The robust stabilization problem for discrete-time descriptor systems*

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Abstract. In this paper the robust stabilization problem for linear discrete-time descriptor systems is investigated. This means that the transfer function matrix of the system at hand is allowed to be improper or even polynomial, as the uncertainty acts on normalized coprime factors. The main results comprising explicit analytical formulas for the maximum stability margin and for the robust stabilizing controller are given in a realization based-setting and involve solutions of two algebraic Riccati equations, thus recapturing the same elegant simplicity and numerical easiness of the standard (proper) case.

Key words. Descriptor systems, Riccati equations, H^∞ control problem.

1. Introduction. With the rise and ubiquity of digital and data technology, the processes that need to be controlled have become more and more complex, and consequently the associated mathematical models more complicated. As, in practice, any system could be subjected to various kinds of perturbations, the problem of stabilizing the system, under the action of these perturbations, could become more intricate, and harder to solve especially for systems that are allowed not to be proper. To this end, the need for finding a single controller that stabilizes a whole class of systems is a current issue that is required to be solved. It is well known that the robust stabilization problem was solved long time ago (see for example [5], [4]), for standard proper systems. In [5], a solution to the suboptimal problem for continuous-time systems is given, and in [4] an optimal solution is built for both continuous-time and discrete-time systems. The attention is focused here in solving this issue for a more general class of systems, namely for descriptor ones in the context of discrete-time systems. From the authors's knowledge, the main drawback of the solutions proposed so far in the literature is that they avoid the improper and polynomial cases, which sometimes can be of great interest.

The main results of this paper can be summarized as follows:

1. Analytical formulas for computing the stability radius, as well as the associated optimal controller, for descriptor discrete-time systems are provided. This means that the assumption of proper systems is removed, and so improper and even polynomial rational matrix functions (**RMF**s) are taken into consideration.
2. All the relations are based on the original datas, using only the stabilizing solutions of two Riccati equations.
3. Finally, the proposed solution avoids preliminary decompositions.

The paper is organized as follows: in *Section 2* some general notations and definitions are given, followed by a brief recall as well as some useful results related to: realization theory, Riccati equations, and normalized coprime factorizations, all for the class of generalized systems under investigation. *Section 3* is devoted to the statement of the robust stabilization problem, as well as to the presentation of the main results of the paper together with their proofs. The author supports his results with one numerical example in *Section 4*, that highlights not only the main novelty of the paper (that is allowing for the given system to be improper, or even polynomial) but also the simplicity of computing the (optimal) solution of the proposed problem,

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and the performance of the robust controller as well. In *Section 5* some conclusions are drawn, and in *Section 6* some auxiliary results (together with their proofs) are presented.

2. Basic notations and definitions.

$\mathbb{D}$ and $\mathbb{D}_c$ stand for the open unit disk and the exterior of the closed-unit disk respectively, and $C_{o;1}$ for the unit circle, centered in origin. $\bar{\mathbb{C}}$ represents the one point compactification of the complex-plane. A constant matrix $A \in \mathbb{C}^{n\times n}$ is said to be hermitic if $A = A^*$, where

$$A^* := \bar{A}^T. \tag{1}$$

A transfer function matrix $G(z)$ is in $\mathbb{RH}^\infty$ if it is stable in discrete-time case, i.e. if it has all its poles insight the open unit disk. Moreover denote by $G^T(z)$ and $G^\#(z)$ the transpose and the para-Hermitian respectively, associated with a general **RMF** $G(z)$; namely

$$G^\#(z) = \bar{G}^T(\frac{1}{\bar{z}}), \tag{2}$$

where the bar above takes the complex conjugate of the respective matrix. $rank_n(\cdot)$ represents the normal rank of the **RMF**, i.e. the rank of $G(z)$ for all $z \in \bar{\mathbb{C}}$ except for a finite number of points. By $\Lambda(A - zE)$ the spectrum of the matrix-pencil $A - zE$ is denoted (for further details see [1]), I stands for the unit matrix, 0 represents any zero-matrix of appropriate dimension, and $\rho(A)$ is the spectral radius of any square complex-valued matrix.

2.1. Centered realizations.

Any $p \times m$ system given by its rational matrix function model $G(z)$ (could it be improper or polynomial) has a realization of the form

$$G(z) = D + C(zE - A)^{-1}B =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right], \tag{3}$$

where $A - zE$ is a regular matrix pencil and all the intervening matrices have complex elements and appropriate dimensions. Although the realization 3 is suited to represent any **RMF** model, it has a couple of drawbacks for the problems under investigation. For example, if ∞ is a pole of $G(z)$, then the minimal order of the realization 3 is strictly greater than the McMillan degree of $G(z)$, while D does not represent the value of $G(z)$ at any particular point. To circumvent this, a slightly more general type of realizations called *centered* will be used (for further details see [10]). To define a centered realization for $G(z)$, one first needs to fix a $z_0 \in \bar{\mathbb{C}}$, and further α, β such that

$$\begin{array}{ll} \alpha = 1, \beta = 0, & if\ z_0 = \infty \\ \alpha = z_0, \beta = 1, & if\ z_0 \in \mathbb{C}. \end{array} \tag{4}$$

A realization centered at z_0 is a representation of the form

$$G(z) = C(zE - A)^{-1}B(\alpha - \beta z) + D =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right]_{z_0}, \tag{5}$$

where $A - zE$ is a regular pencil, $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{p\times m}$. Whenever centered realizations are used throughout the paper, this implicit choice of α and β in 4 will be assumed. The positive order n is called the *order* (or the *dimension*) of the realization 5. In particular, if $z_0 = \infty$ then simply drop the index z_0 from the notation 5, and get precisely the notation in 3. Therefore, realizations 3 are simply realizations centered at $z_0 = \infty$. We say that a realization 5 (or the pair $(A - zE, B)$) is controllable at $z \in \mathbb{C}$ if rank $\begin{bmatrix} A - zE & B \end{bmatrix} = n$, and is controllable at ∞ if rank $\begin{bmatrix} E & B \end{bmatrix} = n$. Analogously, a realization 5 is called observable (or the pair $(C, A - zE)$ is observable) at a certain $z \in \bar{\mathbb{C}}$ provided that the pair $\begin{bmatrix} A^* - zE^* & C^* \end{bmatrix}$ is controllable at z. A realization (or a pair) is called simply controllable (observable) provided that it is controllable (observable) $\forall z \in \bar{\mathbb{C}}$. A realization (or a pair) is called stabilizable (detectable) provided it is controllable (observable) for all $z \in \mathbb{D}_c \cup C_{0;1}$. The realization 5 is called *minimal* if its order is as small as possible among all realizations of this type (i.e. centered at a given z_0). The paramount features of centered realizations are relevant by choosing z_0 such that it is not a pole of $G(z)$-a choice in force throughout the paper. In this case, one recovers all nice properties of standard state-space realizations (for further details see Section 2.4 in [10]):

1. The dimension of a minimal realization equals the McMillan degree;
2. Any two minimal realizations

$$G(z) = \left[\begin{array}{c|c} A_1 - zE_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{z_0} = \left[\begin{array}{c|c} A_2 - zE_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{z_0} \tag{6}$$

   are related by an equaivalence transformation, i.e. there exist invertible matrices Q and Z such that $A_2 - zE_2 = Q(A_1 - zE_1)Z, B_2 = QB_1, C_2 = C_1Z$;
3. Minimality of a realization is equivalent to controllability plus observability;
4. By an appropriate equivalence transformation we can always assume that the realization is *normalized*, i.e. $A - z_0E = I$;
5. Provided that the realizations are normalized, then at 2. we can take $Z = Q^{-1}$;
6. $D = G(z_0)$.

A nice feature of centered realizations, making them as handy to manipulate as standard ones, is the easiness in obtaining them. In [11] a direct method to obtain a centered minimal realization (at any point $z_0 \in \bar{\mathbb{C}}$) is given, starting from the **RMF** description. Alternatively, switching back and forth between realizations centered at infinity and realizations centered at a finite $z_0 \in \mathbb{C}$ can be done by straightforward manipulations (see Section 5 in [8]). Throughout the paper, realizations centered at $z_0 \in \mathbb{C}_{o;1}$ (thus featuring specific symmetries with respect to the unit disk) are considered, where z_0 is not a pole of the underlying **RMF**. Accordingly, let $\alpha \in C_{o;1}$, $\beta := \bar{\alpha}$, which gives $z_0 = \frac{\alpha}{\bar{\alpha}} = \alpha^2 \in C_{0;1}$. This choice will also be in force for the rest of the paper.

Let now $G(z) := \left[\begin{array}{c|c} A_1 - zE_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{z_0}$ and $H(z) := \left[\begin{array}{c|c} A_2 - zE_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{z_0}$ be two **RMF**s. The following relation holds (see Section 2 in [8])

$$G(z)H(z) = \left[\begin{array}{c|c} A_1 - zE_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{z_0} \left[\begin{array}{c|c} A_2 - zE_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{z_0} = \left[\begin{array}{cc|c} A_1 - zE_1 & B_1C_2(\alpha - \bar{\alpha}z) & B_1D_2 \\ 0 & A_2 - zE_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]_{z_0}. \tag{7}$$

Also, from 2 it can be inferred that the para-Hermitian of any **RMF** $G(z)$ as in 5, has the following centered realization

$$G^{\#}(z) = \left[\begin{array}{c|c} E_1^* - zA_1^* & C_1^* \\ \hline B_1^* & D_1^* \end{array}\right]_{z_0}. \tag{8}$$

If, say, $G(z)$ is a square invertible **RMF** and z_0 is also not a zero of it, then it holds (see Section 3 in [8])

$$G^{-1}(z) = \left[\begin{array}{c|c} A_1 - \alpha B_1 D_1^{-1} C_1 - z(E_1 - \bar{\alpha} B_1 D_1^{-1} C_1) & B_1 D_1^{-1} \\ \hline -D_1^{-1} C_1 & D_1^{-1} \end{array}\right]_{z_0}. \tag{9}$$

In the end of this part, the author points out that a nice example which reveals the above mentioned advantages of centered realizations over standard descriptor ones, when dealing with systems that possess poles at infinity, is presented in Section 2 of [12]. There the authors consider a 3×2 polynomial **RMF** $T(z) := \begin{bmatrix} -z^2 + z + 1 & -z + 2 \\ z^3 - z + 4 & z^2 + 1 \\ z^3 - z - 2 & z^2 + 3 \end{bmatrix}$ having two poles at ∞, one with multiplicity 3 and one with multiplicity 2 (hence a McMillan degree $\delta(T) = 5$), and show that when picking up a minimal standard descriptor realization of $T(z)$ the dimension of the realization is $\tilde{n} = 7$ which is strictly greater than the McMillan degree of $T(z)$. On the other hand, writing down a minimal centered realization for $T(z)$, it turns out that it has dimension $n = 5 = \delta(T)$, as depicted in relation 2.6 therein. Also, the authors point out that the D-matrix in the first case has no particular significance, whereas in the second case it equals $T(1)$, where 1 is the centering point.

2.2. The descriptor Riccati equation. Recall from [12] the descriptor discrete - time algebraic Riccati equation (DDTARE)

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB + L)R^{-1}((\bar{\alpha}A - \alpha E)^*XB + L)^* + Q = 0, \tag{10}$$

where $E, A \in \mathbb{C}^{n\times n}$, $L, B \in \mathbb{C}^{n\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$, $Q \geq 0$, $R = R^* \in \mathbb{C}^{m\times m}$ (with R invertible). As stated in [12], the DDTARE 10 may be seen as the 'natural' Riccati equation associated with a dynamical system given by a centered realization 5 (having the states available for measurement $C = I$, $D = 0$) and a quadratic performance index

$$\sum_{k\geq 0} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{11}$$

where x is the state variable and u is the controlled input (also see Chapter 3 [4] for the standard proper case). A solution $X_s = X_s^*$ to 10 is called *stabilizing* provided that

$$\Lambda(A - zE + BF_s(\alpha - \bar{\alpha}z)) \subset \mathbb{D}, \tag{12}$$

where $F_s := -R^{-1}((\bar{\alpha}A - \alpha E)^*XB + L)^*$ is called the *stabilizing* Riccati feedback. The next two results will be instrumental in the sequel.

THEOREM 2.1. *Assume $\bar{\alpha}A - \alpha E$ is invertible. The following statements are equivalent:*

1. *R is invertible and the DDTARE 10 has a stabilizing solution.*
2. *The Descriptor Simplectic Pencil (DSP)*

$$zM - N := z\begin{bmatrix} E & 0 & \bar{\alpha}B \\ \bar{\alpha}Q & -A^* & \bar{\alpha}L \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & \alpha B \\ \alpha Q & -E^* & \alpha L \\ L^* & -B^* & R \end{bmatrix} \tag{13}$$

is regular and there is a $(2n+m)\times n$ matrix

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{14}$$

with invertible V_1, such that $MVS = NVT$, where $\Lambda(S - zT) \subset \mathbb{D}$. Moreover, the stabilizing solution can be computed from

$$X_s = -V_2 V_1^{-1}(\bar{\alpha}A - \alpha E)^{-1} \tag{15}$$

and the stabilizing feedback can be computed from

$$F_s = V_3 V_1^{-1} \tag{16}$$

Proof. The proof is based mainly on three observations.

First notice that the DDTARE 10 has a stabilizing solution X_s if and only if the Riccati equation

$$E^*XE - A^*XA - ((\alpha E - \bar{\alpha}A)^*XB + L)R^{-1}((\alpha E - \bar{\alpha}A)^*XB + L)^* + Q = 0 \tag{17}$$

has a solution $-X_s$, such that $\Lambda(zE - A - BR^{-1}((\alpha E - \bar{\alpha}A)^*(-X_s)B + L)^*(\bar{\alpha}z - \alpha)) \subset \mathbb{D}$.

Hence, from the observation above one can see that if $\alpha = 1$ and $A - E$ is invertible, the theorem follows immediately from *Theorem 10* in [7].

If $\alpha \in C_{o;1} \setminus \{1\}$ and $\bar{\alpha}A - \alpha E$ is invertible the following notations are made $\xi := \bar{z}_0 z = \bar{\alpha}^2 z$, $\tilde{A} := \bar{\alpha}A$, $\tilde{E} := \alpha E$, $\tilde{B} := B$, $\tilde{Q} := Q$, $\tilde{L} := L$, $\tilde{R} := R$. Here comes the second observation.

Denoting $N_1 - \xi M_1 := \begin{bmatrix} \tilde{A} & 0 & \tilde{B} \\ \tilde{Q} & -\tilde{E}^* & \tilde{L} \\ \tilde{L}^* & -\tilde{B}^* & \tilde{R} \end{bmatrix} - \xi \begin{bmatrix} \tilde{E} & 0 & \tilde{B} \\ \tilde{Q} & -\tilde{A}^* & \tilde{L} \\ 0 & 0 & 0 \end{bmatrix}$, it is easy to check that $N_1 - \xi M_1 = diag(\alpha I_n, \alpha I_n, I_m)(N - zM)$, and since $|z| = |\xi|$ it can be seen that the DSP $zM - N$ is regular, and there is an $(2n+m)\times n$ basis matrix $V^T = \begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}$ with V_1 square invertible, such that $MVS = NVT$, where $\Lambda(S - zT) \subset \mathbb{D}$ if and only if the matrix pencil $N_1 - \xi M_1$ is regular, and there is an $(2n+m)\times n$ basis matrix $W^T = \begin{bmatrix} W_1^T & W_2^T & W_3^T \end{bmatrix}$ with W_1 square invertible, such that $M_1 W S_1 = N_1 W T_1$, where $\Lambda(S_1 - zT_1) \subset \mathbb{D}$. Moreover one can choose $W = V$, $S_1 = S$ and $T_1 = T$. Finally, the third observation is made.

$\tilde{X}_s$ is a solution of the Riccati equation

$$\tilde{E}^*\tilde{X}\tilde{E} - \tilde{A}^*\tilde{X}\tilde{A} - ((\tilde{E} - \tilde{A})^*\tilde{X}\tilde{B} + \tilde{L})\tilde{R}^{-1}((\tilde{E} - \tilde{A})^*\tilde{X}\tilde{B} + \tilde{L})^* + \tilde{Q} = 0 \tag{18}$$

such that $\Lambda(\xi\tilde{E}-\tilde{A}-\tilde{B}\tilde{R}^{-1}((\tilde{E}-\tilde{A})^*\tilde{X}\tilde{B}+\tilde{L})^*(\xi-1))\subset\mathbb{D}$ if and only if $\tilde{X}_s$ is a solution of the Riccati equation 17 such that $\Lambda(zE-A-BR^{-1}((\alpha E-\bar{\alpha}A)^*\tilde{X}_sB+L)^*(\bar{\alpha}z-\alpha))\subset\mathbb{D}$

Putting all these together, one has: R is invertible and the DDTARE 10 has a stabilizing solution X_s if and only if R is invertible and the Riccati equation 17 has a solution $-X_s$ such that $\Lambda(zE-A-BR^{-1}((\alpha E-\bar{\alpha}A)^*(-X_s)B+L)^*(\bar{\alpha}z-\alpha))\subset\mathbb{D}$, which in turn is equivalent to $\tilde{R}$ is invertible and the Riccati equation 18 has a solution $-X_s$ such that $\Lambda(\xi\tilde{E}-\tilde{A}-\tilde{B}\tilde{R}^{-1}((\tilde{E}-\tilde{A})^*(-X_s)\tilde{B}+\tilde{L})^*(\xi-1))\subset\mathbb{D}$. Applying now *Theorem 10* in [7] to the Riccati equation 18 with the associated DSP ξM_1-N_1, it follows that R is invertible and the DDTARE 10 has a stabilizing solution X_s if and only if the matrix pencil ξM_1-N_1 is regular, and there is an $(2n+m)\times n$ basis matrix $V^T=\begin{bmatrix}V_1^T & V2^T & V_3^T\end{bmatrix}$ with V_1 invertible, such that $M_1VS=N_1VT$, for some matrices S and T such that $\Lambda(S-zT)\subset\mathbb{D}$. By the second observation, it holds in turn that R is invertible and the DDTARE 10 has a stabilizing solution X_s if and only if the DSP $zM-N$ is regular, and there is an $(2n+m)\times n$ basis matrix $V^T=\begin{bmatrix}V_1^T & V_2^T & V_3^T\end{bmatrix}$ with V_1 invertible, such that $MVS=NVT$, where $\Lambda(S-zT)\subset\mathbb{D}$.

