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**Pareto efficient solutions in multi-objective optimization involving
forbidden regions**¹

by

CHRISTIAN GÜNTHER

Martin Luther University Halle-Wittenberg
Faculty of Natural Sciences II
Institute for Mathematics
06099 Halle (Saale), Germany
christian.guenther@mathematik.uni-halle.de

Abstract: In this paper, the aim is to compute Pareto efficient solutions of multi-objective optimization problems involving forbidden regions. More precisely, we assume that the vector-valued objective function is componentwise generalized-convex and acts between a real topological linear pre-image space and a finite-dimensional image space, while the feasible set is given by the whole pre-image space excepting some forbidden regions that are defined by convex sets. This leads us to a nonconvex multi-objective optimization problem. Using the recently proposed penalization approach by Günther and Tammer (2017), we show that the solution set of the original problem can be generated by solving a finite family of unconstrained multi-objective optimization problems. We apply our results to a special multi-objective location problem (known as point-objective location problem) where the aim is to locate a new facility in a continuous location space (a finite-dimensional Hilbert space) in the presence of a finite number of demand points. For the choice of the new location point, we are taking into consideration some forbidden regions that are given by open balls (defined with respect to the underlying norm). For such a nonconvex location problem, under the assumption that the forbidden regions are pairwise disjoint, we give complete geometrical descriptions for the sets of (strictly, weakly) Pareto efficient solutions by using the approach by Günther and Tammer (2017) and results derived by Jourani, Michelot and Ndiaye (2009).

Keywords: Multi-objective optimization; Pareto efficiency; Generalized-convexity; Forbidden regions; Location theory; Euclidean norm

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1. INTRODUCTION

In multi-objective optimization, several conflicting objective functions $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$, $m \geq 2$, should be simultaneously minimized. Usually one looks for so-called Pareto efficient solutions. A feasible point $x \in X \subseteq \mathbb{R}^n$ of the multi-objective optimization problem

$$\begin{cases} f(x) = (f_1(x), \dots, f_m(x)) \rightarrow \min \\ x \in X \end{cases}$$

is said to be a Pareto efficient solution in X if

$$\nexists y \in X \text{ subject to } \begin{cases} \forall i \in I_m : f_i(y) \leq f_i(x), \\ \exists j \in I_m : f_j(y) < f_j(x), \end{cases}$$

where $I_m = \{1, 2, \dots, m\}$. For certain classes of multi-objective optimization problems it is known how to compute the whole set of Pareto efficient solutions. In most cases one considers a problem in which the goal is to minimize a vector-valued convex function f over a nonempty closed convex feasible set X . In particular, the case when the feasible set is given by the whole pre-image space (i.e., $X = \mathbb{R}^n$) is often considered in the literature since unconstrained problems can be handled more easily in comparison to constrained ones. For instance, in the applied field of location theory, many authors studied unconstrained multi-objective problems (see, e.g., Alzorba, Günther and Popovici [2]; Alzorba et al. [3]; Durier and Michelot [10]; Puerto and Rodríguez-Chía [21]; Thisse, Ward and Wendell [23]). It is known that considering problems without any constraints is a rather inaccurate approximation in many real world location problems (see, e.g., Carrizosa et al. [7]). Constrained multi-objective location problems are considered for instance in the papers by Carrizosa et al. [6], Carrizosa and Plastria [8] and Ndiaye and Michelot [17] for special types of convex objective functions and convex constraints. Jourani, Michelot and Ndiaye [16] studied a multi-objective location problem with nonconvex objective function and a convex feasible set. Planar multi-objective location problems with nonconvex constraints are considered in the work by Carrizosa et al. [7]. However, Puerto and Rodríguez-Chía [22] noted that there is a lack of a common geometrical description of the sets of solutions for constrained versions of multi-objective location problems. For that reason, Günther and Tammer [12] started to investigate relationships between constrained and unconstrained multi-objective optimization. In the work [12], a new approach was presented that shows that the set of solutions of certain classes of (generalized-convex) multi-objective optimization problems with convex constraints can be generated by solving two corresponding multi-objective optimization problems without constraints. Recently Günther and Tammer [13] succeeded to derive a new penalization approach for (generalized-convex) multi-objective optimization problems involving not necessarily convex constraints where the vector-valued objective function is acting between a real topological linear pre-image space and a finite-dimensional image space.

In our article, we will use the approach by Günther and Tammer [13] in order to characterize the sets of (strictly, weakly) Pareto efficient solutions of (generalized-convex) multi-objective optimization problems involving certain types of nonconvex constraints. More precisely, we will consider a feasible set that is given by the whole pre-image space (a real topological linear space) excepting some forbidden regions that are given by convex sets (i.e., the feasible set is an intersection of so-called reverse convex sets). Such a feasible set is of nonconvex type and occurs often in (single-objective) optimization, for instance, in the field of location theory (see, e.g., Hamacher and Nickel [14], Nickel and Puerto [18]).

The article is structured as follows. In Section 2, we recall generalized-convexity and semi-continuity properties as well as solutions concepts for the vector-valued minimization in our multi-objective optimization problems.

The penalization approach by Günther and Tammer [12, 13] is presented in Section 3. We recall some important relationships between the initial constrained multi-objective optimization problem and two corresponding unconstrained problems.

Section 4 is devoted to the study of generalized-convex multi-objective optimization problems where the feasible set is given by the whole pre-image space (a real topological linear space) excepting some forbidden regions that are given by convex sets.

In Section 5, we emphasize the importance of our results by applying it to a special multi-objective location problem (known as point-objective location problem). In this problem the distances are measured by a norm (induced by a scalar product) and the feasible set is given by the whole pre-image space (a finite-dimensional Hilbert space) excepting some forbidden regions that are given by open balls (defined with respect to the underlying norm). For this nonconvex multi-objective location problem, under the assumption that the forbidden regions are pairwise disjoint, we characterize completely the sets of (strictly, weakly) Pareto efficient solutions by using the penalization approach by Günther and Tammer [13] as well as certain results derived by Jourani, Michelot and Ndiaye [16].

We conclude with some remarks in Section 6.

2. PRELIMINARIES

Throughout this article, we denote by \mathbb{N} , \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_{++} the sets of positive integers, real numbers, nonnegative and positive real numbers, respectively. By \mathbb{R}^m we denote the m -dimensional Euclidean space. Let \mathcal{V} be a real topological linear space. For any set $X \subseteq \mathcal{V}$, we denote the interior, the relative interior, the boundary, the closure, the convex hull, the cardinality of X by $\text{int } X$, $\text{rint } X$, $\text{bd } X$, $\text{cl } X$, $\text{conv } X$, and $\text{card } X$, respectively. The closed line segment between two points $x', x'' \in \mathcal{V}$ is denoted by

$$[x', x''] := \{(1 - \lambda)x' + \lambda x'' \mid \lambda \in [0, 1]\},$$

while the open line segment is $]x', x''[:= [x', x''] \setminus \{x', x''\}$ and the half-open line segments are $[x', x''[:= [x', x''] \setminus \{x''\}$ and $]x', x''] := [x', x''] \setminus \{x'\}$.

For any nonempty set $X \subseteq \mathcal{V}$, the following properties hold (see, e.g., Barbu and Precupanu [4], Jahn [15] and Zălinescu [24]):

- If X is a convex set with $\text{int } X \neq \emptyset$, then we have

$$\text{int } X = \{x \in X \mid \forall v \in \mathcal{V} \exists \delta \in \mathbb{R}_{++} : x + [0, \delta] \cdot v \subseteq X\} \quad (2.1)$$

(i.e., the topological interior of X coincides with the algebraic interior of X).

- We have

$$\text{int } X \subseteq \text{rint } X \subseteq X \subseteq \text{cl } X = (\text{int } X) \cup (\text{bd } X).$$

- $\text{rint } X = \text{int } X$ whenever $\text{int } X \neq \emptyset$.

A set $X \subseteq \mathcal{V}$ is called reverse convex if the complement of X (i.e., the set $X^c := \mathcal{V} \setminus X$) is a convex set in \mathcal{V} (i.e., $\lambda \cdot X^c + (1 - \lambda) \cdot X^c \subseteq X^c$ for all $\lambda \in]0, 1[$).

Given a normed space $(\mathcal{V}, \|\cdot\|)$, where $\|\cdot\| : \mathcal{V} \rightarrow \mathbb{R}$, we denote the open ball around the center $d \in \mathcal{V}$ of radius $r \in \mathbb{R}_{++}$ by

$$B_{\|\cdot\|}(d, r) := \{x \in \mathcal{V} \mid \|x - d\| < r\}$$

and the closed ball around the center $d \in \mathcal{V}$ of radius $r \in \mathbb{R}_{++}$ by

$$\overline{B}_{\|\cdot\|}(d, r) := \{x \in \mathcal{V} \mid \|x - d\| \leq r\}.$$

Notice, for any $r \in \mathbb{R}_{++}$, the balls $B_{\|\cdot\|}(d, r)$ and $\overline{B}_{\|\cdot\|}(d, r)$ are convex sets in \mathcal{V} .

For any nonempty set $X \subseteq \mathcal{V}$, the set

$$\text{cone } X := \{\lambda x \in \mathcal{V} \mid (\lambda, x) \in \mathbb{R}_+ \times X\}$$

is called the cone generated by the set X .

In a finite-dimensional normed space $(\mathcal{V}, \|\cdot\|)$, for any convex set $X \subseteq \mathcal{V}$, the equation (2.1) holds (see Barbu and Precupanu [4, Prop. 1.17]), and moreover, we have

$$x \in \text{int } X \iff \text{cone}(X - x) = \mathcal{V}$$

for any $x \in X$ (see Zălinescu [24, Sec. 1.1])

2.1. Generalized-convexity and semi-continuity notions.

In what follows, we will recall certain generalized-convexity and semi-continuity notions (see, e.g., Cambini and Martein [5] for more details).

A real-valued function $h : \mathcal{V} \rightarrow \mathbb{R}$ is said to be

- upper (lower) semi-continuous along line segments if $h \circ l_{x^0, x^1} : [0, 1] \rightarrow \mathbb{R}$ is upper (lower) semi-continuous on $[0, 1]$ for all $x^0, x^1 \in \mathcal{V}$, where $l_{x^0, x^1} : [0, 1] \rightarrow \mathcal{V}$ is defined by $l_{x^0, x^1}(\lambda) := (1 - \lambda)x^0 + \lambda x^1$ for all $\lambda \in [0, 1]$.
- convex if for all $x^0, x^1 \in \mathcal{V}$ and for all $\lambda \in [0, 1]$ we have $h((1 - \lambda)x^0 + \lambda x^1) \leq (1 - \lambda)h(x^0) + \lambda h(x^1)$.
- quasi-convex if for all $x^0, x^1 \in \mathcal{V}$ and for all $\lambda \in [0, 1]$ we have $h((1 - \lambda)x^0 + \lambda x^1) \leq \max\{h(x^0), h(x^1)\}$.
- semi-strictly quasi-convex if for all $x^0, x^1 \in \mathcal{V}$, $h(x^0) \neq h(x^1)$ and for all $\lambda \in]0, 1[$ we have $h((1 - \lambda)x^0 + \lambda x^1) < \max\{h(x^0), h(x^1)\}$.
- explicitly quasi-convex if h is both quasi-convex and semi-strictly quasi-convex.

Moreover, a function $h : \mathcal{V} \rightarrow \mathbb{R}$ is called concave (quasi-concave, semi-strictly quasi-concave, explicitly quasi-concave) if $-h$ is convex (quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex).

We say that a vector-valued function $f = (f_1, \dots, f_m) : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise upper (lower) semi-continuous along line segments / convex / (semi-strictly, explicitly) quasi-convex / semi-strictly quasi-convex or quasi-convex if f_i is upper (lower) semi-continuous along line segments / convex / (semi-strictly, explicitly) quasi-convex / semi-strictly quasi-convex or quasi-convex for all $i \in I_m$.

Remark 1. *It is well-known that each convex function is explicitly quasi-convex and upper semi-continuous along line segments. Furthermore, each semi-strictly quasi-convex function that is lower semi-continuous along line segments is explicitly quasi-convex. Important applications for the field generalized-convexity can be for instance found in fractional programming (see Cambini and Martein [5]). Moreover, in utility and production theory one often maximizes a generalized-concave function (e.g., the well-known Cobb-Douglas function). Notice that this problem is equivalent to the problem that consists of minimizing the negative of a generalized-concave function (hence a generalized-convex function).*

We now define further notions that will be used in the sequel.

Consider a real-valued function $h : \mathcal{V} \rightarrow \mathbb{R}$ and a nonempty set $X \subseteq \mathcal{V}$. For any $s \in \mathbb{R}$, the (strict) lower-level set and the level line of h to the level s are defined by

$$L_{\sim}(X, h, s) := \{x \in X \mid h(x) \sim s\} \quad \text{for all } \sim \in \{<, \leq, =\},$$

while the (strict) upper-level set of h to the level s are

$$L_{>}(X, h, s) := L_{<}(X, -h, -s) \quad \text{and} \quad L_{\geq}(X, h, s) := L_{\leq}(X, -h, -s).$$

Notice, for any $s \in \mathbb{R}$, we have

$$L_{\sim}(X, h, s) = L_{\sim}(\mathcal{V}, h, s) \cap X \quad \text{for all } \sim \in \{<, \leq, =, >, \geq\}.$$

It is well-known that quasi-convex (quasi-concave) functions are characterized by the convexity of its lower-level (upper-level) sets. Semi-strictly quasi-convex functions can also be characterized by a certain level line / level set condition, as given in Lemma 1.

Lemma 1. *Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a function. Then, the following assertions are equivalent:*

1° *h is semi-strictly quasi-convex.*

2° *For all $s \in \mathbb{R}$, $x^0 \in L_=(\mathcal{V}, h, s)$, $x^1 \in L_<(\mathcal{V}, h, s)$, we have $[x^1, x^0[\subseteq L_<(\mathcal{V}, h, s)$.*

According to Popovici [20, Prop. 2], we have the following important property of semi-strictly quasi-convex functions.

Lemma 2 ([20]). *Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a semi-strictly quasi-convex function. Then, for every pair $(x^0, x^1) \in \mathcal{V} \times \mathcal{V}$, the set*

$$L_> (]x^0, x^1[, h, \max\{h(x^0), h(x^1)\})$$

is either a singleton set or the empty set.

Lemma 2 will be used in the proofs of Theorems 1, 2 and 3.

