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Abstract

The development of Disjunctive Conic Cuts (DCCs) for Mixed Integer Second Order Cone Optimization (MISOCO) problems has recently gained significant interest in the optimization community. In this paper, we explore the pathological disjunctions where disjunctive cuts do not tighten the description of the feasible set. We focus on the identification of cases when the generated DCCs are redundant. Avoiding the generation of redundant cuts saves computational time and facilitates efficient implementation of branch and cut algorithms.

1 Introduction

A Mixed Integer Second Order Cone Optimization (MISOCO) problem may be formulated as

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{L} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array} \tag{1}$$

in which $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_k$ is the Cartesian product of second order cones, each of which is defined as follows

$$\mathcal{L}_i = \{(x_0^i, x^i) \in \mathbb{R} \times \mathbb{R}^{n_i-1} \mid \|x^i\|_2 \le x_0^i\}, \quad i = 1, \dots k,$$

with $\sum_{i=1}^k n_i = n$.

In principle, one can solve a MISOCO problem exactly using a branch and cut methodology. One of the key elements of branch and cut is the derivation of effective and efficient cuts to strengthen the formulation. Recent studies have shown the performance improvements that one can obtain in a branch and cut using the lift-and-project approach, which can significantly reduce the solution time [16]. Most of the work on cut generation has been focused on obtaining valid linear inequalities. However, the possibility of generating nonlinear cuts for MISOCO problems have recently received significant attention in the optimization community. Stubbs and Mehrotra [21] extended Balas [6] lift and project procedure to 0-1 mixed integer convex optimization problems. They derive valid inequalities for mixed integer problems by solving a convex optimization sub-problem. Çezik and Iyengar [11] derive convex cuts for mixed 0-1 conic optimization problems. They consider the linear cone, the second-order cone and the cone of positive semidefinite matrices and extend a variety of techniques, used in generating cuts for MILO problems such as Gomory cuts [15] and lift and project cuts.

For MISOCO problems in particular, Atamturk [3, 4] extended the idea of mixed integer rounding cuts developed by Nemhauser and Wolsey [19]. They reformulated a second order cone in terms of two-dimensional polyhedral second order cones and designed a rounding procedure to derive conic cuts for the original MISOCO problem. Kılınç-Karzan and Yıldız [17] consider a two-term disjunction on a second order cone and derive closed-form convex inequalities describing the convex hull of the intersection of the disjunction with the cone. They characterize the cases where one second order cone inequality is enough to describe the mentioned convex hull. Drewes [13] presents lift-and-project based linear and convex quadratic cuts for mixed 0-1 SOCO problems. Dadush et al. [12] extend the idea of split cuts [19] for a full dimensional ellipsoid. They consider parallel disjunctions on ellipsoids and derive a conic cut which describes the convex hull of the ellipsoid intersected with the disjunctive set. Andersen and Jensen [2] extend the idea of intersection cuts [5] to mixed integer conic quadratic sets. Modaresi [18] explains the relationship between mixed integer rounding cuts [3, 4] and split cuts, [12] and discusses the trade-off between computational cost of adding the split cuts and strength of the formulation resulted from adding them to the model.

Belotti et al. [7, 8, 9] consider a disjunction on a general MISOCO problem and generates a class of cuts called *Disjunctive Conic Cuts (DCCs)* and *Disjunctive Cylindrical Cuts (DCyC)*. The DCCs and DCyCs describe the convex hull of the intersection of the disjunction with the feasible set of the continuous relaxation of a MISOCO problem. In other words, the intersection of the DCC with the feasible set of the continuous relaxation of the MISOCO problem is equal to the convex hull of the intersection of the disjunction with the feasible set of the continuous relaxation problem. In this paper we use the approach of Belotti et al. to provide tests to improve the derivation of effective DCCs. The aim of the tests is to save computational time and effort by identifying cases when the DCCs do not provide further tightening of the formulation at hand.

The paper is structured as follows. In Section 2, we overview the derivation of DCCs for the MISOCO problems. In Section 3, we define the pathological disjunctions and propose results on how to identify redundant DCCs. In Section 4, we explore some common instances of the redundant DCCs for MISOCO problems. Finally, Section 5 presents our conclusions.

2 DCCs for MISOCO problems

For the purpose of making this paper self-contained we provide an overview of the derivation of DCCs for the MISOCO problem (1) and a parallel disjunction. In this section, we assume that $\mathcal{L}$, defined in model (1), is a single second order cone.