The expression for the stabilizing solution X_s of the DDTARE 10 as well as for its associated feedback F_s follow by replacing $\tilde{A}$, $\tilde{E}$, $\tilde{B}$, $\tilde{L}$, $\tilde{Q}$ and $\tilde{R}$ by their expressions $\bar{\alpha}A$, αE, B, L, Q and R, respectively in the formula for the stabilizing solution and stabilizing feedback of the Riccati equation 18 provided by *Theorem 10* in [7]. ◻

The next result will be of great importance in constructing the optimal solution.

LEMMA 2.2. *If $G(z)$ is a generalized $p\times m$ system given by the stabilizable and detectable realization*

$$G(z):=\left[\begin{array}{c|c}A-zE & B\\ \hline C & 0\end{array}\right]_{z_0}, \tag{19}$$

where $z_0=\frac{\alpha}{\bar{\alpha}}\in C_{o;1}$ is not a pole of $G(z)$, then the following two statements hold:

1. *The Riccati equation (denoted DDTARE$_o$)*

$$AYA^*-EYE^*-(\bar{\alpha}A-\alpha E)YC^*CY(\bar{\alpha}A-\alpha E)^*+BB^*=0 \tag{20}$$

has a solution $Y_s=Y_s^*\geq 0$ such that $\Lambda(A-zE+K_sC(\alpha-\bar{\alpha}z))\subset D$, where $K_s:=-(\bar{\alpha}A-\alpha E)Y_sC^*$ (for the rest of the paper Y_s will be called the stabilizing solution and K_s will be called the stabilizing feedback, respectively, associated with DDTARE$_o$).

2. *The Riccati equation (denoted DDTARE$_c$)*

$$A^*XA-E^*XE-(\bar{\alpha}A-\alpha E)^*XBB^*X(\bar{\alpha}A-\alpha E)+C^*C=0 \tag{21}$$

has a solution $X_s=X_s^*\geq 0$ such that $\Lambda(A-zE+BF_s(\alpha-\bar{\alpha}z))\subset\mathbb{D}$, where $F_s:=-B^*X_s(\bar{\alpha}A-\alpha E)$ (for the rest of the paper X_s will be called the stabilizing solution and F_s will be called the stabilizing feedback, respectively, associated with DDTARE$_c$).

Proof. 1. Denote $\tilde{E}:=\bar{\alpha}E^*$, $\tilde{A}:=\alpha A^*$, $\tilde{X}:=-Y$, $\tilde{B}:=C^*$, $\tilde{Q}:=BB^*=\tilde{Q}^*\geq 0$, $\tilde{C}:=\begin{bmatrix}B & 0\end{bmatrix}^*$, $\tilde{D}:=\begin{bmatrix}0 & I\end{bmatrix}^*$. With these notations DDTARE$_o$ 20 can be written equivalently as

$$\tilde{E}^*\tilde{X}\tilde{E}-\tilde{A}^*\tilde{X}\tilde{A}-((\tilde{E}-\tilde{A})^*\tilde{X}\tilde{B}+\tilde{C}^*\tilde{D})(\tilde{D}^*\tilde{D})^{-1}(\tilde{B}^*\tilde{X}(\tilde{E}-\tilde{A})+\tilde{D}^*\tilde{C})+\tilde{Q}=0. \tag{22}$$

Define $\tilde{T}(z) := \left[\begin{array}{c|c} \tilde{A} - z\tilde{E} & \tilde{B}(1-z) \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{c|c} \alpha A^* - z\bar{\alpha}E^* & C^*(1-z) \\ \hline B^* & 0 \\ 0 & I \end{array}\right]$.

Since the pair $(A - zE, B)$ is stabilizable, one gets $rank\left(\begin{bmatrix}\bar{\alpha}A - \bar{z}\alpha E & B\end{bmatrix}^*\right) = rank\left(\begin{bmatrix}A - (\bar{z}z_0)E & B\end{bmatrix}^*\right) = n \ \forall \bar{z} \in C_{o;1}$, and consequently $rank(\tilde{T}(z)) = n + p \ \forall z \in C_{o;1}$. Applying *Theorem 12* in [7] it follows that the Riccati equation 22 has a solution $\tilde{X}_s = \tilde{X}_s^*$ such that $\Lambda(z\tilde{E} - \tilde{A} - \tilde{B}\tilde{B}^*\tilde{X}_s(\tilde{E} - \tilde{A})(z-1)) \subset \mathbb{D}$ (notice that the stronger assumption in [7], which on our case reads $rank(\tilde{T}(z)) = n \ \forall z \in \mathbb{C}$, is only required for the invertibility of the stabilizing solution, as can be inferred from the proof of *Theorem 12* therein). To conclude the first part of the proof, it only remains to show that $\Lambda(A - zE - (\bar{\alpha}A - \alpha E)Y_sC^*C(\alpha - \bar{\alpha}z)) \subset \mathbb{D}$ and $Y_s \geq 0$. To this end notice that the following holds true: $\xi \in \Lambda(z\tilde{E} - \tilde{A} - \tilde{B}\tilde{B}^*\tilde{X}_s(\tilde{E} - \tilde{A})(z-1))$ if and only if $\bar{\xi}\bar{z}_0 \in \Lambda(A - zE - (\bar{\alpha}A - \alpha E)Y_sC^*C(\alpha - \bar{\alpha}z))$, and since $|z_0| = 1$ and $\Lambda(z\tilde{E} - \tilde{A} - \tilde{B}\tilde{B}^*\tilde{X}_s(\tilde{E} - \tilde{A})(z-1)) \subset \mathbb{D}$, Y_s is indeed a stabilizing solution. Finally notice that

$$AY_sA^* - EY_sE^* - (\bar{\alpha}A - \alpha E)Y_sC^*CY_s(\bar{\alpha}A - \alpha E)^* + BB^* =$$
$$(A + \alpha K_sC)Y_s(A + \alpha K_sC)^* - (E + \bar{\alpha}K_sC)Y_s(E + \bar{\alpha}K_sC)^* + K_sK_s^* + BB^*,$$

and using the fact that Y_s is a stabilizing solution of (20) one has

$$(A + \alpha K_sC)Y_s(A + \alpha K_sC)^* - (E + \bar{\alpha}K_sC)Y_s(E + \bar{\alpha}K_sC)^* + K_sK_s^* + BB^* = 0, \tag{23}$$

$det(E + \bar{\alpha}K_sC)) \neq 0$, and moreover $\Lambda((E + \bar{\alpha}K_sC)^{-1}(A + \alpha K_sC)) = \Lambda((A + \alpha K_sC) - z(E + \bar{\alpha}K_sC)) \subset \mathbb{D}$. Finally with point 2 of Theorem 1.5.5 in [4] the conclusion follows.

2. The proof follows the same lines as the one above, the only exception being now $\tilde{E} := \alpha E$, $\tilde{A} := \bar{\alpha}A$, $\tilde{X} := -X$, $\tilde{B} := B$, $\tilde{C} := \begin{bmatrix}C^* & 0\end{bmatrix}^*$, $\tilde{D} := \begin{bmatrix}0 & I\end{bmatrix}^*$ and $\tilde{Q} := C^*C = \tilde{Q}^* \geq 0$. In addition one also has

$$A^*X_sA - E^*X_sE - (\bar{\alpha}A - \alpha E)^*X_sBB^*X_s(\bar{\alpha}A - \alpha E) + C^*C =$$
$$(A + \alpha BF_s)^*X_s(A + \alpha BF_s) - (E + \bar{\alpha}BF_s)^*X_s(E + \bar{\alpha}BF_s) + F_s^*F_s + C^*C,$$

and using the same reasoning as for point 1 of the Lemma the conclusion follows. ∎

REMARK 1. *It is immediate to observe that the Riccati equation $DDTARE_c$ is simply a special case of 10 for $L = 0$, $R = I$ and $Q = C^*C = Q^* \geq 0$.*

In the reminder of this section, the notion of *feedback equivalent* quintet, as well as a very nice property related to it will be provided. The author mentions that this concept was previously used in [4] in conjunction with proper linear time-invariant systems (see Definition 3.1.2 and Proposition 3.2.4 for the continuous-time case, and Definition 3.5.1 and Proposition 3.6.4 for the discrete-time case), and quite recently in [12] for solving the suboptimal H^∞ control problem associated with a generalized linear discrete-time system given by a centered realization. The following definition and proposition are taken from [12].

DEFINITION 2.3. *Two algebraic quintets $\Sigma := (A - zE, B; Q, L, R)$ and $\tilde{\Sigma} := (\tilde{A} - z\tilde{E}, \tilde{B}; \tilde{Q}, \tilde{L}, \tilde{R})$ are said to be feedback equivalent if there is a constant matrix F*

such that

$$\begin{aligned}\tilde{A} &:= A + \alpha BF,\\ \tilde{E} &:= E + \bar{\alpha} BF,\\ \tilde{B} &:= B,\\ \tilde{Q} &:= Q + LF + F^*L^* + F^*RF,\\ \tilde{L} &:= L + F^*R,\\ \tilde{R} &:= R.\end{aligned}$$

PROPOSITION 2.4. *Let Σ and $\tilde{\Sigma}$ be as above. There holds:*

1. *The feedback equivalence relation between Σ and $\tilde{\Sigma}$ is indeed an equivalence relation.*
2. *Let $B =: \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and assume $(A - zE; B_2)$ is stabilizable. Also take $F_{s;2}$ to be any constant matrix such that $\Lambda(A - zE + B_2F_{s;2}(\alpha - \bar{\alpha}z)) \subset \mathbb{D}$ (such an $F_{s;2}$ exists whenever $(A - zE; B_2)$ is stabilizable). Define $F := \begin{bmatrix} 0 & F_{s;2}^* \end{bmatrix}^*$. Then the DDTARE*

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB + L)^*R^{-1}((\bar{\alpha}A - \alpha E)^*XB + L) + Q = 0$$

associated with the algebraic quintet Σ has a stabilizing solution $X_s = X_s^* \geq 0$ (with associated stabilizing feedback denoted by F_s) if and only if the DDTARE

$$\tilde{A}^*\tilde{X}\tilde{A} - \tilde{E}^*\tilde{X}\tilde{E} - ((\bar{\alpha}\tilde{A} - \alpha\tilde{E})^*\tilde{X}\tilde{B} + \tilde{L})^*\tilde{R}^{-1}((\bar{\alpha}\tilde{A} - \alpha\tilde{E})^*\tilde{X}\tilde{B} + \tilde{L}) + \tilde{Q} = 0$$

associated with the algebraic quintet $\tilde{\Sigma}$ has a stabilizing solution $\tilde{X}_s = \tilde{X}_s^* \geq 0$ (with the associated feedback denoted by $\tilde{F}_s$). Moreover the two stabilizing solutions satisfy $X_s = \tilde{X}_s$, whereas the two stabilizing feedbacks satisfy $\tilde{F}_s = F_s - F$.

REMARK 2. *Point 2. above assumes $F := \begin{bmatrix} 0 & F_{s;2}^* \end{bmatrix}^*$ is such that $\Lambda(A - zE + B_2F_{s;2}(\alpha - \bar{\alpha}z)) \subset \mathbb{D}$. Actually, one can immediately notice that the same result holds for any constant matrix F of appropriate dimension.*

Whenever the algebraic quintets Σ and $\tilde{\Sigma}$ are feedback equivalent for some matrix F, they will be simply called F-equivalent. This latter terminology will be in force for the rest of the paper.

2.3. Normalized coprime factorizations. This section ends with a brief review on *(normalized) coprime factorization*s.

Let $G(z)$ be a generalized (possibly improper) system. A stable left coprime factorization of $G(z)$ is a fractional representation of the form

$$G(z) = \tilde{M}^{-1}(z)\tilde{N}(z), \tag{24}$$

with $\tilde{M}(z)$ and $\tilde{N}(z)$ stable **RMF**s satisfying

$$\tilde{N}(z)\tilde{X}(z) + \tilde{M}(z)\tilde{Y}(z) \equiv I_p, \tag{25}$$

for certain stable **RMF**s $\tilde{X}(z)$ and $\tilde{Y}(z)$. The above relation is often called the *Bezout identity*. The coprime factorization is called, in addition, *normalized* provided that

$$\tilde{M}(z)\tilde{M}^{\#}(z) + \tilde{N}(z)\tilde{N}^{\#}(z) \equiv I_p. \tag{26}$$

For the standard-proper case, the definition together with analytical formulas for computing normalized coprime factorizations for both continuous-time and discrete-time systems can be found in [4] (Chapter 7.3 and Chapter 7.9, respectively).

LEMMA 2.5. *Let $G(z)$ be a generalized $p \times m$ system, given by the following realization*

$$G(z) := \left[\begin{array}{c|c} A - zE & B \\ \hline C & 0 \end{array}\right]_{z_0}. \tag{27}$$

Suppose in addition that the above realization is stabilizable and detectable. Then a normalized left coprime factorization of $G(z)$ is given by

$$\left[\ \tilde{N}(z) \ \vdots\ \tilde{M}(z)\ \right] = \left[\begin{array}{c|c:c} A_K - zE_K & B & K_s \\ \hline C & 0 & I_p \end{array}\right]_{z_0}, \tag{28}$$

where

$$A_K - zE_K := A - zE + K_s C(\alpha - \bar{\alpha} z), \tag{29}$$

for K_s given in Lemma 2.2.

Proof. As K_s is the stabilizing feedback associated with the DDTARE$_o$ 20, it follows that $\Lambda(A_K - zE_K) \subset \mathbb{D}$, which in turn implies that the **RMF** $\left[\ \tilde{N}(z) \ \vdots\ \tilde{M}(z)\ \right]$ given by 28 is stable. Setting $\Omega := \mathbb{D}$, $\beta := \bar{\alpha}$, and applying *Theorem 3* in [8] one can notice that 28 defines indeed a stable left coprime factorization of $G(z)$. It only remains to prove that the **RMF**s $\tilde{N}(z)$ and $\tilde{M}(z)$ satisfy 26.
Notice that 26 can be alternatively written as $\left[\ \tilde{N}(z) \ \vdots\ \tilde{M}(z)\ \right]\left[\ \tilde{N}(z) \ \vdots\ \tilde{M}(z)\ \right]^{\#} \equiv I_p$. To this end, compute

$$\begin{aligned}
\left[\ \tilde{N}(z) \ \vdots\ \tilde{M}(z)\ \right] \begin{bmatrix} \tilde{N}^{\#}(z) \\ \hdashline \tilde{M}^{\#}(z) \end{bmatrix} &= \left[\begin{array}{c|c:c} A_K - zE_K & B & K_s \\ \hline C & 0 & I \end{array}\right]_{z_0} \left[\begin{array}{c|c} E_K^* - A_K^* & C^* \\ \hline B^* & 0 \\ \hdashline K_s^* & I \end{array}\right]_{z_0} \\
&= \left[\begin{array}{cc|c} A_K - zE_K & (BB^* + K_s K_s^*)(\alpha - \bar{\alpha} z) & K_s \\ 0 & E_K^* - zA_K^* & C^* \\ \hline C & K_s^* & I \end{array}\right]_{z_0},
\end{aligned} \tag{30}$$

where relations 7 and 8 were used. Immediate computations using the definition of the K_s-matrix, as well as the definition of the A_K and E_K matrices lead to the following useful observation $AY_sA^* - EY_sE^* - (\bar{\alpha}A - \alpha E)Y_sC^*CY_s(\bar{\alpha}A - \alpha E)^* + BB^* = A_KY_sA_K^* - E_KY_sE_K^* + (BB^* + K_sK_s^*)$. Hence

$$A_KY_sA_K^* - E_KY_sE_K^* + (BB^* + K_sK_s^*) = 0. \tag{31}$$

Defining $Q := \begin{bmatrix} I & (\bar{\alpha}A - \alpha E)Y_s \\ 0 & I \end{bmatrix}$, $Z := \begin{bmatrix} I & Y_s(\bar{\alpha}A - \alpha E)^* \\ 0 & I \end{bmatrix}$, performing an equivalence transformation on 30 using (Q, Z), and taking into account relation 31 as well as the expression of the K_s-matrix, it follows that

$$\begin{bmatrix} N(z) & M(z) \end{bmatrix} \begin{bmatrix} N(z) & M(z) \end{bmatrix}^{\#} = \left[\begin{array}{cc|c} A_K - zE_K & 0 & 0 \\ 0 & E_K^* - zA_K^* & C^* \\ \hline C & 0 & I \end{array}\right]_{z_0} \equiv I_p. \ \square \tag{32}$$

3. Problem statement and main results. Let $G(z)$ be a generalized $p \times m$ system given by the following *stabilizable* and *detectable* realization

$$G(z) = \left[\begin{array}{c|c} A - zE & B \\ \hline C & 0 \end{array}\right]_{z_0}. \tag{33}$$

A *stabilizing controller* for $G(z)$ is any $m \times p$ system $K(z)$ for which the closed-loop feedback is *internally stable*; namely the **RMF**s

$$\begin{bmatrix} (I - K(z)G(z))^{-1} & K(z)(I - G(z)K(z))^{-1} \\ G(z)(I - K(z)G(z))^{-1} & (I - G(z)K(z))^{-1} \end{bmatrix} \tag{34}$$

are all stable (for further details see [6]). Let $\tilde{M}(z)$ and $\tilde{N}(z)$ be a normalized left coprime factorization 24 of $G(z)$ and take any $\sigma > 0$. Define the following class of uncertain systems

$$\mathcal{D}_\sigma^{cf} := \{G_\Delta(z) : G_\Delta(z) = (\tilde{M}(z) + \Delta_M(z))^{-1}(\tilde{N}(z) + \Delta_N(z))\}, \tag{35}$$

where

$$\Delta(z) := \begin{bmatrix} \Delta_N(z) & -\Delta_M(z) \end{bmatrix} \in \mathbb{RH}^\infty, \ ||\Delta||_\infty < \sigma. \tag{36}$$

The *largest* $\sigma = \sigma_{max}$ for which there exists a single stabilizing controller $K_r(z)$ for all $G_\Delta(z)$ is called the *maximum stability margin* for $G(z)$, while the corresponding controller is dubbed *the robust controller*. We are now in a position to state the main result of the paper giving explicit formulas for computing both the maximum stability margin σ_{max} and the corresponding robust controller $K_r(z)$, both of them for a generalized system $G(z)$.

