n)$, thereby making them implementable only when n is small.

The lex order has also found use in breaking symmetry amongst optimal solutions of a $0\backslash 1$ integer program [Mar10]. This has led to polyhedral study of associated polytopes [KP08; HP17]. These polytopes have quite rich facet structure and arise as (1) orbitopes: convex hulls of $m \times n$ $0\backslash 1$ matrices whose columns are in lex decreasing order under a given permutation, w.l.o.g. the identity permutation, or as (2) symretopes: convex hulls of $0\backslash 1$

n-vectors that are lex bigger than each element in their orbit with respect to some given subgroup of the symmetric group of $\{1,\ldots,n\}$.

There is also a different notion of polytopes arising from the lex order and its variants — convex hulls of all integral vectors in $[0,u]$ that are lex lesser (or bigger) than a given integral vector under a given permutation. Such polytopes are defined by $\mathcal{O}(n)$ many inequalities and have been studied in the 0\1 case [LS92; GK06] and in generality [Gup16; GP16]. The inequality descriptions of such polytopes has been used to construct extended formulations of convex hulls of polytopes from which a given list of vertices is removed [Ang+15].

In this paper, we take a different approach to using the lex order for integer programs. The lex optima in (1) are feasible solutions, and hence can be used to approximate the integer optimum. Another line of thought is to recognize the fact that lex is a total order and for every permutation, consider the convex hull of all integral vectors that are lex between the lex minima and the lex maxima. Then we intersect these convex hulls over all permutations and study how the integer points in this intersection relate to the original feasible set, in particular, when is the intersection equal to the feasible set. Our approaches are applicable to any integer program and do not depend on symmetry considerations, unlike majority of the work in integer programming literature on using lex order. In the next section, we establish the necessary notation to give some background on our approaches, and thereafter, we state the contributions of this work.

1.1. **Background.** For any integer $n \geq 1$, we denote $[n] := \{1,\ldots,n\}$. The set of all permutations of $[n]$ is $\mathfrak{S}_n$.[2] For any $H \subseteq \mathfrak{S}_n$, the collection of lex maxima and lex minima corresponding to permutations in H is denoted by Θ^H_X and Γ^H_X, respectively:

$$\Theta^H_X := \{\theta^\sigma_X \mid \sigma \in H/\sim_{\max}\}, \quad \Gamma^H_X := \{\gamma^\sigma_X \mid \sigma \in H/\sim_{\min}\}, \quad H \subseteq \mathfrak{S}_n. \tag{3}$$

Here, the quotient space $H/\sim_{\max}$ represents the partition of H by the equivalence relation $\sim_{\max}$ on $\mathfrak{S}_n$, which is defined as $\sigma \sim_{\max} \tau$ if and only if $\theta^\sigma_X = \theta^\tau_X$. Similarly for $\sim_{\min}$ and $H/\sim_{\min}$. This ensures that Θ^H_X and Γ^H_X do not have duplicate elements, and in general, they can have much smaller cardinality than H.

The finite set of all integral vectors in $[0,u]$ that are lex-less than a fixed vector and lex-greater than another fixed vector is referred to as a *lex-ordered set*. A majority of this paper considers these fixed vectors to be the lexmax and lexmin vectors θ^σ_X and γ^σ_X, respectively, thereby making our lex-ordered set to be

$$\mathcal{L}^\sigma_X := \left\{x \in [0,u] \cap \mathbb{Z}^n \mid \gamma^\sigma_X \preccurlyeq_\sigma x \preccurlyeq_\sigma \theta^\sigma_X\right\}, \quad \sigma \in \mathfrak{S}_n. \tag{4a}$$

Note that $\mathcal{L}^\sigma_X$ is a implicitly-defined set because we assume that X and the bounding box $[0,u]$ as the input, and the points θ^σ_X and γ^σ_X are computed from this input. In the latter part of this paper, we will also consider explicitly-defined lex-ordered sets:

$$\mathcal{L}^\sigma_{\gamma,\theta} := \{x \in [0,u] \cap \mathbb{Z}^n \mid \gamma \preccurlyeq_\sigma x \preccurlyeq_\sigma \theta\}, \quad \sigma \in \mathfrak{S}_n \tag{4b}$$

where θ and γ are provided as input along with $[0,u]$.

The construction of θ^σ_X and γ^σ_X and the fact that $\preccurlyeq_\sigma$ is a total order gives us

$$\Theta^H_X \cup \Gamma^H_X \subseteq X_I \subseteq \bigcap_{\sigma \in H} \mathcal{L}^\sigma_X, \quad H \subseteq \mathfrak{S}_n. \tag{5}$$

[2]Although the notation $\mathfrak{S}_n$ is typically used for the symmetric group of $[n]$ with the group operation being the composition of two permutations, since we do not need to deal with properties of this group or any of its subgroups, we use $\mathfrak{S}_n$ simply as the set of all permutations.

That is, lex optima yield outer approximations (relaxations) and inner approximations of X_I. The relaxation can also be viewed as

$$\bigcap_{\sigma\in H} \mathcal{L}_X^\sigma = \left(\bigcap_{\sigma\in H} \operatorname{conv} \mathcal{L}_X^\sigma\right) \bigcap \mathbb{Z}^n.$$

Corresponding to these outer and inner approximations of X_I, we obtain lexicographic bounds on the integer optimum. Consider the integer optimization problem

$$z_c^* := \max\{cx \mid x \in X_I\} \tag{6a}$$

for $c \in \mathbb{R}^n$. For $H \subseteq \mathfrak{S}_n$, denote

$$\underline{z}_c(H) := \max\{\underline{z}_c(\Theta_X^H), \underline{z}_c(\Gamma_X^H)\}, \quad \underline{z}_c(\Theta_X^H) := \max_{\sigma\in H} c\,\theta_X^\sigma, \quad \underline{z}_c(\Gamma_X^H) := \max_{\sigma\in H} c\,\gamma_X^\sigma, \tag{6b}$$

$$\bar{z}_c(H) := \max\{cx \mid x \in \cap_{\sigma\in H}\mathcal{L}_X^\sigma\}. \tag{6c}$$

Equation (5) implies that $\underline{z}_c(H) \le z_c^* \le \bar{z}_c(H)$. Moreover, we have

$$\underline{z}_c(H) \le \underline{z}_c(H') \le z_c^* \le \bar{z}_c(H') \le \bar{z}_c(H), \quad H \subset H' \subseteq \mathfrak{S}_n. \tag{6d}$$

We refer to $\underline{z}_c(H)$ as a *lex primal bound* and to $\bar{z}_c(H)$ as a *lex dual bound.*

The concept of lex bounds is illustrated via a simple example in Figure 1 where the polytope $X \subset [0,3]^2$ is defined by three inequalities besides nonnegativity, each having a positive slope. Here, we have two permutations, $\sigma^1 = 2\,1$ and $\sigma^2 = 1\,2$, the lex minima under which are $(0,0)$ and the lex maxima are θ^1 and θ^2, respectively. As can be seen, we have $\mathcal{L}_X^1 \cap \mathcal{L}_X^2 = X_I$, which would make our lex dual bound be tight, and for every $(c_1, c_2) \ge (0,0)$, one of θ^1 or θ^2 attains the optimum, which would make the primal bound be tight. Also note that $\operatorname{conv}\mathcal{L}_X^1 \cap \operatorname{conv}\mathcal{L}_X^2$, which is a relaxation of the dual bound given by $\operatorname{conv}(\mathcal{L}_X^1 \cap \mathcal{L}_X^2)$, is a strict superset of $\operatorname{conv} X_I$ and contains the fractional points denoted by the hashed region.

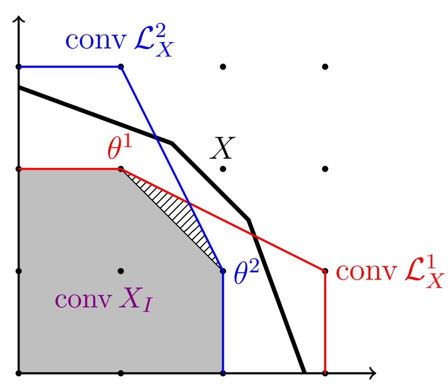

FIGURE 1. Lexicographic bounds

Some basic notation that will be used throughout is as follows. The vector of all ones is $\mathbb{1}$ and for the vector of all zeros, we simply use 0. The lengths of these vectors will be obvious from the context in which they are used. The convex hull of a set $\mathcal{X}$ is $\operatorname{conv}\mathcal{X}$ and its set of vertices is $\operatorname{vert}\mathcal{X}$.

1.2. **Contributions & Outline.** The generality of the lexicographic bounds immediately raises the question of how good can these bounds be. In §2, we provide an analysis of the quality of these bounds with respect to the optimum z_c^*. We also identify conditions on X_I and the objective c under which this these bounds are tight (i.e., equal z_c^*). We begin with the primal bounds in §2.1, and establish $\underline{z}_c(\mathfrak{S}_n)$ to be a weak bound in general, by showing it to be a $\frac{1}{n}$-factor approximation to z_c^*, but also showing it to be tight when $X_I \subseteq \{0,1\}^n$ and $c \geq 0$. In fact, these claims hold for a family of primal bounds, not just $\underline{z}_c(\mathfrak{S}_n)$, where the family is characterized by certain permutations of maximal feasible points. A consequence of this is the equivalence of lex maxima (resp. lex minima) and maximal (resp. minimal) points for $0\backslash1$ sets. In §2.2, we study the strength of the dual bounds for monotone sets. A necessary and sufficient condition for these bounds to be tight is derived, but since this condition is difficult to check, a simple sufficient condition is given subsequently, along with proving $\bar{z}_c(H)$ to be tight for a family of collections H when $X_I \subseteq \{0,1\}^n$. Consequences of this tightness are a stronger polyhedral formulation for integer feasible points, and a new approach for deriving strong valid inequalities to the integer hull. Thus, our analysis of these bounds yields new structural results on integer programs. The main reason why our proofs for showing the primal and dual bounds to be tight work only for $0\backslash1$ programs is that these problems do not have any feasible solutions in the interior of the integer hull.

Since the lex bounds depend on the computation of lex optima of a given set, we next address the question of finding lex optima when X_I is input implicitly as the set of integer solutions to X. It is assumed that X is either input explicitly as a system of inequalities or equipped with a polynomial-time separation oracle. Recursion (2) is the naive method for computing lex optima and is intractable in general. In §3, we use this recursion to provide explicit expressions for lex optima of some special polytopes, such as polytopes that are monotone with respect to each variable, and polymatroids and their base polytopes, which are important sets in combinatorial optimization. An algorithm is presented for computing lex optima, based on coordinate-wise bisection search and solving integer feasibility problems at each step. This algorithm is conceptual in nature, and is used later for obtaining polynomial time cases.

Another contribution is to address the computational complexity of lex bounds. This is the topic of §4. Computing lex optima of X is NP-hard in general since it involves solving integer programs. However, even if every lex optima was computable in polynomial-time, which is the case for example with monotone sets, the strongest lex bounds, $\underline{z}_c(\mathfrak{S}_n)$ and $\bar{z}_c(\mathfrak{S}_n)$, are still NP-hard to compute; this follows from our tightness results in §2. We argue that the hardness persists even when $|H|$ is linear in n. Furthermore, we classify P and NP-hard cases of computing lex optima for explicitly input general X using the integer feasibility decision problem for X. This gives us a generalization of the fact that lex optima of monotone sets can be computed easily, since we show that any set with easy integer feasibility problems has easy lex optimization problems. Finally, we tackle the complexity of finding lex optima of the set defined as the intersection of a polyhedron with finitely many lex constraints.

The paper finishes in §5 with a range of open questions regarding lex bounds.

2. Strength of Lex Bounds

2.1. **Primal bounds.** The primal bound $\underline{z}_c(H)$ is a lower bound on z_c^*, as per (6d). For a given $c \in \mathbb{R}^n$, let

$$X_c^* := \{x \in X_I \mid cx = z_c^*\} \tag{7}$$

denote the set of integral optimal solutions to z_c^*. This set is nonempty due to X being compact and X_I being nonempty. Recall that $\underline{z}_c(H)$ is defined as the maximum of $\underline{z}_c(\Theta_X^H)$ and $\underline{z}_c(\Gamma_X^H)$. Then it is clear that

$$z_c^* = \underline{z}_c(H) \iff X_c^* \cap (\Theta_X^H \cup \Gamma_X^H) \neq \emptyset. \tag{8}$$

Thus, one should not expect the bound $\underline{z}_c(H)$ to be tight for arbitrary H. Assume $c \geq 0$ for the rest of this section[3].

