

# Global Optimisation of Multi-Plant Manganese Alloy Production

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## Abstract

This paper studies the problem of multi-plant manganese alloy production. The problem consists of finding the optimal furnace feed of ores, fluxes, coke, and slag that yields output products which meet customer specifications, and to optimally decide the volume, composition, and allocation of the slag. To solve the problem, a nonlinear pooling problem formulation is presented upon which the bilinear terms are reformulated using the Multiparametric Disaggregation Technique (MDT). This enables global optimisation by means of commercial software for mixed integer linear programs. We demonstrate the model and solution approach through case studies from a Norwegian manganese alloy producer. The computational study shows that the model and proposed optimisation approach can solve problem sizes of up to ten furnaces to a small optimality gap, that global optimization approach with MDT scales well with larger, real problem instances, and that the model outperforms the current operational practice.

**Keywords:** Manganese Alloy Production, Pooling Problem, Multiparametric Disaggregation Technique, Global Optimisation, Multi-plant Production, Mixed Integer Linear Programming

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## 1. Introduction

Manganese is a hard, brittle, silvery metal that occurs in nature in the form of minerals, mainly as oxides. It is an essential element in steel and aluminium alloys, commonly used in railway tracks and safes, and beverage cans and kitchenware, respectively. The total production of manganese alloys has been approximately twenty million tonnes annually in the recent years (*d'Harambure, 2015*). An average price of manganese alloys around 2 USD/kg (*InvestmentMine, 2017*) makes the manganese alloy production a multi-billion dollar industry.

Manganese alloy production can be divided into two categories: extraction and smelting. Extraction constitutes the processes of mining, hauling the ore to a processing plant, crushing, separation and beneficiation at the plant, transportation to sinter plants, and sintering (*Olsen et al., 2007*). The smelting process constitutes the processes of

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10 smelting ores, fluxes, quartz, and coke in furnaces, tapping and casting, refining, crushing, and transportation of  
11 by-products back into processes or to disposal sites (*International Manganese Institute and Hatch, 2015*).

12 The focus of this paper is on the smelting process of the manganese alloy supply-chain. Current operational  
13 practice in this sector is largely based on process operators' experience and process knowledge, where emphasis  
14 is put on stable and efficient production of each individual plant and furnaces, with little integration of the overall  
15 multi-plant structure. The literature on manganese production optimisation is limited. To the authors' knowledge,  
16 only one article exists on the topic. *Jipnang et al. (2013)* present a single high-carbon ferromanganese (HC FeMn)  
17 and medium-carbon silicomanganese (MC SiMn) furnace process-optimisation model, based on mass and energy  
18 balances. The model only focuses on the production specific aspects of the problem, and optimises a target function  
19 such as total operating costs, energy consumption, Mn recovery, or the amount of slag produced from the furnace. The  
20 model relies on proprietary software, thereby hiding the model applied. It is capable of calculating the production  
21 for single HC FeMn and MC SiMn furnaces, respectively, and the paper states that connecting the two processes  
22 and adding possibilities for different production strategies are considered future research (*Jipnang et al., 2013*). An  
23 optimisation model that considers the integrated production of FeMn and SiMn alloys across multiple plants can  
24 hence advance both current practice and targeted optimisation of the production planning. To this end, a multi-plant  
25 manganese alloy production-planning problem consists of optimizing the mixing of raw materials and composition of  
26 end-products satisfying the demand and given quality specifications, while incorporating the possibility of transporting  
27 by-products and other materials between the furnaces and plants to reduce costs and improve profit for the company.  
28 In the remainder of the paper, this considered problem is denoted as the Manganese Alloy Multi-plant Production  
29 (MAMP) problem.

30 Production of manganese alloys resorts to smelting of raw materials in furnaces, with flows of output by-products  
31 between the furnaces. This structure enables the MAMP problem to be formulated as a nonlinear pooling problem,  
32 with intermediate pools present in the form of furnaces and refining stations. The raw material inventories are sources  
33 and end- and by-product inventories are terminals. The pooling problem is a generalisation of the blending problem  
34 introduced by *Haverly (1978)*, and is used to model systems that have intermediate mixing pools in the blending  
35 process of streams with varying qualities and volumes (*Audet et al., 2004*). A complicating element for the MAMP  
36 problem, however, is that intermediate pools are coupled as a result of utilisation of by-products.

37 The pooling problem arises in a variety of industries including oil refining (*Ben-Tal et al., 1994; Amos et al.,*  
38 *1997*), mining industry (*Boland et al., 2015*), and wastewater network problems (*Meyer and Floudas, 2006; Jezowski,*  
39 *2010*). Blending stream qualities results in nonlinear terms in the pooling problem formulation, yielding a nonconvex  
40 nonlinear program (NLP) (*Audet et al., 2004; Alfaki, 2012*). These pooling problems, also classified as bilinear process  
41 networks, are generally difficult to solve to global optimality since bilinear constraints are required to model the mixing  
42 of different streams (*Kolodziej et al., 2013b*). Multiple optimisation formulations of the pooling problem are found in  
43 the literature. Formulating the standard pooling problem in different ways have varying ramifications for the problem  
44 size and relaxation tightness, although the formulations are mathematically equivalent (*Misener and Floudas, 2009*).

45 The most common formulations for the standard and generalised pooling problem are the P-formulation (*Haverly,*  
46 *1978*), the Q-formulation (*Ben-Tal et al., 1994*), and the PQ-formulation (*Quesada and Grossmann, 1995; Sherali et al.,*  
47 *1998; Tawarmalani and Sahinidis, 2002*).

48 Solution methods for the pooling problem can generally be classified into local and global optimisation methods  
49 (*Alfaki and Haugland, 2013*). Guaranteeing global optimality is of major importance, as the objective function typically  
50 is related to an economic metric (*Teles et al., 2012*). A summary of some of the different solution methods can be  
51 found in *Misener and Floudas (2009)*, including Successive Linear Programming (SLP), Lagrangian approaches, the  
52 Reformulation Linearisation Technique (RLT), and different branch-and-bound schemes.

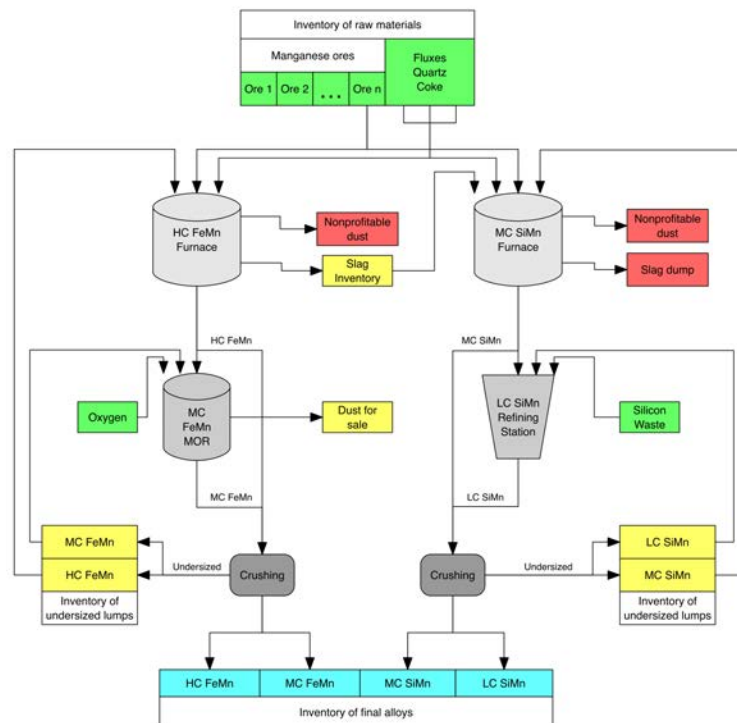
53 A relatively recent solution method to pooling problems is the Multiparametric Disaggregation Technique (MDT)  
54 (*Teles et al., 2012, 2013; Kolodziej et al., 2013a,b*). The method relies on a concept based on the characteristics of the  
55 decimal representation of real numbers. The NLP is transformed into a suitably reformulated problem containing new  
56 sets of continuous and discrete variables. By disaggregating and parameterising the variables in the nonlinear terms, it  
57 is shown how to approximate the original NLP formulation as a mixed integer linear program (MILP). The quality of  
58 the solution depends on the number of significant digits used to represent the number (*Teles et al., 2012*).

59 Nonconvex NLPs with multiple local optima may renders the use of conventional NLP solvers ineffective (*Teles*  
60 *et al., 2012; Wicaksono and Karimi, 2008*). On the other hand, general-purpose global optimization solvers may scale  
61 poorly with larger problem sizes, as they lack the capability of exploiting special structures of the nonconvexities in  
62 classes of problems such as the pooling problem. *Kolodziej et al. (2013a)* show that the MDT relaxation applied to  
63 large problems compares favorably with general global optimisation solvers. They also show that the solution from the  
64 upper and lower bounding formulations converge towards the original nonlinear formulation in the limit of an infinite  
65 number of discretisation intervals. Compared to spatial branch-and-bound involving a continuous relaxation, the MDT  
66 involves a discrete partition of the feasible region. This partitioning means one can use standard MILP solvers to  
67 generate an  $\epsilon$ -optimal solution, given that one exists for the selected accuracy settings. Further, the MDT does not  
68 require the specification of an initial point (*Teles et al., 2012*), and has been demonstrated to scale well with increasing  
69 problems sizes (*Castro and Teles, 2013*). For these reasons, the MDT is selected to be the method for reformulating the  
70 bilinear constraints present in the MAMP.

71 The main contribution of this paper is the formulation of an optimisation model for decision support of multi-plant  
72 manganese alloy production planning. The optimisation model is tested on a case study based on the plant locations  
73 and furnace setup of Eramet Norway. Further contributions include (I) a general, nonlinear formulation of the problem  
74 in consideration, applicable to any alloy production with similar processes as manganese alloy production, and (II) a  
75 demonstration of the MDT to solve a large-scale industrial pooling problem. The remainder of the paper is organised  
76 as follows. First, a brief introduction to the manganese alloy production problem is given. Then, the mathematical  
77 model is presented and the bilinear constraints in the pooling problem are reformulated using the MDT. Finally, a  
78 computational study is conducted based on a realistic case, followed by results, concluding remarks, and considerations  
79 for future research.

## 2. Problem Description

A manganese alloy manufacturer has a set of furnaces located at plants to produce manganese alloys. The alloys produced are given by customer specifications. The production is, therefore, based on contracts that must be satisfied. Customer specifications include order volume and alloy composition, resulting in a wide range of possible order sizes and end-products. To meet the end-product specifications, a set of raw materials, including ores, fluxes, and coke sources, containing different concentrations of various elements and oxides is available to the production. The raw materials are blended in the furnaces and further processed to produce the desired end-products. Any excess end-product produced can be sold on optional contracts in the spot market or held as an inventory. Producing manganese alloys also yields various by-products, where some are valuable and may be sold.



**Figure 1:** Overview of the material flow in manganese alloy production using the duplex method. Green colour: raw materials. Red: wastes. Yellow: inventory of reusable/saleable materials. Blue: end-products. Scales of grey: the furnace, refining, and crushing processes.