We use the approach presented in [9] to rewrite the feasible set of the continuous relaxation of (1) in terms of the null space of the affine constraints. Let $\mathcal{C} = \mathcal{L} \cap \{x \mid Ax = b\}$, and recall from [9] that $\{x \in \mathbb{R}^n \mid Ax = b\} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^\ell, x = x^0 + Hw\}$, where $x^0 \in \mathbb{R}^n$. Here, the columns of $H \in \mathbb{R}^{n\times\ell}$ form a basis for the null space of A, and $\ell = n - m$ is the dimension of the null space. Hence, $\mathcal{C}$ in the null space of matrix A may be represented as

$$\hat{\mathcal{C}} = \{w \in \mathbb{R}^\ell \mid w^\top P w + 2p^\top w + \rho \leq 0,\ x^0 + H_1 w \geq 0\}, \tag{2}$$

where H_1 is the first row of H, $P \in \mathbb{R}^{\ell\times\ell}$ is a symmetric matrix, $p \in \mathbb{R}^\ell$, and $\rho \in \mathbb{R}$. We denote the quadratic set (2) by the triplet (P, p, ρ). Notice that if a set or parameter is defined both in $\mathbb{R}^n$ and the reduced null space $\mathbb{R}^\ell$, then the one defined in the null space is indicated by adding a "ˆ" to the set or parameter. It has been proved in Lemma 4.2 of [14] that the matrix P in (2) has at most one non-positive eigenvalue. That restricts the possible shapes of $\hat{\mathcal{C}}$ to ellipsoids, paraboloids, hyperboloids of two sheets, or second order cones.

Consider now a parallel disjunction of the form

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \leq \alpha\} \bigvee \mathcal{B} = \{x \in \mathbb{R}^n \mid a^\top x \geq \beta\}, \tag{3}$$

where $a \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. We can rewrite the disjunction in the null space of the affine constraints as follows

$$\hat{\mathcal{A}} = \{w \in \mathbb{R}^\ell \mid \hat{a}^\top w \le \hat{\alpha}\} \bigvee \hat{\mathcal{B}} = \{w \in \mathbb{R}^\ell \mid \hat{a}^\top w \ge \hat{\beta}\}, \tag{4}$$

where $\hat{a} \in \mathbb{R}^\ell$, $\alpha, \beta \in \mathbb{R}$, $\hat{a} = H^\top a$, $\hat{\alpha} = \alpha - a^\top x_0$, and $\hat{\beta} = \beta - a^\top x_0$. We assume that $\hat{\mathcal{C}} \cap \hat{\mathcal{A}} \neq \emptyset$ and $\hat{\mathcal{C}} \cap \hat{\mathcal{B}} \neq \emptyset$, and we may assume w.l.o.g. that $\|\hat{a}\|_2 = 1$ and $\hat{\alpha} < \hat{\beta}$. Using this disjunction we now recall the main elements of the derivation of the DCC; for a detailed derivation see [14, 8, 7]. The key result in that procedure is the existence of a uni-parametric family of quadratic sets defined as follows

$$\mathcal{Q}(\tau) = \left\{ w \in \mathbb{R}^\ell \mid w^\top P(\tau) w + 2p(\tau)^\top w + \rho(\tau) \le 0 \right\}, \tag{5}$$

where $P(\tau) = P + \tau \hat{a}\hat{a}^\top$, $p(\tau) = p - \frac{\tau}{2}(\alpha + \beta)\hat{a}^\top$, and $\rho(\tau) = \rho + \tau\alpha\beta$. Observe that $Q(\tau)$ is not necessarily convex. It was shown in [7] that a DCC can be derived using the quadratic sets found at the roots of $p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau)$, which is a quadratic function of τ. That result provides an explicit formula for obtaining DCCs. However, before adding a DCC it is important to know if it does tighten the formulation, or not. For that purpose, in the rest of this paper we provide a set of tests to verify the effectiveness of a DCC.

3 Redundant DCCs and DCyCs

For the classification of the redundant DCCs in this section, we assume that the disjunction used for deriving the DCC is parallel. Also, we use the result of [9], which ensures that a DCC excluding an infeasible point from (2) excludes also an infeasible point from (1). Hence, we focus in this section on sets of the form presented in (2). Now, when we intersect a disjunction of the form (4) with the set $\hat{\mathcal{C}}$ as defined in (2) and derive a DCC, we show that in some instances it does not cut off any part of the feasible region. That implies that the derived DCC is redundant, negatively affecting the effectiveness of the branch and cut procedure.

Definition 1 (Pathological disjunction)**.** *Let $\mathcal{X} \in \mathbb{R}^n$ be a closed convex set, and consider the disjunction $\mathcal{A} \cup \mathcal{B}$ as defined in* (3)*. If* $\operatorname{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$*, then disjunction $\mathcal{A} \cup \mathcal{B}$ is pathological for the set $\mathcal{X}$.*

Notice that if disjunction $\mathcal{A} \cup \mathcal{B}$ is pathological for the closed convex set $\mathcal{X}$, then every convex disjunctive cut will be redundant, since the set $\mathcal{X}$ is the tightest description for $\operatorname{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B}))$. Now, suppose that the DCC exists for set $\mathcal{X}$ and disjunction $\mathcal{A} \cup \mathcal{B}$. In this case, the DCC, which is the tightest possible cut, will be redundant as well.

In this section, we explore pathological disjunctions, and thus redundant DCCs and DCyCs for MISOCO problems. The redundant cases are first defined for a general convex set, and then they are presented for MISOCO problems.

3.1 Redundant DCCs

We first consider sets resulting from the intersection of a closed pointed convex cone and a disjunctive set. In particular, we are interested in the instances when the vertex of the cone is in one of the halfspaces defining the disjunctive set. In this situation we have the following result.