2.2. Multi-objective optimization.

In this paper, our initial multi-objective optimization problem consist in minimizing a vector-valued objective function $f = (f_1, \dots, f_m) : \mathcal{V} \rightarrow \mathbb{R}^m$ over a nonempty set $X \subseteq \mathcal{V}$:

$$\begin{cases} f(x) = (f_1(x), \dots, f_m(x)) \rightarrow \min \\ x \in X. \end{cases} \quad (\mathcal{P}_X)$$

The corresponding unconstrained problem is denoted by

$$\begin{cases} f(x) = (f_1(x), \dots, f_m(x)) \rightarrow \min \\ x \in \mathcal{V}. \end{cases} \quad (\mathcal{P}_{\mathcal{V}})$$

We are going to recall solution concepts for the vector-valued minimization considered in the above problems (see, e.g., Ehrgott [11] and Jahn [15] for more details). Let us denote the image set of f over X by $f[X] := \{f(x) \in \mathbb{R}^m \mid x \in X\}$. Moreover, \mathbb{R}_+^m stands for the standard ordering cone in \mathbb{R}^m .

Definition 1. *The set of Pareto efficient solutions of problem (\mathcal{P}_X) is defined by*

$$\text{Eff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (\mathbb{R}_+^m \setminus \{0\})) = \emptyset\},$$

while that of weakly Pareto efficient solutions is given by

$$\text{WEff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - \text{int } \mathbb{R}_+^m) = \emptyset\}.$$

The set of strictly Pareto efficient solutions is defined by

$$\text{SEff}(X \mid f) := \{x^0 \in \text{Eff}(X \mid f) \mid \text{card}\{x \in X \mid f(x) = f(x^0)\} = 1\}.$$

It can easily be checked that

$$\text{SEff}(X \mid f) \subseteq \text{Eff}(X \mid f) \subseteq \text{WEff}(X \mid f).$$

In preparation of the next lemma, for any $x^0 \in X$, we define the intersections of (strict) lower-level sets / level lines by

$$S_{\sim}(X, f, x^0) := \bigcap_{i \in I_m} L_{\sim}(X, f_i, f_i(x^0)) \quad \text{for all } \sim \in \{<, \leq, =\}.$$

It is known that (strictly, weakly) Pareto efficient solutions can be characterized by certain conditions based on level sets and level lines of the component functions of f .

Lemma 3. *For any $x^0 \in X$, we have*

$$\begin{aligned} x^0 \in \text{Eff}(X \mid f) &\iff S_{\leq}(X, f, x^0) \subseteq S_=(X, f, x^0); \\ x^0 \in \text{WEff}(X \mid f) &\iff S_{<}(X, f, x^0) = \emptyset; \\ x^0 \in \text{SEff}(X \mid f) &\iff S_{\leq}(X, f, x^0) = \{x^0\}. \end{aligned}$$

The geometrical characterizations of (strictly, weakly) Pareto efficient solutions as given in Lemma 3 can be found in the book by Ehrgott [11, Th. 2.30]. Notice that these characterizations were already used in the works by Plastria [19] and Durier and Michelot [10, Prop. 1.1] in the context of location theory.

The next lemma gives useful bounds for the sets of (strictly, weakly) Pareto efficient solutions of the problem (\mathcal{P}_X) under generalized-convexity assumption on f but without convexity assumption on the feasible set X .

Lemma 4 ([13]). *Let $X \subseteq \mathcal{V}$ be a nonempty set and let $Y \subseteq \mathcal{V}$ be a set with $X \subseteq Y$. Then, the following assertions hold:*

1°. *We have*

$$\begin{aligned} X \cap \text{Eff}(Y | f) &\subseteq \text{Eff}(X | f); \\ X \cap \text{WEff}(Y | f) &\subseteq \text{WEff}(X | f); \\ X \cap \text{SEff}(Y | f) &\subseteq \text{SEff}(X | f). \end{aligned}$$

2°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex, then*

$$\begin{aligned} \text{Eff}(X | f) &\subseteq [X \cap \text{Eff}(Y | f)] \cup \text{bd } X; \\ \text{WEff}(X | f) &\subseteq [X \cap \text{WEff}(Y | f)] \cup \text{bd } X. \end{aligned}$$

3°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex, then*

$$\text{SEff}(X | f) \subseteq [X \cap \text{SEff}(Y | f)] \cup \text{bd } X.$$

Proof. The proof is analogous to the proof of Günther and Tammer [13, Lem. 4.4, Cor. 4.5]. Notice that f is a componentwise generalized-convex function on the whole space \mathcal{V} in assertions 2° and 3°. Hence, in the view of the proof of [13, Lem. 4.4, Cor. 4.5], we can omit to assume that Y is convex. \square

3. PENALIZATION APPROACH IN MULTI-OBJECTIVE OPTIMIZATION

In this section, we recall the penalization approach recently derived by Günther and Tammer [12, 13]. This approach can be used for solving a constrained multi-objective optimization problem by using two corresponding unconstrained problems. It should be mentioned that there are also other vectorial penalization approaches known in the literature (see, e.g., Durea, Strugariu and Tammer [9]).

Considering a penalization function $\phi : \mathcal{V} \rightarrow \mathbb{R}$, we can define a new unconstrained multi-objective optimization problem by

$$\begin{cases} f^\oplus(x) := (f_1(x), \dots, f_m(x), \phi(x)) \rightarrow \min \\ x \in \mathcal{V}. \end{cases} \quad (\mathcal{P}_\mathcal{V}^\oplus)$$

In what follows, we will need in certain results some of the following assumptions concerning the lower-level sets / level lines of the function ϕ :

$$\forall x^0 \in \text{bd } X : L_{\leq}(\mathcal{V}, \phi, \phi(x^0)) = X, \quad (\text{A1})$$

$$\forall x^0 \in \text{bd } X : L_{=}(\mathcal{V}, \phi, \phi(x^0)) = \text{bd } X, \quad (\text{A2})$$

$$L_{\leq}(\mathcal{V}, \phi, 0) = X, \quad (\text{A3})$$

$$L_{=}(\mathcal{V}, \phi, 0) = \text{bd } X, \quad (\text{A4})$$

$$\forall x^0 \in \text{bd } X \exists \tilde{x} \in \text{int } X : [\tilde{x}, x^0[\subseteq L_{<}(\mathcal{V}, \phi, \phi(x^0)), \quad (\text{A5})$$

where

$$X \subseteq \mathcal{V} \text{ is a closed set with } X \neq \mathcal{V} \text{ and } \text{int } X \neq \emptyset. \quad (3.1)$$

In the next two lemmata, we present some preliminary results related to the validity of the above assumptions.

Lemma 5. *Let (3.1) be satisfied. Then, we have:*

- 1°. *If ϕ fulfils (A3) and (A4), then ϕ fulfils (A1) and (A2).*
- 2°. *ϕ fulfils (A1) and (A2) if and only if $\widehat{\phi} := h \circ \phi : \mathcal{V} \rightarrow \mathbb{R}$ fulfils (A1) and (A2) (with $\widehat{\phi}$ in the role of ϕ), where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function on the image set $\phi[\mathcal{V}]$.*
- 3°. *ϕ fulfils (A5) if and only if $\widehat{\phi} := h \circ \phi : \mathcal{V} \rightarrow \mathbb{R}$ fulfils (A5) (with $\widehat{\phi}$ in the role of ϕ), where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function on the image set $\phi[\mathcal{V}]$.*
- 4°. *ϕ fulfils (A1) and (A2) if and only if $\widehat{\phi} := \phi - \phi(x^0)$, $x^0 \in \text{bd } X$, fulfils (A1), (A2), (A3) and (A4) (with $\widehat{\phi}$ in the role of ϕ).*

Lemma 6. *Let (3.1) be satisfied. Assume that ϕ is a semi-strictly quasi-convex and continuous function which fulfils Assumption (A3) and $L_{<}(\mathcal{V}, \phi, 0) \neq \emptyset$. Then, ϕ fulfils Assumptions (A1), (A2), (A4) and (A5). Moreover, the set X is convex.*

Proof. Follows by results given in Günther and Tammer [13, Rem. 5.5, Cor. 6.15] and by assertion 1° in Lemma 5. Notice that (A5) follows by the Assumptions (A1) and (A2) and by the semi-strictly quasi-convexity of ϕ . By Günther and Tammer [13, Lem. 6.16], we get the convexity of X . \square

Due to results by Günther and Tammer [13, Th. 5.1, Th. 5.6, Th. 5.13], we can completely characterize the sets of (strictly, weakly) Pareto efficient solutions of the initial constrained multi-objective optimization problem (\mathcal{P}_X) by using the sets of (strictly, weakly) solutions of the corresponding unconstrained problems $(\mathcal{P}_{\mathcal{V}})$ and $(\mathcal{P}_{\mathcal{V}}^{\oplus})$.

Proposition 1 ([13]). *Let (3.1) be satisfied. Suppose that ϕ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:*

- 1°. *We have*

$$[X \cap \text{Eff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{Eff}(\mathcal{V} | f^{\oplus})] \subseteq \text{Eff}(X | f).$$

- 2°. *Let $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise semi-strictly quasi-convex. Then, we have*

$$[X \cap \text{Eff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{Eff}(\mathcal{V} | f^{\oplus})] \supseteq \text{Eff}(X | f).$$

Proposition 2 ([13]). *Let (3.1) be satisfied. Suppose that ϕ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:*

- 1°. *Let $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise upper semi-continuous along line segments. Assume that ϕ fulfils Assumption (A5). Then, we have*

$$[X \cap \text{WEff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{WEff}(\mathcal{V} | f^{\oplus})] \subseteq \text{WEff}(X | f).$$

- 2°. *Let $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise semi-strictly quasi-convex. Then, we have*

$$[X \cap \text{WEff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{WEff}(\mathcal{V} | f^{\oplus})] \supseteq \text{WEff}(X | f).$$

Proposition 3 ([13]). *Let (3.1) be satisfied. Suppose that ϕ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:*

- 1°. *We have*

$$[X \cap \text{SEff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{SEff}(\mathcal{V} | f^{\oplus})] \subseteq \text{SEff}(X | f).$$

- 2°. *Let $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex. Then, we have*

$$[X \cap \text{SEff}(\mathcal{V} | f)] \cup [(\text{bd } X) \cap \text{SEff}(\mathcal{V} | f^{\oplus})] \supseteq \text{SEff}(X | f).$$

4. MULTI-OBJECTIVE OPTIMIZATION PROBLEMS INVOLVING FORBIDDEN REGIONS

In this section, we consider a feasible set X that is given by the whole pre-image space \mathcal{V} excepting some forbidden regions that are given by convex sets. More precisely, we suppose that the following assumption is fulfilled:

$$\begin{cases} \text{Let } D_1, \dots, D_l \subseteq \mathcal{V} \text{ be closed convex sets with } D_i \neq \mathcal{V} \text{ and } \text{int } D_i \neq \emptyset, i \in I_l; \\ \text{let } X := \bigcap_{i \in I_l} X_i \text{ with } X_i := \mathcal{V} \setminus \text{int } D_i, i \in I_l, \text{ and let } X \neq \emptyset. \end{cases} \quad (4.1)$$

Under the assumption (4.1), the feasible set X is an intersection of closed reverse convex sets X_1, \dots, X_l . So, X is a closed set too. Moreover, notice that we have $\text{bd } D_i = \text{bd } X_i$ for all $i \in I_l$.

The sets $D_i, i \in I_l$, are said to be pairwise disjoint if

$$D_i \cap D_j = \emptyset \quad \text{for all } i, j \in I_l, i \neq j. \quad (4.2)$$

Formula (4.2) implies that the sets $\text{int } D_i, i \in I_l$, are pairwise disjoint, i.e.,

$$(\text{int } D_i) \cap (\text{int } D_j) = \emptyset \quad \text{for all } i, j \in I_l, i \neq j. \quad (4.3)$$

Notice that each of the conditions (4.2) and (4.3) implies

$$X \cap \text{bd } X_i = \text{bd } X_i = \text{bd } D_i, \quad (4.4)$$

which is a direct consequence of the next result.

Lemma 7. *Let (4.1) and (4.3) be satisfied. Then, we have*

$$\text{bd } X = \bigcup_{i \in I_l} \text{bd } D_i.$$

Proof. Since I_l is a finite index set, we have

$$\text{int} \left(\bigcap_{i \in I_l} X_i \right) = \bigcap_{i \in I_l} \text{int } X_i = \left(\bigcup_{i \in I_l} D_i \right)^c. \quad (4.5)$$

Now, we are going to prove that

$$(\text{int } D_j)^c \cap D_i = D_i \quad \text{for every } i, j \in I_l, i \neq j. \quad (4.6)$$

Assume the contrary holds, i.e., there exists $x \in D_i \setminus (\text{int } D_j)^c = D_i \cap (\text{int } D_j)$ for some $i, j \in I_l, i \neq j$. Of course, in view of (4.3), we must have $x \in (\text{bd } D_i) \cap (\text{int } D_j)$. Consider some $d \in \text{int } D_i$ (notice that $d \neq x$). By the convexity of D_i , we infer that $]x, d] \subseteq \text{int } D_i$ (see, e.g., Zălinescu [24, Th. 1.1.2]). This means, for every $\delta \in]0, 1]$, we have $x +]0, \delta] \cdot (d - x) \subseteq \text{int } D_i$. Moreover, since $x \in \text{int } D_j$ and D_j is convex, we get $x +]0, \delta'] \cdot (d - x) \subseteq \text{int } D_j$ for some $\delta' \in]0, 1]$. Hence, we have

$$\emptyset \neq x +]0, \delta'] \cdot (d - x) \subseteq (\text{int } D_i) \cap (\text{int } D_j)$$

in contradiction to (4.3). So, (4.6) holds.

Consequently, we have

$$\begin{aligned}
\text{bd } X = X \setminus \text{int } X &= \left(\bigcap_{j \in I_l} X_j \right) \setminus \text{int} \left(\bigcap_{i \in I_l} X_i \right) \\
&\stackrel{(4.5)}{=} \left(\bigcap_{j \in I_l} X_j \right) \setminus \left(\bigcup_{i \in I_l} D_i \right)^c \\
&= \left(\bigcap_{j \in I_l} X_j \right) \cap \left(\bigcup_{i \in I_l} D_i \right) \\
&= \bigcup_{i \in I_l} \bigcap_{j \in I_l} (\text{int } D_j)^c \cap D_i \\
&= \bigcup_{i \in I_l} (\text{bd } D_i) \cap \bigcap_{j \in I_l \setminus \{i\}} (\text{int } D_j)^c \cap D_i \\
&\stackrel{(4.6)}{=} \bigcup_{i \in I_l} (\text{bd } D_i) \cap D_i = \bigcup_{i \in I_l} \text{bd } D_i.
\end{aligned}$$

□

Let us consider, for any $i \in I_l$, a penalization function $\phi_i : \mathcal{V} \rightarrow \mathbb{R}$ that fulfils the Assumptions (A1) and (A2) (with ϕ_i in the role of ϕ and X_i in the role of X). Then, for any $i \in I_l$, we can define a new penalized multi-objective optimization problem by

$$\begin{cases} f^{\oplus i}(x) := (f_1(x), \dots, f_m(x), \phi_i(x)) \rightarrow \min \\ x \in \mathcal{V}. \end{cases} \quad (\mathcal{P}_{\mathcal{V}}^{\oplus i})$$

4.1. Problems with one forbidden region ($l = 1$).

In this section, we analyze an important special case in which we have exactly one (i.e., $l = 1$) forbidden region. For notational convenience, we assume that $\phi := \phi_1$ and $D := D_1$.