THEOREM 3.1. *Let $G(z)$ be a generalized **RMF** given by the stabililzable and detectable realization 33.*

1. *The maximum stability margin is given by*

$$\sigma_{max} = \{1 + \rho[X_s(\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^*]\}^{-1/2}, \tag{37}$$

where $X_s = X_s^* \geq 0$ is the stabilizing solution associated with $DDTARE_c$ 21

$$A^*XA - E^*YE - (\bar{\alpha}A - \alpha E)^*XBB^*X(\bar{\alpha}A - \alpha E) + C^*C = 0, \tag{38}$$

and $Y_s = Y_s^* \geq 0$ is the stabilizing solution associated with $DDTARE_o$ 20

$$AYA^* - EYE^* - (\bar{\alpha}A - \alpha E)YC^*CY(\bar{\alpha}A - \alpha E)^* + BB^* = 0. \tag{39}$$

2. *The robust controller is given by*

$$K_r(z) = \left[\begin{array}{c|c} A_r - zE_r & B_r \\ \hline C_r & D_r \end{array}\right]_{z_0}, \tag{40}$$

where

$$A_r - zE_r := A - zE - [(\bar{\alpha}A - \alpha E)Y_sC^*C + BB^*X_0(\bar{\alpha}A - \alpha E)](\alpha - \bar{\alpha}z), \tag{41}$$

$$B_r := (\bar{\alpha}A - \alpha E)Y_sC^*, \ C_r := -B^*X_0(\bar{\alpha}A - \alpha E), \ D_r := 0 \tag{42}$$

and

$$X_0 := -\sigma_{max}^{-2}X_s\left[(1 - \sigma_{max}^{-2})I + (\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^*X_s\right]^{-1}. \tag{43}$$

.

Proof. 1. Using relations 28, 91, as well as 9, it follows that

$$T^{cf}(z) = \left[\begin{array}{c|c:c} A - zE & -K_s & B \\ \hline 0 & 0 & I \\ C & I & 0 \\ \hdashline C & I & 0 \end{array}\right]_{z_0}, \tag{44}$$

where $K_s := -(\bar{\alpha}A - \alpha E)Y_s C^*$ is the stabilizing feedback associated with DDTARE_o 20. It is immediate to check that assumptions $A.1 : A.4$ of Lemma 6.4 hold for $T(z) = T^{cf}(z)$.

The idea now is to use point 2 of Theorem 6.10 to compute the maximum stability margin for $G(z)$.

To this end let $\Sigma := (A - zE, \left[\begin{array}{c:c} -K_s & B \end{array}\right]; Q, L, R)$, where

$$\begin{aligned} Q &:= C^*C, \\ L &:= \left[\begin{array}{c:c} C^* & 0 \end{array}\right], \\ R &:= \left[\begin{array}{c:c} (1-\gamma^2)I & 0 \\ \hdashline 0 & I \end{array}\right] =: \left[\begin{array}{c:c} R_{11} & R_{12} \\ \hdashline R_{12}^* & R_{22} \end{array}\right]. \end{aligned} \tag{45}$$

Also let $\Sigma_2 := (A - zE, B; C^*C, 0, I)$. According to Proposition 6.9 (or equivalently Lemma 2.2), the DDTARE

$$A^*XA - E^*XE - (\bar{\alpha}A - \alpha E)^* XBB^*X(\bar{\alpha}A - \alpha E) + C^*C = 0, \tag{46}$$

has a stabilizing solution $X_s = X_s^*$ (one can recognize that the Riccati equation associated with Σ_2 coincides in this case with DDTARE_c 21). To this end, the associated stabilizing feedback is $F_s := -B^*X_s(\bar{\alpha}A - \alpha E)$, and consequently it follows that the matrix pencil $A - zE + BF_s(\alpha - \bar{\alpha}z)$ has all its eigenvalues in the open unit disk $\mathbb{D}$. Moreover, it can be noticed that $\bar{\Sigma} = (\bar{A} - z\bar{E}, \bar{B}; \bar{Q}, \bar{L}, \bar{R})$ given by 108 satisfies

$$\begin{aligned} \bar{A} &= A + \alpha BF_s, \\ \bar{E} &= E + \bar{\alpha}BF_s, \\ \bar{B} &= \left[\begin{array}{c:c} -K_s & B \end{array}\right], \\ \bar{Q} &= C^*C + F_s^*F_s = \bar{C}^*\bar{C} \quad \textit{where } \bar{C} := \begin{bmatrix} F_s^* & C^* \end{bmatrix}^*, \\ \bar{L} &= \left[\begin{array}{c:c} C^* & F_s \end{array}\right], \\ \bar{R} &= R = \left[\begin{array}{c:c} (1-\gamma^2)I & 0 \\ \hdashline 0 & I \end{array}\right], \end{aligned} \tag{47}$$

and consequently equation 110 becomes

$$\left[\begin{array}{c:c} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{array}\right] = \left[\begin{array}{c|c:c} \bar{A} - z\bar{E} & -K_s & B \\ \hline F_s & 0 & I \\ C & I & 0 \end{array}\right]_{z_0}. \tag{48}$$

Denote

$$\left[\begin{array}{c|c:c} \bar{A} - z\bar{E} & -K_s & B \\ \hline F_s & 0 & I \\ \hdashline C & I & 0 \end{array}\right]_{z_0} =: \left[\begin{array}{c:c} -X(z) & M(z) \\ \hdashline Y(z) & N(z) \end{array}\right], \tag{49}$$

and observe that the pair $(N(z);M(z))$ is a normalized right coprime factorization of $G(z)$; namely $N^{\#}(z)N(z)+M^{\#}(z)M(z)\equiv I$, and $(N(z);M(z))$ forms a stable right coprime factorization of $G(z)$. Indeed, by relation 49 and equation 48 it follows that $\begin{bmatrix} M^{\#}(z) & N^{\#}(z)\end{bmatrix}^{\#}=\bar{T}_{12}(z)$, which according to point 2. of Theorem 6.10 shows that $\begin{bmatrix} M^{\#}(z) & N^{\#}(z)\end{bmatrix}\begin{bmatrix} M^{\#}(z) & N^{\#}(z)\end{bmatrix}^{\#}=\bar{T}_{12}^{\#}(z)\bar{T}_{12}(z)\equiv I$ (the last equality is also due to the fact that $V=I$ in this case). Also, according to 49 and 47 one can see that $M(z)$ and $N(z)$ are stable **RMF**s. Finally, using Theorem 2 in [9] for $\Omega:=\mathbb{D}$ and $\beta:=\bar{\alpha}$, and taking into account that $D=G(z_0)=0$, it holds that $(N(z);M(z))$ is a right coprime factorization of $G(z)$. Putting all these together, the claim is proved.
Recall that the pair $(\tilde{N}(z);\tilde{M}(z))$ represents a normalized left coprime factorization of $G(z)$, and using the above claim together with relation 49 and Theorem 2 in [9] it follows that the **RMF** $\begin{bmatrix} M(z) & -\tilde{N}^{\#}(z)\\ N(z) & \tilde{M}^{\#}(z)\end{bmatrix}$ is all pass, which in turn implies that that the **RMF** $\hat{T}_{12}^{\perp}(z):=\begin{bmatrix} -\tilde{N}(z) & \tilde{M}(z)\end{bmatrix}^{\#}$ is an orthogonal completion of $\hat{T}_{12}(z)=\bar{T}_{12}(z)$, where $\hat{T}_{12}(z)$ was introduced in point 2. of Theorem 6.10. Further, using again Theorem 2 in [9] one has

$$\bar{T}_{11}^{\#}(z)\hat{T}_{12}^{\perp}(z)\equiv I. \tag{50}$$

Consequently compute

$$\begin{aligned}\bar{T}_{11}^{\#}(z)\hat{T}_{12}(z)&=\bar{T}_{11}^{\#}(z)\bar{T}_{12}(z)=-X^{\#}(z)M(z)+Y^{\#}(z)N(z)\\ &=\left[\begin{array}{cc|c}\bar{E}^*-z\bar{A}^* & (F_s^*F_s+C^*C)(\alpha-\bar{\alpha}z) & F_s^*\\ 0 & \bar{A}-z\bar{E} & B\\ \hline -K_s^* & C & 0\end{array}\right]_{z_0}=:\Omega(z).\end{aligned} \tag{51}$$

Using the definition of $\bar{A}$, $\bar{E}$, F_s and taking into account that X_s is the (stabilizing) solution of DDTARE 46, one obtains

$$\begin{aligned}&\bar{A}^*X_s\bar{A}-\bar{E}^*X_s\bar{E}+F_s^*F_s+C^*C\\ &=A^*X_sA-E^*X_sE-(\bar{\alpha}A-\alpha E)^*X_sBB^*X_s(\bar{\alpha}A-\alpha E)+C^*C=0.\end{aligned} \tag{52}$$

Taking $Q:=\begin{bmatrix} I & (\bar{\alpha}A-\alpha E)^*X_s\\ 0 & I\end{bmatrix}$, $Z:=\begin{bmatrix} I & X_s(\bar{\alpha}A-\alpha E)\\ 0 & I\end{bmatrix}$, noticing that $\bar{\alpha}\bar{A}-\alpha\bar{E}=\bar{\alpha}A-\alpha E$, and performing a state-space transformation on $\Omega(z)$ using $(Q;Z)$ as well as 52 gives

$$\begin{aligned}\Omega(z)&=\left[\begin{array}{cc|c}\bar{E}^*-z\bar{A}^* & (F_s^*F_s+C^*C)(\alpha-\bar{\alpha}z) & F_s^*\\ 0 & \bar{A}-z\bar{E} & B\\ \hline -K_s^* & C & 0\end{array}\right]_{z_0}\\ &=\left[\begin{array}{cc|c}\bar{E}^*-z\bar{A}^* & 0 & 0\\ 0 & \bar{A}-z\bar{E} & B\\ \hline -K_s^* & C-K_s^*X_s(\bar{\alpha}A-\alpha E) & 0\end{array}\right]_{z_0}=\left[\begin{array}{c|c}\bar{A}-z\bar{E} & B\\ \hline C_{\nabla} & 0\end{array}\right]_{z_0},\end{aligned} \tag{53}$$

where $C_{\nabla}:=C-K_s^*X_s(\bar{\alpha}A-\alpha E)=C(I+Y_s(\bar{\alpha}A-\alpha E)^*X_s(\bar{\alpha}A-\alpha E))$.
Using now relations 50, 51, and 113, as well as Theorem 6.2, it follows that

$$\sigma_{max}=\frac{1}{\gamma_{min}}=(1+\rho(\mathbb{H}_{\Omega}\mathbb{H}_{\Omega}^*))^{-\frac{1}{2}}. \tag{54}$$

All it remains now is to express the above Hankel operator associated with $\Omega(z)$ in terms of the original data (for the definition of the (causal) Hankel operator see Definition 6.6). To this end, let ω be the input-output operator associated with $\Omega(z)$

$$\omega : \begin{cases} \bar{E}x_\omega[k+1] = \bar{A}x_\omega[k] + B(\alpha u[k] - \bar{\alpha}u[k+1]), \\ y_\omega[k] = C_\nabla x_\omega[k], \end{cases} \tag{55}$$

where $x_\omega[k]$ is the state vector, $y_\omega[k]$ is the output vector and $u[k] \in l^{2;m}$ is the input vector. As $\Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$, it follows that $\bar{E}$ is invertible and moreover $\Lambda(\bar{A}\bar{E}^{-1}) = \Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$. Making the change of variable $w_\omega[k] = \bar{E}x_\omega[k]$, the equations 55 becomes now

$$\omega : \begin{cases} w_\omega[k+1] = \bar{A}\bar{E}^{-1}w_\omega[k] + B(\alpha u[k] - \bar{\alpha}u[k+1]), \\ y_\omega[k] = C_\nabla \bar{E}^{-1}w_\omega[k]. \end{cases} \tag{56}$$

According to the above equations and Theorem 2.6.1 in [4]

$$y_\omega[k] = (\omega u)[k] = C_\nabla \bar{E}^{-1} \sum_{j=-\infty}^{k-1} (\bar{A}\bar{E}^{-1})^{k-j-1} B(\alpha u[j] - \bar{\alpha}u[j+1]),\ \forall k \in \mathbb{Z}, \tag{57}$$

and using the definition of the Hankel operator, it follows that

$$\begin{aligned} \left(\left(P_+\omega|_{l_-^{2;m}}\right)u\right)[k] &= C_\nabla \bar{E}^{-1} \sum_{j=-\infty}^{-1} \left(\bar{A}\bar{E}^{-1}\right)^{k-j-1} B(\alpha u[j] - \bar{\alpha}u[j+1]) \\ &= C_\nabla \bar{E}^{-1}(\bar{A}\bar{E}^{-1})^k \sum_{j=-\infty}^{-1} \left(\bar{A}\bar{E}^{-1}\right)^{-j-1} B(\alpha u[j] - \bar{\alpha}u[j+1]),\ \forall k \in \mathbb{N}. \end{aligned} \tag{58}$$

Define

$$\Psi_\Omega : l_-^{2;m} \to \mathbb{C}^n;\ (\Psi_\Omega u_-)[k] := \sum_{j=-\infty}^{-1} \left(\bar{A}\bar{E}^{-1}\right)^{-j-1} B(\alpha u_-[j] - \bar{\alpha}u_-[j+1]), \tag{59}$$

$$\Phi_\Omega : \mathbb{C}^n \to l_+^{2;p};\ (\Phi_\Omega \xi)[k] := \begin{cases} C_\nabla \bar{E}^{-1}(\bar{A}\bar{E}^{-1})^k \xi;\ k \in \mathbb{N}, \\ 0;\ k \in \mathbb{Z} \setminus \mathbb{N}, \end{cases} \tag{60}$$

and notice that $\mathbb{H}_\Omega = \Phi_\Omega \Psi_\Omega$. It is easy to check that their adjoints are given by

$$\Psi_\Omega^* : \mathbb{C}^n \to l_-^{2;m};\ (\Psi_\Omega^* \xi)[k] = \begin{cases} -B^*(\bar{E}^{-*}\bar{A}^*)^{-k-1}\bar{E}^{-*}(\bar{\alpha}A - \alpha E)^*\xi;\ k \in \mathbb{Z} \setminus \mathbb{N}, \\ 0;\ k \in \mathbb{N}, \end{cases} \tag{61}$$

$$\Phi_\Omega^* : l_+^{2;p} \to \mathbb{C}^n;\ (\Phi_\Omega^* y_+)[k] = \sum_{j=0}^{\infty} \left(\bar{E}^{-*}\bar{A}^*\right)^j \bar{E}^{-*} C_\nabla^* y_+[j]. \tag{62}$$

Consequently, this implies that

$$\rho(\mathbb{H}_\Omega\mathbb{H}_\Omega^*) = \rho(\Phi_\omega\Psi_\Omega\Psi_\Omega^*\Phi_\Omega^*) = \rho(\Phi_\Omega^*\Phi_\Omega\Psi_\Omega\Psi_\Omega^*). \tag{63}$$

Compute

$$\Psi_\Omega\Psi_\Omega^* = \sum_{i=-\infty}^{-1}\left(\bar{A}\bar{E}^{-1}\right)^{-i-1}B(\alpha\cdot(-B^*\left(\bar{E}^{-*}\bar{A}^*\right)^{-i-1}\bar{E}^{-*}(\bar{\alpha}A-\alpha E)^*)$$
$$-\bar{\alpha}\cdot(-B^*\left(\bar{E}^{-*}\bar{A}^*\right)^{-i-2}\bar{E}^{-*}(\bar{\alpha}A-\alpha E)^*))$$
$$=\alpha\cdot\sum_{i=-\infty}^{-1}\left(\bar{A}\bar{E}^{-1}\right)^{-i-1}BB^*\left(\bar{E}^{-*}\bar{A}^*\right)^{-i-1}\bar{E}^{-*}(\alpha E-\bar{\alpha}A)^*$$
$$-\bar{\alpha}\cdot\sum_{i=-\infty}^{-2}\left(\bar{A}\bar{E}^{-1}\right)^{-i-1}BB^*\left(\bar{E}^{-*}\bar{A}^*\right)^{-i-2}\bar{E}^{-*}(\alpha E-\bar{\alpha}A)^*$$
$$=\alpha\cdot\bar{E}\sum_{i=-\infty}^{-1}\left(\bar{E}^{-1}\bar{A}\right)^{-i-1}\left(\bar{E}^{-1}B\right)\left(B^*\bar{E}^{-*}\right)\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-1}(\alpha E-\bar{\alpha}A)^*$$
$$-\bar{\alpha}\cdot\bar{A}\sum_{i=-\infty}^{-2}\left(\bar{E}^{-1}\bar{A}\right)^{-i-2}\left(\bar{E}^{-1}B\right)\left(B^*\bar{E}^{-*}\right)\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-2}(\alpha E-\bar{\alpha}A)^*$$
$$=\alpha\cdot\bar{E}\sum_{i=-\infty}^{-1}\left(\bar{E}^{-1}\bar{A}\right)^{-i-1}\left(\bar{E}^{-1}B\right)\left(B^*\bar{E}^{-*}\right)\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-1}(\alpha E-\bar{\alpha}A)^*$$
$$-\bar{\alpha}\cdot\bar{A}\sum_{i=-\infty}^{-1}\left(\bar{E}^{-1}\bar{A}\right)^{-i-1}\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-1}(\alpha E-\bar{\alpha}A)^*$$
$$=(\alpha\bar{E}-\bar{\alpha}\bar{A})\sum_{i=-\infty}^{-1}\left(\bar{E}^{-1}\bar{A}\right)^{-i-1}\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-1}(\alpha E-\bar{\alpha}A)^*,$$

and taking into account that (as explained previously) $\bar{\alpha}\bar{A}-\alpha\bar{E}=\bar{\alpha}A-\alpha E$:

$$\Psi_\Omega\Psi_\Omega^* = (\bar{\alpha}A-\alpha E)\sum_{i=-\infty}^{-1}\left(\bar{E}^{-1}\bar{A}\right)^{-i-1}\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*\left(\bar{A}^*\bar{E}^{-*}\right)^{-i-1}(\bar{\alpha}A-\alpha E)^* \tag{64}$$

$$=(\bar{\alpha}A-\alpha E)\sum_{k=0}^{\infty}\left(\bar{E}^{-1}\bar{A}\right)^{k}\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*\left(\bar{A}^*\bar{E}^{-*}\right)^{k}(\bar{\alpha}A-\alpha E)^* \tag{65}$$
$$=:(\bar{\alpha}A-\alpha E)P(\bar{\alpha}A-\alpha E)^*,$$

where $P := \sum_{k=0}^{\infty}\left(\bar{E}^{-1}\bar{A}\right)^{k}\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*\left(\bar{A}^*\bar{E}^{-*}\right)^{k}$. Since $\Lambda(\bar{E}^{-1}\bar{A}) = \Lambda(\bar{A}-z\bar{E})\subset\mathbb{D}$, it follows that the hermitic Stein equation

$$\left(\bar{E}^{-1}\bar{A}\right)X\left(\bar{E}^{-1}\bar{A}\right)^*-X+\left(\bar{E}^{-1}B\right)\left(\bar{E}^{-1}B\right)^*=0 \tag{66}$$

has a unique (hermitic) solution, and using relation 2.125 in [4] it holds that P is exactly the solution of 66. In conclusion P is the unique (hermitic) solution of

$$\bar{A}X\bar{A}^* - \bar{E}X\bar{E}^* + BB^* = 0. \tag{67}$$

In a similar way, compute

$$\Phi_\Omega^*\Phi_\Omega = \sum_{k=0}^{\infty} \left(\bar{E}^{-*}\bar{A}^*\right)^k \bar{E}^{-*}C_\nabla^* C_\nabla \bar{E}^{-1} \left(\bar{A}\bar{E}^{-1}\right)^k =: Q. \tag{68}$$