Theorem 1 (Conic pathological disjunction)**.** *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed pointed convex cone with vertex v, and consider two half spaces $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \le \alpha\}$ and $\mathcal{B} = \{x \in \mathbb{R}^n \mid a^\top x \ge \beta\}$, such that $\alpha < \beta$, $\mathcal{K} \cap \mathcal{A} \neq \emptyset$ and $\mathcal{K} \cap \mathcal{B} \neq \emptyset$. If $v \in \mathcal{A} \cup \mathcal{B}$, then* $\operatorname{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{K}$.

Proof. As $\mathcal{K}$ is convex, we have $\operatorname{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{K}$. Thus, we only need to prove that $\mathcal{K} \subseteq \operatorname{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B}))$. Let $x \in \mathcal{K}$ be given, then we need to show that $x \in \operatorname{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B}))$. If $x \in \mathcal{A} \cup \mathcal{B}$, then $x \in \operatorname{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B}))$. Now suppose $x \notin \mathcal{A} \cup \mathcal{B}$, i.e, $\alpha < a^\top x < \beta$. We know that the vertex v of the cone is in one of the disjunctive half spaces. Without loss of generality, we may assume that $a^\top v \le \alpha$. Let $r = x - v$. Vector r is in fact a ray of the cone $\mathcal{K}$, since $x \in \mathcal{K}$. As $a^\top v \le \alpha$ and $\alpha < a^\top x$, we have $a^\top r > 0$.

We know that $\beta - a^\top x > 0$, therefore, there exists a $\gamma > 0$ such that $\gamma a^\top r = \beta - a^\top x$, and we obtain that $a^\top(x + \gamma r) = \beta$. Let $\bar{x} = x + \gamma r$. As $x = v + r$, we have

$$\bar{x} = v + (1 + \gamma)r.$$

Vector r is a ray of the cone $\mathcal{K}$, so we have $\bar{x} \in \mathcal{K}$. As $v \in \mathcal{A}$ and $\bar{x} \in \mathcal{B}$, we can conclude that $v, \bar{x} \in \mathcal{K} \cap (\mathcal{A} \cup \mathcal{B})$. Let $\eta = \frac{\gamma}{1+\gamma}$, then we have

$$x = \eta v + (1 - \eta)\bar{x},$$

and we obtain that $x \in \text{conv}(\mathcal{K} \cap (\mathcal{A} \cup \mathcal{B}))$, which completes the proof.

□

Notice that the result of Theorem 1 holds for any general closed pointed convex cone $\mathcal{K}$. However, to identify the redundant DCCs for MISOCO problems, in this section we focus on the special case where $\mathcal{K}$ is a second order cone. We use Theorem 1 to characterize the conic redundant DCCs for MISOCO problems.

Corollary 1 (Redundant DCCs for MISOCO)**.** *If the set $\hat{\mathcal{C}}$, as defined in* (2)*, is a cone and its vertex is in one of the disjunctive halfspaces $\hat{\mathcal{A}}$ or $\hat{\mathcal{B}}$, as defined in* (4)*, then the DCC is equal to $\hat{\mathcal{C}}$.*

The main consequence of Corollary 1 is that in this case a DCC does not cut off any part of the feasible set. This redundancy is illustrated in Figure 1. In Figures 1(a) and 1(b), the intersections of the disjunctive hyperplanes with cone $\mathcal{C}$ are hyperboloids. Figure 1(a) is a redundant DCC, since the vertex of the cone is in one of the disjunctive half spaces; while in Figure 1(b) the DCC is not redundant. Figures 1(c) and 1(d) are other instances of the conic redundant DCCs, where the intersection of the cone with the hyperplanes are respectively an ellipsoid and a paraboloid.

Corollary 2 (Identification of a redundant DCC for MISOCO)**.** *If the following two conditions are satisfied for the set $\hat{\mathcal{C}}$ defined in* (2)*, and the disjunctive set defined in* (4)*, then we have a redundant DCC:*

- *the matrix P has exactly $n-1$ positive eigenvalues and one negative eigenvalue, and $p^\top P^{-1} p - \rho = 0$;*
- *the vertex of the cone $v = P^{-1}p$ satisfies either $\hat{a}^\top v \geq \hat{\beta}$, or $\hat{a}^\top v \leq \hat{\alpha}$.*

The first condition of Corollary 2 ensures that set $\hat{\mathcal{C}}$, defined in (2) is a cone and the second condition ensures that the set $\hat{\mathcal{C}}$ and disjunction (4) result in a redundant DCC.

3.2 Redundant DCyCs

We now consider sets resulting from the intersection of a closed convex cylinder and a disjunctive set. We first need to formally define a cylinder.