For nonnegative objectives, Theorem 2.1 proves the following about the strength of lex primal bounds. The strongest bound is a very weak bound in general, as we establish that it is a $\frac{1}{n}$-factor approximation to z_c^*. In fact, we argue this to be true for $\underline{z}_c(H)$ when H is any subset of $\mathfrak{S}_n$ with the property that for every $i \in [n]$ there is some $\sigma \in H$ for which $\sigma(n) = i$, i.e., H belongs to the collection

$$\mathcal{H}_1 := \{H \subseteq \mathfrak{S}_n \mid \cup_{\sigma \in H} \sigma(n) = [n]\}. \tag{9}$$

This $\frac{1}{n}$ approximation factor is shown to be arbitrarily tight in general for every n. For subsets of $\{0,1\}^n$, we argue that the lex bounds close the gap to the integer optimum, and characterize the permutations that allow this. To describe these permutations, we make use of integral maximal points of X with respect to the partial order $\leq$; these are defined as

$$\mathcal{M}_X^+ = \{x \in X_I \mid \nexists\, x' \in X_I \setminus \{x\} \text{ s.t. } x \leq x'\}. \tag{10}$$

Note that for maximal points, there is no dependence on the permutation of variables. We will need the following definition, which will be of use in the next section also.

Definition 1. For $x \in \{0,1\}^n$, $\sigma \in \mathfrak{S}_n$ is called a *monotone* permutation of x if $\sigma \cdot x = (0, \ldots, 0, 1, \ldots 1)$, i.e., for every $i \neq j$, $x_i = 0, x_j = 1$ implies $\sigma^{-1}(i) < \sigma^{-1}(j)$. The set of all such permutations is denoted by $\mathfrak{I}_x$.

Denote

$$\mathcal{H}_2 := \{H \subseteq \mathfrak{S}_n \mid H \cap \mathfrak{I}_x \neq \emptyset,\ \forall x \in \mathcal{M}_X^+\}, \quad \text{when } X_I \subseteq \{0,1\}^n. \tag{11}$$

That is, $\mathcal{H}_2$ is the collection of subsets of $\mathfrak{S}_n$ that have at least one monotone permutation for each maximal point. Using $\mathcal{H}_1$ and $\mathcal{H}_2$, we state our approximation guarantee for the primal bound.

Theorem 2.1. *Assume $c \geq 0$.*[3]

(1) We have

$$\underline{z}_c(H) \geq \underline{z}_c(\Theta_X^H) \geq \frac{1}{n} z_c^*, \quad H \in \mathcal{H}_1$$

and there is a family of polytopes for which this approximation factor is arbitrarily tight for every $n \geq 2$.

(2) When $X_I \subseteq \{0,1\}^n$,

$$z_c^* = \underline{z}_c(\Theta_X^H) = \underline{z}_c(H), \quad H \in \mathcal{H}_2.$$

[3]For optimizing over a compact X, this assumption is w.l.o.g. up to complementing variables because if some $c_i < 0$, then we could simply consider the set X' obtained by replacing x_i with $u_i - x_i$ in X (although the lex optima of X' would be different).

Proof. The relation $\underline{z}_c(H) \geq \underline{z}_c(\Theta_X^H)$ is by definition (6b). Pick any $x^* \in X_c^*$. Set $t := \arg\max\{c_i x_i^* \mid i \in [n]\}$, so that $n c_t x_t^* \geq cx^* = z_c^*$, which yields $c_t x_t^* \geq \frac{1}{n} z_c^*$. Since $H \in \mathcal{H}_1$, we know that there exists some permutation $\tau \in H$ such that $t = \tau(n)$. Since $x^* \in X_c^* \subseteq X_I$, we have $x^* \preccurlyeq_\sigma \theta_X^\tau$ which implies that $x_t^* \leq (\theta_X^\tau)_t$. Nonnegativity of c and θ_X^σ imply that $c\,\theta_X^\tau \geq c_t(\theta_X^\tau)_t \geq c_t x_t^*$. Therefore,

$$\underline{z}_c(H) = \max_{\sigma \in H} \max\{c\,\theta_X^\sigma, c\,\gamma_X^\sigma\} \geq c\,\theta_X^\tau \geq c_t(\theta_X^\tau)_t \geq c_t x_t^* \geq \frac{1}{n} z_c^*.$$

The above bound cannot be improved. Let $c = \mathbb{1}$, $\kappa \geq 2$ be an integer, and X be the convex hull of $\{\kappa\mathbf{e}_1, \kappa\mathbf{e}_2, \dots, \kappa\mathbf{e}_n, (\kappa-1)\mathbb{1}\}$. Note that X is an integral polytope. We have $z_c^* = (\kappa-1)n$ attained uniquely at $x^* = (\kappa-1)\mathbb{1}$. Since θ_X^σ and γ_X^σ are vertices of $\operatorname{conv} X_I$ for every σ, we have $\theta_X^\sigma = \kappa\mathbf{e}_{\sigma(n)}$ and $\gamma_X^\sigma = \kappa\mathbf{e}_{\sigma(1)}$. Thus $c\,\theta_X^\sigma = c\,\gamma_X^\sigma = \kappa$, making every lex primal bound equal to κ. The approximation ratio becomes

$$\frac{\underline{z}_c(H)}{z_c^*} = \frac{\kappa}{(\kappa-1)n},$$

whose limit as $\kappa \to \infty$ is $1/n$.

Now let us assume $X_I \subseteq \{0,1\}^n$. By (8), we know that $X_c^* \cap \Theta_X^H \neq \emptyset$ is a necessary and sufficient condition for $z_c^* = \underline{z}_c(\Theta_X^H) = \underline{z}_c(H)$. Since $X_I \subseteq \{0,1\}^n$, $\operatorname{vert} X_I = X_I$.

Claim 2.1. *There exists some $x^* \in X_c^* \cap \mathcal{M}_X^+$ with the largest cardinality (ℓ_1-norm) in X_c^*.*

Proof of Claim. We need to show that

$$\max\{\sum_{i=1}^n x_i \mid x \in X_c^*\} = \max\{\sum_{i=1}^n x_i \mid x \in X_c^* \cap \mathcal{M}_X^+ \cap \operatorname{vert} X_I\}.$$

The $\geq$ inequality is obvious, and so we need to argue the $\leq$ inequality. First, we claim that $X_c^* \cap \mathcal{M}_X^+ \cap \operatorname{vert} X_I \neq \emptyset$. Since $c \geq 0$, $x \leq y$ implies $cx \leq cy$, which leads to $X_c^* \cap \mathcal{M}_X^+ \neq \emptyset$. Note that $\operatorname{conv} X_c^*$ is a face of $\operatorname{conv} X_I$, and therefore, $\operatorname{vert} X_c^* = \operatorname{vert} X_I \cap X_c^*$. If $\operatorname{vert} X_c^* \cap \mathcal{M}_X^+ = \emptyset$, then $X_c^* \cap \mathcal{M}_X^+ \neq \emptyset$ implies that there is some maximal point which is a convex combination of non-maximal points, which is obviously a contradiction. This proves our claim $X_c^* \cap \mathcal{M}_X^+ \cap \operatorname{vert} X_I \neq \emptyset$. Linearity of $x \mapsto \sum_i x_i$ tells us that some point in $X_c^* \cap \mathcal{M}_X^+ \cap \operatorname{vert} X_I$ achieves the largest cardinality on X_c^*. ◇

Let x^* be the point from the above claim. For every $H \in \mathcal{H}_2$, we know there exists some $\sigma \in H$ such that $\sigma \cdot x^* = (0, \dots, 0, 1, \dots, 1)$. Since $\sigma \in H$, $\theta_X^\sigma \in \Theta_X^H$. We claim that $x^* = \theta_X^\sigma$, which implies $X_c^* \cap \Theta_X^H \neq \emptyset$ and hence finishes our proof. Assume for contradiction that $x^* \neq \theta_X^\sigma$. Since $x^* \in X_I$ and θ_X^σ is lexmax in X_I, $x^* \prec_\sigma \theta_X^\sigma$. Since the permutation σ puts all 1's in x^* at the end, $x_{\sigma(i)}^* = 1$ implies $(\theta_X^\sigma)_{\sigma(i)} = 1$. Therefore, $x^* \lneqq \theta_X^\sigma$. This implies $cx^* \leq c\,\theta_X^\sigma$ since $c \geq 0$, and then $x^* \in X_c^*$ implies that $cx^* = c\,\theta_X^\sigma$ and $\theta_X^\sigma \in X_c^*$. The integrality of x^* and θ_X^σ, and $x^* \lneqq \theta_X^\sigma$, imply that that $\sum_{i=1}^n((\theta_X^\sigma)_i - x_i^*) \geq 1$. After rearranging terms, this becomes $\sum_i (\theta_X^\sigma)_i \geq 1 + \sum_i x_i^*$. Since $\theta_X^\sigma \in X_c^*$, this means that x^* does not have the largest cardinality in X_c^*, but this is a contradiction to the construction of x^*. Therefore, x^* must be equal to θ_X^σ. □

Similar claim holds for $c \leq 0$ after replacing max with min everywhere and θ_X^σ with γ_X^σ. In the above proof, the assumption $X_I \subseteq \{0,1\}^n$ is crucial in the step where we conclude that $x^* \lneqq \theta_X^\sigma$. The same argument does not work for a general discrete set because the interval $\{0, 1, \dots, u_i\}$ contains integer points in its interior for $u_i \geq 2$.

Since $\mathfrak{S}_n \in \mathcal{H}_1, \mathcal{H}_2$, we have the following guarantees for primal bound from all lex optima.

Corollary 2.2. *For $c \geq 0$, $\underline{z}_c(\Theta_X^{\mathfrak{S}_n}) \geq \frac{1}{n} z_c^*$ in general, and for $X_I \subseteq \{0,1\}^n$, $\underline{z}_c(\Theta_X^{\mathfrak{S}_n}) = z_c^*$.*

Remark 1. Theorem 2.1 tells us that there exists some permutation such that a lex maxima under this permutation is a optimal solution to an integer program. Our proof is existential, meaning that it does not construct a permutation explicitly. Hence, we cannot say which permutation provides the optimal lex maxima. If additional information, besides $X_I \subseteq \{0,1\}^n$, on the structure of X_I is known, then one could possibly make more profound claims. For example, if X_I corresponds to a matroid, then Edmonds [Edm70] has shown that an optimal solution is given by the greedy feasible solution that is constructed recursively in the permutation σ that has $c_{\sigma(1)} \geq c_{\sigma(2)} \geq \cdots c_{\sigma(n)}$. In other words, since lex maxima are equivalent to greedy solutions by definition, this classical result of Edmonds tells us which lex optima solves the matroid optimization problem.

A consequence of lex optima being an integer optimum is that it establishes equivalence between lex maxima (resp. minima) and maximal (resp. minimal) points of a 0\1 set. We discuss this next for maximal points; the arguments for minimal points are analogous.

Maximal points were defined in (10). The set of maximal vertices is denoted by

$$\mathcal{MV}_X^+ := \mathcal{M}_X^+ \cap \operatorname{vert} X_I. \tag{12}$$

The following result provides a useful geometric characterization for them. We believe this should be well-known, but since we were unable to find a reference, we provide a proof here for completeness.

Lemma 2.3. *We have*

$$\mathcal{MV}_X^+ = \{v \in \operatorname{vert} X_I \mid \operatorname{rel.int}(\mathcal{N}_{X_I}(v)) \cap \mathbb{R}_+^n \neq \emptyset\},$$

where $\operatorname{rel.int}(\cdot)$ *is the relative interior of a set and* $\mathcal{N}_{X_I}(v)$ *is the normal cone to* $\operatorname{conv} X_I$ *at vertex v.*

Proof. Let v be any vertex of $\operatorname{conv} X_I$ and let the vertices of $\operatorname{conv} X_I$ that are adjacent to v be denoted by $v^1, \ldots, v^k$ for some finite k. Construct the $k \times n$ matrix $\mathcal{B}$ whose rows are $v^i - v$ for $i = 1, \ldots, k$. The polyhedral cone $\mathcal{C}_{X_I}(v) := \{\mathcal{B}^\top y \mid y \geq 0\}$ is the tangent cone to X_I (and also the cone of feasible directions) at v. The polar of this cone is the normal cone $\mathcal{N}_{X_I}(v)$, whose relative interior is equal to $\{c \mid \mathcal{B}c < 0\}$. By definition of maximality, the vertex v is maximal if and only if there does not exist a nonzero nonnegative element in $\mathcal{C}_{X_I}(v)$. By a modified version of Gordan's theorem (which is a special case of Farkas lemma), this is equivalent to saying that there exists a $\xi \geq 0$ such that $\mathcal{B}\xi < 0$. The set $\{z \geq 0 \mid \mathcal{B}z < 0\}$ is precisely $\operatorname{rel.int}(\mathcal{N}_{X_I}(v)) \cap \mathbb{R}_+^n$. □

Since the partial order $\leq$ preserves the total order $\preccurlyeq_\sigma$, and lex optima are vertices of $\operatorname{conv} X_I$, it is obvious that for every $H \subseteq \mathfrak{S}_n$,

$$\Theta_X^H \subseteq \mathcal{MV}_X^+. \tag{13}$$

We next prove that a necessary and sufficient condition for equality between these sets is the lex primal bounds closing the gap to the integer optimum, which leads to an equivalence between lex maxima and maximal points for 0\1 sets.