Figure 1 provides an overview of the processes and material flow in a manganese alloy plant producing several grades of both ferro and silico manganese. The raw material inventory supplies the necessary resources to the HC FeMn and MC SiMn furnaces. The output from the furnaces are Mn alloys, slag, and dust. The Mn alloys are either crushed into end-products or refined into low carbon (LC) or medium carbon (MC) products. Refining by Manganese Oxygen Refining (MOR) also produces metal-oxide dust as a salable by-product. The liquid alloys are solidified before crushing

94 to end-products of different sizes. End-products are stored at sales inventories, while undersized lumps fines are kept  
 95 at inventories and reused or sold at reduced price. The slag produced by HC FeMn furnaces can either be discarded  
 96 or reused to save raw material costs in the MC SiMn furnaces, the latter being the common industry practice. This  
 97 practice is called the duplex method (*Olsen et al., 2007*) and couples the otherwise independent production paths.  
 98 Both slag-to-metal ratio and slag composition can be manipulated through ore combinations and furnace temperatures,  
 99 thereby potentially achieving a more efficient production by blending slags from different HC FeMn furnaces in the  
 100 MC SiMn furnaces. As both furnace types are not necessarily located at the same plant, slag must be transported  
 101 between plants. Thus, the decision-making process is complicated by the slag-to-metal-ratio, the slag composition, and  
 102 the volume of slag to send to each MC SiMn furnace from each HC FeMn furnace. The furnaces used to smelt the raw  
 103 materials can produce both HC FeMn and MC SiMn alloys, but only one alloy type at a time. Each furnace has a mass  
 104 and electrical power capacity that limits the raw material feed to the furnace. The furnaces also have limitations on  
 105 the amount of fines it is possible to feed, since feeding too much fines (or undersized lumps) may lower the furnace  
 106 temperature and thereby impede efficient furnace operations.

107 The MAMP problem must include mass and energy balances for each furnace. Smelting of ores with addition  
 108 of fluxes and coke to produce silico and ferroalloys constitutes highly complex chemical processes occurring over a  
 109 wide range of temperatures. Imposing the full mass and energy balance for all elements present in the ores, slags and  
 110 end-products is therefore intractable for multi-plant production planning. Still, only a subset of all present elements in  
 111 the furnace processes, in particular Mn, Fe, Si, C, Al, Mg, Ca, and their associated oxides, is essential for deciding the  
 112 optimal blend and thereby compute good solutions to the production-planning problem. To this end, we include the  
 113 mass and energy balances for the reactions of these main elements, provided in reactions (70)–(82) in Appendix 8. We  
 114 assume that all reactions are steady state and hence that the chemical reactions in the appendix are complete.

115 The set of reactions taking place in the furnaces are both exothermic and endothermic, while the overall process is  
 116 strongly endothermic. HC FeMn production has a typical overall electrical energy consumption of 2500kWh per tonnes  
 117 of alloy produced. For MC SiMn, the same number is about 4500kWh/tonnes. The electrical power consumption of the  
 118 furnaces is therefore an important cost driver in manganese alloy production, by the raw material composition affecting  
 119 the electrical energy consumption, and with almost twice as high electricity cost for producing MC SiMn compared  
 120 with HC FeMn. The energy consumption in a furnace is determined by the net effect of exothermic and endothermic  
 121 reactions and the enthalpy of the materials entering and leaving the furnace (*Olsen et al., 2007*). The total enthalpy  
 122 consists of formation enthalpy and sensible enthalpy. An approximate electric power consumption is then given by

$$123 \quad \bar{P} = (H_{\text{out}}^{\text{F}} + H_{\text{out}}^{\text{S}}) - (H_{\text{in}}^{\text{F}} + H_{\text{in}}^{\text{S}}) + Q^{\text{loss}} \quad (1)$$

124 where  $H_{\text{out}}^{\text{F}}$  represents the formation enthalpy and  $H_{\text{out}}^{\text{S}}$  the sensible enthalpy of the resultants at the exit temperature,  
 125  $H_{\text{in}}^{\text{F}}$  the formation enthalpy and  $H_{\text{in}}^{\text{S}}$  the sensible enthalpy of the reactants at the entry temperature,  $Q^{\text{loss}}$  the net heat-loss  
 126 to the surroundings, and  $\bar{P}$  the electricity (work) fed to the furnaces. Gas emissions, both from CO, CO<sub>2</sub>, and vaporised  
 127 H<sub>2</sub>O constitute a significant part of the furnace power consumption (*Olsen et al., 2007*).

128 Refining stations are required to produce MC FeMn and LC SiMn products. These refining units have a mass  
129 and energy capacity limiting the feed to each process. Each plant has inventories for storing resources and end-  
130 products, which at each plant are divided into raw material inventories, recycling materials inventories, and end-product  
131 inventories. All inventories have capacity limits. The slag produced by the furnaces is a proportion to the metal  
132 produced, and this ratio can fluctuate between a lower and upper bound dependent on the raw materials used for a  
133 specific alloy and slag. Slag compositions may vary in each HC FeMn furnace. The reuse of HC FeMn slag in MC  
134 SiMn furnaces is the main coupling between the FeMn and SiMn productions. Slag from the MC SiMn furnaces and  
135 the non-profitable dust produced by both furnace types is discarded.

136 Revenues and costs are linked to various parts of the production. The resources used in the production, except  
137 the undersized lumps, are associated with a procurement cost. Smelting the raw materials in the furnaces requires  
138 energy in the form of electricity. Thus, the furnace process incurs electricity costs. Reusing slag produced by HC FeMn  
139 furnaces in MC SiMn furnaces at other plants incurs a transportation cost per tonne slag transported. Slag from the HC  
140 FeMn furnaces can also be discarded instead of reused, which incurs a discard cost. The other by-products that must be  
141 discarded also incur a discard cost. The metallic-oxide dust from the MOR process is associated with a revenue since it  
142 can be sold. Each end-product is associated with a revenue per tonne sold on fixed contracts or in the spot market.

143 The objective of the MAMP problem is to optimise the integrated production of FeMn and SiMn alloys across  
144 multiple plants to maximise profit. The profit is determined by deciding the optimal volumes of end-products to  
145 produce by mixing raw materials, while satisfying given quality specifications. Production costs are considerable, and  
146 the MAMP problem should, therefore, ensure optimal use of raw materials to the furnaces and refining processes. The  
147 solution to the MAMP problem should also describe the optimal slag volume and slag composition to be produced in  
148 the HC FeMn furnaces and the allocation of slag to the MC SiMn furnaces.

### 149 **3. Mathematical Formulation**

150 Multi-plant production, blending, advanced chemistry, and the coupling of the FeMn and SiMn productions that  
151 make up the MAMP problem add a high degree of complexity to the formulation of the problem. To reduce the scope  
152 of the problem, model assumptions are made. The MAMP problem is solved once and the production plan given by  
153 the solution can be used for the given planning period. The end-product demands are assumed to originate from fixed  
154 contracts and a spot market. Fixed contracts are known a priori and the demanded chemical compositions are given.  
155 The excess production can be sold on the spot market up to a limit. The raw material procurement prices, electricity  
156 prices, and end- and by-product sales prices are constant. Switching furnace settings or switching furnaces on and off  
157 are not included in the model.

158 There is an initial inventory of raw materials at each plant. The reuse of undersized lumps must be balanced so the  
159 process cannot consume more undersized lumps than it produces. The feed of undersized lumps is therefore bounded  
160 by the produced volume of undersized lumps. We assume fixed amounts of crushed product that ends up as undersized

161 lumps and the percentage of the total feed to the processes that ends up as by-products.

162 A large set of chemical reactions is involved in the production of manganese alloys. The model includes the main  
 163 reactions occurring in the furnaces to ensure the correct weight fractions of elements in the final alloy. All chemical  
 164 reactions are assumed to be complete. The reactions (70) - (82) in Appendix 8 therefore translate into linear equality  
 165 constraints. In practice, many complicated and incomplete chemical reactions happen in the furnace. However, for  
 166 simplicity, the slag is set to only consist of the most important oxides in the model. These oxides are MnO, FeO, SiO<sub>2</sub>,  
 167 Al<sub>2</sub>O<sub>3</sub>, MgO, and CaO.

168 Process metallurgists typically want to determine a range of slag compositions, given by the lime basicity (*Olsen*  
 169 *et al.*, 2007), to ensure a suitable viscosity of the slag. These considerations are imposed through constraints on the  
 170 slag composition. The specific carbon content of a end-product is only considered within medium and high range for  
 171 FeMn and low and medium range for SiMn. Detailed classifications of product types by percentage carbon content  
 172 within these ranges are disregarded, yielding a reduced set of end-products.

173 The model is flow- and quality-based specifically developed for manganese alloy multi-plant production. We use the  
 174 P-formulation (Haverly, 1978), as this is the most common formulation used in the chemical processing industry and it  
 175 provides an intuitive understanding of the process flows and their qualities for this new problem.

**Table 1:** Sets and indices

Set	
$\mathcal{P}$	Set of plants, indexed by $p, g$
$\mathcal{F}_p$	Set of furnaces at plant $p$ , indexed by $f, t$
$\mathcal{E}$	Set of end-products, indexed by $e$
$\mathcal{B}$	Set of by-products, indexed by $b$
$\mathcal{R}$	Set of raw materials, indexed by $r, \rho$
$\mathcal{K}$	Set of elements and oxides, indexed by $k$
$\mathcal{C}$	Set of chemical reactions, indexed by $c$
$\mathcal{V}$	Set of variables in the chemical reactions, indexed by $v$
$\mathcal{F}_p^{\text{FeMn}}$	Subset of all HC FeMn furnaces at plant $p$ , $\mathcal{F}_p^{\text{FeMn}} \subseteq \mathcal{F}_p$
$\mathcal{F}_p^{\text{SiMn}}$	Subset of all MC SiMn furnaces at plant $p$ , $\mathcal{F}_p^{\text{SiMn}} \subseteq \mathcal{F}_p$
$\mathcal{C}^O$	Subset of original chemical reactions, $\mathcal{C}^O \subset \mathcal{C}$
$\mathcal{C}^C$	Subset of critical chemical reactions, $\mathcal{C}^C \subset \mathcal{C}$
$\mathcal{C}^S$	Subset of slag chemical reactions, $\mathcal{C}^S \subset \mathcal{C}$
$\mathcal{K}^C$	Subset of critical elements and oxides, $\mathcal{K}^C \subset \mathcal{K}$
$\mathcal{K}^G$	Subset of gases, $\mathcal{K}^G \subset \mathcal{K}$
$\mathcal{K}^S$	Subset of elements and oxides in the slag, $\mathcal{K}^S \subset \mathcal{K}$

**Table 2:** Parameters

Parameter	
$A_{fkcv}$	Constant for an element or oxide $k$ in chemical reaction $c$ for variable $v$ in furnace $f$ .