Again, since $\Lambda(\bar{A}\bar{E}^{-1}) = \Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$, it follows that the hermitic Stein equation

$$\left(\bar{A}\bar{E}^{-1}\right)^* Y \left(\bar{A}\bar{E}^{-1}\right) - Y + \bar{E}^{-*}C_\nabla^* C_\nabla \bar{E}^{-1} = 0 \tag{69}$$

has a unique (hermitic) solution, and using the same relation 2.125 in [4] it holds that Q is exactly the solution of 69. In conclusion Q is the unique (hermitic) solution of

$$\bar{A}^*Y\bar{A} - \bar{E}^*Y\bar{E} + C_\nabla^* C_\nabla = 0. \tag{70}$$

Combining relations 63, 64, 67 and 70, one can notice that

$$\rho(\mathbb{H}_\Omega\mathbb{H}_\Omega^*) = \rho((\bar{\alpha}A - \alpha E)P(\bar{\alpha}A - \alpha E)^* Q). \tag{71}$$

Comparing DDTARE$_c$ 21 and DDTARE$_o$ 20 with 67 and 70, and taking into account the definition of the $\bar{A}$, $\bar{E}$, F_s and C_∇ matrices, it follows after some algebra that

$$\begin{aligned} P &= Y_s(I + (\bar{\alpha}A - \alpha E)^* X_s(\bar{\alpha}A - \alpha E)Y_s)^{-1}, \\ Q &= (I + X_s(\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^*)X_s. \end{aligned} \tag{72}$$

This in turn leads to

$$(\bar{\alpha}A - \alpha E)P(\bar{\alpha}A - \alpha E)^* Q = (\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^* X_s, \tag{73}$$

from where one obtains that

$$\begin{aligned} \rho(\mathbb{H}_\Omega\mathbb{H}_\Omega^*) &= \rho((\bar{\alpha}A - \alpha E)P(\bar{\alpha}A - \alpha E)^* Q) \\ &= \rho((\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^* X_s) = \rho(X_s(\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^*), \end{aligned} \tag{74}$$

which combined with equation 54 completes this part of the proof.

2. Define $\epsilon := \gamma - \gamma_{min} \geq 0$, where (recall) $\gamma_{min} := \frac{1}{\sigma_{max}}$. Using Theorem 6.2 it is easy to observe that

$$\gamma_{min} = \inf_{K \ internally \ stabilizing} ||LLFT(T^{cf}; K)||_\infty. \tag{75}$$

Consequently, it follows that for any $\gamma > \gamma_{min}$ the (2-block) γ-suboptimal H^∞ problem formulated for $T^{cf}(z)$ has a solution. Assume now that $\gamma > \gamma_{min}$ and let $\Sigma_\gamma := \left(A - zE, \begin{bmatrix} -K_s & B \end{bmatrix}; C^*C, \begin{bmatrix} C^* & 0 \end{bmatrix}, \begin{bmatrix} (1-\gamma^2)I & 0 \\ 0 & I \end{bmatrix}\right)$.

According to Lemma 6.4 the DDTARE associated with Σ_γ

(76)

$$\begin{aligned}0 &= C^*C + A^*XA - E^*XE \\ &\quad - \left[\begin{array}{c} C - K_s^* X(\bar{\alpha}A - \alpha E) \\ \hdashline B^*X(\bar{\alpha}A - \alpha E)\end{array}\right]^* \left[\begin{array}{c:c} (1-\gamma^2)^{-1}I & 0 \\ \hdashline 0 & I\end{array}\right] \left[\begin{array}{c} C - K_s^* X(\bar{\alpha}A - \alpha E) \\ \hdashline B^*X(\bar{\alpha}A - \alpha E)\end{array}\right].\end{aligned}$$

has a stabilizing solution $X_\gamma = X_\gamma^* \geq 0$. Setting $Q(z) \equiv 0$ in Proposition 21 in [12] and also using the scaling procedure as described in Section 6.4 therein, one can infer that for any $\epsilon > 0$ there exists a controller given by

$$K_\epsilon(z) := \left[\begin{array}{c|c} A_\epsilon - zE_\epsilon & B_\epsilon \\ \hline C_\epsilon & D_\epsilon \end{array}\right], \tag{77}$$

where

$$\begin{aligned} A_\epsilon &:= A + \alpha(K_sC - BB^*X_\gamma(\bar{\alpha}A - \alpha E)), \\ E_\epsilon &:= E + \bar{\alpha}(K_sC - BB^*X_\gamma(\bar{\alpha}A - \alpha E)), \\ B_\epsilon &:= -K_s, \\ C_\epsilon &:= -B^*X_\gamma(\bar{\alpha}A - \alpha E), \\ D_\epsilon &:= 0, \end{aligned} \tag{78}$$

which solves the (2-block) γ-suboptimal H^∞ control problem formulated for $T^{cf}(z)$. In other words, whenever $\gamma > \gamma_{min}$ we are ensured that the system $T_\epsilon^{cl}(z) := LLFT(T^{cf}; K_\epsilon)(z)$ is internally stable and moreover $||T_\epsilon^{cl}||_\infty < \gamma$. In order to prove that 40 is indeed the robust controller that is required to be found, the behavior of $T_\epsilon^{cl}(z)$ as $\epsilon \to 0$ will be examined.
To do so, first notice that

(79)

$$\begin{aligned} T_\epsilon^{cl}(z) &= \left[\begin{array}{cc|c} A - zE & BC_\epsilon(\alpha - \bar{\alpha}z) & -K_s \\ B_\epsilon C(\alpha - \bar{\alpha}z) & A_\epsilon - zE_\epsilon & B_\epsilon \\ \hline 0 & C_\epsilon & 0 \\ C & 0 & I \end{array}\right]_{z_0} \\ &= \left[\begin{array}{cc|c} A - zE & -BB^*X_\gamma(\bar{\alpha}A - \alpha E)(\alpha - \bar{\alpha}z) & -K_s \\ -K_sC(\alpha - \bar{\alpha}z) & A - zE + (K_sC - BB^*X_\gamma(\bar{\alpha}A - \alpha E))(\alpha - \bar{\alpha}z) & -K_s \\ \hline 0 & -B^*X_\gamma(\bar{\alpha}A - \alpha E) & 0 \\ C & 0 & I \end{array}\right]_{z_0}, \end{aligned}$$

and performing a state-space transformation with $Q := \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ and $Z := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ it follows that

(80)

$$\begin{aligned} \left[\begin{array}{c|c} A_{R_\epsilon} - zE_{R_\epsilon} & B_{R_\epsilon} \\ \hline C_{R_\epsilon} & D_{R_\epsilon} \end{array}\right]_{z_0} &:= T_\epsilon^{cl}(z) \\ &= \left[\begin{array}{c|c} A - \alpha BB^*X_\gamma(\bar{\alpha}A - \alpha E) - z(E - \bar{\alpha}BB^*X_\gamma(\bar{\alpha}A - \alpha E)) & -K_s \\ \hline -B^*X_\gamma(\bar{\alpha}A - \alpha E) & 0 \\ C & I \end{array}\right]_{z_0}, \end{aligned}$$

where $\Lambda(A - zE - BB^* X_\gamma(\bar{\alpha}A - \alpha E)(\alpha - \bar{\alpha}z)) \subset \mathbb{D}$ for any $\epsilon > 0$ because $K_\epsilon(z)$ internally stabilizes $T_\epsilon^{cl}(z)$.

Rearranging DDTARE 76 one can obtain

$$0 = A^* X_\gamma A - E^* X_\gamma E - (\bar{\alpha}A - \alpha E)^* X_\gamma BB^* X_\gamma(\bar{\alpha}A - \alpha E) + C_\gamma^* C_\gamma, \tag{81}$$

where $C_\gamma := \left[\begin{array}{c:c} (C^* - (\bar{\alpha}A - \alpha E)^* X_\gamma K_s)(\gamma^2 - 1)^{-1/2} & C^* \end{array}\right]^*$, or yet

$$A_{R_\epsilon}^* X_\gamma A_{R_\epsilon} - E_{R_\epsilon}^* X_\gamma E_{R_\epsilon} + \hat{C}_\gamma^* \hat{C}_\gamma = 0, \tag{82}$$

where:

$\hat{C}_\gamma := \left[\begin{array}{c:c:c} (\bar{\alpha}A - \alpha E)^* X_\gamma B & C^* & (C^* - (\bar{\alpha}A - \alpha E)^* X_\gamma K_s)(\gamma^2 - 1)^{-1/2} \end{array}\right]^*$.

Setting $\epsilon \to 0$ it follows that $A_{R_\epsilon} \to A_{R_0}$, $E_{R_\epsilon} \to E_{R_0}$, $\gamma \to \gamma_{min}$, $\hat{C}_\gamma \to \hat{C}_0$, where

$$\begin{aligned}
A_{R_0} &:= A - \alpha BB^* X_0(\bar{\alpha}A - \alpha E), \\
E_{R_0} &:= E - \bar{\alpha}BB^* X_0(\bar{\alpha}A - \alpha E), \\
\lim_{\epsilon \to 0} X_\gamma &= X_0, \\
\hat{C}_0 &:= \left[\begin{array}{c:c:c} (\bar{\alpha}A - \alpha E)^* X_0 B & C^* & (C^* - (\bar{\alpha}A - \alpha E)^* X_0 K_s)(\gamma_{min}^2 - 1)^{-1/2} \end{array}\right]^*,
\end{aligned} \tag{83}$$

and X_0 is given by 43. For the third equation above Lemma 6.11 has been used, which also reveals that $X_0 = X_0^* \geq 0$.

In order to prove that $\Lambda(A_{R_0} - zE_{R_0}) \subset \mathbb{D}$, first observe that since $\Lambda(A_{R_\epsilon} - zE_{R_\epsilon}) \subset \mathbb{D}$ an inertia argument shows that $\Lambda(A_{R_0} - zE_{R_0}) \subset \bar{\mathbb{D}}$, which in turn implies that $det(E_{R_0}) \neq 0$. Next, notice that the following equalities hold

$$\begin{aligned}
& rank\left(\left[\begin{array}{c} zI - A_{R_0}E_{R_0}^{-1} \\ \hdashline \hat{C}_0 E_{R_0}^{-1} \end{array}\right]\right) = rank\left(\left[\begin{array}{c} A_{R_0} - zE_{R_0} \\ \hdashline \hat{C}_0 \end{array}\right]\right) \\
&= rank\left(\left[\begin{array}{c} A - zE - BB^* X_0(\bar{\alpha}A - \alpha E)(\alpha - \bar{\alpha}z) \\ \hdashline -B^* X_0(\bar{\alpha}A - \alpha E) \\ C \\ (C - K_s^* X_0(\bar{\alpha}A - \alpha E))(\gamma^2 - 1)^{-1/2} \end{array}\right]\right) \\
&= rank\left(\left[\begin{array}{c:ccc} I & B(\alpha - \bar{\alpha}z) & 0 & 0 \\ \hdashline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right]\left[\begin{array}{c} A - zE \\ \hdashline -B^* X_0(\bar{\alpha}A - \alpha E) \\ C \\ (C - K_s^* X_0(\bar{\alpha}A - \alpha E))(\gamma^2 - 1)^{-1/2} \end{array}\right]\right) \\
&= rank\left(\left[\begin{array}{c} A - zE \\ \hdashline C \\ -B^* X_0(\bar{\alpha}A - \alpha E) \\ (C - K_s^* X_0(\bar{\alpha}A - \alpha E))(\gamma^2 - 1)^{-1/2} \end{array}\right]\right) = n \quad \forall z \in C_{o;1},
\end{aligned} \tag{84}$$

since $(C; A - zE)$ is detectable by hypothesis. Thus the pair $(\hat{C}_0 E_{R_0}^{-1}; A_{R_0}E_{R_0}^{-1})$ is detectable and since $X_0 = X_0^* \geq 0$ satisfies the hermitic Stein equation (as can be inferred from 82)

$$(A_{R_0}E_{R_0}^{-1})^* X_0(A_{R_0}E_{R_0}^{-1}) - X_0 + (\hat{C}_0 E_{R_0}^{-1})^*(\hat{C}_0 E_{R_0}^{-1}) = 0, \tag{85}$$

it follows from point 2 of Theorem 1.5.5 in [4] applied to the pair $(E_{R_0}^{-*}A_{R_0}^{*}; E_{R_0}^{-*}\hat{C}_0^{*})$ that $\Lambda(A_{R_0}E_{R_0}^{-1}) \subset \mathbb{D}$ which is equivalent to $\Lambda(A_{R_0} - zE_{R_0}) \subset \mathbb{D}$. The last part of the proof consists in showing that $||T_0^{cl}||_\infty \leq \gamma_{min}$, where

$$T_0^{cl}(z) := \left[\begin{array}{c|c} A_{R_0} - zE_{R_0} & B_{R_0} \\ \hline C_{R_0} & D_{R_0} \end{array}\right]_{z_0}, \tag{86}$$

which together with 75 and Theorem 6.2 will end the proof. Consequently, the relation $||T_0^{cl}||_\infty \leq \gamma_{min}$ follows immediately once the following equality will be proved $\lim_{\epsilon\to 0} ||T_\epsilon^{cl} - T_0^{cl}||_\infty = 0$.

Define

(87)

$$\begin{aligned}\hat{T}_\epsilon^{cl}(z) &:= T_\epsilon^{cl}(z) - T_0^{cl}(z) \\ &= \left[\begin{array}{cc|c} A_{R_\epsilon} - zE_{R_\epsilon} & 0 & B_{R_\epsilon} \\ 0 & A_{R_0} - zE_{R_0} & -B_{R_0} \\ \hline C_{R_\epsilon} & C_{R_0} & D_{R_\epsilon} - D_{R_0} \end{array}\right]_{z_0} =: \left[\begin{array}{c|c} \hat{A}_\epsilon - z\hat{E}_\epsilon & \hat{B}_\epsilon \\ \hline \hat{C}_\epsilon & \hat{D}_\epsilon \end{array}\right]_{z_0},\end{aligned}$$

and notice that $\Lambda(\hat{A}_\epsilon - z\hat{E}_\epsilon) \subset \mathbb{D}, \quad \forall\epsilon \geq 0$. Also denoting $\hat{A}_0 - z\hat{E}_0 := \lim_{\epsilon\to 0} \hat{A}_\epsilon - z\hat{E}_\epsilon$, $\hat{B}_0 := \lim_{\epsilon\to 0} \hat{B}_\epsilon$, $\hat{C}_0 := \lim_{\epsilon\to 0} \hat{C}_\epsilon$, and $\hat{D}_0 := \lim_{\epsilon\to 0} \hat{D}_\epsilon$, it follows that $\hat{T}_0^{cl}(z) := \left[\begin{array}{c|c} \hat{A}_0 - z\hat{E}_0 & \hat{B}_0 \\ \hline \hat{C}_0 & \hat{D}_0 \end{array}\right]_{z_0} \equiv 0$. Applying now Theorem 2.5.1 in [4] to the **RMF**s $\hat{T}_\epsilon^{cl}(z)$ and $\hat{T}_0^{cl}(0)$ it holds that

$$0 = \lim_{\epsilon\to 0} ||\hat{T}_\epsilon^{cl}||_\infty = \lim_{\epsilon\to 0} ||T_\epsilon^{cl} - T_0^{cl}||_\infty, \tag{88}$$

which ends the whole proof. □

4. Numerical result. Consider the following 2×3 improper system

$$G(z) = \begin{bmatrix} z^2+1 & 0 & \frac{1}{1.1z+1} \\ \frac{1}{z+2} & 3 & \frac{2}{z+0.5} \end{bmatrix}$$

which has one pole at ∞ with multiplicity 2, one simple pole in -2, one simple pole in -0.5 and also one simple pole in -0.909. Since $1 \in C_{o;1}$ is not a pole of $G(z)$ one can pick up $z_0 = 1$ and consequently obtain

$$G(z) = \left[\begin{array}{ccccc|ccc} -2-z & 0 & 0 & 0 & 0 & -0.3333 & 0 & 0 \\ 0 & -0.9091-z & 0 & 0 & 0 & 0 & 0 & -0.5238 \\ 0 & 0 & -0.5-z & 0 & 0 & 0 & 0 & -1.3333 \\ 0 & 0 & 0 & -1 & z & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 \\ \hline 0 & -0.9091 & 0 & 0.5 & 0 & 2 & 0 & 0.4762 \\ -1 & 0 & -1 & 0 & 0 & 0.3333 & 3 & 1.3333 \end{array}\right]_1,$$

which is indeed stabilizable and detectable. The stabilizing solutions of DDTARE_c and DDTARE_o are

$$X_s = \begin{bmatrix} 111.4971 & -0.2533 & 0.843 & 14.8923 & -30.4629 \\ -0.2533 & 2.6120 & -0.7901 & 0.3593 & -0.346 \\ 0.843 & -0.7901 & 0.6635 & -0.0758 & -0.085 \\ 14.8923 & 0.3593 & -0.0758 & 2.4264 & -4.0916 \\ -30.4629 & -0.346 & -0.085 & -4.0916 & 8.7434 \end{bmatrix}$$

$$Y_s = \begin{bmatrix} 3.2676 & -1.3691 & -2.2527 & -1.9543 & 2.5029 \\ -1.3691 & 0.705 & 1.2102 & 1.0251 & -1.2979 \\ -2.2527 & 1.2102 & 2.2935 & 1.8908 & -1.6216 \\ -1.9543 & 1.0251 & 1.8908 & 14.6417 & 5.0904 \\ 2.5029 & -1.2979 & -1.6216 & 5.0904 & 10.653 \end{bmatrix}.$$