Proposition 2.4. *$\Theta_X^H = \mathcal{MV}_X^+$ if and only if $z_c^* = \underline{z}_c(\Theta_X^H)$ for any $c \geq 0$. In particular, if $X_I \subseteq \{0,1\}^n$, then $\Theta_X^H = \Theta_X^{\mathfrak{S}_n} = \mathcal{M}_X^+$ for any $H \in \mathcal{H}_2$.*

Proof. We know that $X_c^* \cap \mathcal{MV}_X^+ \neq \emptyset$ for $c \geq 0$. This makes the only if direction obvious. Now suppose $z_c^* = \underline{z}_c(\Theta_X^H)$ for any $c \geq 0$. We have to argue that $\Theta_X^H \supseteq \mathcal{MV}_X^+$ since we know from (13) that the reverse inclusion always holds. Take any $x \in \mathcal{MV}_X^+$. The characterization of $\mathcal{MV}_X^+$ in Lemma 2.3, and the fact that the relative interior of the normal cone at a vertex v contains exactly those vectors for which the maximum value of linear optimization over the polytope is attained uniquely at v, tell us that there exists some $c \geq 0$ for which x is the unique optimum for z_c^*. By assumption, $z_c^* = \underline{z}_c(\Theta_X^H)$, and since x is the unique optimum, it must be that $x \in \Theta_X^H$.

When $X_I \subseteq \{0,1\}^n$, we have vert $X_I = X_I$, so that $\mathcal{M}_X^+ = \mathcal{MV}_X^+$. Theorem 2.1 tells us $z_c^* = \underline{z}_c(\Theta_X^H)$ for $c \geq 0$ and $H \in \mathcal{H}_2$, and then the first part leads to $\Theta_X^H = \mathcal{M}_X^+$. Since $\mathfrak{S}_n \in \mathcal{H}_2$, $\Theta_X^{\mathfrak{S}_n} = \mathcal{M}_X^+$ follows. □

In general, the inclusion in (13) is strict; one can see this from simple examples. However, for integral polymatroids, which are special polytopes arising in combinatorial optimization, we have a bijection between lex maxima and maximal vertices.

Proposition 2.5. $\Theta_{P_f}^{\mathfrak{S}_n} = \mathcal{MV}_{P_f}^+ = \operatorname{vert} EP_f$ *when* P_f *is an integral polymatroid.*

Discussion on polymatroids and a proof of the above result is provided in §3.1. It follows from Propositions 2.4 and 2.5 that the primal lex bound is tight for integral polymatroids.

Corollary 2.6. $z_c^* = \underline{z}_c(\mathfrak{S}_n)$ *when* $c \geq 0$ *and* P_f *is an integral polymatroid.*

2.2. **Dual bounds for monotone sets.** The dual bound $\bar{z}_c(H)$ is an upper bound on z_c^* as per (6d) due to $\cap_{\sigma\in H}\mathcal{L}_X^\sigma$ being a relaxation of X_I. An obvious necessary and sufficient condition for z_c^* to be equal to $\bar{z}_c(H)$ for every c is that these two sets be equal, or equivalently, that every point in $\cap_{\sigma\in H}\mathcal{L}_X^\sigma$ also belong to X_I:

$$z_c^* = \bar{z}_c(H) \ \ \forall c \iff X_I = \cap_{\sigma\in H}\mathcal{L}_X^\sigma \iff X_I \supseteq \cap_{\sigma\in H}\mathcal{L}_X^\sigma. \tag{14}$$

As with the primal bounds, one should not expect the dual bounds to always be tight. For the rest of this section, we restrict our attention to analyzing when the dual bounds can be tight for *monotone* sets.

A set $\mathcal{X}$ is said to be ↓-monotone over $\mathbb{F}^n$, for $\mathbb{F} = \mathbb{R}, \mathbb{Z}$, if it is closed under the partial order $\leq$, i.e., $x \in \mathcal{X}$ and $y \in \mathbb{F}^n$ with $0 \leq y \leq x$ implies that $y \in \mathcal{X}$. This is a generalization of the notion of an *independence system*, which is a family of subsets that is closed under inclusion, and corresponds to a ↓-monotone subset of $\{0,1\}^n$. We will refer to discrete sets being ↓-monotone over $\mathbb{Z}^n$ and convex sets being ↓-monotone over $\mathbb{R}^n$. Similar definitions hold for ↑-monotone sets over $\mathbb{Z}^n$ and $\mathbb{R}^n$. A ↓-monotone (resp. ↑-monotone) polytope is sometimes also referred to as a *packing* (resp. *covering*) polytope. It is well-known that the following are equivalent (analogous statements hold for ↑-monotone sets):

(1) X_I is ↓-monotone over $\mathbb{Z}^n$,
(2) $\operatorname{conv} X_I$ is ↓-monotone over $\mathbb{R}^n$,
(3) all nontrivial facet-defining inequalities of $\operatorname{conv} X_I$ are of the form $\alpha x \leq \alpha_0$ for some $\alpha \geq 0, \alpha_0 > 0$.

Since every lex maxima is a maximal point (cf. (13)), if there exists $x' \in X_I$ such that $X \subseteq x' - \mathbb{R}^n_{\geq 0}$, then $\theta_X^\sigma = x'$ for every σ. Similarly, for γ_X^σ. This means that $\gamma_X^\sigma = 0$ when X_I is ↓-monotone and $\theta_X^\sigma = u$ when X_I is a ↑-monotone set.

The fact that the lex order reverses upon complementing variables and that a bounded set is ↑-monotone if and only if its complement is a ↓-monotone set, leads to the following.

Observation 2.7. *$X_I = \cap_{\sigma\in H}\mathcal{L}_X^\sigma$ for any ↑-monotone X_I if and only if $Y_I = \cap_{\sigma\in H}\mathcal{L}_Y^\sigma$ for $Y_I = u - X_I := \{u - x \mid x \in X_I\}$.*

Equation (14) and Observation 2.7 tell us that when dealing with tightness of lex bounds for monotone sets, it suffices to analyze conditions under which $X_I = \cap_{\sigma\in H}\mathcal{L}_X^\sigma$ for ↓-monotone sets, which is what we do henceforth. Analogous statements of the forthcoming results hold for ↑-monotone sets.

We note that the strongest lex relaxation, $\cap_{\sigma\in\mathfrak{S}_n}\mathcal{L}_X^\sigma$, can be a strict relaxation of X_I when $X_I \not\subseteq \{0,1\}^n$, as seen next on a family of trivial integral ↓-monotone polytopes.

Proposition 2.8. *For any integer $n \geq 4$ and real $\delta \geq 2$, the polytope*

$$X = \left\{ x \in [0,\delta]^n \;\middle|\; \sum_{i=1}^n x_i \leq (\delta-1)n \right\}$$

has $X_I \subsetneq \cap_{\sigma\in\mathfrak{S}_n}\mathcal{L}_X^\sigma$, and hence $z_c^* < \bar{z}_c(\mathfrak{S}_n)$ for some c.

Proof. Note that $(\delta-1)\mathbb{1} \in X_I$. Since $0 \in X$, we have $\gamma_X^\sigma = 0$. Denote $\rho := \left\lfloor \frac{(\delta-1)n}{\delta} \right\rfloor$. We have $\rho \geq 2$ due to $n \geq 4$ and $\delta/(\delta-1) \leq 2$ for $\delta \geq 2$. Then the lexmax points can be characterized as follows:

$$\Big((\theta_X^\sigma)_{\sigma(1)}, \ldots, (\theta_X^\sigma)_{\sigma(n)}\Big) = (\underbrace{0,\ldots,0}_{n-\rho-1},\, (\delta-1)n - \delta\rho,\, \underbrace{\delta,\ldots,\delta}_{\rho}).$$

Consider the point

$$(y_1,\ldots,y_n) = (\underbrace{\delta-1,\ldots,\delta-1}_{n+1-\rho},\, \underbrace{\delta,\ldots,\delta}_{\rho-1}).$$

We have $\sum_{i=1}^n y_i = (\delta-1)(n+1-\rho) + \delta(\rho-1) = n(\delta-1) + \rho - 1 > n(\delta-1)$, and so $y \notin X_I$. For every $\sigma \in \mathfrak{S}_n$ and $i = n+1-\rho,\ldots,n$, we have $y_{\sigma(i)} \leq \delta = (\theta_X^\sigma)_{\sigma(i)}$, with at least one of these inequalities being strict. Therefore, $y \preccurlyeq_\sigma \theta_X^\sigma$ and since $y \geq 0 = \gamma_X^\sigma$, we get $y \in \mathcal{L}_X^\sigma$ for all σ. Thus, $\left(\cap_{\sigma\in\mathfrak{S}_n}\mathcal{L}_X^\sigma\right) \setminus X_I \neq \emptyset$. Equation (14) implies $z_c^* < \bar{z}_c(\mathfrak{S}_n)$ for some c. □

The above example requires $\delta \geq 2$ and cannot be altered to obtain a 0\1 set. In fact, we prove in Theorem 2.12 that there does not exist any ↓-monotone 0\1 set X_I for which $\cap_{\sigma\in\mathfrak{S}_n}\mathcal{L}_X^\sigma$ is a strict relaxation of X_I. Before we get to this, we establish in Proposition 2.11 a necessary and sufficient condition for dual lex bounds to be tight for monotone sets. The condition is stated in terms of maximal points (recall definition of $\mathcal{M}_X^+$ from (10)) of the lex relaxation, making it somewhat more tractable than simply saying that the lex relaxation cannot have infeasible points as per (14). This naturally allows us to characterize monotone sets that are equal to a *single* lex-ordered set, so that taking intersection over many permutations in $\mathfrak{S}_n$ is unnecessary.

We will use the following two claims to derive our condition.

Lemma 2.9. *If X_I is ↓-monotone over $\mathbb{Z}^n$, then any $Y \subseteq \mathbb{Z}_+^n$ satisfies $Y \subseteq X_I$ if and only if $\mathcal{M}_Y^+ \subseteq X_I$.*

Proof. The only if implication is obvious from $Y \subseteq \mathbb{Z}^n$ giving us $\mathcal{M}_Y^+ \subseteq Y$. Suppose $\mathcal{M}_Y^+ \subseteq X_I$. For every $y \in Y$, there exists some $x \in \mathcal{M}_Y^+$ such that $0 \leq y \leq x$. This x belongs to X_I. The monotonicity of X_I implies that $y \in X_I$. □

Lemma 2.10. *$\mathcal{M}^+_{\mathcal{L}^\sigma_X} = \{\theta^\sigma_X\} \cup \{\vartheta^{\sigma,i} \mid i \in [n]$ s.t. $(\theta^\sigma_X)_{\sigma(i)} \geq 1\}$, where*

$$\vartheta^{\sigma,i} := \begin{cases} \theta^\sigma_X - \mathbf{e}_{\sigma(i)} + \sum_{j=1}^{i-1}(u_{\sigma(j)} - (\theta^\sigma_X)_{\sigma(j)})\,\mathbf{e}_{\sigma(j)}, & (\theta^\sigma_X)_{\sigma(i)} \geq 1 \\ \theta^\sigma_X, & (\theta^\sigma_X)_{\sigma(i)} = 0. \end{cases} \tag{15}$$

This is straightforward to verify.

Proposition 2.11. *Let X_I be $\downarrow$-monotone over $\mathbb{Z}^n$. Then $z^*_c = \bar{z}_c(H)$ if and only if $\mathcal{M}^+_{\cap_{\sigma\in H}\mathcal{L}^\sigma_X} \subseteq X_I$. Furthermore, $X_I = \mathcal{L}^\sigma_X$ for some σ if and only if $\vartheta^{\sigma,i} \in X_I$ for all $i \in [n]$.*

Proof. Applying Lemma 2.9 with $Y = \cap_{\sigma\in H}\mathcal{L}^\sigma_X$, and equation (14), leads to the first claim. To know when $X_I = \mathcal{L}^\sigma_X$, we already know that $X_I \subseteq \mathcal{L}^\sigma_X$, and so $X_I = \mathcal{L}^\sigma_X$ if and only if $X_I \supseteq \mathcal{L}^\sigma_X$. Applying Lemma 2.9 with $Y = \mathcal{L}^\sigma_X$ we have that $X_I \supseteq \mathcal{L}^\sigma_X$ if and only if $X_I \supseteq \mathcal{M}^+_{\mathcal{L}^\sigma_X}$. Lemma 2.10 and the fact that $\theta^\sigma_X \in X_I$, finishes our proof. □

Let us illustrate a utility of Proposition 2.11 on the most basic $\downarrow$-monotone set, a knapsack polytope. Consider $\mathcal{K} := \{x \in [0,u] \mid ax \leq b\}$ for $(a,b) \in \mathbb{Z}^{n+1}_{\geq 1}$. If the sequence $\{a_i\}_{i=1}^n$ is superincreasing, meaning that for some σ it satisfies $a_{\sigma(i)} \geq \sum_{j<i} a_{\sigma(j)}u_{\sigma(j)}$ for $i \in [n]$, then $\mathcal{K}_I = \mathcal{L}^\sigma_X$ [cf. Gup16]. The converse of this statement is also true: if $\mathcal{K}_I = \mathcal{L}^\sigma_X$, then there exists some integer π_0 and a superincreasing integer sequence $\{\pi_i\}_{i=1}^n$ such that $\mathcal{K}_I = \{x \in [0,u] \cap \mathbb{Z}^n \mid \pi x \leq \pi_0\}$ (just take $\pi_{\sigma(i)} = 1 + \sum_{j<i}\pi_{\sigma(j)}u_{\sigma(j)}$ and $\pi_0 = \pi\theta^\sigma_X$). Thus the superincreasing property is sufficient for a knapsack to be a lex-ordered set. But it is not necessary, and hence is different than the necessary and sufficient condition in the second part of Proposition 2.11. To see this, consider the following example.