In preparation of the next lemma, we define a new penalization function $\widehat{\phi} : \mathcal{V} \rightarrow \mathbb{R}$ by

$$\widehat{\phi} := -\phi.$$

Lemma 8. *Let (4.1) be satisfied. Then, the following assertions are equivalent:*

- 1°. ϕ fulfils the Assumptions (A3) and (A4).
- 2°. $\widehat{\phi}$ fulfils the Assumptions (A3) and (A4) with $\widehat{\phi}$ in the role of ϕ and D in the role of X .

Proof. First, we are going to prove that

$$\text{int}(\mathcal{V} \setminus \text{int } D) = \mathcal{V} \setminus D. \quad (4.7)$$

Since D is closed, we infer that $\mathcal{V} \setminus D$ is open. Then, then inclusion “ \supseteq ” in (4.7) follows by the fact that $\mathcal{V} \setminus D \subseteq \mathcal{V} \setminus \text{int } D$. Now, we prove the reverse inclusion “ \subseteq ”.

Assume that there is $x \in \text{int}(\mathcal{V} \setminus \text{int } D)$ with $x \notin \mathcal{V} \setminus D$, i.e., $x \in D$. Of course, since $x \in \mathcal{V} \setminus \text{int } D$ we must have $x \in \text{bd } D$. Consider $d \in \text{int } D$. By the convexity of D , we infer that $]x, d[\subseteq \text{int } D$ (see, e.g., Zălinescu [24, Th. 1.1.2]). This means, for every $\delta \in]0, 1[$, we have $x +]0, \delta[\cdot (d - x) \subseteq \text{int } D$. Hence, x is no algebraic interior point of $\mathcal{V} \setminus \text{int } D$, which implies $x \notin \text{int}(\mathcal{V} \setminus \text{int } D)$, a contradiction. We conclude that (4.7) holds.

So, we have

$$\begin{aligned} L_{<}(\mathcal{V}, \phi, 0) &= \text{int } X & (4.8) \\ \iff L_{\geq}(\mathcal{V}, \phi, 0) &= \mathcal{V} \setminus \text{int } X \end{aligned}$$

$$\begin{aligned} \iff L_{\leq}(\mathcal{V}, \widehat{\phi}, 0) &= \mathcal{V} \setminus \text{int}(\mathcal{V} \setminus \text{int } D) \\ \stackrel{(4.7)}{\iff} L_{<}(\mathcal{V}, \widehat{\phi}, 0) &= D & (4.9) \end{aligned}$$

and

$$\begin{aligned} L_{\leq}(\mathcal{V}, \phi, 0) &= X & (4.10) \\ \iff L_{>}(\mathcal{V}, \phi, 0) &= \mathcal{V} \setminus X \\ \iff L_{<}(\mathcal{V}, \widehat{\phi}, 0) &= \mathcal{V} \setminus (\mathcal{V} \setminus \text{int } D) \\ \iff L_{<}(\mathcal{V}, \widehat{\phi}, 0) &= \text{int } D. & (4.11) \end{aligned}$$

Notice that (4.8) follows by (4.10) and

$$L_{=}(\mathcal{V}, \phi, 0) = \text{bd } X, \quad (4.12)$$

while (4.8) and (4.10) imply (4.12). Analogously, (4.11) follows by (4.9) and

$$L_{=}(\mathcal{V}, \widehat{\phi}, 0) = \text{bd } D, \quad (4.13)$$

while (4.11) and (4.9) imply (4.13). The proof is complete. \square

Lemma 9. *Let (4.1) be satisfied. Assume that $\widehat{\phi} = -\phi$ is a semi-strictly quasi-convex and continuous function which fulfils Assumption (A3) (with $\widehat{\phi}$ in the role of ϕ and D in the role of X) and suppose that $L_{<}(\mathcal{V}, \widehat{\phi}, 0) \neq \emptyset$. Then, ϕ is a semi-strictly quasi-concave and continuous function and fulfils the Assumptions (A1), (A2), (A3) and (A4).*

Proof. Follows immediately by Lemma 5 (1°) and Lemmata 6 and 8. \square

Example 1. *Let $D \subseteq \mathcal{V}$ be a closed convex set with $d \in \text{int } D \neq \emptyset$ and $D \neq \mathcal{V}$. Let a Minkowski gauge function $\mu : \mathcal{V} \rightarrow \mathbb{R}$ be given by*

$$\mu(x) := \inf\{\lambda \in \mathbb{R}_{++} \mid x \in \lambda \cdot (-d + D)\} \quad \text{for all } x \in \mathcal{V}.$$

Under our assumptions, the function μ is convex (hence explicitly quasi-convex) and continuous (see, e.g., Zălinescu [24, Prop. 1.1.1]). Hence, the function $\widehat{\phi}$, defined by

$$\widehat{\phi}(x) := \mu(x - d) - 1 \quad \text{for all } x \in \mathcal{V},$$

has these properties too. Since $d \in L_{<}(\mathcal{V}, \widehat{\phi}, 0) \neq \emptyset$ and (A3) holds (with $\widehat{\phi}$ in the role of ϕ and D in the role of X), we get that

$$\phi := -\widehat{\phi}(\cdot) = -\mu(\cdot - d) + 1$$

satisfies the Assumptions (A1), (A2), (A3) and (A4) by Lemma 9. Moreover, the function

$$\bar{\phi}(\cdot) := -\mu(\cdot - d)$$

fulfils the Assumptions (A1) and (A2) (with $\bar{\phi}$ in the role of ϕ) by Lemma 5 (4°).

In assertion 1° of Proposition 2 we need that the function ϕ fulfils the Assumption (A5). In the next lemma, we will show that the penalization function $\bar{\phi}$ given in Example 1 satisfies Assumption (A5) (with $\bar{\phi}$ in the role of ϕ).

Lemma 10. *Let (4.1) be satisfied. Consider any $x^1 \in \text{bd } X$ and $d \in \text{int } D$. Define $\tilde{x} := x^1 + (x^1 - d) \neq x^1$. Then, we have*

$$[\tilde{x}, x^1] \subseteq L_{<}(\mathcal{V}, \bar{\phi}, \bar{\phi}(x^1)) = \text{int } X.$$

Thus, $\bar{\phi}$ fulfils the Assumption (A5) (with $\bar{\phi}$ in the role of ϕ).

Proof. First, notice that $\mu(x^1 - d) = 1 > 0$ since $x^1 \in \text{bd } X = \text{bd } D = L_=(\mathcal{V}, \bar{\phi}, -1)$. Hence, for any $\lambda \in]0, 1]$, we have

$$\begin{aligned} \bar{\phi}((1-\lambda)x^1 + \lambda\tilde{x}) &= -\mu((1-\lambda)x^1 + \lambda(2x^1 - d) - d) \\ &= -\mu((\lambda+1)(x^1 - d)) \\ &= -(\lambda+1)\mu(x^1 - d) \\ &< -\mu(x^1 - d) \\ &= \bar{\phi}(x^1), \end{aligned}$$

which shows the assertion in this lemma. \square

According to Günther and Tammer [13, Th. 5.4, Th. 5.12], we have the inclusions

$$\begin{aligned} \text{SEff}(X | f) &\subseteq \text{SEff}(\mathcal{V} | f^{\oplus 1}), \\ \text{WEff}(X | f) &\subseteq \text{WEff}(\mathcal{V} | f^{\oplus 1}). \end{aligned}$$

However, Günther and Tammer [12, Ex. 1] gave a counter-example for the convex case which shows that the inclusion

$$\text{Eff}(X | f) \subseteq \text{Eff}(\mathcal{V} | f^{\oplus 1}) \quad (4.14)$$

does not hold in general. In the next example, we point out that inclusion (4.14) does not hold in our class of problems.

Example 2. Figure 1 shows a constrained convex multi-objective location problem with functions $f_1, f_2, f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f_i(x) := \|x - a^i\|_1$ for all $x \in \mathbb{R}^2$ and all $i \in I_3$, where $a^1 := (5, 5)$, $a^2 := (2, 2.5)$, $a^3 := (3.5, 3.5) \in \mathbb{R}^2$ and $\|\cdot\|_1$ denotes the Manhattan norm in \mathbb{R}^2 . Consider the feasible set $X := \mathbb{R}^2 \setminus \text{int } D$ with $D := [2, 3.5] \times [3.5, 5]$ and put $d := (3, 4) \in \text{int } D$. Let the penalization function ϕ_1 be given by the function $\bar{\phi}$ considered in Example 1. In the left part of Figure 1 one can see that the point $x^0 \in \text{bd } X = \text{bd } D$ is belonging to both sets $\text{Eff}(X | f)$ and $\text{Eff}(\mathbb{R}^2 | f)$. Notice that we have

$$\text{Eff}(X | f) = \text{Eff}(\mathbb{R}^2 | f) = ([2, 3.5] \times [2.5, 3.5]) \cup ([3.5, 5] \times [3.5, 5])$$

by the Rectangular Decomposition Algorithm in Alzorba et al. [3]. The right part of Figure 1 shows that $x^0 \notin \text{Eff}(\mathbb{R}^2 | f^{\oplus 1})$ since $x^1 \in (\text{int } X) \cap S_=(\mathbb{R}^2, f, x^0)$. Consequently, the inclusion in (4.14) does not hold in this example.

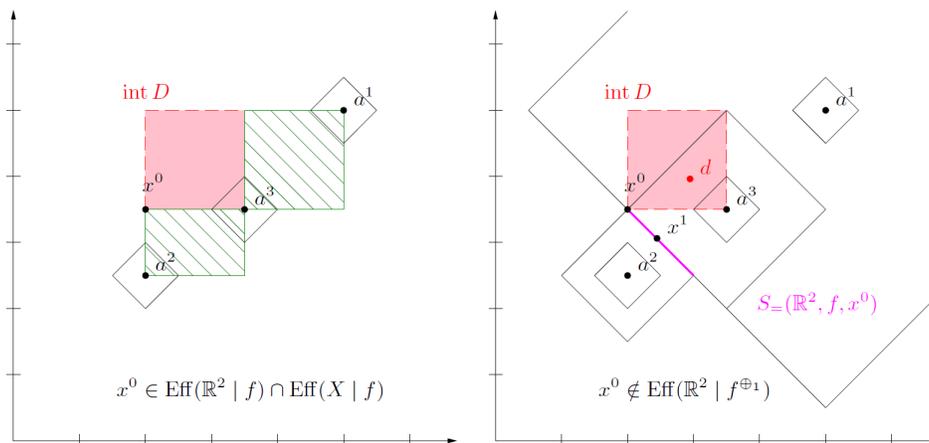


FIGURE 1. Counter-example for the inclusion (4.14).

4.2. Problems with multiple forbidden regions ($l > 1$).

The next theorem is related to the concept of Pareto efficiency and presents relationships between the initial constrained multi-objective optimization problem (\mathcal{P}_X) and a finite family of unconstrained multi-objective optimization problems $(\mathcal{P}_{\mathcal{V}})$, $(\mathcal{P}_{\mathcal{V}^{\oplus i}})$, $i \in I_l$.

Theorem 1. *Let (4.1) be satisfied. Suppose that each function ϕ_i , $i \in I_l$, fulfils Assumptions (A1) and (A2) (with ϕ_i in the role of ϕ and X_i in the role of X). Then, the following assertions hold:*

1°. We have

$$X \cap \text{Eff}(\mathcal{V} | f) \subseteq X \cap \bigcup_{i \in I_l} \text{Eff}(X_i | f) \subseteq \text{Eff}(X | f). \quad (4.15)$$

2°. Assume that (4.2) holds. Let f be componentwise semi-strictly quasi-convex. Then, we have

$$X \cap \bigcup_{i \in I_l} \text{Eff}(X_i | f) \supseteq \text{Eff}(X | f). \quad (4.16)$$

3°. Assume that (4.3) holds. Let f be componentwise explicitly quasi-convex. Then, (4.16) is true.

4°. We have

$$\text{Eff}(X | f) \supseteq [X \cap \text{Eff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_l} X \cap (\text{bd } X_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i}) \right]. \quad (4.17)$$

Now, suppose that (4.2) holds. Let f be componentwise semi-strictly quasi-convex. Then, we have

$$\text{Eff}(X | f) = [X \cap \text{Eff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_l} (\text{bd } D_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i}) \right]. \quad (4.18)$$

5°. Assume that (4.3) holds. Let f be componentwise explicitly quasi-convex. Then, (4.18) is true.

Proof. 1°. Since $X \subseteq X_i \subseteq \mathcal{V}$ for all $i \in I_l$, we get (4.15) directly by Lemma 4 (1°).
2°. Consider $x^0 \in \text{Eff}(X | f)$. On one hand, we can have $x^0 \in \text{Eff}(\mathcal{V} | f)$, hence it follows $x^0 \in \text{Eff}(X_i | f)$ for some $i \in I_l$ by 1° of this theorem. On the other hand, we can have $x^0 \notin \text{Eff}(\mathcal{V} | f)$. Then, there exists $x^1 \in \mathcal{V} \setminus X = \bigcup_{i \in I_l} \text{int } D_i$ with

$$x^1 \in L_{<}(\mathcal{V}, f_j, f_j(x^0)) \cap S_{\leq}(\mathcal{V}, f, x^0) \quad \text{for some } j \in I_m. \quad (4.19)$$

Without loss of generality, we assume $x^1 \in \text{int } D_k$ for some $k \in I_l$. We are going to show that

$$\left[\bigcup_{i \in I_m} L_{<}(\mathcal{V}, f_i, f_i(x^0)) \right] \cap S_{\leq}(\mathcal{V}, f, x^0) \subseteq \text{int } D_k,$$

which implies $x^0 \in \text{Eff}(X_k | f)$.