$A_{fkc\nu}^{LS}$	Constant for an element or oxide $k$ in the left side ratio equation for chemical reaction $c$ for variable $\nu$ in furnace $f$ .
$A_{fkc\nu}^{RS}$	Constant for an element or oxide $k$ in the right side ratio equation for chemical reaction $c$ for variable $\nu$ in furnace $f$ .
$B_{fkc}$	1 if an element or oxide $k$ exist in chemical equation $c$ for furnace $f$ , 0 otherwise.
$C_r$	Procurement cost per tonne raw material $r$ .
$C^E$	Electricity cost per kWh.
$C^{LSiL}$	Cost per tonne LC SiMn undersized lumps used.
$C^{MFeL}$	Cost per tonne MC FeMn undersized lumps used.
$C^O$	Cost per tonne oxygen used. This includes procurement and electricity cost.
$C^S$	Discard cost per tonne slag.
$C^{SiW}$	Cost per tonne silicon waste used. This includes procurement and electricity cost.
$C_{pg}^T$	Transportation cost per tonne slag from plant $p$ to plant $g$ .
$D_e^F$	Fixed contract demand for end-product $e$ .
$D_e^O$	Spot market limit for end-product $e$ .
$H_k^F$	Formation enthalpy for each element or oxide $k$ , in kJ/tonne.
$H_k^S$	Sensible enthalpy for each element or oxide $k$ , in kJ/tonne.
$I_{pr}$	Initial inventory of raw material $r$ at plant $p$ in tonnes.
$I_p^{LSiL}$	Initial inventory of LC SiMn undersized lumps at plant $p$ in tonnes.
$I_p^{MFeL}$	Initial inventory of MC FeMn undersized lumps at plant $p$ in tonnes.
$I_p^O$	Initial inventory of oxygen at plant $p$ in tonnes.
$I_p^{SiW}$	Initial inventory of silicon waste at plant $p$ in tonnes.
$L^H$	Furnace heat loss factor.
$M_k$	Molar mass in moles per tonne for element or oxide $k$ .
$Q_{pf}^F$	Total capacity of furnace $f$ at plant $p$ in tonnes.
$Q_p^{MOR}$	Total MOR capacity at plant $p$ in tonnes.
$Q_p^{REF}$	Total LC SiMn refining station capacity at plant $p$ in tonnes.
$R_b^B$	Revenue or discard cost per tonne of by-product $b$ .
$R_e^F$	Fixed contract revenue per tonne end-product $e$ sold.
$R_e^O$	Spot price per tonne end-product $e$ sold on the spot market.
$T_{kcv}$	1 if element or oxide $k$ exist in chemical equation $c$ for variable $\nu$ , 0 otherwise.
$\underline{\Lambda}$	Lower limit on the weight percentage for slag production in an HC FeMn furnace.
$\overline{\Lambda}$	Upper limit on the weight percentage for slag production in an HC FeMn furnace.
$\Upsilon$	Degree of pre-reduction in the HC FeMn furnaces.
$\underline{\Phi}_k$	Lower limit on the weight percentage for element or oxide $k$ in slag.
$\overline{\Phi}_k$	Upper limit on the weight percentage for element or oxide $k$ in slag.
$\Psi_{fbk}^B$	Weight percentage of element or oxide $k$ in by-product $b$ from furnace $f$ .
$\Psi_b^{CRUSH}$	Weight percentage of by-product $b$ from the crushing process.
$\Psi_k^{FeMn}$	Weight percentage of element or oxide $k$ in HC FeMn.
$\Psi^{LSiL}$	Weight percentage LC SiMn undersized lumps allowed to feed LC SiMn refining station.
$\Psi^{MFeL}$	Weight percentage MC FeMn undersized lumps allowed to feed MOR.
$\Psi_b^{MOR}$	Weight percentage of by-product $b$ from MOR.
$\Psi_{rk}^R$	Weight percentage of element or oxide $k$ in raw material $r$ .
$\Psi_k^{SiMn}$	Weight percentage of element or oxide $k$ in MC SiMn.
$\Psi^{UL}$	Weight percentage of undersized lumps allowed to feed a furnace.

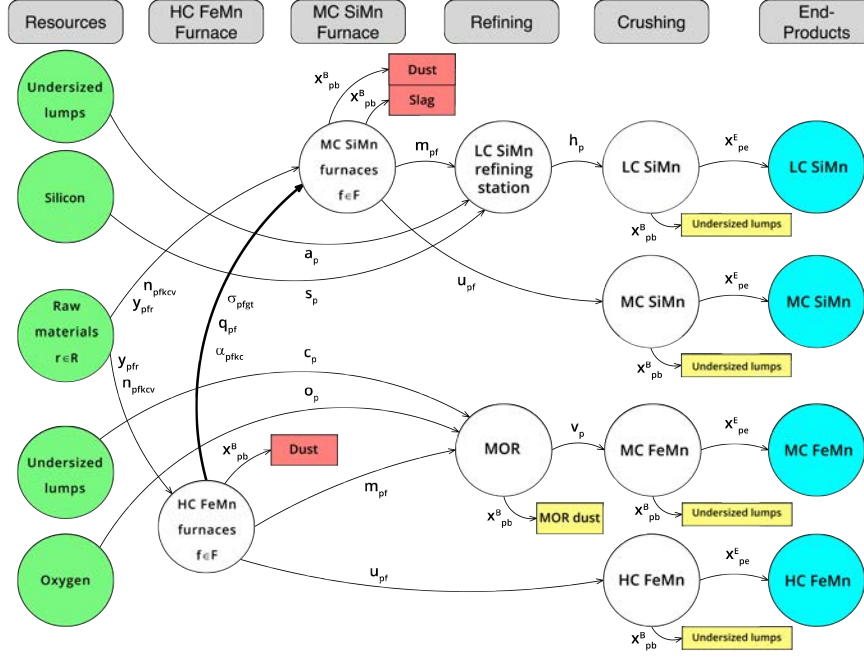


$\Omega^{\text{MOR}}$	Oxygen-HC FeMn weight relationship factor.
$\Omega^{\text{REF}}$	Silicon-MC SiMn weight relationship factor.

**Table 3: Variables**

Variable	
$a_p$	Tonnage of LC SiMn undersized lumps used in the LC SiMn refining station at plant $p$ .
$c_p$	Tonnage of MC FeMn undersized lumps used in MOR at plant $p$ .
$e_{pf}$	Electric power consumed by furnace $f$ at plant $p$ , in kWh.
$g_e^{\text{F}}$	Sale of end-product $e$ made on fixed contracts.
$g_e^{\text{O}}$	Sale of end-product $e$ on the spot market.
$h_p$	Tonnage of LC SiMn produced at plant $p$ sent to crushing.
$m_{pf}$	Tonnage of alloy produced in furnace $f$ at plant $p$ sent to refining processes.
$n_{pfkcv}$	Moles of element or oxide $k$ in furnace $f$ at plant $p$ in equation $c$ for variable $v$ .
$o_p$	Tonnage oxygen fed to the MOR at plant $p$ .
$q_{pf}$	Tonnage slag produced in furnace $f$ at plant $p$ .
$s_p$	Tonnage silicon fed to the LC SiMn refining at plant $p$ .
$u_{pf}$	Tonnage of alloy produced in furnace $f$ at plant $p$ sent to crushing.
$x_{pe}^{\text{E}}$	Tonnage of end-product $e$ produced at plant $p$ .
$x_{pb}^{\text{B}}$	Tonnage of by-product $b$ produced at plant $p$ .
$y_{pfr}$	Tonnage of raw material $r$ fed to furnace $f$ at plant $p$ .
$\alpha_{p f k c}$	Moles of element or oxide $k$ in chemical equation $c$ extracted as slag from furnace $f$ at plant $p$ .
$\sigma_{p f g t}$	Tonnage slag sent from furnace $f$ at plant $p$ to furnace $t$ at plant $g$ .
$\phi_{p f k}$	Weight percentage of element or oxide $k$ in the slag produced by furnace $f$ at plant $p$ .

176 Figure 2 illustrates the material flow within a plant and which processes the variables are describing, using a  
177 simplified superstructure. As an example, the variables  $y_{pfr}$  and  $n_{pfkcv}$  are related to the feeding of the furnaces from  
178 the raw material inventory, while  $a_p$  and  $s_p$  are related to the feeding of the LC SiMn refining station from refining  
179 resources.  $\phi_{p f k}$ ,  $e_{pf}$ ,  $g_e^{\text{F}}$ , and  $g_e^{\text{O}}$  are not included in the figure since these are quality variables and not flow variables.  
180 The MAMP problem is, however, defined for multiple plants. Slag can be sent from an HC FeMn furnace at one plant  
181 to multiple MC SiMn furnaces, at the same plant or other plants.



**Figure 2:** The MAMP superstructure for one plant. The same colour coding is applied as in Figure 1. Green: raw materials. Red: wastes. Yellow: inventory of reusable/saleable materials. Blue: final alloys.

### 182 3.1. Model

$$\begin{aligned}
 \max z = & \sum_{e \in \mathcal{E}} (R_e^F g_e^F + R_e^O g_e^O) + \sum_{p \in \mathcal{P}} \sum_{b \in \mathcal{B}} R_b^B x_{pb}^B - \sum_{p \in \mathcal{P}} \sum_{f \in \mathcal{F}_p} \sum_{r \in \mathcal{R}} C_r y_{pfr} - \sum_{p \in \mathcal{P}} \sum_{f \in \mathcal{F}_p} C^E e_{pf} - \sum_{p \in \mathcal{P}} (C^O o_p + C^{\text{MFeL}} c_p) \\
 & - \sum_{p \in \mathcal{P}} (C^{\text{SiW}} s_p + C^{\text{LSiL}} a_p) - \sum_{p \in \mathcal{P}} \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} \sum_{g \in \mathcal{P}} \sum_{t \in \mathcal{F}_g^{\text{SiMn}}} C_{pg}^T \sigma_{pfgt} - \sum_{p \in \mathcal{P}} \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} C^S (q_{pf} - \sum_{g \in \mathcal{P}} \sum_{t \in \mathcal{F}_g^{\text{SiMn}}} \sigma_{pfgt}) \quad (2)
 \end{aligned}$$

183 The objective function 2 maximises the total profit from selling end- and by-products from manganese alloy  
 184 production. It consists of revenue generated by selling end-products and selling and discarding by-products. It also  
 185 include cost of raw materials, electricity cost, cost of oxygen and MC FeMn undersized lumps added to the MOR  
 186 process and cost of silicon waste and LC SiMn undersized lumps added to the LC SiMn refining station process. Finally,  
 187 total slag transportation cost between plants and cost of discarding the slag that is not re-used are subtracted.

188 To enhance readability and the understanding of which constraints restrict each process stage, the constraints are  
 189 presented in different sections. The sections are presented in order of process stage according to Figure 2.

#### 190 Resource inventory

$$\sum_{f \in \mathcal{F}_p} y_{pfr} \leq I_{pr} \quad p \in \mathcal{P}, r \in \mathcal{R} \quad (3)$$

$$o_p \leq I_p^O \quad p \in \mathcal{P} \quad (4)$$

$$s_p \leq I_p^{\text{SiW}} \quad p \in \mathcal{P} \quad (5)$$

$$c_p \leq I_p^{\text{MFeL}} \quad p \in \mathcal{P} \quad (6)$$

$$a_p \leq I_p^{\text{LSiL}} \quad p \in \mathcal{P} \quad (7)$$

191 Constraints (3) - (7) are resource inventory constraints limiting the feed of particular resources to within the initial  
192 inventories of the respective resources.

