Noname manuscript No.
(will be inserted by the editor)

Variational inequality formulation for the games with random payoffs

Vikas Vikram Singh · Abdel Lisser

Received: date / Accepted: date

Abstract We consider an n-player non-cooperative game with random payoffs and continuous strategy set for each player. The random payoffs of each player are defined using a finite dimensional random vector. We formulate this problem as a chance-constrained game by defining the payoff function of each player using a chance constraint. We first consider the case where the continuous strategy set of each player does not depend on the strategies of other players. If a random vector defining the payoffs of each player follows a multivariate elliptically symmetric distribution, we show that there exists a Nash equilibrium. We characterize the set of Nash equilibria using the solution set of a variational inequality (VI) problem. Next, we consider the case where the continuous strategy set of each player is defined by a shared constraint set. In this case, we show that there exists a generalized Nash equilibrium for elliptically symmetric distributed payoffs. Under certain conditions, we characterize the set of a generalized Nash equilibria using the solution set of a VI problem. As an application, the random payoff games arising from electricity market are studied under chance-constrained game framework.

Keywords Chance-constrained games · Variational Inequality · Elliptically symmetric distribution · Generalized Nash equilibrium · Cournot competition.

Mathematics Subject Classification (2000) 91A10 · 90C15 · 90C33

Vikas Vikram Singh
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, India.
E-mail: vikassingh@iitd.ac.in

Abdel Lisser
Laboratoire de Recherche en Informatique Université Paris Sud, 91405 Orsay, France.
E-mail: abdel.lisser@lri.fr

1 Introduction

In 1950, John Nash [24] introduced the notion of Nash equilibrium for finite strategic games. He showed that there always exists a mixed strategy Nash equilibrium. For two player games, a Nash equilibrium can be obtained by solving an equivalent linear complementarity problem [22]. For two player zero sum games a Nash equilibrium is called a saddle point equilibrium [31] and it can be obtained from the optimal solutions of a primal-dual pair of linear programs [1, 10]. The Nash equilibrium problem has also been studied for continuous strategy sets. However, the existence of a Nash equilibrium is proved only under certain assumptions [2]. A Nash equilibrium problem, where each player has continuous strategy set, is equivalent to a variational inequality (VI) problem [11] if the payoff functions and strategy sets satisfy certain conditions. In many situations, the continuous strategy set of each player depends on the strategies of all other players. The equilibrium concept for these games is called generalized Nash equilibrium (GNE). In general, the existence of a GNE is very difficult to show. A special class of generalized Nash games is considered by Rosen [27] in his fundamental paper where all players share common constraints. Under certain assumptions, he showed the existence of a GNE for these games. Facchinei et al. [12] characterized the generalized Nash equilibria of the games considered in [27] using the solution set of a VI problem. The above mentioned papers considered the games where the payoffs of the players are deterministic. However, in many situations players' payoffs are better modeled by random variables following certain distributions and as a result players compete in a stochastic Nash game.

One way to study stochastic Nash games is by using expected payoff criterion [15, 26, 32]. Ravat and Shanbhag [26] showed the existence and uniqueness of Nash equilibrium in various cases. In [32] and [15], the stochastic Nash equilibrium problems are studied using sample average approximation method and VI approach on probabilistic Lebesgue spaces respectively. The stochastic Nash games under expected payoff criterion using stochastic variational inequalities are considered in [16, 19, 33]. In these papers, the stochastic approximation based schemes to compute the Nash equilibria of stochastic Nash games have been given.

The expected payoff criterion is more appropriate for the cases where the decision makers are risk neutral. The risk averse stochastic Nash games arising from electricity market using risk measures as CVaR and variance are considered in [18, 26] and [8] respectively. A risk averse payoff criterion based on chance constraint programming [5, 25] has also received some attention in electricity market [9, 23]. These games are called chance-constrained games (CCGs). The CCG is suitable for the cases where players are interested in payoffs that can be obtained with certain confidence. Recently, Singh et al. [29, 28, 30] proposed some contributions to develop the theory of CCGs for the case where players have finite action sets. They showed the existence of a mixed strategy Nash equilibrium of an n-player CCG when the payoff vector of each player follows a multivariate elliptically symmetric distribution [28].