Suppose that the contrary holds, i.e., there exists $x^2 \in \text{int } D_{\bar{k}}$ with $\bar{k} \in I_l \setminus \{k\}$ such that

$$x^2 \in L_{<}(\mathcal{V}, f_{\bar{j}}, f_{\bar{j}}(x^0)) \cap S_{\leq}(\mathcal{V}, f, x^0) \quad \text{for some } \bar{j} \in I_m.$$

By (4.2) and the closedness of D_i , $i \in I_l$, we infer that the set $X \cap]x^1, x^2[$ has an infinite number of elements. In particular, we have

$$\text{card}(X \cap]x^1, x^2[) \geq m + 2. \quad (4.20)$$

We are going to prove that

$$\exists x^3 \in]x^1, x^2[: x^3 \in L_{<}(X, f_j, f_j(x^0)) \cap S_{\leq}(X, f, x^0), \quad (4.21)$$

which implies $x^0 \notin \text{Eff}(X | f)$, a contradiction.

Since $\max\{f_i(x^1), f_i(x^2)\} \leq f_i(x^0)$ for every $i \in I_m$, we infer that

$$\text{card} \left(\bigcup_{i \in I_m} L_{>} (]x^1, x^2[, f_i, f_i(x^0)) \right) \leq m \quad (4.22)$$

by Lemma 2. Now, for the specific index j given in (4.19), we consider two cases:

Case 1: If $x^2 \in L_{=}(\mathcal{V}, f_j, f_j(x^0))$, then in view of Lemma 1 we get $]x^1, x^2[\subseteq L_{<}(\mathcal{V}, f_j, f_j(x^0))$. By (4.20), it follows

$$\text{card} (X \cap L_{<} (]x^1, x^2[, f_j, f_j(x^0))) \geq m + 1. \quad (4.23)$$

Case 2: If $x^2 \in L_{<}(\mathcal{V}, f_j, f_j(x^0))$, then we have

$$\text{card} L_{>} (]x^1, x^2[, f_j, s) \leq 1 \quad (4.24)$$

with $s := \max\{f_j(x^1), f_j(x^2)\} < f_j(x^0)$ by Lemma 2. Due to (4.20) and (4.24), it follows (4.23).

So, in both cases (4.23) holds. Consequently, we get the validity of (4.21) by (4.22) and (4.23). This completes the proof of assertion 2°.

3°. The proof is analogous to the proof of assertion 2°. By (4.3), we immediately get $\text{card} (X \cap]x^1, x^2[) \geq 1$ instead of (4.20). Notice, for any $i \in I_m$, the conditions $x^1, x^2 \in L_{\sim}(\mathcal{V}, f_i, f_i(x^0))$ imply $]x^1, x^2[\subseteq L_{\sim}(\mathcal{V}, f_i, f_i(x^0))$ for all $\sim \in \{<, \leq\}$ by the quasi-convexity of f_i . Consequently, it follows

$$\emptyset \neq X \cap]x^1, x^2[\subseteq L_{<}(X, f_j, f_j(x^0)) \cap S_{\leq}(X, f, x^0).$$

4°. By Proposition 1 (1°), for any $i \in I_l$, we have

$$[X_i \cap \text{Eff}(\mathcal{V} | f)] \cup [(bd X_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i})] \subseteq \text{Eff}(X_i | f). \quad (4.25)$$

Notice that $\text{int} X_i \neq \emptyset$ by Lemma 10. Then, due to 1° of this theorem, we get

$$\begin{aligned} \text{Eff}(X | f) &\stackrel{(4.15)}{\supseteq} X \cap \bigcup_{i \in I_l} \text{Eff}(X_i | f) \\ &\stackrel{(4.25)}{\supseteq} X \cap \bigcup_{i \in I_l} ([X_i \cap \text{Eff}(\mathcal{V} | f)] \cup [(bd X_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i})]) \\ &= \left[\bigcup_{i \in I_l} X \cap X_i \cap \text{Eff}(\mathcal{V} | f) \right] \cup \left[\bigcup_{i \in I_l} X \cap (bd X_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i}) \right] \\ &= [X \cap \text{Eff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_l} X \cap (bd D_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i}) \right], \end{aligned}$$

where $X \cap X_i = X$ for every $i \in I_l$, which shows (4.17).

Assume that (4.2) holds. Let f be componentwise semi-strictly quasi-convex. By (4.16) and by Proposition 1 (2°), we get the reverse inclusion, which shows (4.18) in view of (4.4).

5°. This assertion follows by 1° and 3° of this theorem as well as by the ideas given in the proof of assertion 4°.

□

Notice that the assumptions (4.2) in 4° and (4.3) in 5° of Theorem 2 are essential for the validity of (4.18) (see Example 3 in Section 5).

In the next theorem, we derive relationships between the initial constrained multi-objective optimization problem (\mathcal{P}_X) and the corresponding unconstrained problems $(\mathcal{P}_{\mathcal{V}})$ and $(\mathcal{P}_{\mathcal{V}}^{\oplus i})$, $i \in I_l$, for the concept of weak Pareto efficiency.

Theorem 2. *Let (4.1) be satisfied. Suppose that each penalization function ϕ_i , $i \in I_1$, fulfils Assumptions (A1) and (A2) (with ϕ_i in the role of ϕ and X_i in the role of X). Then, the following assertions hold:*

1°. *We have*

$$X \cap \text{WEff}(\mathcal{V} | f) \subseteq X \cap \bigcup_{i \in I_1} \text{WEff}(X_i | f) \subseteq \text{WEff}(X | f).$$

2°. *Assume that (4.2) holds. Let f be componentwise semi-strictly quasi-convex or quasi-convex. Then, we have*

$$X \cap \bigcup_{i \in I_1} \text{WEff}(X_i | f) \supseteq \text{WEff}(X | f). \quad (4.26)$$

3°. *Assume that (4.3) holds. Let f be componentwise quasi-convex. Then, (4.26) is true.*

4°. *Let f be componentwise upper semi-continuous along line segments. Assume that each function ϕ_i , $i \in I_1$, fulfils Assumption (A5). Then, we have*

$$\text{WEff}(X | f) \supseteq [X \cap \text{WEff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } X_i) \cap \text{WEff}(\mathcal{V} | f^{\oplus i}) \right].$$

Now, suppose that (4.2) holds. In addition, assume that f is componentwise semi-strictly quasi-convex. Then, we have

$$\text{WEff}(X | f) = [X \cap \text{WEff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_1} (\text{bd } D_i) \cap \text{WEff}(\mathcal{V} | f^{\oplus i}) \right]. \quad (4.27)$$

5°. *Suppose that (4.3) holds. Let f be componentwise explicitly quasi-convex and upper semi-continuous along line segments. Assume that each function ϕ_i , $i \in I_1$, fulfils Assumption (A5). Then, (4.27) is true.*

Proof. The proof uses similar ideas as given in the proof of Theorem 1.

1°. Follows by Lemma 4 (1°).

2°. Let $x^0 \in \text{WEff}(X | f)$. If $x^0 \in \text{WEff}(\mathcal{V} | f)$, then $x^0 \in X \cap \text{WEff}(X_j | f)$ for some $j \in I_1$ by 1° of this theorem. In what follows, we assume that $x^0 \notin \text{WEff}(\mathcal{V} | f)$. Consequently, there is $x^1 \in S_{<}(\mathcal{V}, f, x^0) \cap \text{int } D_k$ for some $k \in I_1$. We show that $x^0 \in \text{WEff}(X_k | f)$.

Assume the contrary holds, i.e., $x^0 \notin \text{WEff}(X_k | f)$. Then, there exists $x^2 \in S_{<}(\mathcal{V}, f, x^0) \cap \text{int } D_{\bar{k}}$ for some $\bar{k} \in I_1 \setminus \{k\}$. Consider $i \in I_m$. If f_i is semi-strictly quasi-convex, then we get

$$\text{card } L_{\geq} (]x^1, x^2[, f_i, f_i(x^0)) \leq 1$$

by Lemma 2. If f_i is quasi-convex, then it follows

$$\text{card } L_{\geq} (]x^1, x^2[, f_i, f_i(x^0)) = 0.$$

So, we conclude

$$\text{card} \left(\bigcup_{i \in I_m} L_{\geq} (]x^1, x^2[, f_i, f_i(x^0)) \right) \leq m. \quad (4.28)$$

By (4.20) and (4.28), we infer that there exists $x^3 \in]x^1, x^2[$ such that $x^3 \in S_{<}(X, f, x^0)$. This shows $x^0 \notin \text{WEff}(X | f)$, a contradiction.

3°. The proof is analogous to the proof of assertion 2°. Notice that one has

$$\emptyset \neq X \cap]x^1, x^2[\subseteq S_{<}(X, f, x^0).$$

4°. The proof uses Proposition 2, Theorem 2 (1°, 2°), formula (4.4), and the ideas given in the proof of Theorem 1 (4°).

5°. This assertion follows by 1° and 3° of this theorem as well as by the ideas given in the proof of assertion 4°.

□

It is important to mention that the assumptions (4.2) in 4° and (4.3) in 5° of Theorem 2 are essential for the validity of (4.27) (see Example 3 in Section 5).

We now present similar relationships for the concept of strict Pareto efficiency.

Theorem 3. *Let (4.1) be satisfied. Suppose that each function ϕ_i , $i \in I_l$, fulfils Assumptions (A1) and (A2) (with ϕ_i in the role of ϕ and X_i in the role of X). Then, the following assertions hold:*

1°. We have

$$X \cap \text{SEff}(\mathcal{V} | f) \subseteq X \cap \bigcup_{i \in I_l} \text{SEff}(X_i | f) \subseteq \text{SEff}(X | f).$$

2°. Assume that (4.2) holds. Let f be componentwise semi-strictly quasi-convex or quasi-convex. Then, we have

$$X \cap \bigcup_{i \in I_l} \text{SEff}(X_i | f) \supseteq \text{SEff}(X | f). \quad (4.29)$$

3°. We have

$$\text{SEff}(X | f) \supseteq [X \cap \text{SEff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_l} X \cap (\text{bd } X_i) \cap \text{SEff}(\mathcal{V} | f^{\oplus i}) \right].$$

Now, suppose that (4.2) holds. In addition, assume that f is componentwise semi-strictly quasi-convex or quasi-convex. Then, we have

$$\text{SEff}(X | f) = [X \cap \text{SEff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_l} (\text{bd } D_i) \cap \text{SEff}(\mathcal{V} | f^{\oplus i}) \right]. \quad (4.30)$$

Proof. The proof uses similar ideas as given in the proof of Theorem 1.

1°. Follows by Lemma 4 (1°).

2°. Consider $x^0 \in \text{SEff}(X | f)$. In the case that $x^0 \in \text{SEff}(\mathcal{V} | f)$, we conclude $x^0 \in X \cap \text{SEff}(X_j | f)$ for some $j \in I_l$ by 1° of this theorem. In the second case, we can have $x^0 \notin \text{SEff}(\mathcal{V} | f)$, hence there exists $x^1 \in S_{\leq}(\mathcal{V}, f, x^0) \cap \text{int } D_k$ for some $k \in I_l$. Now, we are going to prove that $x^0 \in \text{SEff}(X_k | f)$.

Assume the contrary holds, i.e., $x^0 \notin \text{SEff}(X_k | f)$. Then, there exists a point $x^2 \in S_{\leq}(\mathcal{V}, f, x^0) \cap \text{int } D_{\bar{k}}$ for some $\bar{k} \in I_l \setminus \{k\}$.

Let $i \in I_m$. If f_i is semi-strictly quasi-convex, then we get

$$\text{card } L_{>} (]x^1, x^2[, f_i, f_i(x^0)) \leq 1$$

by Lemma 2. If f_i is quasi-convex, then it follows

$$\text{card } L_{>} (]x^1, x^2[, f_i, f_i(x^0)) = 0.$$

Hence, we infer

$$\text{card} \left(\bigcup_{i \in I_m} L_{>} (]x^1, x^2[, f_i, f_i(x^0)) \right) \leq m. \quad (4.31)$$

Taking into account (4.20) and (4.31), we get that there exists $x^3 \in]x^1, x^2[\setminus \{x^0\}$ such that $x^3 \in S_{\leq}(X, f, x^0)$. This implies $x^0 \notin \text{SEff}(X | f)$, a contradiction.

3°. The proof uses Proposition 3, Theorem 3 (1°, 2°), formula (4.4), and the ideas given in the proof of Theorem 1 (4°).

□

Remark 2. Consider the points $x^0, x^1, x^2 \in \mathcal{V}$ as given in the proof of 2° in Theorem 3. Under the weaker assumption (4.3) (in comparison to (4.2)) and the componentwise quasi-convexity of f , we get

$$\emptyset \neq X \cap]x^1, x^2[\subseteq S_{\leq}(X, f, x^0).$$

We notice, however, that $X \cap]x^1, x^2[$ can be a singleton set. Hence, in the proof of 2° in Theorem 3, we can not ensure that we have

$$X \cap]x^1, x^2[\neq \{x^0\}. \quad (4.32)$$

For the concepts of Pareto efficiency and weak Pareto efficiency, we know that there is $x \in X \cap]x^1, x^2[$ such that $x \in L_{<}(\mathcal{V}, f_j, f_j(x^0)) \cap S_{\leq}(\mathcal{V}, f, x^0)$ for some $j \in I_m$, hence (4.32) holds.

The assumption (4.2) in 4° of Theorem 2 is essential for the validity of (4.30), as shown in Example 3 in Section 5.

In preparation of the next section, we conclude by considering a specific type of penalization functions ϕ_i , $i \in I_l$, that fulfils the Assumptions (A1), (A2) and (A5) (with ϕ_i in the role of ϕ and X_i in the role of X , see Example 1 and Lemma 10).

Corollary 1. Assume that (4.1) holds. Let each penalization function ϕ_i , $i \in I_l$, be defined by

$$\phi_i(x) := -\inf\{\lambda \in \mathbb{R}_{++} \mid x - d^i \in \lambda \cdot (-d^i + D_i)\}$$

for all $x \in \mathcal{V}$, where $d^i \in \text{int } D_i$. Then, the following hold:

1°. We have

$$\begin{aligned} \text{SEff}(X \mid f) &\supseteq [X \cap \text{SEff}(\mathcal{V} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\text{bd } X_i) \cap \text{SEff}(\mathcal{V} \mid f^{\oplus_i}) \right]; \\ \text{Eff}(X \mid f) &\supseteq [X \cap \text{Eff}(\mathcal{V} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\text{bd } X_i) \cap \text{Eff}(\mathcal{V} \mid f^{\oplus_i}) \right]. \end{aligned}$$

Suppose that f is componentwise upper semi-continuous along line segments. Then, it follows

$$\text{WEff}(X \mid f) \supseteq [X \cap \text{WEff}(\mathcal{V} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\text{bd } X_i) \cap \text{WEff}(\mathcal{V} \mid f^{\oplus_i}) \right].$$

Moreover, under the validity of (4.2) or (4.3), one can replace $X \cap (\text{bd } X_i)$ by $\text{bd } D_i$ for every $i \in I_l$.