193 *Furnace constraints*

$$\sum_{r \in \mathcal{R}} y_{pfr} + \sum_{k \in \mathcal{K}^S} \sum_{c \in \mathcal{C}^S} M_k n_{pfkc} v \leq Q_{pf}^F \quad p \in \mathcal{P}, f \in \mathcal{F}_p, v \in \{\text{SLAG}\} \quad (8)$$

$$y_{pfr} \leq \Psi^{\text{UL}} \sum_{\rho \in \mathcal{R}} y_{p\rho} \quad p \in \mathcal{P}, f \in \mathcal{F}_p, r \in \{\text{HC FeMn, MC SiMn}\} \quad (9)$$

$$\begin{aligned} & \sum_{r \in \mathcal{R}} \sum_{k \in \mathcal{K}} \Psi_{rk}^R y_{pfr} + \sum_{k \in \mathcal{K}^S} \sum_{c \in \mathcal{C}^S} M_k n_{pfkc, \text{SLAG}} + \sum_{c \in \mathcal{C}^O} M_{\text{O}n_{pf, \text{CO}_2, c, \text{TOT}}} \\ & + \sum_{c \in \mathcal{C}^O} M_{\text{CO}n_{pf, \text{CO}, c, \text{TOT}}} - \sum_{b \in \mathcal{B}} \sum_{k \in \mathcal{K}} \sum_{r \in \mathcal{R}} \Psi_{fbk}^B \Psi_{rk}^R y_{pfr} \\ & - \sum_{b \in \mathcal{B}} \sum_{k \in \mathcal{K}^S} \sum_{c \in \mathcal{C}^S} \Psi_{fbk}^B M_k n_{pfkc, \text{SLAG}} - \sum_{k \in \mathcal{K}^G} \sum_{c \in \mathcal{C}^O \setminus \{18\}} M_k n_{pfkc, \text{RSRED}} \\ & - m_{pf} - u_{pf} - q_{pf} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (10) \end{aligned}$$

$$x_{pb}^B = \sum_{f \in \mathcal{F}_p} \sum_{r \in \mathcal{R}} \sum_{k \in \mathcal{K}} \Psi_{fbk}^B \Psi_{rk}^R y_{pfr} + \sum_{f \in \mathcal{F}_p} \sum_{k \in \mathcal{K}^S} \sum_{c \in \mathcal{C}^S} \Psi_{fbk}^B M_k n_{pfkc} \quad p \in \mathcal{P}, b \in \mathcal{B}, v \in \{\text{SLAG}\} \quad (11)$$

194 Constraints (8) restrict the feed of raw materials and slag to a furnace to within the capacity of the furnace.

195 Constraints (9) handle the reuse of undersized lumps relative to the feed of other resources used in a furnace.

196 Constraints (10) handle the mass balance in a furnace. The constraints include the mass of the modelled elements  
197 and oxides fed to the furnace, the mass of the slag fed to the furnace, the mass of oxygen accounted for twice due  
198 to the modelling of CO<sub>2</sub> entering the Boudouard reaction, the mass of CO taking part in the prereduction in the  
199 furnace, less the mass of the furnace by-products from the raw material feed and slag feed, the mass of CO and CO<sub>2</sub>  
200 off-gas emissions, the metal output to either the MOR or LC SiMn refining and crushing processes and the mass of  
201 produced slag. For HC FeMn furnaces, the slag terms  $n_{pfkc, \text{SLAG}}$  are zero as no slag is sent to an HC FeMn furnace.  
202 For MC SiMn furnaces, the produced slag terms  $q_{pf}$  are zero as the slag is assumed to be a discard slag. The term  
203  $\sum_{k \in \mathcal{K}^G} \sum_{c \in \mathcal{C}^O \setminus \{18\}} M_k n_{pfkc, \text{RSRED}}$  excludes chemical reaction 18, which is the Boudouard reaction, because it uses the  
204 same variable name, but it accounts for the off-gases leaving the furnace, not the re-entering gas.

205 Constraints (11) state the relationship between the total feed of raw materials and slag sent to the furnaces and  
206 the amount of a discardable by-product produced by the furnaces at a plant. The constraints for electrical power

207 consumption follows the given thermodynamic relations. They have the same structure as (10) using coefficients for  
 208 formation and sensible enthalpies for the furnace temperature. These constraints are omitted for readability.

209 *Furnace-Slag Connection Constraints*

$$\phi_{pfk}q_{pf} = M_k \sum_{c \in \mathcal{C}^S} B_{fkc} \alpha_{p f k c} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (12)$$

$$\sum_{g \in \mathcal{P}} \sum_{t \in \mathcal{F}_g^{\text{SiMn}}} \sigma_{p f g t} \leq q_{p f} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}} \quad (13)$$

$$\sum_{p \in \mathcal{P}} \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} \phi_{p f k} \sigma_{p f g t} = M_k \sum_{c \in \mathcal{C}^S} n_{g t k c v} \quad g \in \mathcal{P}, t \in \mathcal{F}_g^{\text{SiMn}}, k \in \mathcal{K}^S, v \in \{\text{SLAG}\} \quad (14)$$

$$\bar{\Phi}_k \geq \phi_{p f k} \geq \underline{\Phi}_k \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (15)$$

$$\sum_{k \in \mathcal{K}^S} \phi_{p f k} = 1 \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}} \quad (16)$$

$$\bar{\Lambda}(m_{p f} + u_{p f}) \geq q_{p f} \geq \underline{\Lambda}(m_{p f} + u_{p f}) \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}} \quad (17)$$

210 Constraints (12) couple the produced amount of slag  $q_{p f}$  in an HC FeMn furnace and its constituent fractions  $\phi_{p f k}$ ,  
 211 to the mass of the constituents  $M_k \alpha_{p f k c}$  pulled from the chemical reactions occurring in the HC FeMn furnace. Thus,  
 212 the mass of element or oxide  $k$  in the slag extracted from an HC FeMn furnace equals the amount of mass of element or  
 213 oxide  $k$  removed from the redox reactions in the furnace. The left-hand side terms of the constraints are nonlinear and,  
 214 therefore, complicates the problem. The constraints are unique to this problem because there are no pooling problems  
 215 in the manganese alloy industry, to the authors' knowledge, that extracts a proportion of a specific constituent from a  
 216 blending process. The closest similarities may be found in the separation processes in the crude oil industry.

217 Constraints (13) state that sending slag to MC SiMn furnaces from an HC FeMn furnace is optional, by allowing  
 218 less than the produced slag to be sent. This allows the slag to be discarded if it is unfavourable to feed it to MC  
 219 SiMn furnaces. The slag transportation and slag feed to an MC SiMn furnace are coupled by constraints (14). These  
 220 are nonlinear terms common to the pooling problem. Constraints (15) induce lower and upper bounds on the slag  
 221 composition. Constraints (16) enforce that the sum of the weight percentages of all the slag constituents must make  
 222 up the total slag content. Constraints (17) ensure that it is always produced at least a minimum amount of slag in a  
 223 HC FeMn furnace relative to the metal produced and set the upper bound on the slag production relative to the metal  
 224 production.

225 *MOR Constraints*

$$\sum_{f \in \mathcal{F}_p^{\text{FeMn}}} m_{p f} + o_p + c_p \leq Q_p^{\text{MOR}} \quad p \in \mathcal{P} \quad (18)$$

$$\sum_{f \in \mathcal{F}_p^{\text{FeMn}}} m_{pf} + o_p + c_p - x_{pe}^E - \sum_{b \in \mathcal{B}'} x_{pb}^B = 0 \quad p \in \mathcal{P}, \mathcal{B}' = \{\text{MOR dust, MC FeMn}\} \quad (19)$$

$$o_p = \Omega^{\text{MOR}} \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} m_{pf} \quad p \in \mathcal{P} \quad (20)$$

$$c_p \leq \Psi^{\text{MFeL}} \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} m_{pf} \quad p \in \mathcal{P} \quad (21)$$

$$x_{pb}^B = \Psi_b^{\text{MOR}} \left( \sum_{f \in \mathcal{F}_p^{\text{FeMn}}} m_{pf} + o_p \right) \quad p \in \mathcal{P}, b \in \{\text{MOR dust}\} \quad (22)$$

226 Constraints (18) ensure that the feed of HC FeMn, oxygen, and undersized lumps added to the MOR do not surpass  
 227 the MOR capacity. Constraints (19) handle the mass balance in the MOR. Constraints (20) state that the oxygen used  
 228 in the MOR equals a fixed ratio of the added HC FeMn. By calculating this ratio, there is no need to include a chemical  
 229 reaction in the model. Constraints (21) set the upper bound on how much MC FeMn undersized lumps it is possible to  
 230 add to the MOR relative to the feed of metal. This is to prevent too low temperatures in the MOR. Constraints (22)  
 231 state that a certain percentage of the mass fed to the MOR ends up as saleable MOR dust.

### 232 *LC SiMn Refining Station Constraints*

$$\sum_{f \in \mathcal{F}_p^{\text{SiMn}}} m_{pf} + s_p + a_p \leq Q_p^{\text{REF}} \quad p \in \mathcal{P} \quad (23)$$

$$\sum_{f \in \mathcal{F}_p^{\text{SiMn}}} m_{pf} + s_p + a_p - h_p = 0 \quad p \in \mathcal{P} \quad (24)$$

$$s_p = \Omega^{\text{REF}} \sum_{f \in \mathcal{F}_p^{\text{SiMn}}} m_{pf} \quad p \in \mathcal{P} \quad (25)$$

$$a_p \leq \Psi^{\text{LSiL}} \left( \sum_{f \in \mathcal{F}_p^{\text{SiMn}}} m_{pf} + s_p \right) \quad p \in \mathcal{P} \quad (26)$$

233 Constraints (23) handle the capacity of an LC SiMn refining station. The mass balance in the LC SiMn refining  
 234 station is handled by constraints (24). Constraints (25) relate the total amount of silicon waste needed to add, relative to  
 235 the amount of MC SiMn, to alter the product composition. The constraints ensure that it is not possible to get LC SiMn  
 236 out from the refining process without mixing the correct amount of Si with the incoming feed of MC SiMn. The upper  
 237 limit on how much LC SiMn undersized lumps it is possible to add to the LC SiMn refining station process is given by  
 238 constraints (26).

### 239 *Crushing Constraints*

$$\sum_{f \in \mathcal{F}_p^{\text{FeMn}}} u_{pf} = x_{pe}^E + x_{pb}^B \quad p \in \mathcal{P}, e \in \{\text{HC FeMn}\}, b \in \{\text{HC FeMn}\} \quad (27)$$

$$x_{pb}^B = \Psi_b^{\text{CRUSH}}(x_{pe}^E + x_{pb}^B) \quad p \in \mathcal{P}, b \in \{\text{MC FeMn}\} \quad (28)$$

$$\sum_{f \in \mathcal{F}_p} y_{pfr} \leq x_{pb}^B \quad p \in \mathcal{P}, r \in \{\text{HC FeMn, MC SiMn}\}, b \in \{\text{HC FeMn, MC SiMn}\} \quad (29)$$

$$c_p \leq x_{pb}^B \quad p \in \mathcal{P}, b \in \{\text{MC FeMn}\} \quad (30)$$

$$a_p \leq x_{pb}^B \quad p \in \mathcal{P}, b \in \{\text{LC SiMn}\} \quad (31)$$

240 Constraints (27) ensure that the total amount of HC FeMn alloy from a plant's HC FeMn furnaces sent directly to  
 241 crushing equals the HC FeMn end-product and by-products produced at the plant. Constraints (28) ensure that a given  
 242 percentage of the alloy that flows from the HC FeMn furnaces and the MOR ends up as undersized lumps. Similar  
 243 constraints exist for the SiMn production for constraints (27) - (28). To have a sustainable consumption of undersized  
 244 lumps, the volume of undersized lumps used in the process should be less than or equal to the volume of undersized  
 245 lumps exiting the crushing process. Constraints (29) - (31) ensure that this condition is satisfied. Constraints (29) allow  
 246 both HC FeMn and MC SiMn undersized lumps to be used in both furnace types.