Consequently, the maximum stability margin of $G(z)$ is $\sigma_{max} = 0.0197$ and the robust controller is

$$K_r(z) = \left[\begin{array}{c|c} A_r - zE_r & B_r \\ \hline C_r & D_r \end{array}\right]_1,$$

where

$$A_r = \begin{bmatrix} 0.8718 & -0.0046 & 0.0036 & 0.0382 & -0.1172 \\ 0.1736 & -0.0009 & 0.0007 & 0.0076 & -0.0233 \\ 0.4418 & -0.0024 & 0.0018 & 0.0194 & -0.0594 \\ 5.2306 & -0.0279 & 0.0216 & 0.2293 & -0.7034 \\ 5.2306 & -0.0279 & 0.0216 & 0.2293 & -0.7034 \end{bmatrix}$$

$$E_r = \begin{bmatrix} 0.8718 & -0.0046 & 0.0036 & 0.0382 & -0.1172 \\ 0.1736 & -0.0009 & 0.0007 & 0.0076 & -0.0233 \\ 0.4418 & -0.0024 & 0.0018 & 0.0194 & -0.0594 \\ 5.2306 & -0.0279 & 0.0216 & 0.2293 & -0.7034 \\ 5.2306 & -0.0279 & 0.0216 & 0.2293 & -0.7034 \end{bmatrix}$$

$$B_r = \begin{bmatrix} -0.8024 & 3.0447 \\ 0.245 & -0.3033 \\ 0.2321 & 0.0612 \\ -2.6638 & -0.9448 \\ -3.7251 & 0.8813 \end{bmatrix} D_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_r = \begin{bmatrix} -2.6153 & 0.0139 & -0.0108 & -0.1146 & 0.3517 \\ 0 & 0 & 0 & 0 & 0 \\ -0.3314 & 0.0018 & -0.0014 & -0.0145 & 0.0446 \end{bmatrix},$$

and the closed loop generalized eigenvalues have the absolute value $(0.7507, 0.6971, 0.1326, 0.9265, 0.771, 0.9484, 0.2141, 0.9207, 0.9917, 0.902)$.

Next consider the following stable perturbation

$$\Delta(z) := \begin{bmatrix} \frac{0.0003716}{z-0.5} & 0.0003716 & \frac{0.0003716}{z+0.2} & -\frac{0.0003716}{0.3z+0.2} & -\frac{0.0003716}{z} \\ \frac{0.001115}{z-0.1} & \frac{0.0003716}{z} & 0.0007433 & \frac{0.001858}{z+0.9} & \frac{0.0003716}{z-0.1}, \end{bmatrix} \tag{89}$$

which has $||\Delta||_\infty = 0.019$. Finally the generalized eigenvalues of the closed loop connection of $G_\Delta(z)$ and $K_r(z)$ (where $G_\Delta(z)$ was defined in 35, for $\Delta(z)$ given by 89) have absolute value $(0.7513, 0.9998, 0.99, 0.1902, 0.9978, 0.9687, 0.0338, 0.5376, 0.9949, 0.4707, 0.6168, 0.7609, 0.8504, 0.579, 0.7948, 0.9558, 0.7176, 0.8276, 0.9336, 0.9483, 0.8334, 0.5263, 0)$.

5. Conclusions. Analytical formulas have been provided for computing *the maximum stability margin* as well as *the robust controller*, both of them formulated for a general rational matrix function (possibly improper or polynomial). Also, reliable computational formulas are given in terms of the original data, thus regaining the same elegance and simplicity of the standard proper case.

6. Appendix.

6.1. Switching form robust stabilization to disturbance feedforward. In what follows, an important result that links the problem at hand to the new solved (*2-block*) H^∞ one associated with a generalized system 33 is presented. It is useful to notice that each class of uncertainties 35 can be represented as an upper linear fractional transformation (ULFT) of a nominal system T^{cf} and a perturbation Δ:

$$ULFT(T^{cf},\Delta)(z) := T^{cf}_{22}(z) + T^{cf}_{21}(z)\Delta(z)(I - T^{cf}_{11}(z)\Delta(z))^{-1}T^{cf}_{12}(z) \tag{90}$$

where

$$T^{cf}(z) := \left[\begin{array}{c|c} T^{cf}_{11}(z) & T^{cf}_{12}(z) \\ \hline T^{cf}_{21}(z) & T^{cf}_{22}(z) \end{array}\right] := \left[\begin{array}{c|c} 0 & I_m \\ \tilde{M}^{-1}(z) & G(z) \\ \hline \tilde{M}^{-1}(z) & G(z) \end{array}\right] \tag{91}$$

To this end the feedback connection of G_Δ and K is equivalent to the feedback connection between $ULFT(T^{cf};\Delta)$ and K, where Δ is stable by hypothesis. The next lemma is a classical result (see Theorem 7.10.1 in [4]).

LEMMA 6.1 (Small Gain Theorem). *Let $G_1(z)$ and $G_2(z)$ be two stable **RMFs** of dimensions $p\times m$ amd $m\times p$ respectively. Also assume $S := I_m - D_2D_1$ is nonsingular. If $||G_1||_\infty < \frac{1}{\gamma}$ and $||G_2||_\infty \le \gamma$ (for some $\gamma > 0$), then the closed loop connection of $G_1(z)$ and $G_2(z)$ is internally stable.*

The next Theorem can also be found in Chapter 11 in [4]. For convenience the proof will also be provided.

THEOREM 6.2. *A controller $K(z)$ is a solution to the robust stabilization problem with respect to the class of systems $\mathcal{D}^{cf}_\sigma$ defined in 35, $\sigma \le \sigma_{max}$, if and only if $K(z)$ is a solution to the corresponding $\frac{1}{\sigma}$-nonstrict H^∞ control problem formulated for $T^{cf}(z)$ given by 91.*

Proof.

1. (Sufficiency) Since $K(z)$ is a solution to the $\frac{1}{\sigma}$-nonstrict H^∞ problem formulated for $T^{cf}(z)$, it follows that $T^{cl}(z) := LLFT(T^{cf},K)(z)$ is internally stable and, moreover, $||T^{cl}||_\infty \le \frac{1}{\sigma}$, where

$$LLFT(T^{cf},K)(z) := T^{cf}_{11}(z) + T^{cf}_{12}(z)K(z)(I - T^{cf}_{22}(z)K(z))^{-1}T^{cf}_{21}(z). \tag{92}$$

As $\Delta(z)$ was assumed to be a stable **RMF** with $||\Delta||_\infty < \sigma$, it follows from Lemma 6.1 that $K(z)$ is also a solution for the robust stabilization problem formulated for G (see Fig. 1).

2. (Necessity) Suppose now that the closed loop connection of $G_\Delta(z)$ and $\Delta(z)$ is internally stable for all stable **RMFs** $\Delta(z)$ with $||\Delta||_\infty < \sigma$. Then, in particular, it is internally stable for $\Delta(z) \equiv 0$, and hence $LLFT(T^{cf},K)(z) = T^{cl}(z)$ is internally stable. To prove that $||T^{cl}||_\infty \le \frac{1}{\sigma}$, suppose by contradiction that this is not the case; namely assume that $||T^{cl}||_\infty > \frac{1}{\sigma}$. Then, since $T^{cl}(z)$ was proved to be a stable **RMF**, one can invoke the Spectral Mapping Theorem in [2] to conclude that there exists a stable **RMF** $\Delta(z)$ with

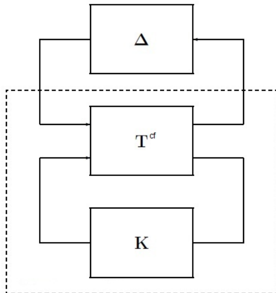

FIG. 1.

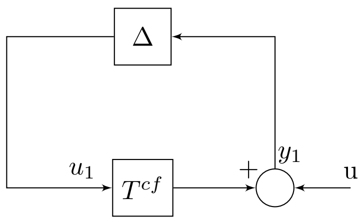

FIG. 2.

□

$||\Delta||_\infty < \sigma$ such that $I - T^{cl}(z)\Delta(z)$ is not a unit in $\mathbb{RH}^\infty$ (i.e. it is not true that both **RMF**s $I - T^{cl}(z)\Delta(z)$ and $(I - T^{cl}(z)\Delta(z))^{-1}$ are stable). Since $I - T^{cl}(z)\Delta(z)$ is clearly stable whenever $K(z)$ internally stabilizes $G_\Delta(z)$, it must be the case that $(I - T^{cl}(z)\Delta(z))^{-1} \notin \mathbb{RH}^\infty$. But, as can be seen from Fig. 2, $(I - T^{cl}(z)\Delta(z))^{-1}$ is the transfer function matrix from $u(z)$ to $y_1(z)$; namely $y_1(z) = (I - T^{cl}(z)\Delta(z))^{-1}u(z)$. This, in turn, leads to a contradiction as $K(z)$ was assumed to internally stabilize $G_\Delta(z)$. Hence the conclusion follows.

DEFINITION 6.3. *A union of five algebraic elements*

$$\Sigma := (A - zE, B; Q, L, R), \tag{93}$$

where $A \in \mathbb{C}^{n\times n}$, $E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$, $L \in \mathbb{C}^{n\times m}$, *and* $R = R^* \in \mathbb{C}^{m\times m}$ *with* $det(R) \neq 0$ *will be called an algebraic quintet.*

LEMMA 6.4. *Consider the* $(p_1 + p_2) \times (m_1 + m_2)$ ***RMF***

$$T(z) =: \left[\begin{array}{c|c:c} A - zE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & 0 \end{array}\right]_{z_0} \tag{94}$$

and suppose it satisfies the following assumptions:

A.1 D_{21} *is square and invertible,*

A.2 $\Lambda(A - zE - B_1 D_{21}^{-1} C_2(\alpha - \bar{\alpha}z)) \subset D$,

A.3 $(A - zE, B_2)$ *is stabilizable,*

A.4 rank $\left(\begin{bmatrix} A - zE & B_2(\alpha - \bar{\alpha}z) \\ C_1 & D_{12} \end{bmatrix}\right) = n + m_2, \ \forall z \in C_{o;1}$.

Also let $J := diag(-I_{m1}; I_{m2})$ *and* $\Sigma := (A - zE, B, Q, L, R)$*, where*

$$B := \left[\begin{array}{c:c} B_1 & B_2 \end{array}\right],$$
$$Q := C_1^* C_1,$$
$$L := \left[\begin{array}{c:c} L_1 & L_2 \end{array}\right] := \left[\begin{array}{c:c} C_1^* D_{11} & C_1^* D_{12} \end{array}\right], \tag{95}$$
$$R := \left[\begin{array}{c:c} R_{11} & R_{12} \\ \hdashline R_{12}^* & R_{22} \end{array}\right] := \left[\begin{array}{c} D_{11}^* \\ \hdashline D_{12}^* \end{array}\right] \left[\begin{array}{c:c} D_{11} & D_{12} \end{array}\right] - \left[\begin{array}{c:c} \gamma^2 I_{m_1} & 0 \\ \hdashline 0 & 0 \end{array}\right].$$

Then the 2-block γ*-suboptimal* H^∞ *problem for* $T(z)$ *has a solution if and only if the DDTARE*

$$A^* X A - E^* X E - ((\bar{\alpha} A - \alpha E)^* X B + L) R^{-1} ((\bar{\alpha} A - \alpha E)^* X B + L)^* + Q = 0 \tag{96}$$

associated with Σ *has a stabilizing solution* $X_s = X_s^* \geq 0$ *and* $sign(R) = J$*.*

Proof. Notice that the DDTARE 96 has a stabilizing solution $X_s = X_s^* \geq 0$ if and only if the Riccati equation

$$0 = \gamma^{-1} C_1^* C_1 + A^* Z A - E^* Z E -$$
$$\left((\bar{\alpha} A - \alpha E)^* Z \left[\gamma^{-\frac{1}{2}} B_1 \quad \gamma^{\frac{1}{2}} B_2\right] + \gamma^{-\frac{1}{2}} C_1^* \left[\gamma^{-1} D_{11} \quad D_{12}\right]\right) \times$$
$$\begin{bmatrix} \gamma^{-2} D_{11}^* D_{11} - I & \gamma^{-1} D_{11}^* D_{12} \\ \gamma^{-1} D_{12}^* D_{11} & D_{12}^* D_{12}^* \end{bmatrix}^{-1}$$
$$\cdot \left((\bar{\alpha} A - \alpha E)^* Z \left[\gamma^{-\frac{1}{2}} B_1 \quad \gamma^{\frac{1}{2}} B_2\right] + \gamma^{-\frac{1}{2}} C_1^* \left[\gamma^{-1} D_{11} \quad D_{12}\right]\right)^*$$

has a stabilizing solution $Z_s = Z_s^* \geq 0$. Moreover:

$$sign\left(\begin{bmatrix} \gamma^{-2} D_{11}^* D_{11} - I & \gamma^{-1} D_{11}^* D_{12} \\ \gamma^{-1} D_{12}^* D_{11} & D_{12}^* D_{12}^* \end{bmatrix}\right)$$
$$= sign\left(\begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{11}^* D_{11} - \gamma^2 I & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} \end{bmatrix} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix}\right) = sign(R).$$

In the end necessity follows from condition (C1) of Theorem 4 in [12], and sufficiency follows from Proposition 21 in [12] also using a scaling procedure as depicted in Section 6.4 therein. ∎

6.2. An auxiliary operator-based result. In this section necessary and sufficient conditions for the existence of a robustly stabilizing controller in terms of linear operators are provided. As a result, a formula for the maximum stability margin arises in operatorial form. First some preparations are made.

Definition 6.5. *Let* X *be a Hilbert space. A linear operator* $\mathcal{R} : X \to X$ *is called coercive (and denoted* $\mathcal{R} >> 0$*) if there exists a* $\delta > 0$ *such that* $\langle \mathcal{R} u, u \rangle_X \geq \delta ||u||_X^2, \ \forall u \in X$*.*

Suppose now that $G(z)$ is a $p \times m$ **RMF**, and let $\mathcal{G}$ be its associated input-output operator. In what follows, the notions of causal/anti-causal Hankel and causal Toeplitz operators will be briefly recalled (for further details, see [3]).

Definition 6.6. *Let* $\mathcal{G}$ *be as above. Define the causal Hankel operator associated with* $\mathcal{G}$ *as* $\mathbb{H}_\mathcal{G} := P_+^p \mathcal{G}|_{l_-^{2;m}}$*, where* P_+^p *denotes the orthogonal projection onto* $l_+^{2;p}$*, and the anti-causal Hankel operator associated with* $\mathcal{G}$ *as* $\hat{\mathbb{H}}_\mathcal{G} := P_-^p \mathcal{G}|_{l_+^{2;m}}$*, where* P_-^p *denotes the orthogonal projection onto* $l_-^{2;p}$*. Also, define the causal Toeplitz operator associated with* $\mathcal{G}$ *as* $\mathbb{T}_\mathcal{G} := P_+^p \mathcal{G}|_{l_+^{2;m}}$*.*

In order to keep the presentation of the results simple, the following convention will be made: whenever $\mathcal{G}$ is the input-output operator associated with a **RMF** $G(z)$,

$\mathbb{T}_G$ will be alternatively used in place of $\mathbb{T}_{\mathcal{G}}$, but keep in mind that the former one is simply a notation for the Toeplitz operator associated to $\mathcal{G}$. The same convention will be made for the Hankel operator associated with $\mathcal{G}$; namely $\mathbb{H}_G$ will be alternatively used in place of $\mathbb{H}_{\mathcal{G}}$, but in the end both represent the same thing.

Consider the algebraic quintet $\Sigma := (A - zE, B; Q, L, R)$, where $\Lambda(A - zE) \subset \mathbb{D}$, and $B \in \mathbb{C}^{n\times(m_1+m_2)}$, $L \in \mathbb{C}^{n\times(m_1+m_2)}$, $R =: \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)}$, for some integers m_1 and m_2. Now associate with it the Popov function partitioned in accordance to R

$$\Pi_\Sigma(z) := \left[\begin{array}{cc|c} A - zE & 0 & B \\ Q(\alpha - \bar{\alpha} z) & E^* - zA^* & L \\ \hline L^* & B^* & R \end{array}\right]_{z_0} =: \begin{bmatrix} \Pi_{\Sigma_{11}} & \Pi_{\Sigma_{12}} \\ \Pi_{\Sigma_{21}} & \Pi_{\Sigma_{22}} \end{bmatrix}, \tag{97}$$

and denote by $\mathcal{R}_{\Sigma_e} =: \begin{bmatrix} \mathcal{R}_{\Sigma_{e_{11}}} & \mathcal{R}_{\Sigma_{e_{12}}} \\ \mathcal{R}_{\Sigma_{e_{12}}}^* & \mathcal{R}_{\Sigma_{e_{22}}} \end{bmatrix}$ its input-output operator. Finally, let $\mathcal{R}_\Sigma := \mathbb{T}_{\mathcal{R}_{\Sigma_e}} =: \begin{bmatrix} \mathcal{R}_{\Sigma_{11}} & \mathcal{R}_{\Sigma_{12}} \\ \mathcal{R}_{\Sigma_{12}}^* & \mathcal{R}_{\Sigma_{22}} \end{bmatrix}$. $\mathcal{R}_\Sigma$ will be called the Σ-Topelitz operator. The latter partitions were considered in accordance to the partition of Π_Σ (or equivalently in accordance to the partition of the R-matrix).