Definition 2. *(Lineality space [20]) Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a closed convex set. The lineality space of $\mathcal{X}$ is defined as*

$$\text{lin}(\mathcal{X}) = \{d \mid x + \alpha d \in \mathcal{X}, \ \forall x \in \mathcal{X}, \ \forall \alpha \in \mathbb{R}\}.$$

Lemma 1. *(Decomposition of a convex set [10]) Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then we have*

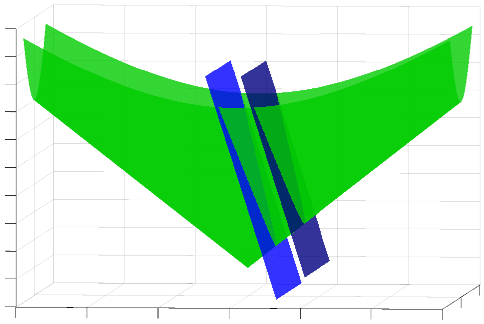
$$\mathcal{X} = \mathcal{S} + \text{lin}(\mathcal{X}),$$

where $\mathcal{S} = \mathcal{X} \cap \text{lin}(\mathcal{X})^\perp$, and $\text{lin}(\mathcal{X})^\perp$ is the orthogonal complement of $\text{lin}(\mathcal{X})$.

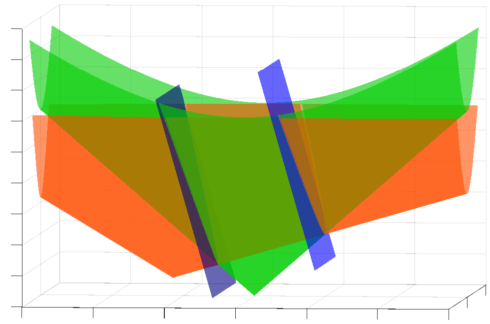
Definition 3. *(Cylinder) Let $\mathcal{X}$ be a nonempty closed convex set, If $\text{lin}(\mathcal{X}) \neq \{0\}$, then set $\mathcal{X}$ is a cylinder, and the set $\mathcal{S} = \mathcal{X} \cap \text{lin}(\mathcal{X})^\perp$ is a base of the cylinder.*

The following theorem formalizes the cylindrical pathological disjunction.

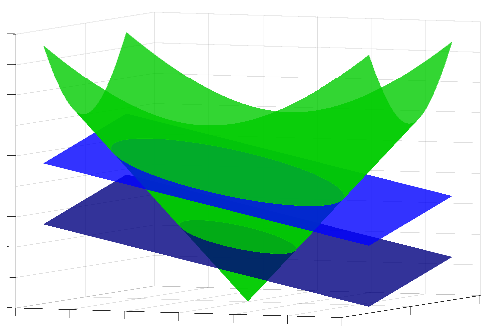
Theorem 2 (Cylindrical pathological disjunction)**.** *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a closed convex cylinder, and consider two half spaces $\mathcal{A}$ and $\mathcal{B}$, defined in* (3)*, such that $\alpha < \beta$, $\mathcal{X} \cap \mathcal{A} \neq \emptyset$, $\mathcal{X} \cap \mathcal{B} \neq \emptyset$. If $a \not\perp \text{lin}(\mathcal{X})$, then $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$.*



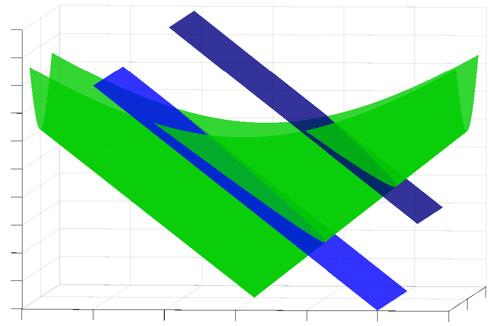
(a) Hyperboloid intersection (Redundant DCC)



(b) Hyperboloid intersection and the DCC (not a redundant DCC)



(c) Ellipsoid intersection (Redundant DCC)



(d) Paraboloid intersection (Redundant DCC)

Figure 1: Illustration of the conic redundant DCCs

Proof. As $\mathcal{X}$ is convex it follows that $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{X}$. Thus, to complete the proof we need to show that $\mathcal{X} \subseteq \text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B}))$. The proof goes by contradiction. Assume to the contrary that there exists an $\bar{x} \in \mathcal{X}$ such that $\bar{x} \notin \text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B}))$. Then, we have that $\bar{x} \notin \mathcal{A} \cup \mathcal{B}$, and we obtain $\alpha < a^\top \bar{x} < \beta$.

We know that $a = \text{Proj}_{\text{lin}(\mathcal{X})}(a) + \text{Proj}_{\text{lin}(\mathcal{X})^\perp}(a)$, where $\text{Proj}_{\text{lin}(\mathcal{X})}(a)$ and $\text{Proj}_{\text{lin}(\mathcal{X})^\perp}(a)$ denote the orthogonal projections of vector a to the subspaces $\text{lin}(\mathcal{X})$ and $\text{lin}(\mathcal{X})^\perp$, respectively. As $a \notin \text{lin}(\mathcal{X})^\perp$, we have $\text{Proj}_{\text{lin}(\mathcal{X})}(a) \neq 0$. Let d be defined as

$$d = \text{sign}\left(a^\top \text{Proj}_{\text{lin}(\mathcal{X})}(a)\right) \text{Proj}_{\text{lin}(\mathcal{X})}(a).$$