#### 247 *Final Inventory and Demand Constraints*

$$g_e^F = D_e^F \quad e \in \mathcal{E} \quad (32)$$

$$g_e^O \leq D_e^O \quad e \in \mathcal{E} \quad (33)$$

248 Constraints (32) handle the demand from fixed contracts while constraints (33) handle the limit on the spot market.

#### 249 *Chemical Balance Constraints*

$$\sum_{c \in \mathcal{C}} T_{kcv} n_{pfkcv} = \frac{1 - \sum_{b \in \mathcal{B}} \Psi_{fbk}^B}{M_k} \sum_{r \in \mathcal{R}} \Psi_{rk}^R y_{pfr} \quad p \in \mathcal{P}, f \in \mathcal{F}_p, k \in \mathcal{K}, v \in \{\text{FED}\} \quad (34)$$

$$\sum_{v \in \mathcal{V} \setminus \{\text{SLAG}\}} \sum_{k \in \mathcal{K}} A_{fkcv} n_{pfkcv} + \sum_{k \in \mathcal{K}^S} (1 - \sum_{b \in \mathcal{B}} \Psi_{fbk}^B) A_{fkc, \text{SLAG}} n_{pfkc, \text{SLAG}} - \sum_{k \in \mathcal{K}} B_{fkc} \alpha_{pfkc} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, c \in \mathcal{C}^O \quad (35)$$

$$\sum_{v \in \mathcal{V} \setminus \{\text{SLAG}\}} \sum_{k \in \mathcal{K}} A_{fkcv}^{\text{LS}} n_{pfkcv} + \sum_{k \in \mathcal{K}^S} (1 - \sum_{b \in \mathcal{B}} \Psi_{fbk}^B) A_{fkc, \text{SLAG}}^{\text{LS}} n_{pfkc, \text{SLAG}} - \sum_{k \in \mathcal{K}} B_{fkc} \alpha_{pfkc} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, c \in \mathcal{C}^O \quad (36)$$

$$\sum_{v \in \mathcal{V}} \sum_{k \in \mathcal{K}} A_{fkcv}^{\text{RS}} n_{pfkcv} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, c \in \mathcal{C}^O \quad (37)$$

$$\sum_{v \in \mathcal{V}} \sum_{k \in \mathcal{K}} A_{fkcv} n_{pfkcv} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, c \in \mathcal{C}^C \quad (38)$$

$$n_{p f k, c+1, \text{RED}} = n_{p f k c, \text{RSRED}} \quad p \in \mathcal{P}, f \in \mathcal{F}_p, k \in \{\text{Mn}_2\text{O}_3, \text{Mn}_3\text{O}_4, \text{MnO}, \text{Fe}_2\text{O}_3, \text{Fe}_3\text{O}_4, \text{FeO}\}, c \in \{1, 2, 3, 6, 7\} \quad (39)$$

250 Constraints (34) connect the chemical processes in a furnace to the raw material feed. The left-hand side of the  
 251 constraints states that the total amount of moles of an element or oxide  $k$  used in the chemical reactions in each furnace  
 252 has to equal the feed of that element or oxide to the furnace. The parameters  $T_{kcv}$  ensure that  $n_{p f k c v}$  cannot take any  
 253 other value than zero where element or oxide  $k \in \mathcal{K}$  is not present in chemical reaction  $c$  for variable  $v \in \{\text{FED}\}$ . The  
 254 right-hand side of the constraints multiplies the weight percentage for each element or oxide  $k$  in raw material  $r$  with  
 255 the total weight of the raw material to find the weight of the element or oxide in the raw material. The sum is taken  
 256 over all raw materials so that the total feed of the respective element or oxide is found. It is then divided by molar mass  
 257 in mole per tonne to determine the amount of mole fed to the furnace for element or oxide  $k$ . The term  $(1 - \sum_{b \in \mathcal{B}} \Psi_{f b k}^{\text{B}})$   
 258 removes the amount of moles that ends up as discardable by-products from the feed since the chemical reactions do not  
 259 account for the production of these.

260 The general form of the chemical reactions is given by constraints (35) - (37). The constraints enforce that the  
 261 mole balances equal zero. Each chemical reaction is represented by three constraints to ensure the correct relationships  
 262 between reactants and resultants. Constraints (35) represent the complete chemical reaction, while constraints (36)  
 263 ensure correct ratios between the reactants, and constraints (37) the resultants.

264 The output of Mn, Fe, Si, and C from the redox reactions and the direct feed of the respective elements from ores  
 265 and undersized lumps are added together in constraints (38) to find the total mass of each element in the furnace output  
 266 alloy. The reactants in some of the chemical reactions originate from a resultant in the previous reaction, therefore the  
 267  $n_{p f k, c+1, \text{RED}}$  variables in these chemical reactions equals the  $n_{p f k c, \text{RSRED}}$  in the previous reaction. This is handled by  
 268 constraints (39).

269 An example of the application of constraints (35) - (37) to model chemical reaction (71),  $3 \text{Mn}_2\text{O}_3(\text{s}) + \text{CO}$   
 270  $(\text{g}) \longrightarrow 2 \text{Mn}_3\text{O}_4(\text{s}) + \text{CO}_2(\text{g})$ , is provided in equations (40a) - (40c).

$$2n_{p f, \text{Mn}_2\text{O}_3, 2, \text{FED}} + 2n_{p f, \text{Mn}_2\text{O}_3, 2, \text{RED}} + 6n_{p f, \text{CO}_2, 2, \text{FED}} - 3n_{p f, \text{Mn}_3\text{O}_4, 2, \text{RSRED}} - 6n_{p f, \text{CO}_2, 2, \text{RSRED}} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (40a)$$

$$n_{p f, \text{Mn}_2\text{O}_3, 2, \text{FED}} + n_{p f, \text{Mn}_2\text{O}_3, 2, \text{RED}} - 3n_{p f, \text{CO}_2, 2, \text{FED}} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (40b)$$

$$n_{p f, \text{Mn}_3\text{O}_4, 2, \text{RSRED}} - 2n_{p f, \text{CO}_2, 2, \text{RSRED}} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (40c)$$

271 Equation (40a) is the representation of reaction (71) with the correct mole ratios between the reactants and resultants.  
 272 Equation (40b) and equation (40c) balance the reactants and the resultants, respectively.

273 *Boudouard Reaction Constraints*

$$n_{pf,C,18,FED} + n_{pf,CO_2,18,TOT} - n_{pf,CO,18,RSRED} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (41a)$$

$$n_{pf,C,18,FED} - n_{pf,CO_2,18,TOT} = 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (41b)$$

$$n_{pf,CO_2,18,TOT} = (1 - \Upsilon) \sum_{c \in \mathcal{C} \setminus \{1,2\}} n_{pf,CO_2,c,RSRED} \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (41c)$$

$$\sum_{c \in \mathcal{C}^O} n_{pf,CO,c,TOT} \leq n_{pf,CO,18,RSRED} + \sum_{c \in \mathcal{C}^S} n_{pf,CO,c,RSRED} \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (41d)$$

274 Constraints (41a) are the Boudouard reaction given in reaction (79). Constraints (41b) ensure correct ratio between  
 275 the left side reactants, no constraints are needed for the right side ratio as only one resultant exists. Constraints (41c)  
 276 handle the degree of pre-reduction in the furnace, i.e. how much CO<sub>2</sub> that is consumed by the Boudouard reaction.  
 277 Following the given definition of pre-reduction, the term has to be formulated as (1 - Υ) to model the amount of CO<sub>2</sub>  
 278 re-entering the process correctly. The chemical reactions involving MnO<sub>2</sub> and Mn<sub>2</sub>O<sub>3</sub> are not normally involved in  
 279 prerelution and are therefore not included in the sum of the right side term. Constraints (41d) ensure that the total CO  
 280 fed to reactions (70) - (72) and (75) - (76) is less than or equal to the CO resulting from the Boudouard reaction and the  
 281 reactions (73), (77), and (78). The CO and CO<sub>2</sub> that do not re-enter the process are released as off-gases.

282 *Chemical Content Constraints*

$$M_k \sum_{c \in \mathcal{C}^C} T_{kcv} n_{pfkcv} = \Psi_k^{\text{FeMn}} (m_{pf} + u_{pf}) \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^C, v \in \{\text{TOT}\} \quad (42)$$

$$M_k \sum_{c \in \mathcal{C}^C} T_{kcv} n_{pfkcv} = \Psi_k^{\text{SiMn}} (m_{pf} + u_{pf}) \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{SiMn}}, k \in \mathcal{K}^C, v \in \{\text{TOT}\} \quad (43)$$

283 Constraints (42) and (43) ensure that the required content of critical elements is satisfied in the HC FeMn and the  
 284 MC SiMn furnace, respectively.

285 *Non-negativity Constraints*

$$a_p, c_p, h_p, o_p, s_p \geq 0 \quad p \in \mathcal{P} \quad (44)$$

$$g_e^F, g_e^O \geq 0 \quad e \in \mathcal{E} \quad (45)$$

$$x_{pb}^B \geq 0 \quad p \in \mathcal{P}, b \in \mathcal{B} \quad (46)$$

$$x_{pe}^E \geq 0 \quad p \in \mathcal{P}, e \in \mathcal{E} \quad (47)$$

$$e_{pf}, m_{pf}, q_{pf}, u_{pf} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p \quad (48)$$

$$\phi_{pfk} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, k \in \mathcal{K}^S \quad (49)$$

$$y_{pfr} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, r \in \mathcal{R} \quad (50)$$



$$\sigma_{pfgt} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, g \in \mathcal{P}, t \in \mathcal{F}_g \quad (51)$$

$$\alpha_{pafc} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, k \in \mathcal{K}, c \in \mathcal{C} \quad (52)$$

$$n_{pfcv} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p, k \in \mathcal{K}, c \in \mathcal{C}, v \in \mathcal{V} \quad (53)$$

#### 286 4. Solution by the Multiparametric Disaggregation Technique

287 The model has bilinear terms to model mixing of the slag quality components and is, therefore, a nonconvex  
 288 NLP. We have used the Multiparametric Disaggregation Technique (MDT), Teles et al. (2012, 2013); Kolodziej et al.  
 289 (2013a,b), to formulate the bilinear terms in the MAMP problem as systems of linear inequalities. In the MDT, a lower  
 290 bound problem (LBP) and an upper bound problem (UBP) are derived. The LBP and UBPs can then be solved by  
 291 standard MILP solvers with increasing accuracy until the global optimality gap  $\varepsilon$  is satisfactory. To parameterise and  
 292 discretise the bilinear constraints, new sets, parameters, and variables are defined. These are found in Tables 4 - 6,  
 293 respectively.