LEMMA 6.7. *Let Σ and Π_Σ be as above. Suppose that R is nonsingular and $\Pi_{\Sigma_{22}}(z) > 0$ for all $z \in C_{o;1}$.*

1. *If there exists a stable **RMF** $H(z)$ such that*

$$\begin{bmatrix} I & H^\#(z) \end{bmatrix} \Pi_\Sigma(z) \begin{bmatrix} I \\ H(z) \end{bmatrix} < 0, \tag{98}$$

for all $z \in C_{o;1}$, then $-\mathcal{R}^X_{\Sigma_{11}} := -\mathcal{R}_{11} + \mathcal{R}_{12}\mathcal{R}_{22}^{-1}\mathcal{R}_{12}^*$ is coercive, where $*$ takes the adjoint of the respective operator.

2. *Provided that, in addition, $\Pi_{\Sigma_{22}} \equiv R_{22}$ for some constant matrix $R_{22} = R_{22}^* > 0$, then if the linear-bounded operator $-\mathcal{R}^X_{\Sigma_{11}}$ is coercive it follows that relation 98 is fulfilled for some stable **RMF** $H(z)$, and for all $z \in C_{o;1}$.*

Proof. 1. Denote by $\hat{h}_2(e^{j\theta})$ the Fourier transform of some $h_2[k] \in l^{2,m_2}$; that is

$$\hat{h}_2(e^{j\theta}) = \sum_{k=-\infty}^{\infty} h_2[k] e^{jk\theta}, \tag{99}$$

for $\forall\ \theta \in [0; 2\pi)$. Since $\Pi_{\Sigma_{22}}(e^{j\theta}) = \left(\Pi_{\Sigma_{22}}(e^{j\theta})\right)^* > 0$ on $[0; 2\pi)$ it follows that for any $\theta \in [0; 2\pi)$ and any $\lambda_i(e^{j\theta}) \in \Lambda(\Pi_{\Sigma_{22}}(e^{j\theta}))$, $\lambda_i(e^{j\theta}) \in (0; +\infty)$. Define $\lambda_{min}(e^{j\cdot}) \equiv \min_{i=1:m_2} \lambda_i(e^{j\cdot})$, and notice that $\lambda_{min}(e^{j\theta}) \in (0; +\infty)\ \forall\theta \in [0; 2\pi)$. Moreover, by Rellich's theorem it follows that $\lambda_{min}(e^{j\theta})$ depends continuously on the real parameter θ, and thus $0 < \lambda_{min}(1) = \lim_{\theta\to 2\pi} \lambda_{min}(e^{j\theta})$. In conclusion, since in addition $C_{o;1}$ is a compact set: $\inf_{\theta\in[0;2\pi)} \lambda_{min}(e^{j\theta}) = \min_{\theta\in[0;2\pi)} \lambda_{min}(e^{j\theta}) \in (0; +\infty)$. Also for any $\theta \in [0; 2\pi)$ one has

$$\langle \hat{h}_2(e^{j\theta}); \Pi_{\Sigma_{22}}(e^{j\theta})\hat{h}_2(e^{j\theta})\rangle_{\mathbb{C}^{m_2}} \geq \lambda_{min}(e^{j\theta})||\hat{h}_2(e^{j\theta})||_2^2$$

and thus using Parseval's identity it holds that

$$\begin{aligned}\langle h_2; \mathcal{R}_{e_{\Sigma_{22}}} h_2\rangle_{l^{2;m_2}} &= \frac{1}{2\pi}\int_0^{2\pi} \langle \hat{h}_2(e^{j\theta}); \Pi_{\Sigma_{22}}(e^{j\theta})\hat{h}_2(e^{j\theta})\rangle_{\mathbb{C}^{m_2}} d\theta \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \lambda_{min}(e^{j\theta})||\hat{h}_2(e^{j\theta})||_2^2 d\theta \geq \left(\min_{\theta\in[0;2\pi)} \lambda_{min}(e^{j\theta})\right) \frac{1}{2\pi}\int_0^{2\pi} ||\hat{h}_2(e^{j\theta})||_2^2 d\theta \\ &= \left(\min_{\theta\in[0;2\pi)} \lambda_{min}(e^{j\theta})\right) ||h||^2_{l^{2;m_2}},\end{aligned}$$

where $\mathcal{R}_{e_{\Sigma_{22}}}$ is the inverse Fourier transform of $\Pi_{\Sigma_{22}}(e^{j\theta})$. In conclusion $\mathcal{R}_{e_{\Sigma_{22}}}$ is coercive, which implies that $\mathcal{R}_{\Sigma_{22}}$ is coercive as well.
Now suppose there is a stable **RMF** $H(z)$ such that 98 holds. Denote by $\mathcal{H}_e$ the input-output operator of the system $H(z)$, and by $\mathcal{H}$ the Toeplitz operator of $\mathcal{H}_e$. Further let

$$(100)\quad \mathcal{T}_{\Sigma_e} := \begin{bmatrix} I & \mathcal{H}_e^* \\ 0 & I\end{bmatrix}\begin{bmatrix}\mathcal{R}_{\Sigma_{e_{11}}} & \mathcal{R}_{\Sigma_{e_{12}}} \\ \mathcal{R}^*_{\Sigma_{e_{12}}} & \mathcal{R}_{\Sigma_{e_{22}}}\end{bmatrix}\begin{bmatrix} I & 0 \\ \mathcal{H}_e & I\end{bmatrix} =: \begin{bmatrix}\mathcal{T}_{\Sigma_{e_{11}}} & \mathcal{T}_{\Sigma_{e_{12}}} \\ \mathcal{T}^*_{\Sigma_{e_{12}}} & \mathcal{T}_{\Sigma_{e_{22}}}\end{bmatrix},$$

and observe that $\mathcal{T}_{\Sigma_{e_{11}}}$ is the input-output operator associated with the left-hand side of 98. Proceeding in a similar way as for $\mathcal{R}_{\Sigma_{22}}$ it follows that $-\mathcal{T}_{\Sigma_{e_{11}}} >> 0$, and $\mathcal{T}_{\Sigma_{e_{22}}} = \mathcal{R}_{\Sigma_{e_{22}}} >> 0$ as well.
It is known that for any linear-bounded operator $\mathcal{G}$ the following relation holds true $\mathbb{T}^*_{\mathcal{G}} = \mathbb{T}_{\mathcal{G}^*}$, and if in addition $\mathcal{G}$ is the input-output operator corresponding to an anti-stable system $G(z)$ or $\mathcal{H}$ is the input-output operator corresponding to a stable system $H(z)$ then $\mathbb{T}_{\mathcal{G}\mathcal{H}} = \mathbb{T}_{\mathcal{G}}\mathbb{T}_{\mathcal{H}}$ (for further details see Chapter 2 in [4]).
Taking all these into account, one can see that

$$(101)\quad \mathcal{T}_{\Sigma} := \mathbb{T}_{\mathcal{T}_{\Sigma_e}} = \begin{bmatrix} I & \mathcal{H}^* \\ 0 & I\end{bmatrix}\begin{bmatrix}\mathcal{R}_{\Sigma_{11}} & \mathcal{R}_{\Sigma_{12}} \\ \mathcal{R}^*_{\Sigma_{12}} & \mathcal{R}_{\Sigma_{22}}\end{bmatrix}\begin{bmatrix} I & 0 \\ \mathcal{H} & I\end{bmatrix} =: \begin{bmatrix}\mathcal{T}_{\Sigma_{11}} & \mathcal{T}_{\Sigma_{12}} \\ \mathcal{T}^*_{\Sigma_{12}} & \mathcal{T}_{\Sigma_{22}}\end{bmatrix}.$$

From the above equation $\mathcal{R}_{\Sigma}$ can be expressed function of $\mathcal{T}_{\Sigma}$ and hence $-\mathcal{R}^X_{\Sigma_{11}} = -\mathcal{T}_{\Sigma_{11}} + \mathcal{T}_{\Sigma_{12}}\mathcal{T}^{-1}_{\Sigma_{22}}\mathcal{T}^*_{\Sigma_{12}}$. Since $-\mathcal{T}_{\Sigma_{e_{11}}} >> 0$ and $\mathcal{T}_{\Sigma_{e_{22}}} >> 0$, it follows that their Toeplitz: $-\mathcal{T}_{\Sigma_{11}}$ and $\mathcal{T}_{\Sigma_{22}}$, respectively, are coercive as well. Thus the conclusion follows.

2. Since $\Pi_{\Sigma_{22}} \equiv R_{22}$ is a constant positive-definite hermitic matrix, it follows that its inverse Fourier transform coincides with R_{22}; namely $\mathcal{R}_{e_{\Sigma_{22}}} = R_{22}$, and consequently $P_+^{m_2}R_{22} = R_{22}|_{l_+^{2;m_2}} = \mathbb{T}_{\mathcal{R}_{e_{\Sigma_{22}}}} = \mathcal{R}_{\Sigma_{22}}$. Define $h := -\mathcal{R}^{-1}_{\Sigma_{22}}\mathcal{R}^*_{\Sigma_{12}}$. It follows that $h = -(R_{22}|_{l_+^{2;m_2}})^{-1}\mathcal{R}^*_{\Sigma_{12}} = -R_{22}^{-1}|_{l_+^{2;m_2}}\mathbb{T}^*_{\mathcal{R}_{e_{\Sigma_{12}}}} = -R_{22}^{-1}|_{l_+^{2;m_2}}\mathbb{T}_{\mathcal{R}^*_{e_{\Sigma_{12}}}} = -R_{22}^{-1}\mathbb{T}_{\mathcal{R}^*_{e_{\Sigma_{22}}}} = -R_{22}^{-1}\mathcal{R}^*_{\Sigma_{12}}$. Moreover, the following relation holds true: $\hat{\mathbb{H}}_h = P_-^{m_2}\left(-\mathcal{R}^{-1}_{\Sigma_{22}}\mathcal{R}^*_{\Sigma_{12}}\right)|_{l_+^{m_1}} = -P_-^{m_2}\left(R_{22}^{-1}\mathbb{T}_{\mathcal{R}^*_{\Sigma_{12}}}\right)|_{l_+^{m_1}} = -P_-^{m_2}\left(P_+^{m_2}R_{22}^{-1}\mathbb{T}_{\mathcal{R}^*_{\Sigma_{12}}}\right) =$
$-P_-^{m_2}P_+^{m_2}R_{22}^{-1}\mathbb{T}_{\mathcal{R}^*_{\Sigma_{12}}} \equiv 0$, which together with Theorem 2.8.3. in [4] lead to the conclusion that $H(z) := \mathcal{Z}\{h\}(z)$ is a stable **RMF** ($\mathcal{Z}\{\cdot\}$ represents the $\mathcal{Z}$-transform). For any $w \in l_+^{2;m}$ one has $\begin{bmatrix} I & 0 \\ \mathcal{R}^{-1}_{\Sigma_{22}}\mathcal{R}^*_{\Sigma_{12}} & I\end{bmatrix}\begin{bmatrix} I \\ h\end{bmatrix} w = \begin{bmatrix} I \\ 0\end{bmatrix} w$, and

consequently

$$\begin{aligned}\langle w; \mathcal{R}^X_{\Sigma_{11}} w\rangle_{l^{2;m_1}} &= \langle w; \begin{bmatrix} I & 0\end{bmatrix} \begin{bmatrix} \mathcal{R}^X_{\Sigma_{11}} & 0 \\ 0 & \mathcal{R}_{\Sigma_{22}}\end{bmatrix} \begin{bmatrix} I \\ 0\end{bmatrix} w\rangle_{l^{2;m_1}} \\ &= \langle w; \begin{bmatrix} I & h^*\end{bmatrix} \begin{bmatrix} I & \mathcal{R}_{\Sigma_{12}}\mathcal{R}^{-1}_{\Sigma_{22}} \\ 0 & I\end{bmatrix} \begin{bmatrix} \mathcal{R}^X_{\Sigma_{11}} & 0 \\ 0 & \mathcal{R}_{\Sigma_{22}}\end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{R}^{-1}_{\Sigma_{22}}\mathcal{R}^*_{\Sigma_{12}} & I\end{bmatrix} \\ &\quad \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}} = \langle w; \begin{bmatrix} I & h^*\end{bmatrix} \mathcal{R}_{\Sigma} \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}}.\end{aligned}$$

On the other hand, since h is the input-output operator corresponding to a stable system $H(z)$, and $w[k] \in l^{2;m}_+$ it holds that

$$\begin{aligned}\langle w; \begin{bmatrix} I & h^*\end{bmatrix} \mathcal{R}_{\Sigma} \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}} &= \langle w; \begin{bmatrix} I & h^*\end{bmatrix} \begin{bmatrix} P_+ I & 0 \\ 0 & P_+ I\end{bmatrix} \mathcal{R}_{\Sigma_e}|_{l^{2;m}_+} \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}} \\ &= \langle w; \begin{bmatrix} I & h^*\end{bmatrix} \mathcal{R}_{\Sigma_e} \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}},\end{aligned}$$

where for the last equality Theorem 2.6.1 in [4], applied to the linear-bounded operator h, was in addition used. Combining the last two identities above together with the coercivity of $-\mathcal{R}^X_{\Sigma_{11}}$, it follows that

$$\langle w; \begin{bmatrix} I & h^*\end{bmatrix} (-\mathcal{R}_{\Sigma_e}) \begin{bmatrix} I \\ h\end{bmatrix} w\rangle_{l^{2;m_1}_+} > C||w||^2_{l^{2;m_1}_+}, \tag{102}$$

for some real constant $C > 0$. The last part of the proof consists in showing that given relation 102, then

$$\hat{w}^*(e^{j\theta}) \begin{bmatrix} I & H^*(e^{j\theta})\end{bmatrix} \left(-\Pi_{\Sigma}(e^{j\theta})\right) \begin{bmatrix} I \\ H(e^{j\theta})\end{bmatrix} \hat{w}(e^{j\theta}) \geq \mu||\hat{w}(e^{j\theta})||^2_2, \tag{103}$$

for any $\theta \in [0; 2\pi)$, where $\hat{w}(e^{j\theta}) := \sum_{k=0}^{\infty} w_k e^{jk\theta}$ is the Fourier transform of $w[k]$, and $\mu > 0$ is some real constant.

Suppose by contradiction that there is a $\theta_0 \in [0; 2\pi)$ such that

$$\hat{w}_0^*(e^{j\theta_0}) \begin{bmatrix} I & H^*(e^{j\theta_0})\end{bmatrix} \left(-\Pi_{\Sigma}(e^{j\theta_0})\right) \begin{bmatrix} I \\ H(e^{j\theta_0})\end{bmatrix} \hat{w}_0(e^{j\theta_0}) < \mu||\hat{w}_0(e^{j\theta_0})||^2_2, \tag{104}$$

where $\mu > 0$ is to be determined. By continuity, there is an $\epsilon > 0$ such that

$$\hat{w}_0^*(e^{j\theta}) \begin{bmatrix} I & H^*(e^{j\theta})\end{bmatrix} \left(-\Pi_{\Sigma}(e^{j\theta})\right) \begin{bmatrix} I \\ H(e^{j\theta})\end{bmatrix} \hat{w}_0(e^{j\theta}) < \mu||\hat{w}_0(e^{j\theta})||^2_2, \tag{105}$$

for any $\theta \in (\theta_0 - \epsilon/2; \theta_0 + \epsilon/2)$. Take $\mu := C$ and define $\hat{w}^{\epsilon}_0(e^{j\theta}) := \hat{w}_0(e^{j\theta_0})$ for any $\theta \in (\theta_0 - \epsilon/2; \theta_0 + \epsilon/2)$ and 0 otherwise. Also let $w^{\epsilon}_0[k]$ be its inverse

Fourier-transform. Using inequality 102, it follows that

$$\begin{aligned}
&\langle w_0^\epsilon; \begin{bmatrix} I & h^* \end{bmatrix} (-\mathcal{R}_{\Sigma_e}) \begin{bmatrix} I \\ h \end{bmatrix} w_0^\epsilon \rangle_{l_+^{2;m_1}} > C||w_0^\epsilon||^2_{l_+^{2;m_1}} = C(2\pi)^{-1}||\hat{w}_0^\epsilon||^2_{l_+^{2;m_1}} \\
&= \epsilon C(2\pi)^{-1}||\hat{w}_0(e^{j\theta_0})||_2^2 = \epsilon(2\pi)^{-1}\mu||\hat{w}_0(e^{j\theta_0})||_2^2 \\
&> \epsilon(2\pi)^{-1}\hat{w}_0^*(e^{j\theta_0}) \begin{bmatrix} I & H^*(e^{j\theta_0}) \end{bmatrix} \left(-\Pi_\Sigma(e^{j\theta_0})\right) \begin{bmatrix} I \\ H(e^{j\theta_0}) \end{bmatrix} \hat{w}_0(e^{j\theta_0}) \\
&= \langle \hat{w}_0^\epsilon; \begin{bmatrix} I & H^* \end{bmatrix} (-\Pi_\Sigma) \begin{bmatrix} I \\ H \end{bmatrix} \rangle_{l_+^{2;m_1}} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \langle \hat{w}_0^\epsilon(e^{j\theta}); \begin{bmatrix} I & H^*(e^{j\theta}) \end{bmatrix} \left(-\Pi_\Sigma(e^{j\theta})\right) \begin{bmatrix} I & H^*(e^{-j\theta}) \end{bmatrix}^* \rangle_{\mathbb{C}^{m_1}} d\theta,
\end{aligned}$$

which is obviously a contradiction, by Parseval's identity. ∎

The next result is an immediate consequence of Lemma 6.7.