Example 1. Let $a = (2, 8, 40, 150, 310)$, $b = 825$, and $u = (1, 5, 4, 1, 2)$. Since $a_i < a_{i+1}$, the only permutation that can have a_i's be superincreasing is the identity permutation, but this has $a_i < \sum_{j<i} a_j u_j$ for $i = 3, 4, 5$, and so $\mathcal{K}_I$ is not a superincreasing knapsack. For $\sigma = id$, we have $\theta^{id}_X = (1, 1, 1, 1, 2)$, and the related maximal points, as per Lemma 2.10, are $\vartheta^{id,1} = (0, 1, 1, 1, 2)$, $\vartheta^{id,2} = (1, 0, 1, 1, 2)$, $\vartheta^{id,3} = (1, 5, 0, 1, 2)$, $\vartheta^{id,4} = (1, 5, 4, 1, 1)$, and $\vartheta^{id,5} = (1, 5, 4, 1, 1)$. Since the largest value of $a\vartheta^{id,i}$ is 822, achieved by $\vartheta^{id,2}$, and is less than the right hand side 825, Proposition 2.11 tells us that this knapsack is a lex-ordered set under the identity permutation. We do not know if there are other permutations that satisfy the condition $\vartheta^{\sigma,i} \in \mathcal{K}_I$ for all i.

The second claim in Proposition 2.11 does not tell us anything about identifying a σ for which $\vartheta^{\sigma,i} \in X_I$ for all i, or certifying it does not exist. If such a σ can be found in polynomial time, then linear optimization over X_I is polynomial time since the convex hull of a lex-ordered set has linearly many facet-defining inequalities [Gup16].

Now we state and prove our result on strength of lex dual bounds for monotone sets. Recall that for arguing tightness of lex primal bounds, we defined the set $\mathcal{H}_2$ in (11) as consisting of at least one monotone permutation for each maximal feasible point. Similarly, we denote $\mathcal{H}_3$ as consisting of at least one monotone permutation for each *minimal infeasible* point:

$$\mathcal{H}_3 := \{H \subseteq \mathfrak{S}_n \mid H \cap \mathfrak{I}_x \neq \emptyset,\ \forall x \in \mathcal{M}^-_{\{0,1\}^n\setminus X_I}\}, \quad \text{when } X_I \subseteq \{0,1\}^n. \tag{16}$$

Theorem 2.12. *Let X_I be $\downarrow$-monotone over $\mathbb{Z}^n$.*

(1) $z^*_c = \bar{z}_c(H)$ for all c if for some $\sigma \in H$, $\vartheta^{\sigma,i} \in X_I$ for all $i \in [n]$.

(2) If $X_I \subseteq \{0,1\}^n$, then $z^*_c = \bar{z}_c(H)$ for all c and all $H \in \mathcal{H}_3$.

Proof. By equation (14), for both claims, it suffices to show that $X_I \supseteq \cap_{\sigma\in H}\mathcal{L}^\sigma_X$. If $\vartheta^{\sigma,i} \in X_I$ for all i and some $\sigma \in H$, then Proposition 2.11 tells us that $X_I = \mathcal{L}^\sigma_X$, which obviously implies $X_I \supseteq \cap_{\tau\in H}\mathcal{L}^\tau_X$.

Now let $X_I \subseteq \{0,1\}^n$. Since X_I is a $\downarrow$-monotone set, $\gamma^\sigma_X = 0$ and

$$\mathcal{L}^\sigma_X = \{x \in \{0,1\}^n \mid x \preccurlyeq_\sigma \theta^\sigma_X\}, \quad \forall\sigma. \tag{17}$$

Note that $\mathcal{L}^\sigma_X$ is $\downarrow$-monotone and hence, so is $\cap_\sigma\mathcal{L}^\sigma_X$. We show that for any $H \in \mathcal{H}_3$, $x \in \{0,1\}^n \setminus X_I$ implies that $x \notin \cap_{\sigma\in H}\mathcal{L}^\sigma_X$, thereby proving $X_I \supseteq \cap_{\sigma\in H}\mathcal{L}^\sigma_X$ by contraposition. We first argue that it suffices to show this for every minimal x in $\{0,1\}^n \setminus X_I$. Suppose x belongs to $\{0,1\}^n \setminus X_I$ but is not a minimal point. The set of minimal points of $\{0,1\}^n \setminus X_I$ is denoted by $\mathcal{M}^-_{\{0,1\}^n\setminus X_I}$. Then there is a $x' \in \mathcal{M}^-_{\{0,1\}^n\setminus X_I}$ such that $x' \neq x$. Since $x' \notin \cap_{\sigma\in H}\mathcal{L}^\sigma_X$ by assumption, the $\downarrow$-monotone property of $\cap_\sigma\mathcal{L}^\sigma_X$ and $x' \leq x$ imply that $x \notin \cap_{\sigma\in H}\mathcal{L}^\sigma_X$. Therefore, we only need to prove for every minimal point of $\{0,1\}^n \setminus X_I$.

Fix an arbitrary $x \in \mathcal{M}^-_{\{0,1\}^n\setminus X_I}$. Since $H \in \mathcal{H}_3$, we can choose some $\sigma \in \mathfrak{I}_x \cap H$ to obtain

$$x_{\sigma(1)} = x_{\sigma(2)} = \cdots = x_{\sigma(k)} = 0, \quad x_{\sigma(k+1)} = x_{\sigma(k+2)} = \cdots = x_{\sigma(n)} = 1,$$

for some $0 \leq k \leq n$. We argue $x \notin \mathcal{L}^\sigma_X$. Take any $y \in \{0,1\}^n$ such that $y \succcurlyeq_\sigma x$. The construction of σ enforces $y_{\sigma(i)} = 1$ for $k+1 \leq i \leq n$. Hence $y \geq x$ and then the assumption that X_I is $\downarrow$-monotone implies that $y \notin X_I$. Thus $\{y \in \{0,1\}^n \mid x \preccurlyeq_\sigma y\} \subseteq \{0,1\}^n \setminus X_I$. Since $\preccurlyeq_\sigma$ is a total order, it follows that $X_I \subseteq \{y \in \{0,1\}^n \mid y \prec_\sigma x\}$. This leads to $\theta^\sigma_X \prec_\sigma x$ since $\theta^\sigma_X \in X_I$. Therefore $x \notin \mathcal{L}^\sigma_X$, and we are done proving $X_I \supseteq \cap_{\sigma\in H}\mathcal{L}^\sigma_X$. □

Remark 2. In the "if" part of the above proof, the $[0,1]^n$ box is crucial in arriving at the implication $y \geq x$, which further allowed us to conclude $y \notin X_I$. For a general box, one may analogously construct σ as the permutation that orders all the zeros in the beginning, all the upper bounds at the end, and everything else in between. However, this will not guarantee $y \geq x$.

Since $\mathfrak{S}_n \in \mathcal{H}_3$, the dual bound from all lex optima is tight.

Corollary 2.13. *$\bar{z}_c(\mathfrak{S}_n) = z^*_c$ for any X_I that is either $\downarrow$-monotone or $\uparrow$-monotone over $\{0,1\}^n$.*

Unlike Theorem 2.1, where we bounded the approximation quality of lex primal bounds, we do not know of any bounds on the quality of lex dual bounds for general discrete sets.

We now discuss two main consequences of Theorem 2.12 on independence systems.

2.2.1. *Stronger Polyhedral Representation.* For any independence system X_I, a well-known *implicit* polyhedral representation is

$$X_I = \left\{ x \in \{0,1\}^n \mid \sum_{i:\, v_i=1} x_i \leq n - k_v - 1, \ \forall v \in \mathcal{M}^-_{\{0,1\}^n\setminus X_I} \right\}. \tag{18}$$

This was originally proposed as a "minimal cover formulation" in the context of a 0\1 knapsack [BJ72]; its generalization to any $\downarrow$-monotone 0\1 set is immediate. It is referred to as being implicit because it requires enumeration of all the minimally infeasible points. A first consequence of Theorem 2.12 is to improve upon this classical representation. To describe this new and stronger formulation, we need the following terminology.

Definition 2. For $v \in \mathcal{M}^-_{\{0,1\}^n \setminus X_I}$, denote k_v to be the number of 0's in v, and σ_v to be some monotone permutation from $\mathfrak{I}_v$ (cf. Definition 1). Partition $[k_v+1]$ using the lexmax point $\theta_X^{\sigma_v}$ as

$$I_v^0 := \{i \in [k_v+1] \mid (\theta_X^{\sigma_v})_{\sigma_v(i)} = 0\}, \quad I_v^1 := [k_v+1] \setminus I_v^0 = \{i \in [k_v+1] \mid (\theta_X^{\sigma_v})_{\sigma_v(i)} = 1\}.$$

Proposition 2.14 (Implicit Formulation)**.** *Let X_I be a $\downarrow$-monotone $0\backslash 1$ set. Then X_I is exactly the set of all $0\backslash 1$ solutions satisfying the inequality*

$$x_{\sigma_v(i)} + \sum_{j \in I_v^1 \colon j > i} x_{\sigma_v(j)} + \sum_{j=k_v+2}^{n} x_{\sigma_v(j)} \leq n - k_v - 1 + |\{j \in I_v^1 \mid j > i\}| \tag{19}$$

for every $v \in \mathcal{M}^-_{\{0,1\}^n \setminus X_I}$ and $i \in I_v^0$. Furthermore, the formulation provided by (19) is stronger than the minimal cover formulation (18).

Proof. Theorem 2.12 and equations (14) and (17) tell us that for $\downarrow$-monotone $0\backslash 1$ sets, we have

$$X_I = \left\{ x \in \{0,1\}^n \mid x \preccurlyeq_\sigma \theta_X^\sigma, \ \forall v \in \mathcal{M}^-_{\{0,1\}^n \setminus X_I}, \text{ some } \sigma \in \mathfrak{I}_v \right\}. \tag{20a}$$

Therefore,

$$X_I = \{0,1\}^n \bigcap \bigcap_{v \in \mathcal{M}^-_{\{0,1\}^n \setminus X_I}} \operatorname{conv}\{x \in \{0,1\}^n \mid x \preccurlyeq_{\sigma_v} \theta_X^{\sigma_v}\}, \tag{20b}$$

where $\sigma_v \in \mathfrak{I}_v$ as per Definition 2. For fixed points v and $\theta_X^{\sigma_v}$ and permutation σ_v, the set $\{x \in \{0,1\}^n \mid x \preccurlyeq_{\sigma_v} \theta_X^{\sigma_v}\}$ is a $0\backslash 1$ lex-ordered set. The convex hull of $\{x \in \{0,1\}^n \mid x \preccurlyeq \theta\}$, for fixed θ, is defined by $\mathcal{O}(n)$ facet-defining inequalities [LS92; Gup+13] that are of the form:

$$x_i + \sum_{j > i \colon \theta_j = 1} x_j \leq |\{j > i \colon \theta_j = 1\}|, \quad \forall i \colon \theta_i = 0.$$

Since the set $\{x \in \{0,1\}^n \mid x \preccurlyeq_{\sigma_v} \theta_X^{\sigma_v}\}$ is equal to $\{x \in \{0,1\}^n \mid \sigma_v \cdot x \preccurlyeq \sigma_v \cdot \theta_X^{\sigma_v}\}$, it follows that the convex hull of this set is defined by the inequalities

$$x_{\sigma_v(i)} + \sum_{j > i \colon (\theta_X^{\sigma_v})_{\sigma_v(j)} = 1} x_{\sigma_v(j)} \leq |\{j > i \colon (\theta_X^{\sigma_v})_{\sigma_v(j)} = 1\}|, \quad \forall i \colon (\theta_X^{\sigma_v})_{\sigma_v(i)} = 0.$$

The above inequality is exactly (19) for every $i \in I_v^0$ because minimality of v and σ_v being a monotone permutation tells us that

$$\sigma_v \cdot v = (\underbrace{0, \ldots, 0}_{k_v}, \underbrace{1, \ldots, 1}_{n-k_v}) \implies \sigma_v \cdot \theta_X^{\sigma_v} = (\underbrace{*, \ldots, *}_{k_v}, 0, \underbrace{1, \ldots, 1}_{n-k_v-1}). \tag{21}$$

This, combined with (20b), leads to our first claim.