**Table 4:** Definition of sets for the MDT

Set	
$\mathcal{M}$	Set of single digit integers, (0 - 9), indexed by $m$
$\mathcal{L}$	Set of negative integers, indexed by $l$

**Table 5:** Definition of parameters for the MDT

Parameter	
$j$	The last significant number's position.
$\underline{q}_{pf}$	Lower bound on the slag produced by furnace $f$ at plant $p$ .
$\bar{q}_{pf}$	Upper bound on the slag produced by furnace $f$ at plant $p$ .
$\varepsilon$	Optimality gap between the lower bound problem objective value and the best bound of the upper bound problem.

**Table 6:** Definition of variables for the MDT

Variable	
$\hat{q}_{pfcml}$	The disaggregated flow variables of the product $q_{pfc} \mu_{pfcml}$ .
$\mu_{pfcml}$	1 if the decimal power $l$ is active for integer $m$ for element or oxide $k$ in furnace $f$ at plant $p$ , 0 otherwise.
$\lambda_{pfcml}$	Discretisation variable for use in reformulating $\phi_{pfc}$ .
$\Delta\phi_{pfc}$	Slack variable for the continuous representation of the discretised variable $\phi_{pfc}$ .

##### 294 4.1. Lower Bound Problem

This section describes how the bilinear constraints (12) are formulated as a system of linear constraints in the LBP. Of the two variables appearing in a bilinear term, one variable is parameterised and the other disaggregated (Teles

et al., 2012). A continuous variable can be disaggregated into a set of non-negative continuous variables, which can only assume positive values up to the upper bound of the original variable, (Teles et al., 2013). The  $\phi_{pfk}$  variables are chosen to be parameterised since they are limited between zero and one. This reduces the feasible region of the problem compared with disaggregating  $\phi_{pfk}$  and parameterising the variables  $q_{pfl}$  instead, which have a range between zero and the maximum slag production possible in a furnace. The reformulation of constraints (12) is based on the MDT found in Kolodziej et al. (2013a) and yields the following constraints:

$$\phi_{pfk}q_{pfl} = \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{M}} 10^l m \cdot \hat{q}_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (54)$$

$$\phi_{pfk} = \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{M}} 10^l m \cdot \mu_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (55)$$

$$q_{pfl} = \sum_{m \in \mathcal{M}} \hat{q}_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, l \in \mathcal{L} \quad (56)$$

$$\bar{q}_{pfl} \mu_{pflkml} \geq \hat{q}_{pflkml} \geq \underline{q}_{pfl} \mu_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, m \in \mathcal{M}, l \in \mathcal{L} \quad (57)$$

$$\sum_{m \in \mathcal{M}} \mu_{pflkml} = 1 \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, l \in \mathcal{L} \quad (58)$$

$$\mu_{pflkml} \in \{0, 1\} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, m \in \mathcal{M}, l \in \mathcal{L} \quad (59)$$

295 In constraints (54), the bilinear term  $\phi_{pfk}q_{pfl}$  is replaced by a weighted sum of continuous variables. Constraints  
 296 (55) parameterise the variable  $\phi_{pfk}$  as a sum of binary variables where  $\mu_{pflkml} = 1$  if the  $l^{\text{th}}$  decimal place contains  
 297 the number  $m$ , i.e  $\phi_{pfk} = 0.43 \rightarrow \mu_{pflk4,-1} = 1, \mu_{pflk3,-2} = 1$ . In constraints (56), the other variable in the bilinear  
 298 expression is connected to the new continuous variables, which are bounded in constraints (57). Constraints (58)  
 299 are convexity constraints and constraints (59) are variable restrictions. Constraints (14) are reformulated in the same  
 300 manner. Implementing the reformulations of constraints (12) and (14) in the MAMP problem yields the LBP.

#### 301 4.2. Upper Bound Problem

This section describes the reformulation of constraints (12) for the UBP. The same derivation as found in Kolodziej et al. (2013a) is used, leading to a continuous representation of the discretised variables. The result is the following constraints, which replaces constraints (12):

$$\phi_{pfk}q_{pfl} = \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{M}} 10^l m \cdot \hat{q}_{pflkml} + \Delta\phi_{pfk}q_{pfl} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (60)$$

$$\phi_{pfk} = \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{M}} 10^l m \cdot \mu_{pflkml} + \Delta\phi_{pfk} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (61)$$

$$q_{pfl} = \sum_{m \in \mathcal{M}} \hat{q}_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, l \in \mathcal{L} \quad (62)$$

$$\bar{q}_{pfl} \mu_{pflkml} \geq \hat{q}_{pflkml} \geq \underline{q}_{pfl} \mu_{pflkml} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, m \in \mathcal{M}, l \in \mathcal{L} \quad (63)$$

$$\sum_{m \in \mathcal{M}} \mu_{pflkml} = 1 \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, l \in \mathcal{L} \quad (64)$$

$$\mu_{p f k m l} \in \{0, 1\} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S, m \in M, l \in \mathcal{L} \quad (65)$$

$$\bar{q}_{p f} \Delta \phi_{p f k} \geq \Delta \phi_{p f k} q_{p f} \geq \underline{q}_{p f} \Delta \phi_{p f k} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (66)$$

$$\Delta \phi_{p f k} q_{p f} \geq (q_{p f} - \bar{q}_{p f}) 10^j + \bar{q}_{p f} \Delta \phi_{p f k} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (67)$$

$$\Delta \phi_{p f k} q_{p f} \leq (q_{p f} - \underline{q}_{p f}) 10^j + \underline{q}_{p f} \Delta \phi_{p f k} \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (68)$$

$$10^j \geq \Delta \phi_{p f k} \geq 0 \quad p \in \mathcal{P}, f \in \mathcal{F}_p^{\text{FeMn}}, k \in \mathcal{K}^S \quad (69)$$

302 The structure of this reformulation is the same as in the LBP, but a slack variable is added to the discretisation. This  
 303 relaxes the problem and gives an upper bound on the optimal objective value. Constraints (14) are reformulated in the  
 304 same manner for the UBP. Implementing the reformulations of constraints (12) and (14) in the MAMP problem, yields  
 305 the UBP.

## 306 5. Computational Study

307 We have evaluated the applicability and limitations of the MAMP through a computational study. First, the test  
 308 case is presented, based on data provided by Eramet Norway, before testing the applicability of the MDT technique  
 309 presented in Section 4. Further, we compare the results of applying the model described in Section 3, with today's  
 310 operational practice. Finally, we investigate how the slag composition, and the slag-to-metal ratio, changes with  
 311 increasing demand for all products.

312 The mathematical model is written in the algebraic modelling language Mosel and run in FICO<sup>®</sup> Xpress Optimisa-  
 313 tion Suite 7.9 using an HP bl685c G7 computer with an 2.2GHz AMD Opteron 6274 16 core CPU and 128 GB RAM  
 314 on a Linux operation system.

### 315 5.1. Test Case

316 The test case used in this computational study is based on the production set-up and data provided by Eramet  
 317 Norway, supplemented by data found in the literature. Eramet Norway operates seven furnaces distributed across three  
 318 plants, where three are HC FeMn furnaces and four are MC SiMn furnaces. Plant 1 has one HC FeMn furnace and one  
 319 MC SiMn furnace, Plant 2 has three MC SiMn furnaces, and Plant 3 has two HC FeMn furnaces.

320 All furnaces are set to have 22% pre-reduction, as used in *Olsen et al. (2007)*, thus  $\Upsilon = 0.22$ . Each furnace lose  
 321 35% of the heat generated to the surroundings, thus  $L^H = 1.35$ . The electricity price is set lower than the current  
 322 market price, as alloy companies often have lucrative price agreements. The electricity price is, therefore, set to  
 323  $C^E = 0.0118$  USD/kWh. The input mass and electrical power capacity of each furnace are set to 1000 tonnes/40 MW  
 324 and 750 tonnes/30 MW for HC FeMn and MC SiMn furnaces, respectively. In practice, it is usually the electrical  
 325 power that limits furnace capacity. The refining processes' input capacities are set sufficiently high not to be limiting  
 326 factors. The feed limit for each type of undersized lump is given as a weight fraction of the raw material feed to the  
 327 furnaces, or the liquid metal feed to the MORs and LC SiMn refining stations. The feed limit fractions are set to 0.10.

328 The demands and revenues for fixed and spot contracts per production period can be found in Table 7. It is assumed  
329 that the total market demand is greater than the total furnace capacity. The production is thus limited by the mass and  
330 electrical power capacity of the furnaces. Estimated slag transportation costs between plants are set to 8.4 USD/tonne  
331 between Plant 1 and Plant 2, 14.0 USD/tonne between Plant 1 and Plant 3 and, 4.2 USD/tonne between Plant 2 and  
332 Plant 3. The production period is set to  $\Delta T = 30$  days. It should be noted that the cost and revenue data provided by  
333 Eramet Norway are perturbed for confidentiality purposes.

**Table 7:** End-products with demands and revenues for fixed and spot contracts.  
Demand and spot market limit are given in tonnes, revenue is given in USD/tonne.

End-product $e$	Demand $D_e^F$	Revenue $R_e^F$	Spot market limit $D_e^O$	Revenue $R_e^O$
HC FeMn	13500	771	6000	810
MC FeMn	15000	899	6000	944
MC SiMn	10500	783	6000	822
LC SiMn	12000	853	6000	896

334 Each end-product is produced to satisfy certain content specifications. Explicit specifications are only set for  
335 the contents of HC FeMn and MC SiMn as these products are made in the furnaces where chemical composition is  
336 modelled. The content specifications for HC FeMn are: 0.790 Mn, 0.136 Fe, 0.004 Si, and 0.070 C. The MC SiMn  
337 specifications are 0.712, 0.081, 0.192, and 0.015 for the same elements, respectively. Notice that the composition of  
338 each end-product sum up to one. Correct content specifications of MC FeMn and LC SiMn are given implicitly by  
339 predetermined parameters for the refining stations. The weight percentage of every other constituent in MC FeMn and  
340 LC SiMn changes proportionally to the reduction of carbon as a result of the altered composition.