Thus we obtain $a^\top d > 0$. As $a^\top \bar{x} - \alpha > 0$, there exists $\gamma_\alpha > 0$ such that $\gamma_\alpha a^\top d \geq a^\top \bar{x} - \alpha$, i.e., we have $a^\top(\bar{x} - \gamma_\alpha d) \leq \alpha$. Let $\bar{x}_1 = \bar{x} - \gamma_\alpha d$, then $\bar{x}_\alpha \in \mathcal{A}$. Similarly, there exists $\gamma_\beta > 0$ such that $\gamma_\beta a^\top d \geq \beta - a^\top \bar{x}$, i.e., we have $a^\top(\bar{x} + \gamma_\beta d) \geq \beta$. Let $\bar{x}_\beta = \bar{x} + \gamma_\beta d$, then $\bar{x}_\beta \in \mathcal{B}$. Additionally, we have $\bar{x}_\alpha, \bar{x}_\beta \in \mathcal{X}$, since $\bar{x} \in \mathcal{X}$ and $d \in \text{lin}(\mathcal{X})$. Hence, we obtain that $\bar{x}_\alpha, \bar{x}_\beta \in \mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})$. Now, let $\eta = \frac{\gamma_\beta}{\gamma_\beta + \gamma_\alpha}$, then we have

$$\bar{x} = \eta \bar{x}_\alpha + (1 - \eta)\bar{x}_\beta.$$

Therefore, $\bar{x} \in \text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B}))$, which is a contradiction. This proves that $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$. $\square$

From Theorem 2 we obtain the following result for the special case where the lineality space of cylinder $\mathcal{X}$ is one-dimensional.

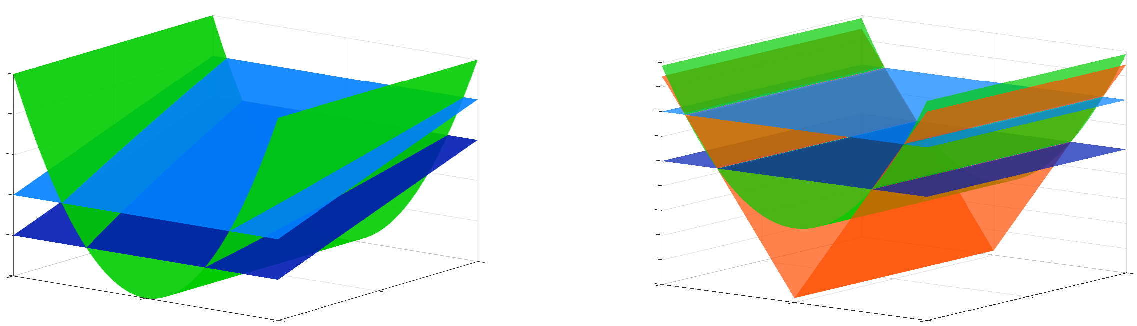
Remark 1. *In Theorem 2, assume that $\mathcal{X}$ is a closed convex cylinder such that* $\dim(\text{lin}(\mathcal{X})) = 1$*, i.e.,* $\text{lin}(\mathcal{X}) = \{\alpha d \mid \alpha \in \mathbb{R}\}$*. If $a^\top d \neq 0$, then* $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$*.*

Notice that the results of Theorem 2 and Remark 1 hold for a general closed convex cylinder $\mathcal{X}$. However, to identify the redundant DCyCs for MISOCO problems, in this section we focus on the special case where $\mathcal{X}$ is a quadratic cylinder.

We use Theorem 2 and Remark 1 to characterize the redundant DCyCs for MISOCO problems.

Corollary 3 (Redundant DCyC for MISOCO)**.** *Let $\hat{\mathcal{C}}$, as defined in* (2)*, be a cylinder, and consider the two half spaces defined in* (4)*. If $\hat{a} \not\perp \text{lin}(\hat{\mathcal{C}})$, then the DCyC is equal to $\hat{\mathcal{C}}$.*

Now, consider the special case of Corollary 3 where $\text{lin}(\hat{\mathcal{C}})$ is one-dimensional and defined as $\text{lin}(\hat{\mathcal{C}}) = \{\alpha d \mid \alpha \in \mathbb{R}\}$. Then, the condition $\hat{a} \notin \text{lin}(\hat{\mathcal{C}})$ simplifies to $\hat{a}^\top d \neq 0$. The redundant DCyC is illustrated in Figure 2(a), where the DCyC is equal to the cylindrical set. However, in Figure 2(b), the DCyC is not equal to the original cylinder. In that case we may derive a DCyC that does tighten the original cylinder.



(a) A cylindrical redundant DCyC

(b) Not a cylindrical redundant DCyC

Figure 2: Illustration of the cylindrical redundant DCyC

Corollary 4. (Identification of a redundant DCyC for MISOCO) *Consider the set $\hat{\mathcal{C}}$, as defined in* (2)*, and a disjunction as defined in* (4)*. We have a cylindrical redundant DCyC if the following two conditions are satisfied:*

- *System* $\begin{bmatrix} P & p \end{bmatrix}^\top d = 0$*, for $d \neq 0$, has a solution.*
- *System* $\begin{bmatrix} P & p \end{bmatrix} y = \hat{a}$*, for $y \in \mathbb{R}^{\ell+1}$, does not have a solution.*

Proof. From $\begin{bmatrix} P & p \end{bmatrix}^\top d = 0$, we have $Pd = 0$ and $p^\top d = 0$. So for all $w \in \hat{\mathcal{C}}$, we have

$$d^\top P d \alpha^2 + 2d^\top (Pw + p)\alpha = 0. \tag{6}$$

We know, for all $w \in \hat{\mathcal{C}}$, that $w^\top P w + 2p^\top w + \rho \leq 0$. So from (6), we have

$$(w + \alpha d)^\top P(w + \alpha d) + 2p^\top (w + \alpha d) + \rho \leq 0, \quad \forall \alpha \in \mathbb{R}.$$

Hence, $d \in \text{lin}(\hat{\mathcal{C}})$. As $d \neq 0$, we conclude that $\hat{\mathcal{C}}$ is a cylinder.