To argue the dominance of the new formulation, it suffices to show that every minimal cover inequality in (18) also appears in (19). Equation (21) tells us that $k_v + 1 \in I_v^0$. Therefore, inequality (19) with $i = k_v + 1$ reads as $\sum_{j=k_v+1}^n x_{\sigma_v(j)} \leq n - k_v - 1$, which is exactly the inequality in (18) since $v_{\sigma_v(j)} = 1$ if and only if $k_v + 1 \leq j \leq n$. □

Remark 3. For every minimal infeasible v, constructing inequality (19) requires the computation of $\theta_X^{\sigma_v}$, which, for an independence system, can be done in polynomial time using the formula in Proposition 3.1.

The size of this stronger formulation, where size is measured in terms of the number of inequalities, depends on the cardinality of $\mathcal{M}^-_{\{0,1\}^n\setminus X_I}$. Thus, this formulation is compact if and only if $|\mathcal{M}^-_{\{0,1\}^n\setminus X_I}|$ is polynomial in the encoding size of X, which is not the case in general (knapsack polytopes have exponentially many minimal covers), but is the case, for set packing problems such as the independent set problem.

Observation 2.15. *Let $X_I = \{x \in \{0,1\}^n \mid x_i + x_j \le 1,\ (i,j) \in E\}$ be the set of characteristic vectors of independent sets in $G = (V,E)$. Then $v \in \mathcal{M}^-_{\{0,1\}^n\setminus X_I}$ if and only if $v_i = v_j = 1$ for some $(i,j) \in E$, and $v_k = 0$ for $k \neq i,j$.*

Using this characterization of minimal infeasible points, Proposition 2.14 implies a new compact formulation for the independent set problem. To avoid notational overload, we do not describe this formulation in detail.

Finally, as opposed to a *subset* of the lex maxima yielding a formulation in the *original space* as per Proposition 2.14, we show that *all* the lex maxima yield an *extended* formulation for the convex hull of an independence system. We will first prove an extended formulation for general $\downarrow$-monotone sets and then specialize this extension to the case of $0\backslash 1$ $\downarrow$-monotone sets. Recall the definition of maximal vertices, $\mathcal{MV}^+_X$, from (12).

Proposition 2.16. *Let X_I be $\downarrow$-monotone over $\mathbb{Z}^n_+$. Then*

$$\operatorname{conv} X_I = (\operatorname{conv} \mathcal{M}^+_X - \mathbb{R}^n_+) \cap \mathbb{R}^n_+ = (\operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+) \cap \mathbb{R}^n_+, \tag{22a}$$

so that

$$\operatorname{conv} X_I = \operatorname{Proj}_x \left\{ (x,\alpha) \in \mathbb{R}^{n+|\mathcal{MV}^+_X|}_+ \;\middle|\; x \le \sum_{y \in \mathcal{MV}^+_X} \alpha_y y,\ \sum_{y \in \mathcal{MV}^+_X} \alpha_y = 1 \right\}. \tag{22b}$$

The first equality in (22a) can be shown easily using simple arguments, it is the second equality that makes the proof nontrivial. Note that for $0\backslash 1$ sets, $\mathcal{M}^+_X = \mathcal{MV}^+_X$, and so only the first equality in (22a) needs to be shown, making the proof simple in this special case.

Proof. The case $\operatorname{conv} X_I = [0,u]$ for some $u \in \mathbb{Z}^n_{\ge 0}$, being trivial, can be discarded. So assume $\operatorname{conv} X_I$ is not a box. The main thing to prove is (22a), since it immediately implies the extended formulation in (22b) after applying the definition of $\operatorname{conv} \mathcal{MV}^+_X$.

Monotonicity of X_I, and hence of $\operatorname{conv} X_I$, and $\mathcal{MV}^+_X \subseteq \mathcal{M}^+_X \subseteq X_I$, imply the $\supseteq$ inclusions in (22a). Equality is obtained throughout if $\operatorname{conv} X_I \subseteq \operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+$. Since $(\operatorname{conv} X_I)_I = X_I$, and so $\mathcal{M}^+_{\operatorname{conv} X_I} = \mathcal{M}^+_X$, it is trivial to verify, similar to Lemma 2.9, that $\mathcal{M}^+_X \subseteq \operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+$ is equivalent to $\operatorname{conv} X_I \subseteq \operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+$.

So it remains to prove $\mathcal{M}^+_X \subseteq \operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+$. Take any $x \in \mathcal{M}^+_X$. Recall that $\operatorname{conv} X_I$ was assumed to not be a box. This means that if x is in the relative interior of $\operatorname{conv} \mathcal{MV}^+_X$, then there exists some $y \ge x, y \notin \mathbb{Z}^n$ that belongs to a nontrivial facet of $\operatorname{conv} \mathcal{MV}^+_X$. Otherwise, x itself is on a nontrivial facet of $\operatorname{conv} \mathcal{MV}^+_X$. Therefore, $x \in F - \mathbb{R}^n_+$ for some nontrivial facet F. To argue $x \in \operatorname{conv} \mathcal{MV}^+_X - \mathbb{R}^n_+$ in either case, it suffices to prove that $\operatorname{vert} F \subset \mathcal{MV}^+_X$, since $F = \operatorname{conv} \operatorname{vert} F$. Since F is nontrivial and $\operatorname{conv} X_I$ is a $\downarrow$-monotone polytope, it must be that the normal vector ξ of the hyperplane defining F satisfies $\xi \ge 0$ with $\xi_i, \xi_j > 0$ for some distinct i,j. For every $v \in \operatorname{vert} F$, the definition of normal cone gives us that ξ belongs to the normal cone to $\operatorname{conv} X_I$ at v, denoted by $\mathcal{N}_{\operatorname{conv} X_I}(v)$. Since the normal cone is a convex cone and ξ is nonnegative with at least two positive elements, it follows that there exists some nonnegative vector that is arbitrarily close to ξ and belongs to the relative interior of $\mathcal{N}_{\operatorname{conv} X_I}(v)$. The characterization of $\mathcal{MV}^+_X$ in Lemma 2.3 then gives us $v \in \mathcal{MV}^+_X$, and our proof for (22a) is complete. □

Applying the above result to an independence system gives us an implicit extension that uses all the lex maxima.

Corollary 2.17 (Implicit Extension)**.** *If X_I is a $\downarrow$-monotone $0\backslash 1$ set, then*

$$\operatorname{conv} X_I = \operatorname{Proj}_x \left\{ (x,\alpha) \in \mathbb{R}^{n+|\Theta^{\mathfrak{S}_n}_X|}_+ \;\middle|\; x_i \le \sum_{\substack{\theta \in \Theta^{\mathfrak{S}_n}_X : \\ \theta_i = 1}} \alpha_\theta \;\; \forall i, \;\; \sum_{\theta \in \Theta^{\mathfrak{S}_n}_X} \alpha_\theta = 1 \right\}. \tag{22c}$$

Proof. When $X_I \subseteq \{0,1\}^n$, we have $\operatorname{vert} X_I = X_I$, and Proposition 2.4 tells us $\Theta^{\mathfrak{S}_n}_X = \mathcal{MV}^+_X$. The claim then follows from Proposition 2.16. □

The $0\backslash 1$ case of Proposition 2.16, and the corresponding extended formulation in (22c), was proved by [BB15, Lemma 1] (although in terms of maximal points and not lex maxima), but as we have shown, a more general statement holds for any $\downarrow$-monotone polytope.

2.2.2. *Polyhedral study of convex hull.* A second consequence of Theorem 2.12 is on the characterization of convex hulls of independence systems. These polytopes are NP-hard to separate over and a explicit algebraic representation is unknown in general, although there is significant literature on strong valid inequalities for certain independence systems such as knapsack and stable set polytopes. The second claim in Theorem 2.12 and equation (14) tell us that an independence system X_I is equal to the intersection of finitely many lex-ordered sets of the form (17). Therefore, an inequality is valid to $\operatorname{conv} X_I$ if and only if it is valid to $\operatorname{conv}(\cap_{\sigma \in H} \mathcal{L}^\sigma_X)$. Although Proposition 2.14 provides an exponential class of valid inequalities — inequality (19) is valid to $\operatorname{conv} X_I$ for every $v \in \mathcal{M}^-_{\{0,1\}^n \setminus X_I}$, $\sigma_v \in \mathfrak{I}_v$, and $i \in I^0_v$, these inequalities are not strong and unlikely to be facet-defining, since they are essentially covering inequalities. For example, for the stable set polytope, several classes of valid inequalities are known, such as odd holes, cliques, wheels, etc., but following the characterization of minimal infeasible points in Observation 2.15, none of these are dominated by any inequality in (19). This motivates a thorough polyhedral study of polytopes arising from intersections of multiple lex-ordered sets taken under different permutations, where the lex ordering is with respect to a given fixed integer vector. This has not been studied yet; literature only contains polyhedral studies of single lex-ordered sets [LS92; GK06; Gup16; GP16]. We leave this study as a topic of future research.

3. Computing Lex Optima

Recursion (2) is the naive method for computing lex optima. It is intractable in general since it depends on solving an integer program at each step, and hence is implemented in practice by relaxing integrality on some or all variables and backtracking when infeasibility occurs. However, it can lead to explicit formulae for lex optima of some special polytopes, which is what we derive next and in §3.1. Later, in §3.2, we give an algorithm for computing lex optima that is conceptual in nature and will be used in §4 for obtaining some polynomial time cases.

Consider the family of polytopes defined by linear inequalities in which variables do not take conflicting signs. This means that such a polytope can be defined as

$$\tilde{P} := \{(x^+, x^-) \in [0, u] \mid A^+x^+ + A^-x^- \leq b\}, \tag{23}$$

where $A^+ \geq 0$ and $A^- \leq 0$. Note that $\tilde{P}$ is monotone with respect to each variable, where the direction of monotonicity is allowed to be different for different variables. This family includes ↓-monotone (packing) and ↑-monotone (covering) polytopes, and knapsack polytopes (i.e., polytopes defined by a single linear inequality). Lex optima of this family can be computed recursively as follows.

Proposition 3.1. *Consider $\tilde{P}$ and let $A = \left[A^+,\ A^-\right] = (a_{ij}) \in \mathbb{R}^{m\times n}$, with N^+ indexing the columns of A^+ and N^- indexing the columns of A^-. Assume for convenience that $u_j = \max\{x_j \mid x \in X_I\}$ for all j. Denote $\xi_i := \sum_{j\in N^-} a_{ij}u_j$ for all i. Then for $j = n, \ldots, 1$,*

$$(\theta^{\sigma}_{\tilde{P}})_{\sigma(j)} = \begin{cases} \min\left\{ u_{\sigma(j)},\ \min\limits_{i:\ a_{i,\sigma(j)}>0} \left\lfloor \dfrac{b_i - \xi_i - \sum\limits_{k>j:\ a_{i,\sigma(k)}>0} a_{i,\sigma(k)}(\theta^{\sigma}_{\tilde{P}})_{\sigma(k)}}{a_{i,\sigma(j)}} \right\rfloor \right\}, & \sigma(j) \in N^+, \\ u_{\sigma(j)}, & \sigma(j) \in N^-. \end{cases}$$

The point $\gamma^{\sigma}_{\tilde{P}}$ is obtained similarly.

The proof is omitted since it is obvious from (2). The above formulas can be made more explicit if additional structure is available. We show this for a polymatroid and its base polytope, which are important sets in combinatorial optimization.

3.1. **Polymatroids.** For a set function $f\colon 2^{[n]} \mapsto \mathbb{R}$, the extended polymatroid and polymatroid of f are, respectively, the polyhedra

$$EP_f := \{x \in \mathbb{R}^n \mid x(S) \leq f(S),\ S \subseteq [n]\}, \quad P_f := EP_f \cap \mathbb{R}^n_+, \tag{24}$$

where $x(S) := \sum_{i\in S} x_i$. Clearly, P_f is a ↓-monotone polytope. Note that $EP_f \neq \emptyset$ if and only if $f(\emptyset) \geq 0$, and $P_f \neq \emptyset$ if and only if $f(\cdot) \geq 0$. A set function f is called submodular if $f(S) + f(S') \geq f(S \cup S') + f(S \cap S')$ for all $S, S' \subseteq [n]$, and is said to be nondecreasing if $f(S) \leq f(S')$ for all $S \subseteq S' \subseteq [n]$. Since both these properties are translation invariant, we may assume for convenience that $f(\emptyset) = 0$. Many of the polyhedral results about polymatroids were obtained by Edmonds [Edm70], and are also explained in Schrijver [Sch03, chap. 44]. One such result is the characterization of polymatroids whose extreme points are integral vectors: a nonempty polymatroid P_f is integral if and only if f is a nondecreasing integer-valued submodular set function. The lex optima of such polymatroids is described next.

Theorem 3.2. *Let P_f be a nonempty integral polymatroid. Then for any $\sigma \in \mathfrak{S}_n$, $\gamma^{\sigma}_{P_f} = 0$ and*

$$(\theta^{\sigma}_{P_f})_{\sigma(i)} = f(T_i) - f(T_{i+1}), \quad i = 1, \ldots, n$$

where $T_i := \{\sigma(i), \sigma(i+1), \ldots, \sigma(n)\}$ for $i = 1, \ldots, n$ and $T_{n+1} = \emptyset$.