341 The mass output of slag is in relation to the total output of metal in an HC FeMn furnace. The maximum slag-to-  
342 metal ratio is set to  $\bar{\Lambda} = 1.00$  and the minimum value  $\underline{\Lambda} = 0.50$ . The slag exiting the HC FeMn furnaces has set quality  
343 specification intervals for oxides with metal-bearing capabilities. The upper and lower bound on the slag quality are  
344 defined by the parameters  $\bar{\Phi}_k$  and  $\underline{\Phi}_k$ , respectively.  $\bar{\Phi}_k$  is 0.50 for MnO, 0.02 for FeO, 0.35 for SiO<sub>2</sub>, 0.20 for Al<sub>2</sub>O<sub>3</sub>  
345 and MgO, and 0.30 for CaO.  $\underline{\Phi}_k$  is 0.30, 0.00, 0.15, 0.10, 0.5, and 0.10 for the same oxides, respectively. The sum of  
346 the lower bounds on the slag composition  $\underline{\Phi}_k$  means that this amount of the slag composition is predetermined. As in  
347 the base instance, 70% of the slag composition is already determined. 30% of the slag composition is then left for the  
348 MAMP problem to decide. The greater the sum of the upper bound on the slag composition  $\bar{\Phi}_k$ , the greater the number  
349 of possible combinations of oxides with which to fill the remaining 30% of the slag composition. At every stage of the  
350 production, except at the LC SiMn refining station, by-products are produced as a fixed amount of the total feed to the  
351 process stage. These values are set to 0.02 for by-products produced in HC FeMn furnaces and 0.10 in the MC SiMn  
352 furnaces. The oxides Al<sub>2</sub>O<sub>3</sub>, MgO, and CaO completely exit the MC SiMn furnace as slag and thus the associated  
353 by-product parameters are 1.00 for these. In the MOR, the by-product fraction is 0.08. Values are based on *Olsen et al.*

354 (2007).

355 A set of 19 raw materials is at disposal at each plant. These raw materials contain various elements and oxides of  
356 different concentrations. Oxygen, silicon waste, MC FeMn undersized lumps, and LC SiMn undersized lumps, named  
357 refining resources, are separated from the raw materials since these feed other processes than the furnace process. The  
358 inventories of these raw materials are assumed to be large enough to satisfy any demand.

### 359 5.2. Testing the applicability of the MDT

360 To be able to test the scalability of the MDT, some additional test instances are created. For these instances, all the  
361 data is equal to the test case presented above, except for the number of plants and furnaces. We name each instance  
362  $PX-YFeZSi$ , where  $X$  denote the number of plants, and  $Y$  and  $Z$  the number of HC FeMn and MC SiMn furnaces,  
363 respectively. The Base case is thus named  $P3-3Fe4Si$ , while we introduce three additional instances:  $P1-1Fe1Si$ ,  
364  $P2-2Fe2Si$  and  $P5-5Fe5Si$ .

365 In addition, we need to decide on the values of some of the parameters used in the MDT, based on the values given  
366 for the base case. The parameters  $\bar{q}_{pf}$  greatly affect the run time as they set the solution space for the volume of slag  
367 produced for each furnace.  $\bar{q}_{pf}$  should therefore be set as tight as possible to avoid a too large feasible region. The  
368 parameters are naturally limited by the total amount of slag a furnace can produce per day. They are easily scaled  
369 by multiplying with  $\Delta T$ . For the defined HC FeMn furnace capacities, a suitable mathematical upper bound is set to  
370  $\bar{q}_{pf} = 500\Delta T$ , to not be a limiting factor. The parameters  $q_{pf}$  are set to zero as the HC FeMn furnace possibly can  
371 produce zero output. A similar parameter is defined for constraints (14) through the MDT.

372 For each test instance, both the LBP and the UBP were run for a maximum of 7200 seconds. Both the LBP and  
373 UBP is first solved with a discretization of  $-2$ , and if this problem is solved to optimality the remaining time is used to  
374 solve the same problem with a discretization of  $-3$ . The results of these tests are presented in Table 8. For each test  
375 instance, the number of variables and constraints for each discretization, the optimal objective value, and the computing  
376 time of the LBP, and the optimality gap and computing time of the UBP, as well as the total computing time and global  
377 optimality gap  $\epsilon$  for the test instances, are given. The value of  $\epsilon$  is calculated as  $\epsilon = (\bar{z}(UBP) - \underline{z}(LBP)) / \underline{z}(LBP)$ , where  
378  $\bar{z}(p)$  and  $\underline{z}(p)$  is the dual and primal bound, respectively, from problem  $p$ .

379 As Table 8 shows, the number of variables and constraints grows rapidly with small increases in the number of  
380 furnaces. Adding a furnace adds a new set of chemical balance constraints and furnace restrictions to the problem. This  
381 increase in the number of variables and constraints make the larger problems much more difficult to solve, which can  
382 be seen by the time it takes to solve the UBP and LBP, and the deterioration of the UBP gap as well as  $\epsilon$ . The base  
383 instance  $P3-3Fe4Si$  is solved to an  $\epsilon = 2.49\%$  within the time limit. The applied precision has a significant effect on the  
384 required computational time. For larger instances, the  $-2$  precision is never completed, and thus some problems are  
385 never solved with a precision of  $-3$ . Generally, the solution to the LBP problem is better and located faster than for the  
386 UBP. It is worth noting that the MAMP problem can find feasible solutions even for large instances such as  $P5-5Fe5Si$ ,  
387 however, it is not possible to know exactly how good the solution is, due to the large value of  $\epsilon$ .

**Table 8:** Statistics from the computational experiments. For the instances that were solved for both precision levels, the data are given as  $\{a, b\}$ , where  $a$  and  $b$  are the values for precision of  $-2$  and  $-3$ , respectively. For the other instances, the data obtained when solving the problem with a precision of  $-2$  is presented. The objective value is given in thousands of USD, while the computing times are given in seconds.

Instance	LBP				UBP		MAMP Main	
	# Variables	# Constraints	Objective value	Time	Gap	Time	Time	$\epsilon$
<b>P1-1Fe1Si</b>	{1167, 1347}	{17927, 18065}	5652	{1,9}	0.00 %	{1,6}	17	0.09 %
<b>P2-2Fe2Si</b>	{4202, 4802}	{70929, 71709}	20912	{1839, 5359}	0.72 %	7200	14409	1.21 %
<b>P3-3Fe4Si</b>	7985	125477	35383	7200	2.05 %	7200	14470	2.49 %
<b>P5-5Fe5Si</b>	13280	183102	52406	7200	5.96 %	7200	14405	6.76 %

### 388 5.3. Comparison of the MAMP to Single Furnace Optimisation

389 The current industry practice is to optimise the production for individual furnaces based on software and expert  
390 judgement made by metallurgists. This practice is denoted single furnace optimisation and is the practice of optimising  
391 the profit of each single furnace, and consequently the metal it produces, without regards to the overall production. This  
392 may be sub-optimal compared with planning the production when taking the multi-plant production into consideration.  
393 In this section we investigate if, and how, solving the MAMP problem can improve production planning.

394 To simulate single furnace optimisation, the following process is used. An instance only containing one HC FeMn  
395 furnace and an instance only containing one MC SiMn furnace are created. The authors assume that the single furnace  
396 optimisation process is done by satisfying the fixed contracts first, then the spot contracts with the highest profit. In the  
397 first iteration, the HC FeMn instance is solved with the total demand as input. The demand is then reduced with the  
398 production in the first iteration and the instance is solved again. This is done for three iterations with the HC FeMn  
399 instance, followed by four iterations with the MC SiMn instance. The results from the HC FeMn production iterations  
400 are shown in Table 9, together with the results from applying the MAMP formulation. It should be noted that the  
401 objective values for each furnace given for the MAMP problem is an approximation, since not all costs can easily be

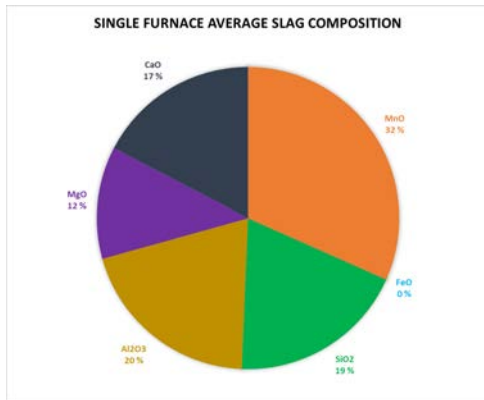
402 allocated to a single furnace.

**Table 9:** Comparison of the MAMP formulation to single furnace optimisation for the FeMn production. Production volumes are given in tonnes. Objective values are given in thousand USD. \* denotes that objective value is an approximation.

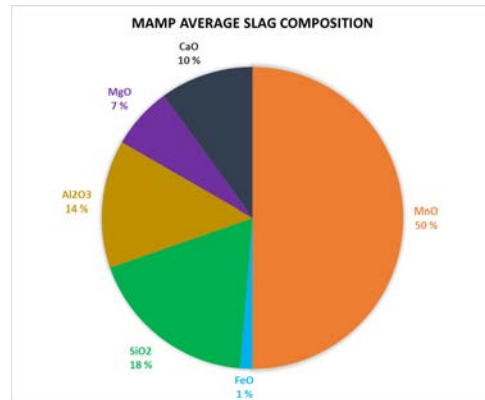
Plant $p$ , Furnace $f$	Single Furnace Optimisation				MAMP Optimisation			
	1, 1	3,6	3,7	Total	1, 1	3,6	3,7	Total
Furnace Type	FeMn	FeMn	FeMn		FeMn	FeMn	FeMn	
$u_{pf}$ , to crushing	0	8314	4175	12489	0	10688	3021	13709
$m_{pf}$ , to refining	10439	2599	6519	19557	10998	0	8559	19557
$q_{pf}$ , produced slag	5220	5456	5347	16023	5499	5344	5790	16633
$x_{pe}^E$ , HC FeMn	0	7482	3758	11240	0	9620	2719	12339
$x_{pe}^E$ , MC FeMn	9608	2392	6000	18000	10122	0	7878	18000
<b>Obj. Val. FeMn</b>	6668	5975	6741	19384	6967*	5225*	7092*	19284

403 The total demand of MC FeMn from fixed and spot contracts are satisfied for both production methods, while  
 404 not all of the spot contract demand for HC FeMn is satisfied. This indicates that MC FeMn is the most profitable  
 405 FeMn end-product. Considering only the production of FeMn alloys, the total profit of the single furnace optimization  
 406 approach is actually better than the solution obtained from the MAMP problem. Even though the production of HC  
 407 FeMn is lower, giving lower revenues, this is offset by cheaper raw materials. Thus, even though the slag-to-metal ratio  
 408 is optimal at the lower bound of 0.50 in both production methods (can be verified by consulting  $\frac{q_{pf}}{u_{pf}+m_{pf}}$ ), the amount  
 409 and composition of the slag is different.

410 Figures 3 and 4 show the average slag composition of single furnace optimisation and MAMP optimisation,  
 411 respectively. The most notable differences are the changes in the MnO, CaO, MgO, and Al<sub>2</sub>O<sub>3</sub> concentrations. When  
 412 optimising single HC FeMn furnaces, it is favourable to keep the MnO concentration in the slag to a minimum to  
 413 maximise the HC FeMn output. Consequently, the oxides that are not substances of HC FeMn metal are maximised in  
 414 the slag output. In MAMP optimisation, these concentrations are changed to optimise the overall production in the  
 415 entire system. It is worth noting that even though the slag to metal ratio is at the lower bound also for the MAMP  
 416 optimisation in this instance, it could increase the slag-to-metal ratio for an instance with different parameter settings,  
 417 while the single furnace optimization always keeps the ratio at its lower bound.