Let col(.), row(.), and null(.) denote respectively the column space, row space, and null space of a matrix. If system $[P\ p]y = \hat{a}$, for $y \in \mathbb{R}^{\ell+1}$, does not have a solution, then $\hat{a} \notin \text{col}([P\ p])$; thus, $\hat{a} \notin \text{row}([P\ p]^\top)$. Therefore, there exists a $d_0 \neq 0$ such that $d_0 \in \text{null}([P\ p]^\top)$ and $\hat{a}^\top d_0 \neq 0$. Hence, we obtain that $\hat{a} \not\perp \text{lin}(\hat{\mathcal{C}})$. From Corollary 3, we can conclude that this is a redundant DCyC. $\square$

Notice in Corollary 4 that the first condition ensures that set $\hat{\mathcal{C}}$, defined in (2), is a cylinder and the second condition ensures that $\hat{C}$ and disjunction (4) define a redundant DCyC.

Corollary 4 is in fact a sufficient condition to identify the cylindrical pathological disjunction, as defined in Theorem 2, for a MISOCO problem. In Corollary 5, we provide a necessary and sufficient condition to identify when the normal vector of the disjunctive hyperplanes is orthogonal to the lineality space of the cylinder. The difference between Corollaries 4 and 5 is that the former considers a condition in the null space of the affine constraints of the MISOCO problem, while the latter one is defined in the original space of the decision variables of the MISOCO problem. The following lemma is needed to prove Corollary 5.

Lemma 2 (Lineality space of intersection of two convex sets [10])**.** *If $\mathcal{X}_1$ and $\mathcal{X}_2$ are convex sets such that $\mathcal{X}_1 \cap \mathcal{X}_2 \neq \emptyset$, then $\text{lin}(\mathcal{X}_1 \cap \mathcal{X}_2) = \text{lin}(\mathcal{X}_1) \cap \text{lin}(\mathcal{X}_2)$.*

Corollary 5. (Identification of a redundant DCyC for MISOCO) *Consider the MISOCO problem* (1) *and disjunction* (3)*. We may assume w.l.o.g. that we derive the DCyC for $\mathcal{L}_1 \in \mathbb{R}^{n_1}$. Condition $a \not\perp \text{lin}(\mathcal{X})$ holds if and only if*

$$\begin{pmatrix} A \\ a^\top \end{pmatrix} \begin{pmatrix} 0_{n_1} \\ x \end{pmatrix} = \begin{pmatrix} 0_m \\ 1 \end{pmatrix}, \tag{7}$$

where $x \in \mathbb{R}^{n-n_1}$.

Proof. Let $\mathcal{K}^{\mathcal{L}_1}$ be the cone $\mathcal{L}_1$ lifted to $\mathbb{R}^n$. So we have $\mathcal{K}^{\mathcal{L}_1} = \{(x^c, x^r) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n-n_1} | x^c \in \mathcal{L}_1\}$. We know that $\text{lin}(\mathcal{L}_1) = \{0\}$, hence, $\text{lin}(\mathcal{K}^{\mathcal{L}_1}) = \{(0_{n_1}, x^r) \mid x^r \in \mathbb{R}^{n-n_1}\}$. Let $\mathcal{C} = \mathcal{K}^{\mathcal{L}_1} \cap \{x \in \mathbb{R}^n \mid Ax = b\}$. We know that $\text{lin}(\{x \in \mathbb{R}^n \mid Ax = b\}) = \text{null}(A)$. So from Lemma 2, we have $\text{lin}(\mathcal{C}) = \text{lin}(\mathcal{K}^{\mathcal{L}_1}) \cap \text{null}(A)$, and we can conclude that

$$\text{lin}(\mathcal{C}) = \left\{ \begin{pmatrix} 0_{n_1} \\ x^r \end{pmatrix} \;\middle|\; x^r \in \mathbb{R}^{n-n_1},\ A \begin{pmatrix} 0_{n_1} \\ x^r \end{pmatrix} = 0 \right\}. \tag{8}$$

From Theorem 2, we know that if $a \not\perp \text{lin}(\mathcal{C})$, then we have a cylindrical redundant DCyC. Condition $a \not\perp \text{lin}(\mathcal{C})$ holds if and only if there exists $\bar{x} \in \text{lin}(\mathcal{C})$ such that $a^\top \bar{x} \neq 0$. As $\text{lin}(\mathcal{C})$ is a subspace, w.l.o.g. we can acquire $a^\top \bar{x} = 1$ for $a \not\perp \text{lin}(\mathcal{C})$. Combining this condition with equation (8), we can conclude that $a \not\perp \text{lin}(\mathcal{C})$ if and only if system $\begin{pmatrix} A \\ a^\top \end{pmatrix} \begin{pmatrix} 0_{n_1} \\ x^r \end{pmatrix} = \begin{pmatrix} 0_m \\ 1 \end{pmatrix}$ has a solution, which completes the proof.