An equivalent complementarity problem for a CCG is given in [29]. Singh et al. [30] used distributionally robust approach to formulate the games with partially known probability distributions and showed the existence of a mixed strategy Nash equilibrium in two different cases. For each case they proposed an equivalent mathematical program to compute the Nash equilibria of a distributionally robust CCG. The zero sum CCGs with finite action sets of the players are considered in [3, 4, 6, 7].

The case of finite strategy games form an important class of game problems, but to capture more realistic situations the case of continuous strategy sets is needed. In this paper, we consider an n-player non-cooperative game with random payoffs where each player has a continuous strategy set. The payoffs of each player are defined using a random vector. We formulate this problem as a CCG by defining the payoff of each player using a chance constraint. We show that there exists a Nash equilibrium of a CCG if the random vector defining the payoffs of each player follows a multivariate elliptically symmetric distribution. We characterize the set of Nash equilibria of a CCG using the solution set of an equivalent VI problem. We also consider the case where a strategy set of each player is defined by a shared constraint set. We show the existence of a GNE for these games if a random vector defining the payoffs of each player follows a multivariate elliptically symmetric distribution. We characterize the set of a certain types of generalized Nash equilibria using the solution set of an equivalent VI problem. As applications of these games, we consider two stochastic Nash games from electricity Markets and formulate them as CCGs. For illustration purpose we solve few instances of these CCGs.

The structure of the rest of the paper is as follows: Section 2 contains the existence of a Nash equilibrium of a CCG and its equivalent VI formulation. Section 3 contains the existence of a GNE of a CCG and its equivalent VI formulation. We give two examples from electricity markets in Section 4. Section 5 presents the numerical results. We conclude the paper in Section 6.

2 The model

We consider an n-player non-cooperative game with continuous strategy sets and random payoffs defined by a tuple $(I, (X^i)_{i\in I}, (r^i)_{i\in I})$, where

(*i*) $I = \{1, 2, \cdots, n\}$ is a set of players.

(*ii*) $X^i \subset \mathbb{R}^{m_i}$ denote the set of all strategies of player i. Let $X = \times_{i=1}^{n} X^i \subset \mathbb{R}^m$, $m = \sum_{i=1}^{n} m_i$, be the set of strategy profiles, and X^{-i} be a set of vectors of strategies x^j, $j \neq i$. The generic elements of X^i, X, and X^{-i} are denoted by x^i, x, and x^{-i} respectively.

(*iii*) r^i is a random payoff function of player i. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $i \in I$, let $\xi^i = \left(\xi^i_k\right)_{k=1}^{l_i}$ be a random vector defined by $\xi^i : \Omega \to \mathbb{R}^{l_i}$. For a given strategy profile $x = (x^1, x^2, \cdots, x^n)$, and for an

$\omega \in \Omega$ the realization of a random payoff of player i, $i \in I$, is given by

$$r^i(x,\omega) = \sum_{k=1}^{l_i} f_k^i(x)\xi_k^i(\omega),$$

where $f_k^i : \mathbb{R}^m \to \mathbb{R}$ for all $k = 1, 2, \cdots, l_i$, $i \in I$. Denote, $f^i(x) = (f_k^i(x))_{k=1}^{l_i}$, $i \in I$.

We consider the competition among risk averse players where each player is interested in payoffs that can be obtained with certain confidence. A chance constraint programming approach is adequate to handle such situation. We formulate this problem as a chance-constrained game (CCG) by defining the payoff function of each player using a chance constraint. Let $\alpha_i \in [0,1]$ be the confidence (risk) level of player i, and $\alpha = (\alpha_i)_{i \in I}$. The confidence level of each player is given a priori and it is known to other players. For a given strategy profile $x \in X$, and a given confidence level vector α the payoff function of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(x) = \sup\{\gamma | P\left(\{\omega | r^i(x,\omega) \geq \gamma\}\right) \geq \alpha_i\}. \quad (1)$$

We assume that the probability distribution of random vector ξ^i, $i \in I$ is known to all the players. Then, for a fixed $\alpha \in [0,1]^n$, the payoff function of a player defined by (1) is known to all the players. Therefore, for a fixed $\alpha \in [0,1]^n$ the above CCG is a non-cooperative game with complete information. In non-cooperative games, each player seeks for a strategy which gives him the maximum payoffs for a fixed strategy profile of other players. The set of such best response strategies of player i, against a given strategy profile x^{-i} of other players is given by

$$BR_i^{\alpha_i}(x^{-i}) = \left\{\bar{x}^i \in X^i | u_i^{\alpha_i}(\bar{x}^i, x^{-i}) \geq u_i^{\alpha_i}(x^i, x^{-i}),\ \forall\ x^i \in X^i\right\}.$$