**Figure 3:** The average slag composition produced by the three HC FeMn furnaces using single furnace optimisation.



**Figure 4:** The average slag composition produced by the three HC FeMn furnaces using the MAMP formulation.

418 In the single furnace optimization we assume that the slag transportation between furnaces in the single furnace  
 419 optimisation is based on minimising transportation costs, to make the two production planning methods as comparable  
 420 as possible. This implies that as much slag as possible is sent internally at a plant if a plant has both HC FeMn and MC  
 421 SiMn furnaces, as is the case at Plant 1. When the internal capacity of slag is reached, slag is transported to the plant  
 422 with MC SiMn furnaces incurring the lowest transportation costs. The results from the single furnace optimisation  
 423 method and the MAMP optimisation for the SiMn production are provided in Table 10.

424 All the produced slag is consumed in both production planning methods and all the LC SiMn demand is satisfied.  
 425 The difference lies in the produced volume of MC SiMn, where an additional 673 tonnes of MC SiMn is produced  
 426 by solving the MAMP problem due to different composition and allocation of the slag. This is an 8.52% increase  
 427 in production of MC SiMn alloy. One can argue that the slag could have been distributed in another manner for the  
 428 single furnace optimisation planning. However, regardless of what rule of thumb is used to distribute the slag it will be  
 429 inferior to solving the MAMP problem which implicitly considers all possible ways of distributing it.



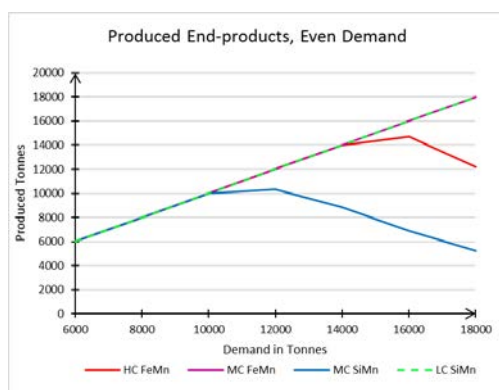
**Table 10:** Comparison of the MAMP problem to single furnace optimisation for the SiMn production and total profit. Production values are given in tonnes. Costs and objective values are given in thousand USD. *Cumulative Objective Value* is excluded the *Transportation Costs*. *Total profit* is the *Cumulative Objective Value* less the *Transportation Costs*. \* denotes that objective value is an approximation.

Plant $p$ , Furnace $f$	Single Furnace Optimisation					MAMP Optimisation				
	1, 2	2,3	2,4	2,5	Total	1, 2	2,3	2,4	2,5	Total
Furnace Type	SiMn	SiMn	SiMn	SiMn		SiMn	SiMn	SiMn	SiMn	
$u_{pf}$ , to crushing	0	2786	5547	447	8780	2338	0	1381	5809	9528
$m_{pf}$ , to refining	5021	2175	42	4755	11993	2856	5246	3892	0	11994
$\sigma_{gtpf}$ , used slag	5220	5456	2674	2673	16023	5499	4095	3421	3618	16633
Leftover slag	0	0	0	0	0	0	0	0	0	0
$x_{pe}^E$ , MC SiMn	0	2508	4992	402	7902	2104	0	1243	5228	8575
$x_{pe}^E$ , LC SiMn	6280	2720	53	5947	15000	3571	6588	4841	0	15000
Obj. Val. SiMn	4418	3565	2929	4599	15511	3979*	4803*	4200*	3162*	16145
Cum. Obj. Val.					34895					35430
Transport. Costs	0	23	11	11	45	0	17	14	15	47
Discard Slag Cost					0					0
<b>Total Profit</b>					<b>34850</b>					<b>35383</b>

430 The solution to the MAMP problem yields more slag transportation than the single furnace optimisation, thus,  
431 transportation costs are greater. Using the MAMP formulation yields 1.53% higher profit compared to single furnace  
432 optimisation in the case of base instance *P3-3Fe4Si*. Two significant factors in making the MAMP formulation superior  
433 to single furnace optimisation are the volume of slag produced and the composition of the slag.

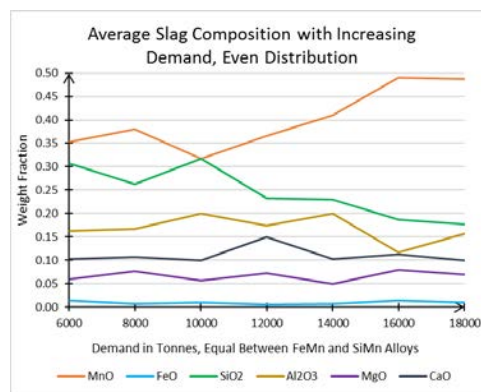
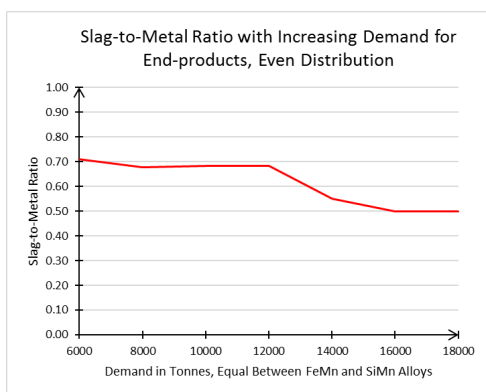
#### 434 5.4. Optimal Slag Composition as a Function of Demand

435 The measure slag-to-metal ratio is widely used in the manganese alloy production industry. In the previous section,  
436 it can be observed that the slag-to-metal ratio is at the lower bound of 0.50 in all furnaces. The ratio has an upper limit  
437 of 1.00, i.e. one tonne slag is produced per tonne metal produced. We therefore investigate how this ratio changes  
438 with variations in the demand volume of the end-products. In this study we assume that the end-product demand is  
439 equal for all products, and distributed equally between FeMn and SiMn alloys. The fixed demand for all end-products  
440 are set as optional to make the solution to the MAMP problem select the most profitable ones. The end-product  
441 production volumes are also provided as it may better visualise the changes occurring in the slag-to-metal ratio and  
442 slag composition. The volumes produced of each end-product for increasing demand are illustrated in Figure 5.



**Figure 5:** End-product production volumes for increasing demand when the demand is evenly distributed between FeMn and SiMn alloys.

443 Observe from Figure 5 that MC SiMn and HC FeMn are the first products in each production path to be reduced  
 444 when reaching furnace capacities at 12 000 tonnes and 16 000 tonnes, respectively. The slight increase in productions  
 445 before the steady decrease at the furnace capacities are due to the alteration of slag composition. A plot of the average  
 446 slag-to-metal ratio as a function of demand is shown in Figure 6. A plot of the average slag composition as a function  
 447 of demand is shown in Figure 7. Between 8 000 - 12 000 tonnes demand, a slight increase in slag-to-metal ratio can be  
 448 observed. This is to better accommodate the lack of capacity in the furnace producing MC SiMn. At 12 000 tonnes, the  
 449 HC FeMn furnace needs the capacity to produce more HC FeMn, and the slag-metal-ratio starts to decline. At 16 000  
 450 tonnes, it reaches the lower limit of the ratio and starts sending more of the metal output to MC FeMn refining, which  
 451 is more profitable.



**Figure 6:** Average slag-to-metal ratio across all HC FeMn furnaces for increasing demand. Equal demand for each end-product.  
**Figure 7:** Average slag composition across all HC FeMn furnaces for increasing demand. Equal demand for each end-product.

452 The slag-to-metal ratio is 0.68 - 0.71 for demands less than 12 000 tonnes. For greater demands, the slag-to-metal  
 453 ratio decreases to the lower bound of 0.50. Here, production of the least profitable alloy MC SiMn cannot satisfy  
 454 demand as the slag-to-metal ratio decreases to release capacity in the FeMn furnaces to satisfy the increasing demand  
 455 of FeMn alloys which are more profitable. At a demand of 12 000 tonnes, the FeMn production needs the capacity to

456 satisfy the FeMn alloy demand. Thus, less capacity is available to produce slag. This trend continues for increasing  
457 demand until the lower bound of the slag-to-metal ratio is reached. As the slag-to-metal ratio decreases, it becomes  
458 more favourable to send slag with a higher content of MnO and lower content of SiO<sub>2</sub>. Thus, as the volume of slag  
459 goes down, the amount of MnO increases to make the slag carry more of the important oxides to form pure metals in  
460 the MC SiMn furnace, as can be seen from Figure 7.

461 The results from Table 10 are for an instance where the slag-to-metal ratio is at its lower bound of 0.50. The authors  
462 suspect that the value of using the MAMP formulation is even greater in situations where the optimal slag-to-metal  
463 ratio is above the lower bound. The single furnace optimisation practice always minimises the slag-to-metal ratio, as it  
464 optimises the profit for single furnaces, consequently produces as much metal as possible. Solving the MAMP problem,  
465 on the other hand, enables optimising the metal-to-slag ratio relative to the overall profit. A larger volume of slag is  
466 therefore produced and allocated efficiently to the MC SiMn furnaces. The decision to maximise metal output in single  
467 furnace optimisation could of course also be changed to maximise slag output, but this is not a trivial decision to make  
468 simply by consulting raw material costs and end-product revenues.

## 469 **6. Concluding Remarks**

470 This paper has presented an optimisation model for manganese alloy multi-plant production (MAMP) planning as a  
471 nonlinear pooling problem, and outlined a global optimisation algorithm by means of a Multiparametric Disaggregation  
472 Technique (MDT). The computational study from Eramet Norway shows that the proposed solution approach can  
473 solve MAMP problem sizes of up to seven furnaces spread across three plants to within a global optimality gap of  
474 3% for a run time of four hours. These results demonstrate that the MDT scales well with larger problem instances,  
475 thus supporting earlier computational results of the MDT as a promising method for large-scale nonconvex bilinear  
476 problems. Yet, for large problem sizes, the precision level of the discretisation is decisive for the numerical efficiency.

477 Solving the production planning holistically as a multi-plant problem outperforms the current operational practice  
478 of single furnace optimisation. Moreover, the incorporation of a spot market visualises which products are most  
479 profitable since these are the products produced after the fixed demand is met. By minimising power consumption  
480 and accounting for CO and CO<sub>2</sub> emissions, the proposed production-planning model may facilitate new manganese  
481 production strategies that ensure sustainable resources utilisation and lower emissions, thereby aligning with the  
482 increasing focus on environmental impacts caused by production from energy-intensive industries (*United Nations,*  
483 *2016; Figueres et al., 2017*). Further work on the problem and solution approach studied in the paper includes  
484 accounting for uncertainty in market demand and raw material availability, as well as including furnace configuration  
485 as a decisions variable.

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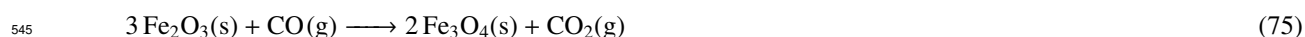
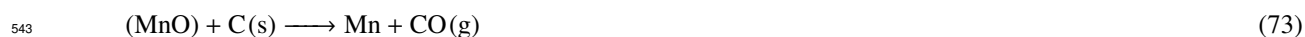
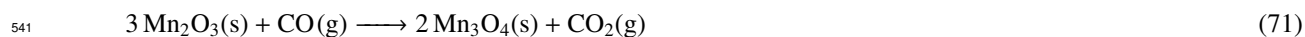
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### 537 A. Main furnace reactions

538 The main chemical reactions taking place in the furnaces during producing of HC FeMn and MC SiMn, respectively,  
539 are (Olsen et al., 2007)



553 The parentheses denote the slag phase and underlines the metal phase.