2°. If f is componentwise semi-strictly quasi-convex or quasi-convex, then we have

$$\text{SEff}(X \mid f) \subseteq [X \cap \text{SEff}(\mathcal{V} \mid f)] \cup \text{bd } X.$$

3°. If f is componentwise semi-strictly quasi-convex, then

$$\begin{aligned} \text{Eff}(X \mid f) &\subseteq [X \cap \text{Eff}(\mathcal{V} \mid f)] \cup \text{bd } X; \\ \text{WEff}(X \mid f) &\subseteq [X \cap \text{WEff}(\mathcal{V} \mid f)] \cup \text{bd } X. \end{aligned}$$

4°. Assume that (4.2) holds. Let f be componentwise semi-strictly quasi-convex or quasi-convex. Then, we have

$$\text{SEff}(X \mid f) = [X \cap \text{SEff}(\mathcal{V} \mid f)] \cup \left[\bigcup_{i \in I_l} (\text{bd } D_i) \cap \text{SEff}(\mathcal{V} \mid f^{\oplus_i}) \right].$$

5°. Assume that (4.3) holds. Let f be componentwise explicitly quasi-convex. Then, we have

$$\text{Eff}(X | f) = [X \cap \text{Eff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_f} (\text{bd } D_i) \cap \text{Eff}(\mathcal{V} | f^{\oplus i}) \right].$$

In addition, suppose that f is componentwise upper semi-continuous along line segments. Then, it follows

$$\text{WEff}(X | f) = [X \cap \text{WEff}(\mathcal{V} | f)] \cup \left[\bigcup_{i \in I_f} (\text{bd } D_i) \cap \text{WEff}(\mathcal{V} | f^{\oplus i}) \right].$$

Proof. Follows by Lemma 4, Theorems 1, 2 and 3 and formula (4.4). \square

5. APPLICATION TO A NONCONVEX MULTI-OBJECTIVE LOCATION PROBLEM

In this section, we apply our results to multi-objective location problems. Assume that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Consider m a priori given facilities located at the points $a^1, \dots, a^m \in \mathcal{V}$. For notational convenience, we define the set of all existing facilities by

$$\mathcal{A} := \{a^1, \dots, a^m\}.$$

Let $X \subseteq \mathcal{V}$ be a nonempty closed set. Our aim is to find a point $x \in X$ for a new facility such that the distances (associated with the norm $\|\cdot\|$) between x and the given points a^1, \dots, a^m are to be simultaneously minimized. Such a problem can be modeled as follows:

$$\begin{cases} g(x) := (\|x - a^1\|, \dots, \|x - a^m\|) \rightarrow \min \\ x \in X. \end{cases} \quad (\mathcal{LP}_X)$$

If, in addition, X is convex and $(\mathcal{V}, \|\cdot\|)$ is a Hilbert space, then it is known (see Ndiaye and Michelot [17, Cor. 4.2]) that we have

$$\text{SEff}(X | g) = \text{Eff}(X | g) = \text{WEff}(X | g) = \text{Proj}_X(\text{conv } \mathcal{A}),$$

where the projection of $\text{conv } \mathcal{A}$ onto X is defined with respect to the norm $\|\cdot\|$.

By applying the weighted-sum scalarization method to the problem (\mathcal{LP}_X) , we infer that minimal solutions of the well-known generalized Fermat-Weber problem are actually Pareto efficient solutions for the problem (\mathcal{LP}_X) , i.e., we have

$$\text{argmin}_{x \in X} \sum_{i=1}^m \|x - a^i\| \subseteq \text{Eff}(X | g).$$

Taking into account the literature in multi-objective location theory, one can see that most papers are dealing with closed and convex feasible sets. In particular, the case $X = \mathcal{V}$ is well-studied in the literature. In contrast to that, the nonconvex case is less studied. For instance, Carrizosa et al. [7] considered the point-objective location problem (\mathcal{LP}_X) involving the Euclidean norm and used a geometrical construction in the plane (i.e., $\mathcal{V} = \mathbb{R}^2$) based on their concept of a closed and convex decomposition of the not necessarily convex feasible set X , in order to obtain a characterization for the set of weakly Pareto efficient solutions. To the best of our knowledge, it is unknown how to compute the set of (strictly, weakly) Pareto efficient solutions for the problem (\mathcal{LP}_X) involving forbidden regions. Since in practical problems, there often exist regions where it is forbidden to locate a new facility, it is interesting to study the classical problem (\mathcal{LP}_X) in the presence of forbidden regions.

So, let the feasible set X of (\mathcal{LP}_X) be given by the whole pre-image space \mathcal{V} excepting some forbidden regions that are defined by open balls with respect to the norm $\|\cdot\|$. More

precisely, throughout this section, we assume that the following assumption is fulfilled:

$$\begin{cases} \text{Let } (\mathcal{V}, \|\cdot\|) \text{ be a real finite-dimensional Hilbert space;} \\ \text{let } D_i := \overline{B}_{\|\cdot\|}(d^i, r_i) \text{ with } d^i \in \mathcal{V}, r_i \in \mathbb{R}_{++}, i \in I_l, l \in \mathbb{N}; \\ \text{let } X := \bigcap_{i \in I_l} X_i \text{ with } X_i := \mathcal{V} \setminus \text{int } D_i, i \in I_l. \end{cases} \quad (5.1)$$

As one can see in (5.1), the feasible set X is given by an intersection of reverse convex sets X_1, \dots, X_l . For convenience the reader may assume that $\mathcal{V} = \mathbb{R}^n$ and that $\|\cdot\|$ is given by the Euclidean norm (denoted by $\|\cdot\|_2$) defined in \mathbb{R}^n .

Notice that the Hilbert space $(\mathcal{V}, \|\cdot\|)$ is strictly convex. Hence, for any $i \in I_l$, we have

$$]x', x''[\subseteq \text{int } D_i \quad \text{for all } x', x'' \in \text{bd } D_i, x' \neq x''.$$

Moreover, we have

$$\|d^i - d^j\| > r_i + r_j \quad \text{for all } i, j \in I_l, i \neq j \quad (5.2)$$

if and only if the balls D_1, \dots, D_l are pairwise disjoint. Furthermore, we have

$$\|d^i - d^j\| \geq r_i + r_j \quad \text{for all } i, j \in I_l, i \neq j \quad (5.3)$$

if and only if the interiors $\text{int } D_1, \dots, \text{int } D_l$ of the balls D_1, \dots, D_l are pairwise disjoint. Obviously, 5.3 follows by (5.2). In general, we have $\text{bd } X \subseteq \bigcup_{i \in I_l} \text{bd } D_i$. Under the assumption (5.3), we actually have

$$\text{bd } X = \bigcup_{i \in I_l} \text{bd } D_i$$

by Lemma 7.

For every $i \in I_l$, we consider a penalized point-objective location problem by

$$\begin{cases} g^{\oplus i}(x) := (\|x - a^1\|, \dots, \|x - a^m\|, -\|x - d^i\|) \rightarrow \min \\ x \in \mathcal{V}, \end{cases} \quad (\mathcal{LP}_{\mathcal{V}}^{\oplus i})$$

where we define the penalization function $\phi_i : \mathcal{V} \rightarrow \mathbb{R}$ by

$$\phi_i(x) := -\|x - d^i\| \quad \text{for all } x \in \mathcal{V}.$$

Notice that (\mathcal{LP}_X) involves a convex objective function g and a nonconvex feasible set X . In contrast to that, $(\mathcal{LP}_{\mathcal{V}}^{\oplus i})$ involves a nonconvex objective function $g^{\oplus i}(x)$ and a convex feasible set \mathcal{V} for every $i \in I_l$. According to Jourani, Michelot and Ndiaye [16], the problem $(\mathcal{LP}_{\mathcal{V}}^{\oplus i})$ can be seen as the problem of locating a new facility $x \in \mathcal{V}$ in presence of attracting points a^1, \dots, a^m and a repulsive demand point d^i in a continuous location space \mathcal{V} .

Remark 3. By Example 1 and Lemma 10, we know that the function $\widehat{\phi}_i : \mathcal{V} \rightarrow \mathbb{R}$, defined for every $x \in \mathcal{V}$ by

$$\widehat{\phi}_i(x) := -\frac{1}{r_i} \|x - d^i\| = -\inf\{\lambda \in \mathbb{R}_{++} \mid x - d^i \in \lambda \cdot (-d^i + D_i)\},$$

fulfils Assumptions (A1), (A2) and (A5) (with $\widehat{\phi}_i$ in the role of ϕ and X_i in the role of X) for every $i \in I_l$. In view of Lemma 5 (2°, 3°), we actually get that $\phi_i(\cdot) = -\|\cdot - d^i\|$ fulfils Assumptions (A1), (A2) and (A5) (with ϕ_i in the role of ϕ and X_i in the role of X) for every $i \in I_l$.

By Durier and Michelot [10, Prop. 1.3], we know that one can determine completely the set of (strictly, weakly) Pareto efficient solutions for the problem $(\mathcal{LP}_{\mathcal{V}})$ (defined as (\mathcal{LP}_X) with \mathcal{V} in the role of X) as stated in the next lemma.

Lemma 11. *Assume that (5.1) holds. Then, we have*

$$\text{SEff}(\mathcal{V} | g) = \text{Eff}(\mathcal{V} | g) = \text{WEff}(\mathcal{V} | g) = \text{conv}\mathcal{A}.$$

By Jourani, Michelot and Ndiaye [16] we get the following characterizations of the sets of (strictly, weakly) Pareto efficient solutions for the nonconvex location problem $(\mathcal{LP}_{\mathcal{V}}^{\oplus i})$.

Lemma 12. *Assume that (5.1) holds. For every $i \in I_1$, the following assertions hold:*

- 1°. $\text{SEff}(\mathcal{V} | g^{\oplus i}) = \text{conv}\mathcal{A} + \text{cone}(\text{conv}\mathcal{A} - d^i)$.
- 2°. $d^i \in \text{int}(\text{conv}\mathcal{A})$ if and only if $\text{SEff}(\mathcal{V} | g^{\oplus i}) = \mathcal{V}$.
- 3°. If $d^i \notin \text{conv}\mathcal{A}$, then $\text{SEff}(\mathcal{V} | g^{\oplus i}) = \text{Eff}(\mathcal{V} | g^{\oplus i}) = \text{WEff}(\mathcal{V} | g^{\oplus i}) \neq \mathcal{V}$.
- 4°. $d^i \in \text{conv}\mathcal{A}$ if and only if $\text{WEff}(\mathcal{V} | g^{\oplus i}) = \mathcal{V}$.
- 5°. $d^i \notin \text{rint}(\text{conv}\mathcal{A})$ if and only if $\text{Eff}(\mathcal{V} | g^{\oplus i}) = \text{SEff}(\mathcal{V} | g^{\oplus i}) \neq \mathcal{V}$.
- 6°. $d^i \in \text{rint}(\text{conv}\mathcal{A})$ if and only if $\text{Eff}(\mathcal{V} | g^{\oplus i}) = \mathcal{V}$.
- 7°. $\text{WEff}(\mathcal{V} | g^{\oplus i}) = \{x \in \mathcal{V} | (\text{conv}\mathcal{A}) \cap \text{conv}\{x, d^i\} \neq \emptyset\}$.
- 8°. $\text{rint}(\text{SEff}(\mathcal{V} | g^{\oplus i})) = \{x \in \mathcal{V} | \text{rint}(\text{conv}\mathcal{A}) \cap \text{rint}(\text{conv}\{x, d^i\}) \neq \emptyset\}$.

Proof. First, notice that $\text{cone}(\text{conv}\mathcal{A} - d^i) = \mathcal{V}$ if and only if $d^i \in \text{int}(\text{conv}\mathcal{A})$ (see Section 2). Now, 1° follows by [16, Cor. 4.1]; 2° follows by 1°; 3° follows by [16, Th. 4.5] and by 2°; 4° follows by [16, Prop. 4.2]; 5° follows by [16, Th. 4.3]; 6° follows by [16, Prop. 4.1]; 7° follows by [16, Th. 4.4]; 8° follows by [16, Th. 4.2]. \square

Remark 4. *Notice that Lemma 11 is actually true for infinite-dimensional Hilbert spaces (see Durier and Michelot [10, Prop. 1.3]) taking into account that $\text{conv}\mathcal{A}$ is compact for the finite set \mathcal{A} (see Aliprantis and Border [1, Cor. 5.30]). According to Jourani, Michelot and Ndiaye [16], the results given in Lemma 12 are valid for finite-dimensional inner product spaces (hence finite-dimensional Hilbert spaces). For that reason, we assume in our main assumption (5.1) that \mathcal{V} is a finite-dimensional Hilbert space.*

Since \mathcal{A} is finite, the set $\text{conv}\mathcal{A}$ is a polytope. In the case $d^i \notin \text{int}(\text{conv}\mathcal{A})$, for any $i \in I_1$, the set $\text{cone}(\text{conv}\mathcal{A} - d^i)$ is a (closed and convex) polyhedral cone and $\text{conv}\mathcal{A} + \text{cone}(\text{conv}\mathcal{A} - d^i)$ is a polyhedral set. Otherwise, if $d^i \in \text{int}(\text{conv}\mathcal{A})$, then both sets are equal to \mathcal{V} . In addition, we have

$$T(\text{conv}\mathcal{A}, d^i) = \text{cl}\left(\text{cone}(\text{conv}\mathcal{A} - d^i)\right) = \text{cone}(\text{conv}\mathcal{A} - d^i),$$

where $T(\text{conv}\mathcal{A}, d^i)$ stands for the contingent cone of $\text{conv}\mathcal{A}$ at the point d^i . For more details, see the books by Aliprantis and Border [1] and Jahn [15].

As mentioned by Jourani, Michelot and Ndiaye [16], these complete geometrical descriptions of the sets of (strictly, weakly) Pareto efficient solutions given in Lemma 12 are surprising due to the nonconvexity of the objective function $g^{\oplus i}$, $i \in I_1$.