A strategy profile $x^* \in X$ is said to be a Nash equilibrium of a CCG at α if for each $i \in I$, the following inequality holds

$$u_i^{\alpha_i}(x^{i*}, x^{-i*}) \geq u_i^{\alpha_i}(x^i, x^{-i*}),\ \forall\ x^i \in X^i. \quad (2)$$

Hence, a strategy profile x^* is a Nash equilibrium if and only if $x^{i*} \in BR_i^{\alpha_i}(x^{-i*})$ for all $i \in I$.

2.1 Payoffs following multivariate elliptical distributions

We consider the case where the random vector ξ^i, $i \in I$, follows a multivariate elliptically symmetric distribution with location parameter vector $\mu_i = (\mu_{i,k})_{k=1}^{l_i}$ and $l_i \times l_i$ scale matrix Σ_i which is positive definite. We use the notation $\xi^i \sim Ellip(\mu_i, \Sigma_i)$. We denote the positive definite matrix Σ_i by $\Sigma_i \succ 0$. The probability distributions belonging to the class of multivariate elliptically symmetric distributions generalize the multivariate normal distribution. Some famous multivariate distributions like normal, Cauchy, t, Laplace, and logistic distributions belong to the family of elliptically symmetric distributions [13].

From the definition of a multivariate elliptically symmetric distribution, any linear combination of the components of ξ^i is a univariate elliptically symmetric distribution. Then, for a given $x \in X$, $r^i(x) = (f^i(x))^T \xi^i$ follows a univariate elliptically symmetric distribution with parameters $\mu_i^T f^i(x)$ and $(f^i(x))^T \Sigma_i f^i(x)$. Let $||\cdot||$ be the Euclidean norm. Then, $\sqrt{(f^i(x))^T \Sigma_i f^i(x)} = ||\Sigma_i^{1/2} f^i(x)||$ because $\Sigma_i \succ 0$. Hence, $Z_i = \frac{r^i(x)-\mu_i^T f^i(x)}{||\Sigma_i^{1/2} f^i(x)||}$, $i \in I$, follows a univariate spherically symmetric distribution with parameters 0 and 1 [13]. Let $\phi_{Z_i}^{-1}(\cdot)$ be a quantile function of a spherically symmetric distribution. For a given $x \in X$ and α, we have

$$\begin{aligned} u_i^{\alpha_i}(x) &= \sup\{\gamma| \ P\left(\{\omega | r^i(x,\omega) \geq \gamma\}\right) \geq \alpha_i\} \\ &= \sup\left\{\gamma | P\left(\left\{\omega \Big| \ \frac{r^i(x,\omega) - \mu_i^T f^i(x)}{||\Sigma_i^{1/2} f^i(x)||} \leq \frac{\gamma - \mu_i^T f^i(x)}{||\Sigma_i^{1/2} f^i(x)||}\right\}\right) \leq 1-\alpha_i\right\} \\ &= \sup\left\{\gamma | \gamma \leq \mu_i^T f^i(x) + ||\Sigma_i^{1/2} f^i(x)|| \phi_{Z_i}^{-1}(1-\alpha_i)\right\}. \end{aligned}$$

Therefore,

$$u_i^{\alpha_i}(x) = \mu_i^T f^i(x) + ||\Sigma_i^{1/2} f^i(x)|| \phi_{Z_i}^{-1}(1-\alpha_i), \quad i \in I. \tag{3}$$

2.2 Existence of Nash equilibrium

Define a set valued map

$$G : X \to \mathcal{P}(X)$$

such that

$$G(x) = \prod_{i\in I} BR_i^{\alpha_i}(x^{-i}).$$

A point $x \in X$ is a fixed point of $G(\cdot)$ if $x \in G(x)$. It is clear that a fixed point of $G(\cdot)$ is a Nash equilibrium. Under Assumption 1 on strategy sets and payoff functions, we show that there exists a Nash equilibrium for a chance-constrained game by showing the existence of a fixed point for $G(\cdot)$.