COROLLARY 6.8. *Let Σ and Π_Σ be as in Lemma 6.7. Suppose that R is non-singular and that there is a constant positive-definite hermitic matrix R_{22} such that $\Pi_{\Sigma_{22}}(z) = R_{22}$ for all $z \in \mathbb{C}$. Then there is a stable **RMF** $H(z)$ such that inequality 98 is fulfilled for any $z \in \mathbb{C}$ if and only if $-\mathcal{R}^X_{\Sigma_{11}}$ is coercive.*

Consider again the $(p_1 + p_2) \times (m_1 + m_2)$ system 94

$$T(z) := \left[\begin{array}{c|c:c} A - zE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & 0 \end{array}\right]_{z_0} =: \left[\begin{array}{c:c} T_{11}(z) & T_{12}(z) \\ \hdashline T_{21}(z) & T_{22}(z) \end{array}\right], \tag{106}$$

and suppose that assumptions $A.1 : A.4$ in Lemma 6.4 hold for it. Let $\Sigma := (A - zE, B; Q, L, R)$ be the associated algebraic quintet given by 95. Define $\Sigma_2 := (A - zE, B_2; Q, L_2, R_{22})$, and associate with it the following DDTARE

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB_2 + L_2)R_{22}^{-1}((\bar{\alpha}A - \alpha E)^*XB_2 + L_2)^* + Q = 0, \tag{107}$$

where the invertibility of $R_{22} = D_{12}^* D_{12}$ follows from $A.4$ by setting $z = z_0$.

PROPOSITION 6.9. *The DDTARE 107 has a stabilizing solution $X_{s;2} = X_{s;2}^*$.*

Proof. The proof follows mutatis mutandis the lines of the proof of Theorem 12 in [7], using assumption $A.4$ and Theorem 2.1. ∎

Denote by $F_{s;2}$ the stabilizing feedback associated with the DDTARE 107; namely $F_{s;2} = -R_{22}^{-1}(B_2^* X_{s;2}(\bar{\alpha}A - \alpha E) + L_2^*)$. Also define $\bar{\Sigma} := (\bar{A} - z\bar{E}, \bar{B}; \bar{Q}, \bar{L}, \bar{R})$, where

$$\begin{aligned}
\bar{A} &:= A + \alpha B_2 F_{s;2}, \\
\bar{E} &:= E + \bar{\alpha} B_2 F_{s;2}, \\
\bar{B} &:= B, \\
\bar{Q} &:= \bar{C}_1^* \bar{C}_1 := (C_1 + D_{12}F_{s;2})^*(C_1 + D_{12}F_{s;2}), \\
\bar{L} &:= \bar{C}_1^* \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \\
\bar{R} &:= R.
\end{aligned} \tag{108}$$

Since $F_{s;2}$ is the stabilizing feedback associated to DDTARE 107, it follows that $\Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$. Also let $\Pi_{\bar{\Sigma}}(z)$ be the Popov function associated with $\bar{\Sigma}$. Immediate computations using relation 7 lead to

$$\Pi_{\bar{\Sigma}}(z) = \begin{bmatrix} \bar{T}_{11}^{\#}(z) \\ \hdashline \bar{T}_{12}^{\#}(z) \end{bmatrix} \left[\begin{array}{c:c} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{array}\right] - \left[\begin{array}{c:c} \gamma^2 I_{m_1} & 0 \\ \hdashline 0 & 0 \end{array}\right], \tag{109}$$

where

$$\left[\begin{array}{c:c} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{array}\right] := \left[\begin{array}{c|c:c} \bar{A} - z\bar{E} & B_1 & B_2 \\ \hline \bar{C}_1 & D_{11} & D_{12} \end{array}\right]_{z_0}. \tag{110}$$

The next result provides necessary and sufficient conditions for the existence of a solution to the γ-suboptimal H^∞ problem in terms of $\mathcal{R}^X_{\bar{\Sigma}_{11}}$. It shall be seen that, using this result, one can obtain the value of the maximum stability margin $\sigma_{max} = \frac{1}{\gamma_{min}}$ in operatorial form. Consequently, this formula will be used further for computing σ_{max} in 37.

THEOREM 6.10. *Consider the system $T(z)$ given by 106, and suppose that the assumptions $A.1 : A.4$ hold for it. Let*

$$\mathcal{R}_{\bar{\Sigma}} =: \begin{bmatrix} \bar{\mathcal{R}}_{11} & \bar{\mathcal{R}}_{12} \\ \bar{\mathcal{R}}_{12}^* & \bar{\mathcal{R}}_{22} \end{bmatrix} \tag{111}$$

be the $\bar{\Sigma}$-Toeplitz operator partitioned in accordance with $\Pi_{\bar{\Sigma}}$. Then the following statements hold true:

1. *The 2-block γ-suboptimal H^∞ control problem formulated for $T(z)$ has a solution if and only if*

$$-\mathcal{R}^X_{\bar{\Sigma}_{11}} >> 0. \tag{112}$$

2. *The least achievable $\gamma = \gamma_{min}$ (the optimal value) such that the γ -suboptimal H^∞ problem still has a solution is given by*

$$\gamma_{min} = \rho\left(\mathbb{H}_{\bar{T}_{11}^*\hat{T}_{12}}\mathbb{H}^*_{\bar{T}_{11}^*\hat{T}_{12}} + \mathbb{T}_{(\bar{T}_{11}^*\hat{T}_{12}^{\perp})(\bar{T}_{11}^*\hat{T}_{12}^{\perp})^*}\right)^{1/2}, \tag{113}$$

where $\hat{T}_{12}^{\perp} \in \mathbb{RL}^\infty$ is the orthogonal completion of the (inner) ***RMF*** *$\hat{T}_{12}(z) := \bar{T}_{12}V^{-1}$ in such manner that $\hat{T}_{12}^a(z) := \begin{bmatrix} \hat{T}_{12}(z) & \hat{T}_{12}^{\perp}(z) \end{bmatrix}$ is all pass. Here V is the Cholesky factor of $D_{12}^*D_{12}$.*

Proof. As explained earlier, setting $z = z_0$ in the assumption $A.4$, it follows that D_{12} has full column rank, and consequently V as defined above is invertible. The next step is to show that indeed $\hat{T}_{12}(z)$ is inner. To prove this claim notice that

$$\hat{T}_{12}(z) = \left[\begin{array}{c|c} \bar{A} - z\bar{E} & B_2V^{-1} \\ \hline \bar{C}_1 & D_{12}V^{-1} \end{array}\right]_{z_0}, \tag{114}$$

and that the following relations hold

$$\bar{C}_1^* D_{12} + (\bar{\alpha}A - \alpha E)^* X_{s;2} B_2 = 0, \tag{115}$$

$$\bar{A}^* X_{s;2}\bar{A} - \bar{E}^* X_{s;2}\bar{E} + \bar{C}^*\bar{C} = 0, \tag{116}$$

where (recall) $X_{s;2}$ is the stabilizing solution of 107. Actually the second relation is a mere rewriting of DDTARE 107 using the definition of $\bar{A}$, $\bar{E}$ and $\bar{C}_1$, whereas the first one follows by using the definition of $\bar{C}_1$ and the fact that $R_{22} = D_{12}^* D_{12}$. Compute

$$\hat{T}_{12}^{\#}(z)\hat{T}_{12}(z) = \left[\begin{array}{c|c} \bar{E}^* - z\bar{A}^* & \bar{C}_1^* \\ \hline V^{-*}B_2^* & V^{-*}D_{12}^* \end{array}\right]_{z_0} \left[\begin{array}{c|c} \bar{A} - z\bar{E} & B_2 V^{-1} \\ \hline \bar{C}_1 & D_{12}V^{-1} \end{array}\right]_{z_0}$$

$$= V^{-*}\left[\begin{array}{cc|c} \bar{E}^* - z\bar{A}^* & \bar{C}_1^*\bar{C}_1(\alpha - \bar{\alpha}z) & \bar{C}_1^* D_{12} \\ 0 & \bar{A} - z\bar{E} & B_2 \\ \hline B_2^* & D_{12}^*\bar{C}_1^* & V^*V \end{array}\right]_{z_0} V^{-1}, \tag{117}$$

where for the first equality relation 8 was used. Notice now that $\bar{\alpha}\bar{A} - \alpha\bar{E} = \bar{\alpha}A - \alpha E$. Defining $Q := \begin{bmatrix} I & (\bar{\alpha}A - \alpha E)^* X_{s;2} \\ 0 & I \end{bmatrix}$, $Z := \begin{bmatrix} I & X_{s;2}(\bar{\alpha}A - \alpha E) \\ 0 & I \end{bmatrix}$, performing an equivalence transformation on 117 using $(Q;Z)$, and using equations 115 as well as 116, one obtains

$$\hat{T}_{12}^{\#}(z)\hat{T}_{12}(z) = V^{-*}\left[\begin{array}{cc|c} \bar{E}^* - z\bar{A}^* & 0 & 0 \\ 0 & A - zE & B_2 \\ \hline B_2^* & 0 & V^*V \end{array}\right]_{z_0} V^{-1} \equiv I. \tag{118}$$

Combining the above identity with the fact that $\Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$, the conclusion follows: $\hat{T}_{12}(z)$ is an inner **RMF**.

The fact that one can always find an orthogonal completion of $\hat{T}_{12}(z)$ such that $\hat{T}_{12}^a(z)$ is all pass follows in a straightforward way from some standard arguments regarding Hilbert spaces (see for instance Lemma 13.31 in [5]).

1. According to Lemma 6.4, the 2-block γ-suboptimal H^∞ control problem formulated for $T(z)$ has a solution if and only if the DDTARE

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB + L)R^{-1}((\bar{\alpha}A - \alpha E)^*XB + L)^* + Q = 0 \tag{119}$$

associated with $\Sigma := (A - zE, B; Q, L, R)$ given by 95 has a stabilizing solution $X_s = X_s^* \geq 0$ and $sign(R) = J$, where $J := diag(-I_{m_1}; I_{m_2})$. Straightforward computations using Definition 2.3 show that $\bar{\Sigma} = (\bar{A} - z\bar{E}, \bar{B}; \bar{Q}, \bar{L}, \bar{R})$ as defined in 108 is actually the $\begin{bmatrix} 0 & F_{s;2}^* \end{bmatrix}^*$-equivalent of Σ. Combining this with Proposition 2.4, it follows that the DDTARE 119 associated with Σ has a stabilizing solution $X_s = X_s^* \geq 0$ if and only if the DDTARE

$$\bar{A}^*\bar{X}\bar{A} - \bar{E}^*\bar{X}\bar{E} - ((\bar{\alpha}\bar{A} - \alpha\bar{E})^*\bar{X}\bar{B} + \bar{L})\bar{R}^{-1}((\bar{\alpha}\bar{A} - \alpha\bar{E})^*\bar{X}\bar{B} + \bar{L})^* + \bar{Q} = 0 \tag{120}$$

associated with $\bar{\Sigma}$ has a stabilizing solution $\bar{X}_s = X_s$. In conclusion, the 2-block γ-suboptimal H^∞ control problem formulated for $T(z)$ has a solution if and only if the DDTARE 120 associated with $\bar{\Sigma}$ has a stabilizing solution $X_s = X_s^* \geq 0$ and $sign(R) = J$. Moreover, denoting by F_s the stabilizing feedback of 119, and defining $\bar{F}_s := F_s - \begin{bmatrix} 0 & F_{s;2}^* \end{bmatrix}^*$, it follows also by Proposition 2.4 that $\bar{F}_s$ is the stabilizing feedback of the DDTARE 120. The advantage is that the new DDTARE 120 has $\Lambda(\bar{A} - z\bar{E}) \subset \mathbb{D}$. As we shall see immediately, this property is instrumental in the proof.

Necessity: Suppose DDTARE 120 has a stabilizing solution $X_s = X_s^* \geq 0$ and $sign(R) = J$. The latter assumption implies, in particular, that R is nonsingular. Now let $\Pi_{\bar{\Sigma}}(z)$ given in 109 be the Popov function associated with $\bar{\Sigma}$; namely

$$\left[\begin{array}{c:c} \Pi_{\bar{\Sigma}_{11}} & \Pi_{\bar{\Sigma}_{12}} \\ \hdashline \Pi_{\bar{\Sigma}_{21}} & \Pi_{\bar{\Sigma}_{22}} \end{array}\right] := \Pi_{\bar{\Sigma}}(z) = \left[\begin{array}{c} \bar{T}_{11}^{\#}(z) \\ \hdashline \bar{T}_{12}^{\#}(z) \end{array}\right] \left[\begin{array}{c:c} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{array}\right] \tag{121}$$

$$-\left[\begin{array}{c:c} \gamma^2 I_{m_1} & 0 \\ \hdashline 0 & 0 \end{array}\right], \text{ where}$$

$$\left[\begin{array}{c:c} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{array}\right] := \left[\begin{array}{c|c:c} \bar{A} - z\bar{E} & B_1 & B_2 \\ \hline \bar{C}_1 & D_{11} & D_{12} \end{array}\right]_{z_0}. \tag{122}$$

It follows that $\Pi_{\bar{\Sigma}_{22}}(z) = \bar{T}_{12}^{\#}(z)\bar{T}_{12}(z) = V^*\hat{T}_{12}^{\#}(z)\hat{T}_{12}(z)V \equiv V^*V > 0$. Since the 2-block γ-suboptimal H^∞ problem formulated for $T(z)$ has a solution, there is a controller $K(z)$ such that $T^{cl}(z) := LLFT(T,K)(z)$ is internally stable and moreover $||T^{cl}||_\infty < \gamma$. Take $H(z) := K(z)(I_{p_2} - T_{22}(z)K(z))^{-1}T_{21}(z)$ and observe that this is exactly the transfer function matrix from the vector of external outputs to the vector of controlled inputs. Since $K(z)$ internally stabilizes $T(z)$, it must be true that $H(z) \in \mathbb{RH}^\infty$. Immediate computations show that

$$\begin{bmatrix} I & H^{\#}(z) \end{bmatrix} \Pi_\Sigma(z) \begin{bmatrix} I \\ H(z) \end{bmatrix} = (T^{cl})^{\#}(z)T^{cl}(z) - \gamma^2 I < 0, \tag{123}$$

where the inequality arises from the fact that $K(z)$ is γ-contracting for $T(z)$. Invoking now relations (6.10) and (6.11) in [12] one obtains

$$\begin{bmatrix} I & \bar{H}^{\#}(z) \end{bmatrix} \Pi_{\bar{\Sigma}}(z) \begin{bmatrix} I \\ \bar{H}(z) \end{bmatrix} = \begin{bmatrix} I & H^{\#}(z) \end{bmatrix} \Pi_\Sigma(z) \begin{bmatrix} I \\ H(z) \end{bmatrix} \; \forall z \in C_{o;1}, \tag{124}$$

where Π_Σ is the Popov function associated with Σ, and $\bar{H}(z)$ is such that

$$\left[\begin{array}{c} I \\ \hdashline \bar{H}(z) \end{array}\right] = \left[\begin{array}{c|cc} A - zE & B_1 & B_2 \\ \hline 0 & I & 0 \\ \hdashline -F_{s;2} & 0 & I \end{array}\right]_{z_0} \begin{bmatrix} I \\ H(z) \end{bmatrix},$$

from where it follows that $\bar{H}(z) \in \mathbb{RH}^\infty$. To sum up: R is nonsingular, $\Pi_{\bar{\Sigma}_{22}}(z) \equiv R_{22} > 0$ and there is a stable **RMF** $\bar{H}(z)$ such that

$$\begin{bmatrix} I & \bar{H}^{\#}(z) \end{bmatrix} \Pi_{\bar{\Sigma}}(z) \begin{bmatrix} I \\ \bar{H}(z) \end{bmatrix} \; \forall z \in C_{o;1}.$$

Applying Lemma 6.7 to $\bar{\Sigma}$ and its associated Popov function $\Pi_{\bar{\Sigma}}(z)$ the necessity part follows.

Sufficiency: Assume now that $-\mathcal{R}_{\bar{\Sigma}}^X$ is coercive.
Notice that $sign(\Pi_{\bar{\Sigma}}(z)) = J$ for any $z \in \mathbb{C}$ since $\Pi_{\bar{\Sigma}_{22}}(z) \equiv V^*V > 0$ and its Schur complement satisfies $\Pi_{\bar{\Sigma}_{11}}^X(z) = \bar{T}_{11}^{\#}(z)\bar{T}_{11}(z) - \gamma^2 I - \bar{T}_{11}^{\#}(z)\hat{T}_{12}^{\#}(z)\hat{T}_{12}(z)\bar{T}_{11}(z) \equiv -\gamma^2 I < 0$. Consequently this implies that $sign(R) = J$ since $R = \Pi_{\bar{\Sigma}}(z_0)$, and in particular R turns out to be nonsingular.
In conclusion, Lemma 6.7 (or alternatively Corollary 6.8) can be applied to $\bar{\Sigma}$ and its associated Popov function $\Pi_{\bar{\Sigma}}(z)$ to conclude that there exists a stable **RMF** $\bar{H}(z)$ such that

$$\begin{bmatrix} I & \bar{H}^{\#}(z) \end{bmatrix} \Pi_{\bar{\Sigma}}(z) \begin{bmatrix} I \\ \bar{H}(z) \end{bmatrix} < 0 \;\; \forall z \in C_{o;1}.$$

The proof of part 1. ends now by applying Lemma 16 in [12].