Proof. We have that $f\colon 2^{[n]} \mapsto \mathbb{Z}$ is a nondecreasing integer-valued submodular set function. Applying the expressions from Proposition 3.1, we obtain $\gamma^{\sigma}_{P_f} = 0$ and

$$(\theta^{\sigma}_{P_f})_{\sigma(i)} = \min\left\{\lfloor f(\sigma(i))\rfloor,\ \min_{S\subseteq[n]\colon\,\sigma(i)\in S}\left\lfloor f(S) - \sum_{k>i,\sigma(k)\in S}(\theta^{\sigma}_{P_f})_{\sigma(k)}\right\rfloor\right\}.$$

Observe the following three things about the above formula: (i) $f(\cdot)\in\mathbb{Z}$ implies that we can dispense with the rounding operator; (ii) monotonicity of f implies that in the inner minimization, we only need to consider subsets S such that $\sigma(i)\in S\subseteq T_i$; (iii) the first term $\lfloor f(\sigma(i))\rfloor$ also appears in the second term for $S=\sigma(i)$. Thus, the recursion becomes

$$(\theta^{\sigma}_{P_f})_{\sigma(i)} = \min_{\sigma(i)\in S\subseteq T_i}\left[f(S) - \sum_{k>i,\sigma(k)\in S}(\theta^{\sigma}_{P_f})_{\sigma(k)}\right].$$

We use backward induction to verify that the above formula yields $(\theta^{\sigma}_{P_f})_{\sigma(i)} = f(T_i) - f(T_{i+1})$. For the base case, we have $(\theta^{\sigma}_{P_f})_{\sigma(n)} = \min\{f(\sigma(n)), \min_{\sigma(n)\in S\subseteq T_n} f(S)\}$, which is equal to $f(\sigma(n))$ because $T_n = \{\sigma(n)\}$. Assume the hypothesis for some $i+1\le n$ and consider i. We have

$$(\theta^{\sigma}_{P_f})_{\sigma(i)} = \min_{\sigma(i)\in S\subseteq T_i}\left[f(S) - \sum_{k>i,\sigma(k)\in S}\big(f(T_k) - f(T_{k+1})\big)\right].$$

Define $g_i(S) = f(S) - \sum_{k>i,\sigma(k)\in S}(f(T_k) - f(T_{k+1}))$. We claim the following.

Claim 3.1. *$g_i(T_i)\le g_i(S)$ for all $S\subseteq T_i$ with $\sigma(i)\in S$.*

Proof of Claim. For nontriviality, assume $S\subset T_i$. Consider subsets $S_t := \{i_t, i_t+1,\ldots,j_t\}$ with $i_t\le j_t$, for $t=1,\ldots,r+1$ and some $r\ge 0$, so that S can be partitioned as $S = \cup_{t=1}^{r+1}\{\sigma(i)\mid i\in S_t\}$. Since $\sigma(i)\in S$, we have $i_1 = i$. Consider subsets $R_t := \{p_t, p_t+1,\ldots,q_t\}$ for $t=1,\ldots,r$, with $p_t\le q_t$ and $q_t = i_{t+1}-1 < p_{t+1}-1$. The induction hypothesis gives us $\sum_{k\in R_t}(\theta^{\sigma}_{P_f})_{\sigma(k)} = f(T_{p_t}) - f(T_{q_t+1})$. Also note that $T_i\setminus S = \cup_{t=1}^{r} R_t$. Therefore

$$g_i(T_i)\le g_i(S) \iff f(T_i) + \sum_{j=1}^{r} f(T_{q_t+1}) \le f(S) + \sum_{j=1}^{r} f(T_{p_t}). \tag{25}$$

Submodularity of f gives us $f(\cup_{t=l}^{r+1}S_t) + f(T_{p_j}) \ge f(T_{i_l}) + f(\cup_{t=l+1}^{r+1}S_t)$ for $l=1,\ldots,r$. Summing over all l, canceling terms, and using $q_t+1 = i_{t+1}$ and $f(S_{r+1}) = f(T_{i_{r+1}}) = f(T_{q_r+1})$, leads us exactly to the equivalent condition for $g_i(T_i)\le g_i(S)$ in (25). ◇

The above claim gives us the desired equality $(\theta^{\sigma}_{P_f})_{\sigma(i)} = g_i(T_i) = f(T_i) - f(T_{i+1})$ and completes our inductive proof for $\theta^{\sigma}_{P_f}$. □

Recall that $\Theta_X^{\mathfrak{S}_n}\subseteq\mathcal{MV}_X^{+}$. An immediate consequence of Theorem 3.2 is the equivalence of lex maxima and maximal vertices for integral polymatroids, as stated earlier in Proposition 2.5.

Proof of Proposition 2.5. We know that $v\in\operatorname{vert} EP_f$ if and only if the elements of v are given by $v_{\sigma(i)} = f(\{\sigma(1),\ldots,\sigma(i)\}) - f(\{\sigma(1),\ldots,\sigma(i-1)\})$ for $\sigma\in\mathfrak{S}_n$ and $i=1,\ldots,n$. Denote these points by v^{σ}. By Theorem 3.2, $\theta^{\sigma}_{P_f}$ corresponds to $v^{\sigma'}$, where σ' is the reverse permutation of σ, i.e., $\sigma'(i) = \sigma(n+1-i)$. Since $\mathfrak{S}_n$ and the set of reverse permutations are in a bijection, we get $\Theta^{\mathfrak{S}_n}_{P_f} = \operatorname{vert} EP_f$. Since f is nondecreasing, $\operatorname{vert} EP_f \subseteq \operatorname{vert} P_f$; in particular, the vertices of P_f are given by $v_{\sigma(i)} = f(\{\sigma(1),\dots,\sigma(i)\}) - f(\{\sigma(1),\dots,\sigma(i-1)\})$ for all $i \le k$, and $v_{\sigma(i)} = 0$ for all $i > k$, where $\sigma \in \mathfrak{S}_n$ and $k = 0,\dots,n$. Therefore, $\mathcal{MV}^+_{P_f}$ is also equal to $\operatorname{vert} EP_f$. □

Another polytope related to submodular functions is the base polytope B_f, which is the face of EP_f induced by the inequality $x([n]) \le f([n])$ and is defined as $B_f := EP_f \cap \{x \mid x([n]) = f([n])\}$. This polytope is not a monotone set. It is clear that every lex maxima of P_f that belongs to the hyperplane $x([n]) = f([n])$ is also a lex maxima of B_f. It is possible that P_f has some lex maxima that do not belong to B_f. However, we show next that this is not the case with an integral polymatroid. We also note *both* lex maxima and lex minima for the special case where B_f is the permutahedron P_n, the base polytope of the submodular function $f(S) = g(|S|)$ for $g(k) = \binom{n+1}{2} - \binom{n+1-k}{2}$. We do not know formulas for lex minima of B_f in general.

Corollary 3.3. *$\Theta^{\mathfrak{S}_n}_{B_f} = \Theta^{\mathfrak{S}_n}_{P_f} = \operatorname{vert} EP_f$ when P_f is an integral polymatroid. Furthermore, for the permutahedron P_n, we have $\Theta^{\mathfrak{S}_n}_{P_n} = \mathcal{MV}^+_{P_n}$ and $\Gamma^{\mathfrak{S}_n}_{P_n} = \mathcal{MV}^-_{P_n}$.*

Proof. The characterization of $\theta^\sigma_{P_f}$ in Theorem 3.2 leads to $\sum_{i=1}^n (\theta^\sigma_{P_f})_{\sigma(i)}$ being a telescoping sum and equal to $f([n])$. Therefore, every $\theta^\sigma_{P_f}$ lies on the hyperplane $x([n]) = f([n])$, implying that $\Theta^{\mathfrak{S}_n}_{P_f} \subseteq \Theta^{\mathfrak{S}_n}_{B_f}$. The reverse inclusion is due to $\Theta^{\mathfrak{S}_n}_{B_f} \subseteq \operatorname{vert} B_f = \operatorname{vert} EP_f$, and Proposition 2.5 giving us $\Theta^{\mathfrak{S}_n}_{P_f} = \operatorname{vert} EP_f$.

Since P_n is the convex hull of $\mathfrak{S}_n$, its lex optima are easy to characterize via a direct application of recursion (2) and the fact that every lex optima of P_n is a vertex of P_n. We get $(\theta^\sigma_{P_n})_{\sigma(i)} = i$ and $(\gamma^\sigma_{P_n})_{\sigma(i)} = n+1-i$ for all i, σ. Therefore, the sets $\Theta^{\mathfrak{S}_n}_{P_n}$, $\Gamma^{\mathfrak{S}_n}_{P_n}$, and $\mathfrak{S}_n$ are in bijection with each other. This leads to $\Theta^{\mathfrak{S}_n}_{P_n} = \mathcal{M}^+_{P_n}$ and $\Gamma^{\mathfrak{S}_n}_{P_n} = \mathcal{M}^-_{P_n}$. □

3.2. **Bisection search.** We present an algorithm for computing lex optima that invokes the integer feasibility question for X over an integral sub-box.

Problem $\mathsf{FEAS}_X(B)$. Given $X \subset [0,u]$ and a box $B := [l', u'] \subseteq [0,u]$, is $X_I \cap B \neq \emptyset$?

We describe here the version of the algorithm that finds a lex maxima of X and note that lex minima can be found similarly *mutatis mutandis*.

Algorithm 1: Given X and $\sigma \in \mathfrak{S}_n$, compute θ^σ_X.

(1) Solve $\mathsf{FEAS}_X([0,u])$. If the answer is no, then LexMax^σ_X is infeasible and θ^σ_X does not exist. Else, set $i = 1$ and $B_0 = [0,u]$.
(2) For $k_i := \max\{k \mid \exists\, x, y \in B_{i-1} \text{ s.t. } x_{\sigma(k)} \neq y_{\sigma(k)}\}$, set $u_{k_i} := \max\{x_{\sigma(k_i)} \mid x \in B_{i-1}\}$, $l_{k_i} := \min\{x_{\sigma(k_i)} \mid x \in B_{i-1}\}$, and $m_{k_i} := \left\lceil \frac{u_{k_i} + l_{k_i}}{2} \right\rceil$.
(3) Consider
$$B^{\geq} := \{x \in B_{i-1} \mid x_{\sigma(k_i)} \geq m_{k_i}\}, \quad B^{\leq} := \{x \in B_{i-1} \mid x_{\sigma(k_i)} \leq m_{k_i} - 1\}.$$
Solve $\mathsf{FEAS}_X(B^{\geq})$. If the answer is yes, set $B_i = B^{\geq}$, otherwise set $B_i = B^{\leq}$.
(4) If $|B_i| = 1$, return B_i. Otherwise, set $i = i+1$ and go to (2).

Proposition 3.4. *Algorithm 1 finds θ^σ_X at termination in finite time.*

Proof. Since X is compact, enumeration solves $\mathsf{FEAS}_X(B)$ in finite time. Since the sequence $\{B_i\}_{i\geq 0}$ generated by the algorithm performs a bisection on the maximum and minimum values of each dimension in $[0,u]$, this sequence is finite and hence, it follows that the algorithm has finite termination. It is also clear that $|B_{i+1}| \leq |B_i|$. Now we argue that when B_i is a singleton, then this point is the lexmax point θ^σ_X. To prove this, it suffices to show that $\theta^\sigma_X \in B_i$ for all $i \geq 0$. We perform induction on i. Denote $x^* := \theta^\sigma_X$ for notational convenience. Trivially, $x^* \in B_0$. Assume $x^* \in X \cap B_{i-1}$ for $i \geq 1$.

Claim 3.2. *A yes answer to* $\mathsf{FEAS}_X(B^{\geq})$ *implies* $x^* \in B^{\geq}$ *and a no answer implies* $x^* \in B^{\leq}$.