In the next lemma, we will see that Lemmata 11 and 12 are very important results in order to obtain complete geometrical descriptions of the sets of (strictly, weakly) Pareto efficient solutions (under the validity of (5.2) or (5.3)) for the nonconvex problem (\mathcal{LP}_X) .

Lemma 13. *Let (5.1) be fulfilled. Then, the following assertions hold:*

1°. *We have*

$$\begin{aligned} \text{SEff}(X | g) &\supseteq [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{SEff}(\mathcal{V} | g^{\oplus i}) \right]; \\ \text{Eff}(X | g) &\supseteq [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{Eff}(\mathcal{V} | g^{\oplus i}) \right]; \\ \text{WEff}(X | g) &\supseteq [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{WEff}(\mathcal{V} | g^{\oplus i}) \right] \end{aligned}$$

and

$$\text{SEff}(X | g) \subseteq \text{Eff}(X | g) \subseteq \text{WEff}(X | g) \subseteq [X \cap \text{conv}\mathcal{A}] \cup \text{bd } X.$$

2°. *Assume that (5.2) holds. Then, we have*

$$\text{SEff}(X | g) = [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} (\text{bd } D_i) \cap \text{SEff}(\mathcal{V} | g^{\oplus i}) \right].$$

3°. *Assume that (5.3) holds. Then, we have*

$$\begin{aligned} \text{Eff}(X | g) &= [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} (\text{bd } D_i) \cap \text{Eff}(\mathcal{V} | g^{\oplus i}) \right]; \\ \text{WEff}(X | g) &= [X \cap \text{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_1} (\text{bd } D_i) \cap \text{WEff}(\mathcal{V} | g^{\oplus i}) \right]. \end{aligned}$$

Proof. Follows by Corollary 1 and Lemma 11. \square

The reverse inclusions in 1° of Lemma 13 do not hold in general, as shown in the next example.

Example 3. *Consider the space $\mathcal{V} = \mathbb{R}^2$, the set $\mathcal{A} = \{a^1\} = \{(0, 0)\}$, and three Euclidean balls in \mathbb{R}^2 , namely*

$$D_1 \text{ with center } d^1 = (-2, 0) \text{ and radius } r_1 = 3,$$

$$D_2 \text{ with center } d^2 = (2, 0) \text{ and radius } r_2 = 3,$$

$$D_3 \text{ with center } d^3 = (0, 2) \text{ and radius } r_3 = 3.$$

For the problem $(\mathcal{L}\mathcal{P}_X)$ (with $m = 1$), we suppose that $X = X_1 \cap X_2 \cap X_3$ with $X_i = \mathbb{R}^2 \setminus \text{int } D_i$ for every $i \in I_3$. Then, we have $\text{conv}\mathcal{A} = \{(0, 0)\}$, hence

$$X \cap \text{conv}\mathcal{A} = \emptyset.$$

Moreover, we get for $d^1, d^2, d^3 \notin \text{conv}\mathcal{A}$,

$$\text{Eff}(\mathbb{R}^2 | g^{\oplus 1}) = -\text{cone} \{d^1\} = [0, \infty) \times \{0\},$$

$$\text{Eff}(\mathbb{R}^2 | g^{\oplus 2}) = -\text{cone} \{d^2\} = (-\infty, 0] \times \{0\},$$

$$\text{Eff}(\mathbb{R}^2 | g^{\oplus 3}) = -\text{cone} \{d^3\} = \{0\} \times (-\infty, 0]$$

by Lemma 12. We thus infer

$$X \cap (\text{bd } D_1) \cap \text{Eff}(\mathbb{R}^2 | g^{\oplus 1}) = X \cap \{(1, 0)\} = \emptyset,$$

$$X \cap (\text{bd } D_2) \cap \text{Eff}(\mathbb{R}^2 | g^{\oplus 2}) = X \cap \{(-1, 0)\} = \emptyset,$$

$$X \cap (\text{bd } D_3) \cap \text{Eff}(\mathbb{R}^2 | g^{\oplus 3}) = X \cap \{(0, -1)\} = \emptyset.$$

Notice, in view of Lemma 12, we have

$$\text{SEff}(\mathbb{R}^2 | g^{\oplus i}) = \text{Eff}(\mathbb{R}^2 | g^{\oplus i}) = \text{WEff}(\mathbb{R}^2 | g^{\oplus i}) \quad \text{for all } i \in I_3.$$

However, it can easily be checked that

$$\emptyset \neq \{(0, -\sqrt{5})\} = \text{argmin}_{x \in X} \|x\|_2 = \text{SEff}(X | g) = \text{Eff}(X | g) = \text{WEff}(X | g).$$

This means that the reverse inclusions in 1° of Lemma 13 do not hold for this example problem. Notice that (5.2) and (5.3) are not fulfilled.

In preparation of the next theorem, we define the following three sets of indices:

$$\begin{aligned} I^{\text{conv}} &:= \{i \in I_l \mid d^i \in \text{conv} \mathcal{A}\}; \\ I^{i\text{-conv}} &:= \{i \in I_l \mid d^i \in \text{int}(\text{conv} \mathcal{A})\}; \\ I^{ri\text{-conv}} &:= \{i \in I_l \mid d^i \in \text{rint}(\text{conv} \mathcal{A})\}. \end{aligned}$$

We now present the main theorem of this section where we give complete geometrical descriptions for the sets of (strictly, weakly) Pareto efficient solutions of (\mathcal{LP}_X) that are valid under the assumptions (5.1) and (5.2) (or (5.3)).

Theorem 4. *Let (5.1) be fulfilled. Then, the following assertions hold:*

1°. *Assume that (5.2) holds. Then, we have*

$$\begin{aligned} \text{SEff}(X | g) &= X \cap \text{conv} \mathcal{A} \\ &\cup \left[\bigcup_{i \in I_l \setminus I^{i\text{-conv}}} (\text{bd } D_i) \cap (\text{conv} \mathcal{A} + \text{cone}(\text{conv} \mathcal{A} - d^i)) \right] \\ &\cup \left[\bigcup_{i \in I^{i\text{-conv}}} \text{bd } D_i \right] \\ &\supseteq X \cap \text{conv} \mathcal{A} \\ &\cup \left[\bigcup_{i \in I_l \setminus I^{ri\text{-conv}}} \left\{ x \in \text{bd } D_i \mid \text{rint}(\text{conv} \mathcal{A}) \cap \text{rint}(\text{conv} \{x, d^i\}) \neq \emptyset \right\} \right] \\ &\cup \left[\bigcup_{i \in I^{ri\text{-conv}}} \text{bd } D_i \right]. \end{aligned}$$

2°. Assume that (5.3) holds. Then, we have

$$\begin{aligned}
\text{Eff}(X | g) &= X \cap \text{conv}\mathcal{A} \\
&\cup \left[\bigcup_{i \in I_l \setminus I^{ri\text{-conv}}} (\text{bd } D_i) \cap \left(\text{conv}\mathcal{A} + \text{cone}(\text{conv}\mathcal{A} - d^i) \right) \right] \\
&\cup \left[\bigcup_{i \in I^{ri\text{-conv}}} \text{bd } D_i \right]; \\
\text{WEff}(X | g) &= X \cap \text{conv}\mathcal{A} \\
&\cup \left[\bigcup_{i \in I_l \setminus I^{\text{conv}}} (\text{bd } D_i) \cap \left(\text{conv}\mathcal{A} + \text{cone}(\text{conv}\mathcal{A} - d^i) \right) \right] \\
&\cup \left[\bigcup_{i \in I^{\text{conv}}} \text{bd } D_i \right] \\
&= X \cap \text{conv}\mathcal{A} \\
&\cup \left[\bigcup_{i \in I_l \setminus I^{\text{conv}}} \left\{ x \in \text{bd } D_i \mid (\text{conv}\mathcal{A}) \cap \text{conv}\{x, d^i\} \neq \emptyset \right\} \right] \\
&\cup \left[\bigcup_{i \in I^{\text{conv}}} \text{bd } D_i \right].
\end{aligned}$$

Proof. Follows by Lemmata 12 and 13. \square

Corollary 2. Let (5.1) be fulfilled. Then, the following assertions hold:

- 1°. Assume that (5.2) holds. Then, $\text{SEff}(X | g)$ is a compact set.
- 2°. Assume that (5.3) holds. Then, $\text{Eff}(X | g)$ and $\text{WEff}(X | g)$ are compact sets.

Proof. The sets D_i , $i \in I_l$, and $\text{conv}\mathcal{A}$ are compact sets. In addition, the sets X and $\text{cone}(\text{conv}\mathcal{A} - d^i)$, $i \in I_l$, are closed. Hence, we easily obtain that $\text{SEff}(X | g)$, $\text{Eff}(X | g)$ and $\text{WEff}(X | g)$ are closed and bounded sets by Theorem 4. Notice that the sum of a compact set and a closed set in \mathcal{V} is closed. Since \mathcal{V} is a finite-dimensional normed space, both assertions of this corollary follow immediately. \square

Next, we present two examples in order to illustrate (for the case $l = 1$ as well as for the case $l = 2$) the geometrical descriptions given for the sets of (strictly, weakly) Pareto efficient solutions of the problem (\mathcal{LP}_X) in Theorem 4.

Example 4. We consider a point-objective location problem (\mathcal{LP}_X) involving the Euclidean norm $\|\cdot\|_2$ where the set of existing facilities is given by

$$\mathcal{A} = \{a^1, a^2, a^3\} \subseteq \mathbb{R}^2 = \mathcal{V}$$

and the feasible set is given by $X = X_1 = \mathbb{R}^2 \setminus \text{int } D_1$. Figure 2 shows the location problem as well as the procedure for computing the set $\text{Eff}(X | g)$. Notice that $d^1 \in (\text{conv}\mathcal{A}) \setminus \text{int}(\text{conv}\mathcal{A})$. Due to Lemma 11 and Theorem 4, we have

$$\text{SEff}(\mathbb{R}^2 | g) = \text{Eff}(\mathbb{R}^2 | g) = \text{WEff}(\mathbb{R}^2 | g) = \text{conv}\mathcal{A}$$

and

$$\begin{aligned}
\text{SEff}(X | g) &= [X \cap \text{conv}\mathcal{A}] \cup [(\text{bd } D_1) \cap (\text{conv}\mathcal{A} + \text{cone}(\text{conv}\mathcal{A} - d^1))]; \\
\text{Eff}(X | g) &= \text{SEff}(X | g); \\
\text{WEff}(X | g) &= [X \cap \text{conv}\mathcal{A}] \cup \text{bd } D_1.
\end{aligned}$$

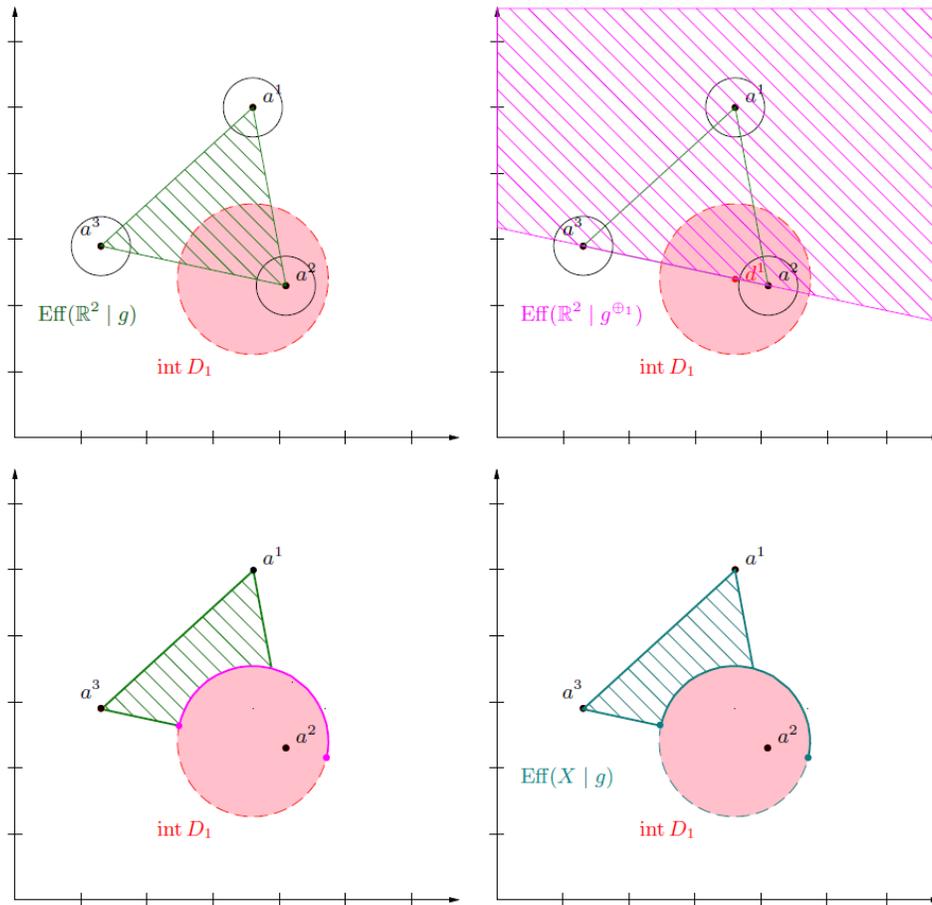


FIGURE 2. Construction of the set of Pareto efficient solutions of the problem (\mathcal{LP}_X) with $m = 3$ and $l = 1$.