□

Remark 2. *The redundancy of a DCyC is independent of the base of the cylinder.*

4 Discussion

The cases discussed in Sections 3.1 and 3.2 form the fundamental units for analyzing the redundant DCCs and DCyCs. In this section, we explore some instances where one can find the redundant cases embedded in more complex configurations. The cases presented in this section are built on the quadratic set $\hat{\mathcal{C}}$, as defined in (2), and a disjunction defined in (4), such that $\alpha < \beta$, $\mathcal{X} \cap \mathcal{A} \neq \emptyset$, $\mathcal{X} \cap \mathcal{B} \neq \emptyset$.

4.1 Conic cylinders

In this section, we define a conic cylinder, and we consider sets resulting from the intersection of a conic cylinder and a disjunctive set. Then, we formalize a special case of redundant DCyCs.

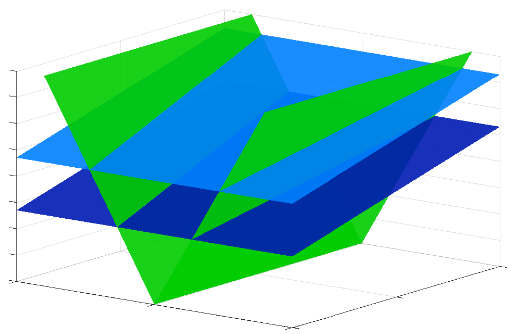
Definition 4. *(Conic cylinder) Let $\mathcal{X}$ be a closed convex set, and let $\mathcal{K} = \mathcal{X} \cap \text{lin}(\mathcal{X})^\perp$. Set $\mathcal{X}$ is a conic cylinder if $\mathcal{K}$ is a convex cone, and $\text{lin}(\mathcal{X}) \neq \{0\}$.*

The following corollary formalizes a special case where the DCyC is redundant.

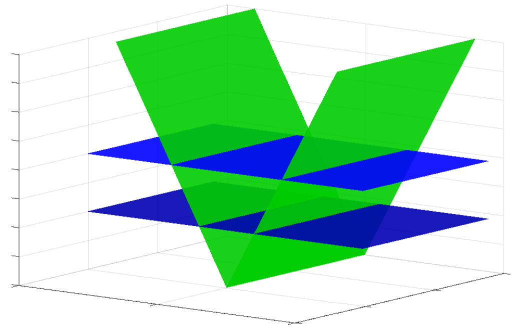
Corollary 6. *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a closed convex cylinder such that $\mathcal{X} = \text{lin}(\mathcal{X}) + \mathcal{K}$, and $\mathcal{K} = \mathcal{X} \cap \text{lin}(\mathcal{X})^\perp$. Suppose that $\mathcal{K}$ is a convex cone with vertex v. Let $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \leq \alpha\}$ and $\mathcal{B} = \{x \in \mathbb{R}^n \mid a^\top x \geq \beta\}$ such that $\alpha < \beta$, $\mathcal{X} \cap \mathcal{A} \neq \emptyset$, $\mathcal{X} \cap \mathcal{B} \neq \emptyset$, and $a \perp \text{lin}(\mathcal{X})$. If $v \in \mathcal{A} \cup \mathcal{B}$, then $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$.*

Proof. The proof is analogous to that of Theorem 1. □

Notice in Corollary 6 that if $a \not\perp \text{lin}(\mathcal{X})$, then from Corollary 3, we know that we have a redundant DCyC. However, in Corollary 6 we assume that $a \perp \text{lin}(\mathcal{X})$, and derive another case of pathological disjunction and redundant DCyC. Figure 3 illustrates this result showing two different cases of redundant DCyCs. In Figure 3(a) we have a case where $a \not\perp \text{lin}(\mathcal{X})$, which complies with Corollary 3, thus we have a redundant DCyC. In Figure 3(b) we have a case where $a \perp \text{lin}(\mathcal{X})$, so it does not satisfy the conditions of Corollary 3. However, it complies with Corollary 6, thus we have a redundant DCyC. The classification of Figure 3(b) may be obtained noting that the base of the cylinder is a convex cone and its vertex is in one of the half spaces defining the disjunction. Henceforth, the original cylinder configures a redundant DCyC.



(a) A cylindrical redundant DCyC



(b) A conic redundant DCC case for the base of the cylinder

Figure 3: Illustration of redundant cases for conic cylinders

4.2 Branching on a higher dimensional subspace

In some cases one may have to make a split disjunction on an integer variable that does not appear in the quadratic set $\mathcal{C} \in \mathbb{R}^n$. This is in fact one common instance of redundant DCyCs, where the cylinder is given by $\{(\xi, x) \in \mathbb{R} \times \mathbb{R}^n \mid (0, x) + (\xi, 0), x \in \mathcal{C}\}$, and we want to make a disjunction on the variable ξ. In this case, the disjunction is pathological and we have a redundant DCyC. This case is illustrated in Figure 4, where the quadratic set defines a cylinder with an ellipsoid base defined in the space of (x_1, x_2), and we make a disjunction on variable ξ.