Proof of Claim. Clearly, $B_{i-1}\cap\mathbb{Z}^n = (B^{\geq}\cap\mathbb{Z}^n)\cup(B^{\leq}\cap\mathbb{Z}^n)$. If $\mathsf{FEAS}_X(B^{\geq})$ returns no, then the induction hypothesis gives us $x^* \in B^{\leq}$. Suppose that $\mathsf{FEAS}_X(B^{\geq})$ is yes but $x^* \notin B^{\geq}$. Since $x^* \in B_{i-1}$, we have $x^* \in B^{\leq}$ and so $x^*_{\sigma(k_i)} \leq m_{k_i} - 1$. We have assumed $\mathsf{FEAS}_X(B^{\geq})$ is yes and so denote $y := \theta^\sigma_{X\cap B^{\geq}}$. The construction of k_i tells us that $y_{\sigma(k)} = x^*_{\sigma(k)}$ for $k > k_i$. Now $y_{\sigma(k_i)} \geq m_{k_i}$ and $x^*_{\sigma(k_i)} \leq m_{k_i} - 1$ implies $x^* \prec_\sigma y$. But this is a contradiction to the maximality of x^*. Therefore $x^* \in B^{\geq}$. ⋄

This finishes our induction step. By construction of B_i, the above claim implies $x^* \in B_i$, which is sufficient to prove the correctness of our algorithm. □

4. Complexity Status

We begin by trying to understand the computational complexity of the lex bounds. These bounds rely on the computation of the lex optima. Even if every lex optima can be computed in polynomial time, the strongest lex bounds, which are obtained by using $H = \mathfrak{S}_n$, whose size is $n!$, are still NP-hard to compute. This follows from Theorems 2.1 and 2.12, which also tell us that the smallest size of H corresponding to the strongest lex bound is bounded from above by $|\mathcal{M}^+_X|$ for primal bound and $|\mathcal{M}^-_{\{0,1\}^n\setminus X_I}|$ for dual bound. In general, finding the number of maximal feasible and minimal infeasible points is $\#P$-complete and these numbers are exponential in the input. Hence, the hardness of $\bar{z}_c(\mathfrak{S}_n)$ and $\underline{z}_c(\mathfrak{S}_n)$ may not be all that surprising. We note that, in fact, the situation is much worse — the hardness persists even for small-sized H.

Proposition 4.1. *Let X be a $\downarrow$-monotone polytope. Computing $\bar{z}_c(\mathfrak{S}_n)$ and $\underline{z}_c(\mathfrak{S}_n)$ is* NP-hard *in general, and $\bar{z}_c(H)$ remains* NP-hard *even when $|H| = \mathcal{O}(n)$.*

Proof. Corollaries 2.2 and 2.13 give us $\underline{z}_c(\mathfrak{S}_n) = \bar{z}_c(\mathfrak{S}_n) = z^*_c$ for any $\downarrow$-monotone $X_I \subseteq \{0,1\}^n$. Since optimization over such a X_I is NP-hard, computing $\underline{z}_c(\mathfrak{S}_n)$ and $\bar{z}_c(\mathfrak{S}_n)$ is NP-hard. For the second claim, we consider the independent set problem INDSET on a bounded degree graph $G = (V,E)$ on n vertices. Theorem 2.12 tells us $\bar{z}_c(H) = z^*_c$ for any $H \in \mathcal{H}_3$. The smallest size of such an H is bounded from above by $|\mathcal{M}^-_{\{0,1\}^n\setminus X_I}|$, and then the characterization of minimally infeasible points in Observation 2.15 gives us that this smallest size is equal to $|E|$, which is equal to $\mathcal{O}(n)$ for a bounded degree graph. The fact that computing z^*_c for INDSET is NP-hard finishes our proof. □

There are polytopes with exponentially many lex optima, the computation of each of which is in P, and for which computing the strongest primal lex bound is also in P. One such class of polytopes is integral polymatroids for which we know from Corollary 2.6 that $z^*_c = \underline{z}_c(\mathfrak{S}_n)$, and so one can use Edmonds greedy algorithm to compute $\underline{z}_c(\mathfrak{S}_n)$ in polynomial time.

It is clear that if the lex optima are NP-hard to compute, i.e., LexMax_X^σ or LexMin_X^σ are NP-hard to solve, for some $\sigma \in H$, then $\bar{z}_c(H)$ and $\underline{z}_c(H)$ are also NP-hard to compute. Hence, a important part of understanding the complexity of the lex bounds is to classify P and NP-hard cases of computing lex optima of X, which is what we devote our attention to henceforth. Throughout, we assume that X is either input explicitly as a system of inequalities or equipped with a polynomial-time separation oracle. We will adopt the following notation: for two decision problems D_1 and D_2, let $\mathsf{D}_1 \propto \mathsf{D}_2$ denote polynomial reducibility of D_1 to D_2.

4.1. **A connection to integer feasibility.** For the polytope family from (23), Proposition 3.1 provides explicit expressions for its lex optima, and these are clearly computable in polynomial time. To analyze the complexity of computing lex optima for more general sets, we use the decision problem $\mathsf{FEAS}_X(B)$ defined in §3.2. It is obvious that if $\mathsf{FEAS}_X([0,u])$ is NP-hard, then LexMax_X^σ and LexMin_X^σ are NP-hard for every σ. But what about the opposite question: does an easy integer feasibility problem over $[0,u]$ imply that lex optimization is also easy? This is certainly not true for optimizing an arbitrary linear function over integer points in a polytope. The answer to our question is yes for the polytopes in (23), but we next prove it to be in the negative in general by constructing an instance of X for which $\mathsf{FEAS}_X([0,u])$ is in P but LexMin_X^σ is NP-hard for some σ. We also argue that hardness of the feasibility question is a necessary condition for lex optimization to be hard by giving a polynomial reduction from both LexMax_X^σ and LexMin_X^σ to $\mathsf{FEAS}_X(B)$.

Theorem 4.2. *In general, if* $\mathsf{FEAS}_X(B)$ *is* NP-hard *for some* B*, then* LexMax_X^σ *is* NP-hard *for some* σ*. Similarly for* LexMin_X^σ*.*

On the contrary, $\mathsf{LexMax}_X^\sigma \propto \mathsf{FEAS}_X(B)$ *and* $\mathsf{LexMin}_X^\sigma \propto \mathsf{FEAS}_X(B)$*, so that if* $\mathsf{FEAS}_X(B)$ *is in* P *for every* B*, then* LexMax_X^σ *and* LexMin_X^σ *are in* P *for every* σ*.*

Proof. We prove the first statement for the cutset polytope X^{tsp} of a traveling salesman problem defined on the complete graph K_n. This polytope has a well-known polynomial-time separation oracle for its inequalities, and is trivially integer feasible. Consider the problem HC of deciding whether a given graph $G = (V, E)$ is Hamiltonian. Finding a Hamiltonian cycle in G is equivalent to answering $\mathsf{FEAS}_{X^{tsp}}(B)$ with a yes for $B = \{x \in [0,1]^{\binom{n}{2}} \mid x_e = 0, e \in E'\}$.

Claim 4.1. $\mathsf{HC} \propto \mathsf{LexMin}_{X^{tsp}}^\sigma$ *for some* σ*.*

Proof of Claim. Let $|V| = n$ and let K_n be defined on the same vertex set V so that $X^{tsp} \subset [0,1]^{\binom{n}{2}}$. Denote E' to be the complement of E in G. Let $\sigma^* \in \mathfrak{S}_{\binom{n}{2}}$ be such that it orders the variables in x as follows: all x_e with $e \in E$ first and then all x_e with $e \in E'$, so that $\sigma^* \cdot x$ can be partitioned as $(x_E, x_{E'})$. Denote $x^* := \gamma_{X^{tsp}}^{\sigma^*}$. We prove that

$$G \text{ is Hamiltonian } \iff (x^*_E, x^*_{E'}) \preccurlyeq_{\sigma^*} (\mathbb{1}, 0). \tag{26}$$

Since $x^* \in X^{tsp} \cap \{0,1\}^{\binom{n}{2}}$, x^* defines a Hamiltonian cycle in K_n. If $x^* \preccurlyeq_{\sigma^*} (\mathbb{1}, 0)$, then $x^*_{E'} = 0$ and since G and K_n have the same vertex set V, x^*_E defines a Hamiltonian cycle in G. If G contains a Hamiltonian cycle whose incidence vector is $y_E \in \{0,1\}^{|E|}$, then $(y_E, 0) \in X^{tsp} \cap \{0,1\}^{\binom{n}{2}}$ and so $x^* \preccurlyeq_{\sigma^*} (y_E, 0) \preccurlyeq_{\sigma^*} (\mathbb{1}, 0)$. This proves our claim in (26).

Since $(x^*_E, x^*_{E'}) \preccurlyeq_{\sigma^*} (\mathbb{1}, 0)$ can be checked in linear time, it follows that our reduction from HC to $\mathsf{LexMin}_{X^{tsp}}^\sigma$ is polynomial. ⋄

It follows from Claim 4.1 that $\mathsf{LexMin}^{\sigma}_{X^{tsp}}$ is NP-hard. Similar arguments can be used for showing that $\mathsf{LexMax}^{\sigma}_{\mathbb{1}-X^{tsp}}$ is NP-hard, where $\mathbb{1} - X^{tsp} := \{\mathbb{1} - x \mid x \in X^{tsp}\}$ is the complemented polytope.

For the second statement, we recall Algorithm 1 from Proposition 3.4. Due to the bisection procedure, the sequence $\{B_i\}_{i\geq 0}$ has $\mathcal{O}(n \log u_{\max})$ many elements, and the encoding of each box in the sequence is obviously polynomial in the encoding of the input. Therefore we have a polynomial reduction from $\mathsf{LexMax}^{\sigma}_{X}$ to $\mathsf{FEAS}_X(B)$. A modified version of Algorithm 1 gives a polynomial reduction from $\mathsf{LexMin}^{\sigma}_{X}$ to $\mathsf{FEAS}_X(B)$. □

Proposition 3.1 provided a polynomial time algorithm to compute lex optima of certain polytopes $\tilde{P}$ for which $\mathsf{FEAS}_{\tilde{P}}(B)$ is easy. The above theorem, via Algorithm 1, generalizes this polynomial time computability to any bounded set whose integer feasibility question is always easy.

4.2. **Explicit lex constraints.** Now we address the complexity of finding lex optima of the intersection of a polyhedron with finitely many lex constraints over a box. As mentioned in the introduction, such sets and their lex optima arise in branch-and-cut algorithms that employ lexicographic disjunctions to solve integer programs, and also in optimization problems that impose preference criteria.

The lex constraints are assumed to be given explicitly, meaning that they are regarded as part of the input to the problem. We incorporate this assumption by working with the explicitly-input lex-ordered set:

$$\mathcal{L}_{m_1,m_2} := \left\{ x \in [0,u] \mid x \preccurlyeq_{\sigma_i} \theta^i,\ i = 1,\dots,m_1,\ x \succcurlyeq_{\tau_j} \gamma^j,\ j = 1,\dots,m_2 \right\}. \tag{27}$$

Here, it is assumed that $\theta^i, \gamma^j \in [0,u] \cap \mathbb{Z}^n$ and $\sigma_i, \tau_j \in \mathfrak{S}_n$, for $i = 1,\dots,m_1, j = 1,\dots,m_2$, are given as inputs. Thus, this set can be thought of as the intersection of $m_1 + m_2$ many explicitly input lex sets from (4b), in contrast to the set $\mathcal{L}^{\sigma}_X$ from (4a) where γ^{σ}_X and θ^{σ}_X are to be computed from the inputs X and σ. For nontriviality, we assume that $\sigma_i \neq \sigma_j$ and $\tau_i \neq \tau_j$, for $i \neq j$. Each lex constraint defines a convex cone in $\mathbb{R}^n$ that is neither open nor closed. However, this will not matter much since we are interested in integer points inside $\mathcal{L}_{m_1,m_2}$ and the convex hull of $\mathcal{L}_{m_1,m_2} \cap \mathbb{Z}^n$ is a polytope.

Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a given polyhedron. We prove that the complexity of computing lex optima of $P \cap \mathcal{L}_{m_1,m_2}$ can be classified as follows.

Theorem 4.3. *Consider $X = P \cap \mathcal{L}_{m_1,m_2}$.*

(1) $\mathsf{LexMax}^{\sigma}_{X}$ *and* $\mathsf{LexMin}^{\sigma}_{X}$ *are* NP-hard *when at least one of m_1 or m_2 is arbitrary.*
(2) If m_1+m_2 is bounded by a constant and $\mathsf{FEAS}_P([l',u'])$ *is in* P *for any $[l',u'] \subseteq [0,u]$, then* $\mathsf{LexMax}^{\sigma}_{X}$ *and* $\mathsf{LexMin}^{\sigma}_{X}$ *are in* P.
(3) If $A \geq 0$ (resp. $A \leq 0$) in the description of P, then $\mathsf{LexMax}^{\sigma}_{X}$ *(resp.* $\mathsf{LexMin}^{\sigma}_{X}$*) is* NP-hard *if and only if m_2 (resp. m_1) is arbitrary.*

Before proving this theorem, we make some remarks. The claims on polynomial time solvability, under the stated assumptions, are shown by a enumeration algorithm. The first claim says that the problems immediately become hard as soon as one of m_1 or m_2 is allowed to be arbitrary. We do not know the complexity status when both m_1 and m_2 are bounded by a constant but the feasibility check is not easy. When $A \geq 0$, $\mathsf{FEAS}_P(B)$ is in P and so the second claim would imply that boundedness of both m_1 and m_2 is sufficient to make the lex problems easy. The third claim states that the dependence on m_1 can be removed, and is stronger than the observation that when $A \geq 0$ and $m_2 = 0$, since every

$\preccurlyeq_\sigma$ constraint is equivalent to a $\leq$ constraint with positive coefficients, the lex optima can be found in polynomial time using Proposition 3.1.