Example 5. Again, let us consider a point-objective location problem (\mathcal{LP}_X) involving the Euclidean norm $\|\cdot\|_2$ where the set of existing facilities is given by

$$\mathcal{A} = \{a^1, a^2, a^3\} \subseteq \mathbb{R}^2 = \mathcal{V}.$$

We assume that X is an intersection of two reverse convex sets, i.e., we have

$$X = X_1 \cap X_2 = (\mathbb{R}^2 \setminus \text{int } D_1) \cap (\mathbb{R}^2 \setminus \text{int } D_2).$$

Figure 3 illustrates this problem and shows how the set $\text{Eff}(X | g)$ can be computed. Notice that $d^1 \in \text{int}(\text{conv } \mathcal{A})$ and $d^2 \notin \text{conv } \mathcal{A}$. In view of Lemma 11 and Theorem 4, we infer

$$\text{SEff}(\mathbb{R}^2 | g) = \text{Eff}(\mathbb{R}^2 | g) = \text{WEff}(\mathbb{R}^2 | g) = \text{conv } \mathcal{A}$$

and

$$\begin{aligned} \text{SEff}(X | g) &= [X \cap \text{conv } \mathcal{A}] \cup \text{bd } D_1 \\ &\quad \cup [(\text{bd } D_2) \cap (\text{conv } \mathcal{A} + \text{cone}(\text{conv } \mathcal{A} - d^2))]; \\ \text{Eff}(X | g) &= \text{WEff}(X | g) = \text{SEff}(X | g). \end{aligned}$$

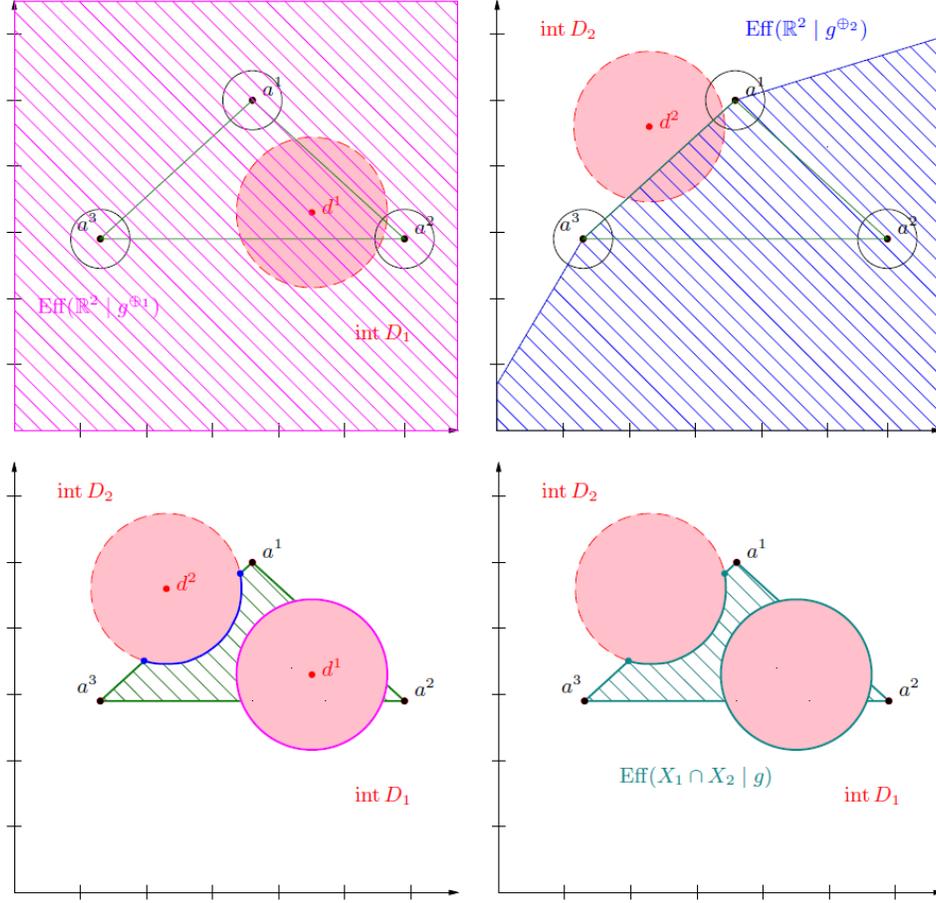


FIGURE 3. Construction of the set of Pareto efficient solutions of the problem (\mathcal{LP}_X) with $m = 3$ and $l = 2$.

In Proposition 4, we present some characterizations related to the sets of (strictly, weakly) Pareto efficient solutions.

Proposition 4. *Let (5.1) and (5.3) be fulfilled. Then, the following assertions are true:*

1°. *Assume that (5.2) holds. Then, we have*

$$I^{i\text{-conv}} = I_l \iff \text{SEff}(X | g) = [X \cap \text{conv}\mathcal{A}] \cup \text{bd } X.$$

2°. *Assume that (5.2) or $\dim \mathcal{V} \geq 2$ holds. Then, we have*

$$I^{ri\text{-conv}} = I_l \iff \text{Eff}(X | g) = [X \cap \text{conv}\mathcal{A}] \cup \text{bd } X.$$

3°. *Assume that (5.2) or $\dim \mathcal{V} \geq 2$ holds. Then, we have*

$$I^{\text{conv}} = I_l \iff \text{WEff}(X | g) = [X \cap \text{conv}\mathcal{A}] \cup \text{bd } X.$$

4°. *Assume that (5.2) holds. Then, we have*

$$I^{ri\text{-conv}} = I^{i\text{-conv}} \iff \text{SEff}(X | g) = \text{Eff}(X | g),$$

or, equivalently, we have

$$\text{int}(\text{conv}\mathcal{A}) \neq \emptyset \vee I^{ri\text{-conv}} = \emptyset \iff \text{SEff}(X | g) = \text{Eff}(X | g).$$

5°. Assume that (5.2) or $\dim \mathcal{V} \geq 2$ holds. Then, we have

$$I^{\text{conv}} = I^{r^i\text{-conv}} \iff \text{Eff}(X | g) = \text{WEff}(X | g).$$

6°. Assume that (5.2) holds. Then, we have

$$I^{\text{conv}} = I^{i\text{-conv}} \iff \text{SEff}(X | g) = \text{Eff}(X | g) = \text{WEff}(X | g).$$

7°. Assume that (5.2) holds. Then, we have

$$\emptyset = I^{i\text{-conv}} \subsetneq I^{r^i\text{-conv}} \subsetneq I^{\text{conv}} \iff \text{SEff}(X | g) \subsetneq \text{Eff}(X | g) \subsetneq \text{WEff}(X | g).$$

To prove Proposition 4, we need the following key lemma.

Lemma 14. *Let (5.1) be fulfilled. The following assertions hold:*

1°. For any $j \in I_l$, we have

$$j \in I^{i\text{-conv}} \iff (\text{bd } D_j) \cap \left(\text{conv} \mathcal{A} + \text{cone} \left(\text{conv} \mathcal{A} - d^j \right) \right) = \text{bd } D_j.$$

2°. Let $j \in I_l \setminus I^{i\text{-conv}}$. Then, we have

$$(\text{bd } D_j) \cap (\text{conv} \mathcal{A}) \subseteq (\text{bd } D_j) \cap \left(\text{conv} \mathcal{A} + \text{cone} \left(\text{conv} \mathcal{A} - d^j \right) \right) \subsetneq \text{bd } D_j.$$

Hence, the set

$$(\text{bd } D_j) \setminus \left(\text{conv} \mathcal{A} + \text{cone} \left(\text{conv} \mathcal{A} - d^j \right) \right)$$

is nonempty, and if $\dim \mathcal{V} \geq 2$, has an infinite number of elements.

3°. Assume that (5.3) holds. For any $i, j \in I_l$, $i \neq j$, the set $(\text{bd } D_i) \cap (\text{bd } D_j)$ is a singleton set or the empty set. Hence, for any $j \in I_l$, the set $(\text{bd } D_j) \cap \bigcup_{i \in I_l \setminus \{j\}} \text{bd } D_i$ has at most l elements.

4°. Assume that (5.2) holds. For any $i, j \in I_l$, $i \neq j$, the set $(\text{bd } D_i) \cap (\text{bd } D_j)$ is the empty set.

Proof. For notational convenience, we define $C_j := \text{cone}(\text{conv} \mathcal{A} - d^j)$ for all $j \in I_l$.

1°. Since for $j \in I^{i\text{-conv}}$ we have $C_j = \mathcal{V}$ (see Section 2), the implication “ \implies ” follows immediately. Now, let us establish the reverse implication “ \impliedby ”.

Let $j \in I_l$. Since $\text{bd } D_j \subseteq \text{conv} \mathcal{A} + C_j$ and $\text{conv} \mathcal{A} + C_j$ is a convex set, we get

$$d^j \in \text{int } D_j \subseteq D_j \subseteq \text{conv} \mathcal{A} + C_j,$$

hence

$$d^j \in \text{int}(\text{conv} \mathcal{A} + C_j). \quad (5.4)$$

Assume that the contrary holds, i.e., $j \in I_l \setminus I^{i\text{-conv}}$, hence $d^j \notin \text{int}(\text{conv} \mathcal{A})$. First, we show that

$$\exists v \in \mathcal{V} \setminus \{0\} \forall \delta \in \mathbb{R}_{++} : d^j + \delta v \notin \text{conv} \mathcal{A} \quad (5.5)$$

by considering two cases:

Case 1: Assume that $d^j \notin \text{conv} \mathcal{A}$. By a separation theorem (see e.g., Barbu and Precupanu [4, Cor 1.45]), we infer that there exists a linear functional $\psi : \mathcal{V} \rightarrow \mathbb{R}$ such that

$$\sup_{a \in \text{conv} \mathcal{A}} \psi(a) < \psi(d^j). \quad (5.6)$$

Assume that the contrary of (5.5) holds. Then, for $v := d^j - a$ with $a \in \text{conv} \mathcal{A}$, there exists $\delta \in \mathbb{R}_{++}$ such that $d^j + \delta v \in \text{conv} \mathcal{A}$. So, we have

$$\begin{aligned} \psi(d^j) &\stackrel{(5.6)}{>} \psi(d^j + \delta v) \\ &= \psi(d^j) + \delta \psi(d^j - a) \\ &= \psi(d^j) + \delta \psi(d^j) - \delta \psi(a), \end{aligned}$$

which implies $\psi(a) > \psi(d^j)$, a contradiction to (5.6). Thus, (5.5) holds.

Case 2: Assume that $d^j \in \text{bd}(\text{conv}\mathcal{A})$. Since $d^j \in \text{conv}\mathcal{A}$ is not an interior point of $\text{conv}\mathcal{A}$, it follows

$$\exists \bar{v} \in \mathcal{V} \setminus \{0\} \forall \delta \in \mathbb{R}_{++} \exists \theta \in]0, \delta] : d^j + \theta \bar{v} \notin \text{conv}\mathcal{A} \quad (5.7)$$

in the finite-dimensional normed space $(\mathcal{V}, \|\cdot\|)$ (see Section 2). If we suppose that $d^j + \bar{\delta} \bar{v} \in \text{conv}\mathcal{A}$ for some $\bar{\delta} \in \mathbb{R}_{++}$, then

$$d^j + [0, \bar{\delta}] \cdot \bar{v} \subseteq \text{conv}\mathcal{A}$$

by the convexity of $\text{conv}\mathcal{A}$, a contradiction to (5.7). This shows (5.5) with $v := \bar{v}$.

In both cases (5.5) holds.

In view of (5.4), for $v \in \mathcal{V} \setminus \{0\}$ given in (5.5), we get that there exists $\widehat{\delta} \in \mathbb{R}_{++}$ such that $d^j + \widehat{\delta} v \in \text{conv}\mathcal{A} + C_j$. So, there exist $k \in \mathbb{R}_+$, $a', a'' \in \text{conv}\mathcal{A}$, such that $d^j + \widehat{\delta} v = a' + k(a'' - d^j)$. This means that

$$d^j + \frac{\widehat{\delta}}{1+k} v = \frac{1}{1+k} a' + \frac{k}{1+k} a'' = \left(1 - \frac{k}{1+k}\right) a' + \frac{k}{1+k} a'' \in \text{conv}\mathcal{A},$$

a contradiction to (5.5).

The proof of assertion 1° is complete.

2°. We have $0 \in C_j$, hence $\text{conv}\mathcal{A} \subseteq \text{conv}\mathcal{A} + C_j$, which shows the first inclusion in assertion 2°. By 1° of this lemma, we get the second strict inclusion. Hence, we infer that $(\text{bd } D_j) \setminus (\text{conv}\mathcal{A} + C_j) \neq \emptyset$.

Now, we show that $(\text{bd } D_j) \setminus (\text{conv}\mathcal{A} + C_j)$ has an infinite number of elements. Let us consider two cases:

Case 1: Assume that $d^j \notin \text{conv}\mathcal{A}$. Then, we get $d^j \notin \text{conv}\mathcal{A} + C_j$. Indeed, if there exist $k \in \mathbb{R}_+$, $a', a'' \in \text{conv}\mathcal{A}$, such that $d^j = a' + k(a'' - d^j)$, then

$$d^j = \frac{1}{1+k} a' + \frac{k}{1+k} a'' = \left(1 - \frac{k}{1+k}\right) a' + \frac{k}{1+k} a'' \in \text{conv}\mathcal{A},$$

a contradiction.

Since $\text{conv}\mathcal{A} + C_j$ is closed and convex, we infer that there exists a linear functional $\psi : \mathcal{V} \rightarrow \mathbb{R}$ such that

$$\sup_{c \in \text{conv}\mathcal{A} + C_j} \psi(c) < \psi(d^j) \quad (5.8)$$

by a separation theorem (see, e.g., Barbu and Precupanu [4, Cor 1.45]). Without loss of generality, assume that \mathcal{V} is n -dimensional ($n \geq 2$). The sum of the dimensions of the kernel of ψ ($\ker \psi$ for short) and the image of ψ ($\text{img } \psi$ for short) is equal to n . More precisely, we have $\dim(\ker \psi) = n - 1$ and $\dim(\text{img } \psi) = 1$. Consider $\bar{x} \in (\ker \psi) \setminus \{0\}$. Since $d^j \in \text{int } D_j$ and D_j is convex, it exists $\bar{\delta} \in \mathbb{R}_{++}$ such that $S := d^j + [0, \bar{\delta}] \cdot \bar{x} \subseteq \text{int } D_j$. Notice that S has an infinite number of elements. Define $v := d^j - c$ for some $c \in \text{conv}\mathcal{A} + C_j$. For any $y \in S$, we define a function $h_y : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_y(t) := \|y + tv - d^j\| \quad \text{for all } t \in \mathbb{R}. \quad (5.9)$$

Consider $y \in S$. Since D_j is bounded and $v \neq 0$, there exists $t_y \in \mathbb{R}_{++}$ such that $y + t_y v \notin D_j$. By the continuity of h_y and by $h_y(0) < r_j < h_y(t_y)$, we get some $t_y^* \in]0, t_y[\subseteq \mathbb{R}_{++}$ such that $h_y(t_y^*) = r_j$, hence $x_y := y + t_y^* v \in \text{bd } D_j$. Since $y \in S$ we know that $y = d^j + \delta \bar{x}$ for some $\delta \in [0, \bar{\delta}]$. Then, due to $\bar{x} \in \ker \psi$ and formula (5.8), we infer

$$\psi(x_y) = \psi(d^j) + \delta \psi(\bar{x}) + t_y^* (\psi(d^j) - \psi(c)) > \psi(d^j),$$

which implies $x_y \notin \text{conv}\mathcal{A} + C_j$ in view of (5.8). We conclude that

$$x_y \in (\text{bd } D_j) \setminus (\text{conv}\mathcal{A} + C_j).$$

Moreover, the map $y \mapsto x_y$ is injective. Assume the contrary holds, i.e., there exist $y', y'' \in S$, $y' \neq y''$, such that

$$x_{y'} = y' + t_{y'}^* v = y'' + t_{y''}^* v = x_{y''}.$$

Of course, $t_{y'}^* = t_{y''}^*$ implies $y' = y''$, a contradiction. Without loss of generality, assume that $t_{y'}^* > t_{y''}^*$. We get $y' - y'' = (t_{y'}^* - t_{y''}^*)v$, hence

$$0 = \psi(y' - y'') = (t_{y'}^* - t_{y''}^*)\psi(v) = (t_{y'}^* - t_{y''}^*)(\psi(d^j) - \psi(c)) > 0$$

taking into account the definition of S and formula (5.8), a contradiction.