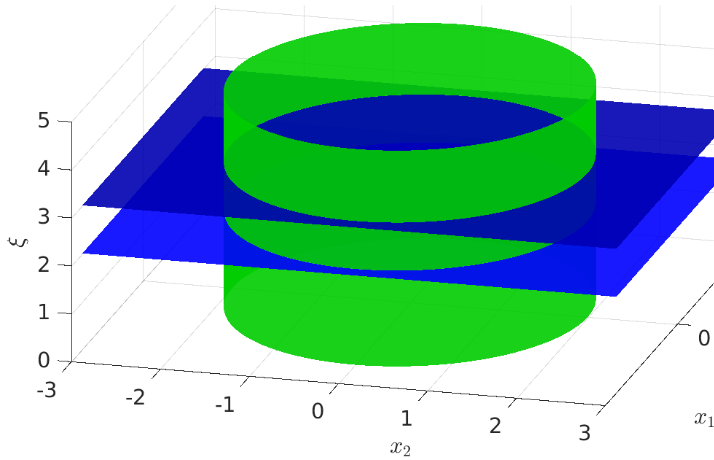


Figure 4: A redundant DCyC

4.3 Eliminating pathology by branching

Suppose that deriving the DCC for the set $\hat{\mathcal{C}} \cap (\hat{\mathcal{A}} \cup \hat{\mathcal{B}})$, as defined in (2) and (4), one ends at a redundant cut. This situation does not necessarily render the DCC approach useless. In particular, further down the branch and bound tree, effective DCCs may be generated.

Figure 5 illustrates this case. Suppose that $\hat{\mathcal{C}}$ is a cone, with a vertex v, and one wants to make a disjunction on the binary variable x_2. Notice that the vertex of the cone is in one of the disjunctive half spaces in Figure 5. Then, the DCC will be equal to $\hat{\mathcal{C}}$, which is a redundant DCC. In this case, one can branch on the binary variable x_2 to obtain new quadratic sets in each branch. Consider first the branch $x_2 = 0$, the new quadratic set is obtained from the intersection of $\hat{\mathcal{C}}$ with the hyperplane $x_2 = 0$, which defines a two-dimensional second order cone. In this branch, making a disjunction on the binary variable x_1 again leads to a redundant DCC. Now consider the branch $x_2 = 1$; the new quadratic set is obtained from the intersection of $\hat{\mathcal{C}}$ with the hyperplane $x_2 = 1$, which is one branch of a hyperboloid. Considering a disjunction on the binary variable x_1, one can derive a useful DCC in this branch.

The case presented in this section shows how one can identify opportunities down the branch and bound tree for using DCCs to improve the performance of a solver. This is useful to complement existing branching rules [1] to define new rules capable of exploiting the structure of a MISOCO problem, which calls for further research in this area.

5 Conclusions

In this paper we presented two fundamental pathological disjunctions, which help to identify redundant DCCs and DCyCs for MISOCO problems. We know that if the DCC is redundant, then any other disjunctive cut will be redundant too, since the DCC is the tightest disjunctive cut describing the convex hull of the disjunctive set. We have also shown how those two cases are the building blocks of more complex instances. We illustrated that by analyzing some instances in Section 4, and showing how those were combinations of the two redundant cases considered in this paper. The identification of the redundant DCCs is important for an efficient implementation of DCCs or derivation of them. For example, in a branch and cut approach, one wants to keep under control the growth of the problem. For that reason, identifying whether a DCC is redundant before adding it to the formulation is essential to obtain an efficient implementation of this technology. Therefore, identification of the redundant DCCs for MISOCO highlights both the limitations and the opportunities for the efficient implementation of the DCCs.

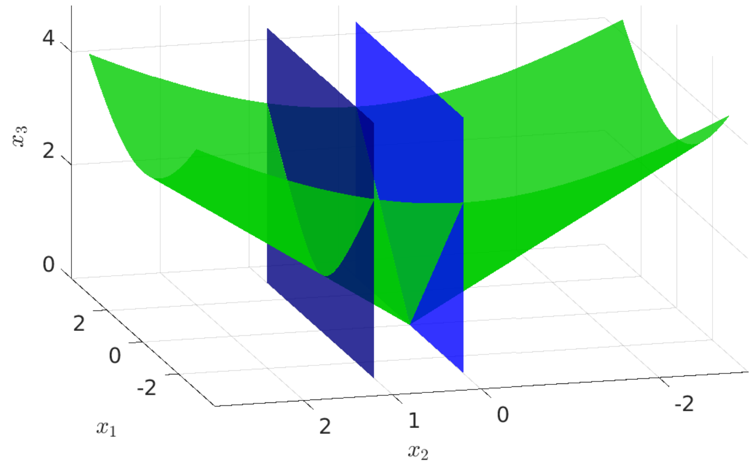


Figure 5: An instance of the conic redundant DCC

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