The main component of our proof of Theorem 4.3 is the complexity of checking integer feasibility of $\mathcal{L}_{m_1,m_2}$. This is NP-hard in general, and remains hard even when considering points in a halfspace satisfying only $\preccurlyeq_\sigma$ ($m_2 = 0$) or only $\succcurlyeq_\sigma$ ($m_1 = 0$) orders.

Lemma 4.4. *Let $H := \{x \in \mathbb{R}^n \mid ax \leq a_0\}$ for $(a, a_0) \in \mathbb{Z}^{n+1}$.* $\mathsf{FEAS}_X([0,1]^n)$ *is* NP-hard *when X is any of the following sets:*

(1) $\mathcal{L}_{m_1,m_2}$ for arbitrary m_1 and m_2.
(2) $\mathcal{L}_{m_1,0} \cap H$ for arbitrary m_1 and $a \leq 0$.
(3) $\mathcal{L}_{0,m_2} \cap H$ for arbitrary m_2 and $a \geq 0$.

Proof. We give a polynomial reduction from the boolean satisfiability problem $\mathsf{3SAT}$ to each of the above feasibility problems. For convenience, we adopt the following simplified notation for lex constraints in our reduction.

Notation 1. Consider $x \preccurlyeq_\sigma \theta$ and suppose there exists some $2 \leq k \leq n-1$ such that $\theta_{\sigma(i)} = 1$ for all $1 \leq i \leq n-k$. Let $i_t = \sigma(n-k+t)$ for $1 \leq t \leq k$, so that the k-most significant variables under the permutation σ are $(x_{i_1}, x_{i_2}, \ldots, x_{i_k} := x_{\sigma(n)})$. Then we express $x \preccurlyeq_\sigma \theta$ in its equivalent form $(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \preccurlyeq (\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_k})$, where the order of comparing variables is assumed to be from right to left (since the variables $x_{\sigma(1)}, \ldots, x_{\sigma(n-k)}$ do not matter, this allows the permutation σ in $\preccurlyeq_\sigma$ to be suppressed). Similarly for $\succcurlyeq_\sigma$ constraints.

Consider any instance of $\mathsf{3SAT}$ with n booleans $(a_1, \ldots, a_n)$ and m clauses $\mathcal{C}_i = \ell^i_1 \vee \ell^i_2 \vee \ell^i_3$ for $i \in [m]$, where each literal ℓ^i_t is either some boolean a_j or its negation $\neg a_j$. Represent the binary variables as x_{a_i} and $x_{\neg a_i}$. We construct instances of the three feasibility problems as follows, and denote them, respectively, by X_1, X_2, and X_3.

X_1**:** Set $m_1 = n+m$ and $m_2 = n$. For $i \in [n]$, the i^{th} $\preccurlyeq$ constraint is $(x_{a_i}, x_{\neg a_i}) \preccurlyeq (0,1)$ and the i^{th} $\succcurlyeq$ constraint is $(x_{a_i}, x_{\neg a_i}) \succcurlyeq (1,0)$. For $i \in [m]$, the $(n+i)^{th}$ $\preccurlyeq$ constraint is $(x_{\neg \ell^i_1}, x_{\neg \ell^i_2}, x_{\neg \ell^i_3}) \preccurlyeq (0,1,1)$.

X_2**:** Set $m_1 = n+m$ and same $\preccurlyeq$ constraints as above, and $H = \{x \in \mathbb{R}^{2n} \mid -\mathbb{1}^\top x \leq -n\}$.

X_3**:** Set $m_2 = n+m$ and $H = \{x \in \mathbb{R}^{2n} \mid \mathbb{1}^\top x \leq n\}$. The $\succcurlyeq$ constraints are $(x_{a_i}, x_{\neg a_i}) \succcurlyeq (1,0)$ for $i \in [n]$, and $(x_{\ell^i_1}, x_{\ell^i_2}, x_{\ell^i_3}) \succcurlyeq (1,0,0)$ for $i \in [m]$.

Now we claim that any $x \in (X_1 \cup X_2 \cup X_3) \cap \{0,1\}^{2n}$ satisfies $x_{a_i} + x_{\neg a_i} = 1$ for all i. For integral $x \in X_1$, this is immediate from

$$\{(x_{a_i}, x_{\neg a_i}) \in \{0,1\}^2 \mid x_{a_i} + x_{\neg a_i} = 1\} = \{(x_{a_i}, x_{\neg a_i}) \in \{0,1\}^2 \mid (x_{a_i}, x_{\neg a_i}) \preccurlyeq (0,1),\ (x_{a_i}, x_{\neg a_i}) \succcurlyeq (1,0)\}. \quad (28)$$

For integral $x \in X_2$, we have $x_{a_i} + x_{\neg a_i} \leq 1$ from $(x_{a_i}, x_{\neg a_i}) \preccurlyeq (0,1)$. This leads to

$$-\sum_{j=1}^{n} \left(x_{a_j} + x_{\neg a_j}\right) + \sum_{j \neq i} \left(x_{a_j} + x_{\neg a_j}\right) \leq -n + \sum_{j \neq i} 1, \quad \forall i.$$

This inequality simplifies to $x_{a_i} + x_{\neg a_i} \geq 1$ and hence we get $x_{a_i} + x_{\neg a_i} = 1$ for all i. For integral $x \in X_3$, we have $x_{a_i} + x_{\neg a_i} \geq 1$, which implies $\mathbb{1}^\top x \geq n$. Since $x \in H$, we get $\mathbb{1}^\top x = n$ and $x_{a_i} + x_{\neg a_i} = 1$ for all i.

There is a trivial bijection between n-dimensional boolean vectors and $\{0,1\}^{2n}$: $x_{a_i} = 1$ if and only if a_i is true, $x_{\neg a_i} = 1$ if and only if a_i is false. Since we argued that $x_{a_i} + x_{\neg a_i} = 1$ for all i, this mapping maps every binary feasible point in any one of X_1, X_2, X_3 to some boolean vector. It remains to show that under this mapping, 3SAT has a feasible truth assignment if and only if any one of X_1, X_2, X_3 is integer feasible. To argue this, note that the constraint $(x_{\neg\ell_1^i}, x_{\neg\ell_2^i}, x_{\neg\ell_3^i}) \preccurlyeq (0,1,1)$ in X_1 and X_2 is equivalent to saying that some literal ℓ_t^i has to be true in the boolean vector a. Same equivalence holds for $(x_{\ell_1^i}, x_{\ell_2^i}, x_{\ell_3^i}) \succcurlyeq (1,0,0)$ in X_3. Therefore, for any binary feasible point in $X_1 \cup X_2 \cup X_3$, the corresponding boolean vector satisfies every clause in 3SAT and hence, is a feasible truth assignment. The other direction works similarly and uses (28) and the fact that every truth assignment has $x_{a_i} + x_{\neg a_i} = 1$ for all i. □

Proof of Theorem 4.3. The NP-hardness of the different cases follows from Lemma 4.4. Now let us argue the polynomial time guarantees. Every $\preccurlyeq_\sigma$ (resp. $\succcurlyeq_\sigma$) order over $\mathbb{Z}^n$ can be written as the union of $n \le$ (resp. $\ge$) inequality comparisons. In particular, $x \preccurlyeq_\sigma \theta$ if and only if $x = \theta$ or $x \in \cup_{i=1}^n B_i$ where $B_i := \{x \in \mathbb{Z}^n \mid x_{\sigma(i)} \le \theta_{\sigma(i)} - 1, x_{\sigma(j)} = \theta_{\sigma(j)}, j = i+1, \dots, n\}$ represents the i^{th} fixing. Since $\mathcal{L}_{m_1,m_2}$ has $m_1 + m_2$ lex orders, there are a total of $n^{m_1+m_2}$ fixings of $\mathcal{L}_{m_1,m_2}$ and so $\mathsf{FEAS}_{\mathcal{L}_{m_1,m_2}}(B)$ can be solved by enumeration in $\mathcal{O}(n^{m_1+m_2})$ time. Furthermore, for each of these fixings, LexMax_X^σ is equivalent to $\mathsf{LexMax}_{P\cap[l',u']}^\sigma$ for some $[l', u'] \subseteq [0, u]$. Thus this simple enumeration algorithm solves LexMax_X^σ by making $\mathcal{O}(n^{m_1+m_2})$ many calls to an algorithm for finding the lexmax point in P over some box. When both m_1 and m_2 are bounded by a constant, the number of calls is polynomial in n. When $\mathsf{FEAS}_P(B)$ is in P for any B, Theorem 4.2 tells us that finding the lexmax and lexmin points in P is also in P. Hence, the overall running time is polynomial. A similar enumeration argument works for the case $A \ge 0$ and m_2 bounded. □

5. DISCUSSION

This paper has introduced a hierarchy of lexicographic primal and dual bounds for an integer program. Properties of these bounds were derived, including their strengths relative to the integer optimum. In particular, it was shown that a specific family of the primal bounds is tight for any 0\1 problem and a specific family of the the dual bounds is tight for any packing- or covering-type 0\1 problem. A major component of constructing these lex bounds is the computation of lex optima of the given set, which inspired us to establish explicit formulas for lex optima of certain polytopes. We also conducted a complexity analysis of the computation of lex bounds and lex optima.

The ideas in this paper are a first step in the direction of using lex order to construct approximations of integer programs without necessarily requiring symmetry considerations. There are several open questions and future research directions worth mentioning.

(1) *Approximation analysis*: We do not know any theoretical guarantees, for general discrete sets, on the quality of the dual bound $\bar{z}_c(H)$ with respect to the integer optimum, unlike the $1/n$-factor approximation provided by the primal bound $\underline{z}_c(H)$ in Theorem 2.1. This bound corresponds to the relaxation $\operatorname{conv}(\cap_{\sigma\in\mathfrak{S}_n}\mathcal{L}_X^\sigma)$.

(2) *Valid inequalities*: Due to $\operatorname{conv}(\cap_{\sigma\in H}\mathcal{L}_X^\sigma)$ being a relaxation of $\operatorname{conv} X_I$, a immediate consequence is that it provides a new approach to deriving valid inequalities for an integer program. These inequalities are obtained by conducting a polyhedral study of the intersection of many lex-ordered sets under different permutations. So far, the literature has focused only on $\operatorname{conv}\mathcal{L}_X^\sigma$, the convex hull of a single lex-ordered set, although a different notion, called an orbitope, of the lex order under many

permutations has been studied in the context of symmetry breaking (cf. §1). Since Theorem 2.12 provides a tightness guarantee of our relaxation for packing-type $0\backslash 1$ programs, the valid inequalities from such a polyhedral study have the potential of being very strong, perhaps even facet-defining in some cases. Note that, as observed in §2.2.2, the valid inequalities derived in this paper for reformulating packing-type $0\backslash 1$ programs are likely not going to be facet-defining, but they can serve as a basis for deriving stronger inequalities through lifting and tilting.

(3) *Complexity analysis*: Proposition 4.1 tells us that the relaxation $\operatorname{conv}(\cap_{\sigma\in H}\mathcal{L}_X^\sigma)$ is NP-hard to optimize over, even when H has linearly many permutations. The set $\operatorname{conv}\mathcal{L}_X^\sigma$ has linearly many facet-defining inequalities, but that does not say anything about how to optimize/separate over the weaker relaxation $\cap_{\sigma\in\mathfrak{S}_n}\operatorname{conv}\mathcal{L}_X^\sigma$. In particular, the complexity status of the question:

$$\textit{given a } x^* \notin X_I\textit{, find a } \sigma\in\mathfrak{S}_n \textit{ such that } x^*\notin \operatorname{conv}\mathcal{L}_X^\sigma,$$

remains open. Note that although a linear function can be optimized over the permutahedron $\operatorname{conv}\mathfrak{S}_n$ in polynomial time, the function $\psi_c(\sigma) := \max_x\{cx \mid x\in \operatorname{conv}\mathcal{L}_X^\sigma\}$ is likely not linear in σ, and so the answer to this question does not seem obvious.

Another complexity question worth tackling is related to Proposition 2.11 which gives a necessary and sufficient condition for X_I to be equal to a single lex-ordered set. This condition is easy to check when the permutation is given, but it is not known how to identify a permutation for which this condition may hold, or certifying such a permutation does not exist.

(4) *Computational analysis*: The lex bounds proposed in this paper can be used in a branch-and-cut algorithm for solving integer programs. One could first find the lex optima for a random collection of permutations H, using either recursion (2) with backtracking, formula in Proposition 3.1, or Algorithm 1. Once this is done, then the primal bound $\underline{z}_c(H)$ is a simple linear time calculation. The dual bound $\bar{z}_c(H)$ would still require solving a integer program, which one may give a prescribed time limit. To guarantee that the dual bound provides an improvement, it is necessary to ensure that the collection H is such that the current fractional point x^* does not belong to $\operatorname{conv}(\cap_{\sigma\in H}\mathcal{L}_X^\sigma)$. Since this set is hard to separate over even when $|H|$ is small, a heuristic procedure would need to be devised for constructing H.

Finally, there still seems to be a need to develop more sophisticated algorithms, besides the three methods ((2), Proposition 3.1, Algorithm 1) mentioned here, for finding lex optima of a set, or lex discrete optimization, in general.

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