This completes the proof in the first case.

Case 2: Assume that $d^j \in \text{bd}(\text{conv } \mathcal{A})$. We must have $d^j \notin \text{int}(\text{conv } \mathcal{A} + C_j)$, otherwise $d^j \in \text{int}(\text{conv } \mathcal{A})$ by the ideas given in the proof of assertion 1° in this lemma. Notice that the case $d^j \notin \text{conv } \mathcal{A} + C_j$ is considered in Case 1 (assertion 2°). Now, assume that $d^j \in \text{bd}(\text{conv } \mathcal{A} + C_j)$. Then, similar to the proof given in 1° of this lemma, there exists $v \in \mathcal{V} \setminus \{0\}$ such that $d^j + \delta v \notin \text{conv } \mathcal{A} + C_j$ for all $\delta \in \mathbb{R}_{++}$. Since $d^j \in \text{int } D_j$ and D_j is a convex set, there is $\bar{\delta} \in \mathbb{R}_{++}$ such that $x^0 := d^j + \bar{\delta}v \in \text{int } D_j$. So, we get $x^0 \in (\text{int } D_j) \setminus (\text{conv } \mathcal{A} + C_j)$. Now, the proof is analogous to the proof given in Case 1 (assertion 2°) where x^0 is in the role of d^j (except in the definition of the function h_y given in (5.9)).

The proof of assertion 2° is complete.

3°, 4°. Directly follow by the assumptions (5.3) and (5.2), respectively. □

Now, we are going to show Proposition 4.

Proof. 1°. In view of assertion 1° in Theorem 4, the implication “ \implies ” is obvious. Let us prove the reverse implication “ \impliedby ”. Let

$$\text{SEff}(X \mid g) = [X \cap \text{conv } \mathcal{A}] \cup \text{bd } X. \quad (5.10)$$

Assume that the contrary holds, i.e, there exists $j \in I_l \setminus I^{i\text{-conv}}$. Then, due to Theorem 4 (1°) and formula (5.10), we must have

$$\begin{aligned} \text{bd } D_j &\subseteq \bigcup_{i \in I_l \setminus \{j\}} \text{bd } D_i \\ &\quad \cup (\text{bd } D_j) \cap \text{conv } \mathcal{A} \\ &\quad \cup \left((\text{bd } D_j) \cap (\text{conv } \mathcal{A} + \text{cone}(\text{conv } \mathcal{A} - d^j)) \right) \\ &= \bigcup_{i \in I_l \setminus \{j\}} \text{bd } D_i \\ &\quad \cup \left((\text{bd } D_j) \cap (\text{conv } \mathcal{A} + \text{cone}(\text{conv } \mathcal{A} - d^j)) \right). \end{aligned}$$

Then, it can easily be seen that we get a contradiction by Lemma 14 (2°, 4°).

2°, 3°. Analogous to the proof of 1° in this proposition by using Theorem 4 (2°) and Lemma 14 (2°, 3°, 4°).

4°. By Theorem 4 (1°, 2°), the implication “ \implies ” holds. Now, we prove the reverse implication “ \impliedby ”. Assume that the contrary holds, i.e, there exists $j \in I^{i\text{-conv}} \setminus I^{i\text{-conv}}$. Then, in view of Theorem 4 (1°, 2°) and because of the assumption $\text{SEff}(X \mid g) = \text{Eff}(X \mid g)$, we obtain a contradiction by Lemma 14 (2°, 4°) and by the ideas given in the proof of 1° of this proposition.

5°. Analogous to the proof of 4° in this proposition by using Theorem 4 (2°) and Lemma 14 (2°, 3°, 4°).

6°, 7°. Follow by assertions 4° and 5° of this proposition. □

Usually, the new facility $x \in \mathcal{V}$ should be located as close a possible to the existing facilities a^i , $i \in I_m$. In our model, each existing facility is located at one single point a^i in \mathcal{V} and has no expansion around this point. In particular, in the field of town planning, a given facility has a certain expansion. Hence, it is convenient to consider a forbidden region around a^i defined by a certain open ball centered at a^i with positive radius. So, it is possible to include information about the sizes of the existing facilities in the model. This means we are going to study the special case

$$l = m \text{ and } d^i = a^i \text{ for all } i \in I_l = I_m. \quad (5.11)$$

Corollary 3. *Let (5.1), (5.3) and (5.11) be fulfilled. Then, the following assertions are true:*

1°. *Assume that (5.2) holds. Then, we have*

$$\begin{aligned} \text{SEff}(X | g) &= X \cap \text{conv}\mathcal{A} \\ &\cup \left[\bigcup_{i \in I_m \setminus I^{i\text{-conv}}} (\text{bd } D_i) \cap \left(\text{conv}\mathcal{A} + \text{cone} \left(\text{conv}\mathcal{A} - d^i \right) \right) \right] \\ &\cup \left[\bigcup_{i \in I^{i\text{-conv}}} \text{bd } D_i \right]. \end{aligned}$$

2°. *Assume that (5.2) or $\dim \mathcal{V} \geq 2$ holds. Then, we have*

$$\begin{aligned} \text{Eff}(X | g) &= X \cap \text{conv}\mathcal{A} \\ &\cup \left[\bigcup_{i \in I_m \setminus I^{ri\text{-conv}}} (\text{bd } D_i) \cap \left(\text{conv}\mathcal{A} + \text{cone} \left(\text{conv}\mathcal{A} - d^i \right) \right) \right] \\ &\cup \left[\bigcup_{i \in I^{ri\text{-conv}}} \text{bd } D_i \right]; \\ \text{WEff}(X | g) &= [X \cap \text{conv}\mathcal{A}] \cup \text{bd } X. \end{aligned}$$

Proof. Follows by Theorem 4. □

Corollary 4. *Let (5.1), (5.3) and (5.11) be fulfilled. Then, the following assertions are true:*

1°. *Assume that (5.2) holds. Then, we have*

$$\text{int}(\text{conv}\mathcal{A}) \neq \emptyset \vee I^{ri\text{-conv}} = \emptyset \iff \text{SEff}(X | g) = \text{Eff}(X | g).$$

2°. *Assume that (5.2) or $\dim \mathcal{V} \geq 2$ holds. Then, we have*

$$\text{card } \mathcal{A} = 1 \iff \text{Eff}(X | g) = \text{WEff}(X | g).$$

Proof. Directly follows by 4° and 5° in Proposition 4. □

Next, we present an applied example of a location problem of type (\mathcal{LP}_X) in which the conditions (5.2), (5.3) and (5.11) are fulfilled.

Example 6. *A new central taxi station should be located in the district around La Habana on Cuba. We assume that the new location will be located as close as possible to each center of the cities La Habana, Guanabo, San José de las Lajas, Santiago de las Vegas, and Playa Baracoa. Due to the high car traffic in the centers of the cities we want to avoid to place the new facility in the near of the city centers. This means that we consider some forbidden regions around the given city centers. Figure 4 illustrates the example problem and shows the whole set of Pareto efficient solutions for this nonconvex location problem.*

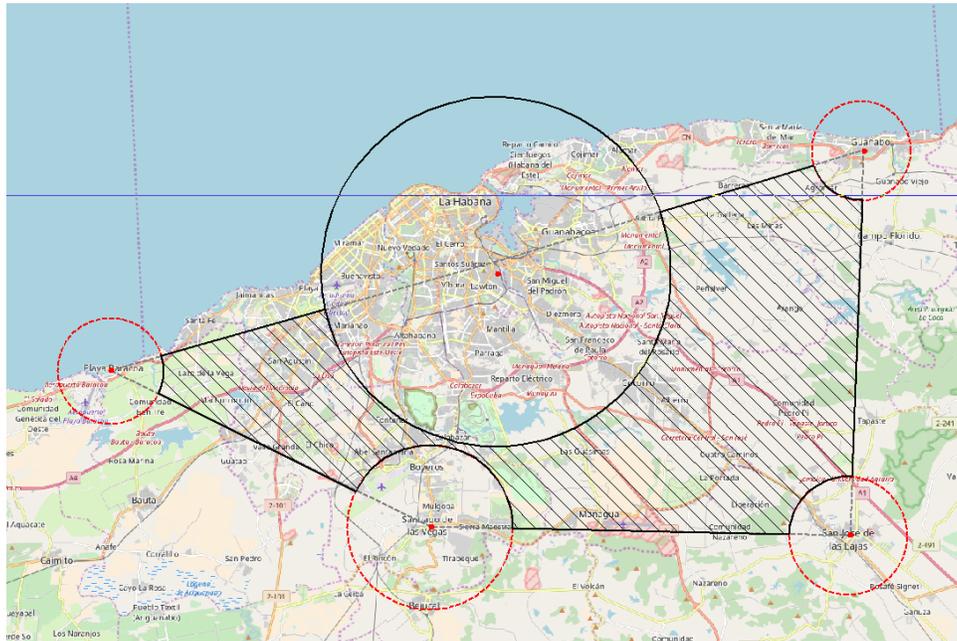


FIGURE 4. The set of Pareto efficient solutions (black colored region) of the problem (\mathcal{LP}_X) for Example 6 illustrated on a map (from OpenStreetMap) of La Habana in Cuba.

Remark 5. Let (5.1) be satisfied. We consider the problem of locating a new facility in presence of attracting and repulsive demand points. Such problems are discussed by Jourani, Michelot and Ndiaye [16] and can be modeled as follows:

$$\begin{cases} \widehat{g}(x) := (\|x - a^1\|, \dots, \|x - a^m\|, -\|x - b^1\|, \dots, -\|x - b^q\|) \rightarrow \min \\ x \in X \end{cases} \quad (5.12)$$

for attraction points $a^1, \dots, a^m \in \mathcal{V}$, $m \in \mathbb{N}$, and repulsion points $b^1, \dots, b^q \in \mathcal{V}$, $q \in \mathbb{N}$. Then, for any $i \in I_1$, the penalized problem is given by

$$\begin{cases} \widehat{g}^{\oplus i}(x) := (\widehat{g}(x), -\|x - d^i\|) \rightarrow \min \\ x \in \mathcal{V}. \end{cases} \quad (5.13)$$

It is important to mention that \widehat{g} is neither componentwise semi-strictly quasi-convex nor componentwise quasi-convex. However, in view of Corollary 1, we can obtain the following useful lower bounds for sets of (strictly, weakly) Pareto efficient solutions of (5.12):

$$\begin{aligned} \text{SEff}(X \mid \widehat{g}) &\supseteq [X \cap \text{SEff}(\mathcal{V} \mid \widehat{g})] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{SEff}(\mathcal{V} \mid \widehat{g}^{\oplus i}) \right]; \\ \text{Eff}(X \mid \widehat{g}) &\supseteq [X \cap \text{Eff}(\mathcal{V} \mid \widehat{g})] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{Eff}(\mathcal{V} \mid \widehat{g}^{\oplus i}) \right]; \\ \text{WEff}(X \mid \widehat{g}) &\supseteq [X \cap \text{WEff}(\mathcal{V} \mid \widehat{g})] \cup \left[\bigcup_{i \in I_1} X \cap (\text{bd } D_i) \cap \text{WEff}(\mathcal{V} \mid \widehat{g}^{\oplus i}) \right]. \end{aligned}$$

Notice, under the assumption (5.3), it follows $X \cap \text{bd } D_i = \text{bd } D_i$ for every $i \in I_1$. The set of (strictly, weakly) Pareto efficient solutions of the unconstrained problem (5.13) can be completely characterized by using results in Jourani, Michelot and Ndiaye [16].

6. CONCLUSION

In this paper, we considered a multi-objective optimization problem in which the vector-valued objective function is componentwise generalized-convex and acts between a real topological linear pre-image space and a finite-dimensional image space, while the feasible set is given by the whole pre-image space excepting some forbidden regions that are defined by convex sets. We succeeded to characterize the set of (strictly, weakly) Pareto efficient solutions of such a problem by using a finite family of unconstrained multi-objective optimization problems.

Then, we applied our results to a special multi-objective location problem (known as point-objective location problem) that consists of locating a new facility in a continuous location space (a finite-dimensional Hilbert space) in the presence of a finite number of demand points. For the choice of the new location point, we took into consideration some forbidden regions that are given by open balls (defined with respect to the underlying norm). For such a nonconvex location problem, under the assumption that the forbidden regions are pairwise disjoint, we characterized completely the set of (strictly, weakly) Pareto efficient solutions by using the penalization approach by Günther and Tammer [13] and results obtained by Jourani, Michelot and Ndiaye [16].

It is important to mention that our approach relies essentially on the fact that the objective function in (\mathcal{LP}_X) as well as the unit balls D_1, \dots, D_l (see the assumptions given in (5.1)) are defined with respect to a norm induced by a scalar product. This ensures that we can apply the results derived by Jourani, Michelot and Ndiaye [16].

It would be interesting to study other types of balls D_1, \dots, D_l , for instance balls defined with respect to a polyhedral norm $\mu : \mathcal{V} \rightarrow \mathbb{R}$. Therefore, in a forthcoming work, we will analyze a planar point-objective location problem involving a polyhedral norm $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$. It is known that such a problem without considering constraints can be solved completely. In order to solve a corresponding constrained problem with a feasible set that is given by the complement of a finite union of open balls with respect to a polyhedral norm $\mu : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have to compute the set of (strictly, weakly) Pareto efficient solutions of the problem

$$\begin{cases} (\eta(x - a^1), \dots, \eta(x - a^m), -\mu(x - d^i)) \rightarrow \min \\ x \in \mathcal{V} = \mathbb{R}^2. \end{cases}$$

Notice that this problem includes only one repulsive demand point, namely the point d^i . Hence, as mentioned by Jourani, Michelot and Ndiaye [16] in their conclusion, some results for the polyhedral case in presence of only one repulsive demand point could be expected.

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