2. To obtain the expression of γ_{min} in 113 one has to evaluate the operator $\mathcal{R}^X_{\bar{\Sigma}}$. With the relation 121, it follows that

$$\begin{aligned}\mathcal{R}^X_{\bar{\Sigma}} :&= \bar{\mathcal{R}}_{11} - \bar{\mathcal{R}}_{12}\bar{\mathcal{R}}_{22}^{-1}\bar{\mathcal{R}}_{12}^* = \mathbb{T}_{\mathcal{R}_{e_{\bar{\Sigma}_{11}}}} - \mathbb{T}_{\mathcal{R}_{e_{\bar{\Sigma}_{12}}}}\mathbb{T}^{-1}_{\mathcal{R}_{e_{\bar{\Sigma}_{22}}}}\mathbb{T}^*_{\mathcal{R}_{e_{\bar{\Sigma}_{12}}}}\\ &=: \mathbb{T}_{\Pi_{\bar{\Sigma}_{11}}} - \mathbb{T}_{\Pi_{\bar{\Sigma}_{12}}}\mathbb{T}^{-1}_{\Pi_{\bar{\Sigma}_{22}}}\mathbb{T}^*_{\Pi_{\bar{\Sigma}_{12}}},\end{aligned}$$

where (recall) the last equality is just a simpler way of representing the previous one. Taking now into account that $\Pi_{\bar{\Sigma}_{11}} = \bar{T}_{11}^{\#}(z)\bar{T}_{11}(z) - \gamma^2 I$, $\Pi_{\bar{\Sigma}_{22}}(z) \equiv V^*V$, $\hat{T}_{12}(z) = \bar{T}_{12}(z)V^{-1}$, as well as the following relations (the first two of which hold for any linear-bounded operator, as shown for instance in [4], Chapter 2)

$$\begin{aligned}\mathbb{T}_{\mathcal{G}^*} &= \mathbb{T}^*_{\mathcal{G}},\\ \mathbb{T}_{\mathcal{G}\mathcal{G}^*} &= \mathbb{T}_{\mathcal{G}}\mathbb{T}^*_{\mathcal{G}} + \mathbb{H}_{\mathcal{G}}\mathbb{H}^*_{\mathcal{G}},\\ I \equiv \hat{T}^a_{12}(z)(\hat{T}^a_{12})^{\#}(z) &= \begin{bmatrix}\hat{T}_{12}(z) & \hat{T}^{\perp}_{12}(z)\end{bmatrix}\begin{bmatrix}\hat{T}_{12}(z) & \hat{T}^{\perp}_{12}(z)\end{bmatrix}^{\#},\end{aligned}$$

one gets

$$\begin{aligned}\mathcal{R}^X_{\bar{\Sigma}} &= \mathbb{T}_{\bar{T}^*_{11}\bar{T}_{11}} - \gamma^2 I - \mathbb{T}_{\bar{T}^*_{11}\hat{T}_{12}}\mathbb{T}^*_{\bar{T}^*_{11}\hat{T}_{12}}\\ &= \mathbb{T}_{\left(\bar{T}^*_{11}\left[\hat{T}_{12} \quad \hat{T}^{\perp}_{12}\right]\left[\hat{T}_{12} \quad \hat{T}^{\perp}_{12}\right]^*\bar{T}_{11}\right)} - \mathbb{T}_{\bar{T}^*_{11}\hat{T}_{12}}\mathbb{T}^*_{\bar{T}^*_{11}\hat{T}_{12}} - \gamma^2 I\\ &= \mathbb{H}_{\bar{T}^*_{11}\hat{T}_{12}}\mathbb{H}^*_{\bar{T}^*_{11}\hat{T}_{12}} + \mathbb{T}_{(\bar{T}^*_{11}\hat{T}^{\perp}_{12})(\bar{T}^*_{11}\hat{T}^{\perp}_{12})^*} - \gamma^2 I.\end{aligned}$$

In conclusion, condition 112 is equivalent in this case to

$$\gamma > \rho\left(\mathbb{H}_{\bar{T}^*_{11}\hat{T}_{12}}\mathbb{H}^*_{\bar{T}^*_{11}\hat{T}_{12}} + \mathbb{T}_{(\bar{T}^*_{11}\hat{T}^{\perp}_{12})(\bar{T}^*_{11}\hat{T}^{\perp}_{12})^*}\right)^{1/2}. \tag{125}$$

Finally, the conclusion follows by the above inequality and point 1. of the Theorem. ∎

6.3. An auxiliary result. In this section relation 43 will be proved. The result below is interesting in its own.

Lemma 6.11. *Let X_s and Y_s be the stabilizing solutions associated with the DDTARE$_c$ 21*

$$A^*XA - E^*XE - (\bar{\alpha}A - \alpha E)^*XBB^*X(\bar{\alpha}A - \alpha E) + C^*C = 0, \tag{126}$$

and DDTARE$_o$ 20

$$AYA^* - EYE^* - (\bar{\alpha}A - \alpha E)YC^*CY(\bar{\alpha}A - \alpha E)^* + BB^* = 0, \tag{127}$$

respectively. Consequently, define

$$\Sigma_\gamma := \left(A - zE, \begin{bmatrix} -K_s & B\end{bmatrix}; C^*C, \begin{bmatrix} C^* & 0\end{bmatrix}, \begin{bmatrix}(1-\gamma^2)I & 0\\ 0 & I\end{bmatrix}\right)$$

, and consider the associated DDTARE 76

$$\begin{aligned}0 = C^*C &+ A^*XA - E^*XE\\ &- \begin{bmatrix}C - K_s^*X(\bar{\alpha}A - \alpha E)\\ B^*X(\bar{\alpha}A - \alpha E)\end{bmatrix}^* \begin{bmatrix}(1-\gamma^2)^{-1}I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}C - K_s^*X(\bar{\alpha}A - \alpha E)\\ B^*X(\bar{\alpha}A - \alpha E)\end{bmatrix}.\end{aligned} \tag{128}$$

Suppose 128 has a stabilizing solution $X_\gamma = X_\gamma^* \geq 0$, and $\gamma > 1$. Then

$$X_\gamma = -\gamma^2 X_s \left[(1-\gamma^2)I + (\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^* X_s\right]^{-1}. \tag{129}$$

Proof. The proof will be splitted in two parts: in the first part it will be assumed (for simplicity) taht $\bar{\alpha}A - \alpha E = I$ and the result will pe proved under this assumption, whereas in the second part the assumption will be removed and the final result will be proved using the first part.

1. As explained above, assume $\bar{\alpha}A - \alpha E = I$. The idea is to link the DSPs corresponding to DDTARE 128 and DDTARE$_c$. Then, using Theorem 2.1 the claim for this part of the proof will immediately follow. Notice that the stabilizing feedbacks associated with DDTARE$_c$ and DDTARE$_o$ are given now by $F_s := -B^* X_s$ and $K_s := -Y_s C^*$, respectively. Let $F =: \begin{bmatrix} F_1^* & F_2^* \end{bmatrix}^*$ be the stabilizing feedback associated with DDTARE 128, and define $zM_\gamma - N_\gamma :=$

$$z\left[\begin{array}{c:c:cc} E & 0 & \bar{\alpha}Y_sC^* & \bar{\alpha}B \\ \hdashline \bar{\alpha}C^*C & -A^* & \bar{\alpha}C^* & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c:c:cc} A & 0 & \alpha Y_sC^* & \alpha B \\ \hdashline \alpha C^*C & -E^* & \alpha C^* & 0 \\ \hdashline C & -CY_s & (1-\gamma^2)I & 0 \\ 0 & -B & 0 & I \end{array}\right].$$

According to Theorem 2.1, this matrix pencil is exactly the DSP associated with DDTARE 128, and since it was assumed that 128 has a stabilizing solution X_γ with the associated stabilizing feedback F, it can be inferred from the same Theorem 2.1 that

$$\begin{aligned} &\left[\begin{array}{c:c:cc} E & 0 & \bar{\alpha}Y_sC^* & \bar{\alpha}B \\ \hdashline \bar{\alpha}C^*C & -A^* & \bar{\alpha}C^* & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} I \\ \hdashline -X_\gamma \\ \hdashline F_1 \\ F_2 \end{array}\right] S \\ &= \left[\begin{array}{c:c:cc} A & 0 & \alpha Y_sC^* & \alpha B \\ \hdashline \alpha C^*C & -E^* & \alpha C^* & 0 \\ \hdashline C & -CY_s & (1-\gamma^2)I & 0 \\ 0 & -B & 0 & I \end{array}\right] \left[\begin{array}{c} I \\ \hdashline -X_\gamma \\ \hdashline F_1 \\ F_2 \end{array}\right], \end{aligned}$$

for some stable matrix S. This in turn is equivalent to

$$(E + \bar{\alpha}Y_sC^*F_1 + \bar{\alpha}BF_2)S = A + \alpha Y_sC^*F_1 + \alpha BF_2, \tag{130}$$

$$(\bar{\alpha}C^*C + A^*X_\gamma + \bar{\alpha}C^*F_1)S = \alpha C^*C + E^*X_\gamma + \alpha C^*F_1, \tag{131}$$

$$C + CY_sX_\gamma + (1-\gamma^2)F_1 = 0, \tag{132}$$

$$B^*X_\gamma + F_2 = 0. \tag{133}$$

From the last two equations above, one has

$$F_1 = -(1-\gamma^2)^{-1}C(I + Y_sX_\gamma), \tag{134}$$

$$F_2 = -B^*X_\gamma \tag{135}$$

Denote $\bar{M} := (1-\gamma^2)^{-1}(I+Y_sX_\gamma)$, which shows that $\bar{M}$ is a hermitic negative-definite matrix; this is a consequence of the fact that $Y_s = Y_s^* \geq 0$ (see Lemma 2.2) and $X_\gamma = X_\gamma^* \geq 0$. With this notation, relation 134 becomes $F_1 = -C\bar{M}$. Also observe that

$$I - \bar{M} = \frac{1}{1-\gamma^2}(-\gamma^2 I - Y_sX_\gamma). \tag{136}$$

Now compute

$$\bar{\alpha} Y_s E^* = Y_s(\alpha E)^* = -(\alpha E - \bar{\alpha} A) Y_s (\alpha E)^* = -E Y_s E^* + \bar{\alpha}^2 A Y_s E^*$$
$$= (\bar{\alpha} A) Y_s (\alpha E - \bar{\alpha} A)^* + (\bar{\alpha} A) Y_s (\bar{\alpha} A)^* - E Y_s E^* = A Y_s A^* - E Y_s E^* - \bar{\alpha} A Y_s,$$

and thus $\bar{\alpha} Y_s E^* = A Y_s A^* - E Y_s E^* - \bar{\alpha} A Y_s$. In the same way $\alpha Y_s A^* = A Y_s A^* - E Y_s E^* - \alpha E Y_s$, and using DDTARE$_o$

$$\begin{aligned} \bar{\alpha} Y_s E^* &= Y_s C^* C Y_s - B B^* - \bar{\alpha} A Y_s, \\ \alpha Y_s A^* &= Y_s C^* C Y_s - B B^* - \alpha E Y_s, \end{aligned}$$

or equivalently

$$\begin{aligned} E Y_s &= \bar{\alpha} Y_s C^* C Y_s - \bar{\alpha} B B^* - Y_s A^*, \\ A Y_s &= \alpha Y_s C^* C Y_s - \alpha B B^* - Y_s E^*. \end{aligned}$$

Pre-multiplying the above two relations on the right by X_γ it follows that

$$E Y_s X_\gamma = \bar{\alpha} Y_s C^* C Y_s X_\gamma - \bar{\alpha} B B^* X_\gamma - Y_s A^* X_\gamma, \tag{137}$$
$$A Y_s X_\gamma = \alpha Y_s C^* C Y_s X_\gamma - \alpha B B^* X_\gamma - Y_s E^* X_\gamma. \tag{138}$$

Using 134, one may notice that relation 131 is equivalent to

$$(\bar{\alpha} C^* C (I - \bar{M}) + A^* X_\gamma) S = \alpha C^* C (I - \bar{M}) + E^* X_\gamma. \tag{139}$$

Putting together 130, 133, 139, as well as $F_1 = -C\bar{M}$ it follows that

$$\begin{bmatrix} E - \bar{\alpha} Y_s C^* C \bar{M} & 0 & \bar{\alpha} B \\ \bar{\alpha} C^* C (I - \bar{M}) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix} S$$

$$= \begin{bmatrix} A - \alpha Y_s C^* C \bar{M} & 0 & \alpha B \\ \alpha C^* C (I - \bar{M}) & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix} \begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix}. \tag{140}$$

After some algebra, it can be shown that:
$E(I - \bar{M}) + \bar{\alpha} B F_2 = -\frac{\gamma^2}{1-\gamma^2}(E - \bar{\alpha} Y_s C^* C) + \frac{1}{1-\gamma^2} Y_s A^* X_\gamma + \bar{\alpha} Y_s C^* C (I - \bar{M}) - \frac{\gamma^2}{1-\gamma^2} \bar{\alpha} B F_2$,
$A(I - \bar{M}) + \alpha B F_2 = -\frac{\gamma^2}{1-\gamma^2}(A - \alpha Y_s C^* C) + \frac{1}{1-\gamma^2} Y_s E^* X_\gamma + \alpha Y_s C^* C (I - \bar{M}) - \frac{\gamma^2}{1-\gamma^2} \alpha B F_2$. Pre-multiplying 140 on the left side by the product:
$\begin{bmatrix} I & Y_s & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -\frac{\gamma^2}{1-\gamma^2} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -Y_s & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ one obtains

(141)
$$- \begin{bmatrix} -\frac{\gamma^2}{1-\gamma^2}(E - \bar{\alpha} Y_s C^* C) + \bar{\alpha} Y_s C^* C (I - \bar{M}) & -\frac{\gamma^2}{1-\gamma^2} Y_s A^* & -\frac{\gamma^2}{1-\gamma^2} \bar{\alpha} B \\ \bar{\alpha} C^* C (I - \bar{M}) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix} S$$
$$= \begin{bmatrix} -\frac{\gamma^2}{1-\gamma^2}(A - \alpha Y_s C^* C) + \alpha Y_s C^* C (I - \bar{M}) & -\frac{\gamma^2}{1-\gamma^2} Y_s E^* & -\frac{\gamma^2}{1-\gamma^2} \alpha B \\ \alpha C^* C (I - \bar{M}) & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix} \begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix},$$

and notice that the first equation above can be equivalently rewritten as

$$(E(I-\bar{M})+\bar{\alpha}BF_2)S = A(I-\bar{M})+\alpha BF_2, \tag{142}$$

which finally gives

$$\begin{aligned}&\begin{bmatrix} E(I-\bar{M}) & 0 & \bar{\alpha}B \\ \bar{\alpha}C^*C(I-\bar{M}) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix} S \\ &= \begin{bmatrix} A(I-\bar{M}) & 0 & \alpha B \\ \alpha C^*C(I-\bar{M}) & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix}\begin{bmatrix} I \\ -X_\gamma \\ F_2 \end{bmatrix},\end{aligned} \tag{143}$$

or equivalently

$$\begin{bmatrix} E & 0 & \bar{\alpha}B \\ \bar{\alpha}C^*C & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (I-\bar{M}) \\ -X_\gamma \\ F_2 \end{bmatrix} S = \begin{bmatrix} A & 0 & \alpha B \\ \alpha C^*C & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix}\begin{bmatrix} (I-\bar{M}) \\ -X_\gamma \\ F_2 \end{bmatrix}. \tag{144}$$

Taking now into account that the $I-\bar{M}$ is invertible, 144 becomes equivalent to:

$$\begin{aligned}&\begin{bmatrix} E & 0 & \bar{\alpha}B \\ \bar{\alpha}C^*C & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I \\ -X_\gamma(I-\bar{M})^{-1} \\ F_2(I-\bar{M})^{-1} \end{bmatrix}(I-\bar{M})S(I-\bar{M})^{-1} \\ &= \begin{bmatrix} A & 0 & \alpha B \\ \alpha C^*C & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix}\begin{bmatrix} I \\ -X_\gamma(I-\bar{M})^{-1} \\ F_2(I-\bar{M})^{-1} \end{bmatrix}.\end{aligned} \tag{145}$$

But the matrix pencil $z\begin{bmatrix} E & 0 & \bar{\alpha}B \\ \bar{\alpha}C^*C & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & \alpha B \\ \alpha C^*C & -E^* & 0 \\ 0 & -B^* & I \end{bmatrix}$ is exactly the DSP associated with DDTARE$_c$, from where it can be inferred, using Theorem 2.1, that

$$-X_s = -X_\gamma(I-\bar{M})^{-1},$$

or equivalently

$$X_\gamma = -\gamma^2\left[(1-\gamma^2)I + X_sY_s\right]^{-1}X_s, \tag{146}$$

or yet

$$X_\gamma = -\gamma^2X_s\left[(1-\gamma^2)I + Y_sX_s\right]^{-1}, \tag{147}$$

which is exactly what was to be proved for this part.

2. If $\bar{\alpha}A-\alpha E \neq I$, then notice that X_s and Y_s verify

$$\hat{A}^*X_s\hat{A} - \hat{E}^*X_s\hat{E} - X_sBB^*X_s + \hat{C}^*\hat{C} = 0, \tag{148}$$

$$\hat{A}\hat{Y}_s\hat{A}^* - \hat{E}\hat{Y}_s\hat{E}^* - \hat{Y}_s\hat{C}^*\hat{C}\hat{Y}_s + BB^* = 0, \tag{149}$$

where $\hat{A} := A(\bar{\alpha}A - \alpha E)^{-1}$, $\hat{E} := E(\bar{\alpha}A - \alpha E)^{-1}$, $\hat{C} := C(\bar{\alpha}A - \alpha E)^{-1}$ and $\hat{Y}_s := (\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^*$. Moreover, the associated stabilizing feedbacks satisfy $\hat{F}_s = -B^* X_s = F_s(\bar{\alpha}A - \alpha E)^{-1}$, and $\hat{K}_s := -\hat{Y}_s\hat{C}^* = K_s$. In addition, X_γ is also the stabilizing solution of

$$\begin{aligned} \hat{A}^* X_\gamma \hat{A} - \hat{E}^* X_\gamma \hat{E} - \left[\begin{array}{c} \hat{C} - K_s^* X_\gamma \\ \hline B^* X_\gamma \end{array}\right]^* \left[\begin{array}{c|c} (1-\gamma^2)^{-1} I & 0 \\ \hline 0 & I \end{array}\right] \\ \cdot \left[\begin{array}{c} \hat{C} - K_s^* X_\gamma \\ \hline B^* X_\gamma \end{array}\right] + \hat{C}^* \hat{C} = 0, \end{aligned} \tag{150}$$

with the associated stabilizing feedback
$\hat{F} := \left[\begin{array}{c|c} -K_s & B \end{array}\right] \left[\begin{array}{c|c} (1-\gamma^2)^{-1} I & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} \hat{C} - K_s^* X_\gamma \\ \hline B^* X_\gamma \end{array}\right] = F(\bar{\alpha}A - \alpha E)^{-1}.$
Through this trick, the new case reduced to the previous one since now $\bar{\alpha}\hat{A} - \alpha\hat{E} = I$. Hence applying point 1 of the Lemma, it follows that

$$X_\gamma = -\gamma^2 X_s \left[(1-\gamma^2)I + \hat{Y}_s X_s\right]^{-1}, \tag{151}$$

from where conclude that

$$X_\gamma = -\gamma^2 X_s \left[(1-\gamma^2)I + (\bar{\alpha}A - \alpha E)Y_s(\bar{\alpha}A - \alpha E)^* X_s\right]^{-1} \tag{152}$$

which ends the whole proof. □

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