Assumption 1 *For each player i, $i \in I$, the following conditions hold:*

1. *X^i is a non-empty, convex and compact subset of $\mathbb{R}^{m_i}$.*
2. *$f_k^i : \mathbb{R}^m \to \mathbb{R}$ is a continuous function of x, for all $k = 1, 2, \cdots, l_i$.*
3. *(a) For every $x^{-i} \in X^{-i}$, $f_k^i(\cdot, x^{-i})$, $k = 1, 2, \cdots, l_i$, is an affine function of x^i.*
   or
   (b) For every $x^{-i} \in X^{-i}$, $f_k^i(\cdot, x^{-i})$, $k = 1, 2, \cdots, l_i$, is non-positive and a concave function of x^i, and $\mu_{i,k} \geq 0$ for all $k = 1, 2, \cdots, l_i$, and all the elements of Σ_i are non-negative.

Lemma 1 *If condition 3 of Assumption 1 holds, for every $x^{-i} \in X^{-i}$, $BR_i^{\alpha_i}(x^{-i})$ is a convex set for all $\alpha_i \in (0.5, 1]$.*

Proof It is enough to show that the payoff function $u_i^{\alpha_i}(\cdot, x^{-i})$ is a concave function of x^i. Under condition 3 of Assumption 1, the concavity of $u_i^{\alpha_i}(\cdot, x^{-i})$ directly follows from Proposition 2.1. of [14].

Remark 1 Lemma 1 holds for $\alpha_i \in [0.5, 1]$, if ξ^i has strictly positive density [14].

Theorem 1 *Consider an n-player non-cooperative game with random payoffs. Let a random vector $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, where $\Sigma_i \succ 0$. If Assumption 1 holds, there exists a Nash equilibrium of a CCG for all $\alpha \in (0.5, 1]^n$.*

Proof As discussed above, a fixed point of $G(\cdot)$ is a Nash equilibrium. We use Kakutani fixed point theorem [17] to show that there exists a fixed point of $G(\cdot)$. The conditions of Kakutani fixed point theorem are as follows:

- X is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
- $G(x)$ is non-empty and convex for all $x \in X$.
- $G(\cdot)$ has closed graph.

The first condition follows from the fact that X^i, $i \in I$, is a non-empty, convex, and compact. Since, $\xi^i \sim Ellip(\mu_i, \Sigma_i)$ with $\Sigma_i \succ 0$, then the payoff function $u_i^{\alpha_i}(\cdot)$, $i \in I$, is given by (3). It is easy to see that $u_i^{\alpha_i}(\cdot)$, $i \in I$, is a continuous function of x. Then, the best response sets $BR_i^{\alpha_i}(x^{-i})$, $i \in I$, are non-empty because a continuous function over compact set always attains maxima. This implies $G(x)$ is non-empty for all x. The convexity of $G(x)$ follows from Lemma 1. The closed graph condition can be proved using the continuity of the payoff functions $u_i^{\alpha_i}(\cdot)$, $i \in I$. For detailed proof of closed graph condition see Theorem 3.2 of [28]; see also [2]. Hence, there exists a Nash equilibrium for a CCG.

Remark 2 Theorem 1 holds for $\alpha \in [0.5, 1]^n$, if ξ^i, $i \in I$, has strictly positive density [14].

2.3 Variational inequality formulation

We recall the definition of variational inequality (VI) from [11]. Given a closed, convex set K and a continuous function G, solving a VI is to find a vector $z \in K$ such that

$$(y - z)^T G(z) \geq 0, \ \forall \ y \in K.$$

We denote the above VI problem as VI(K, G). It is well known that the Nash equilibrium problem of a deterministic non-cooperative game can be formulated as a VI problem [11] if the strategy set of each player is closed and convex, and the payoff function of each player is concave (convex in minimization setting), continuous, and differentiable in its own strategies for the fixed strategies of other players. The continuity and differentiability of the payoff function of a player is understood to mean on an open set containing his strategy set.

We formulate the Nash equilibrium problem of a CCG as a VI problem. The differentiability of the payoff function of each player, in its own strategies, defined by (3) is required in VI formulation. It exists under the Assumption 2 given below.

Assumption 2 *For each $i \in I$, the following conditions hold:*

1. *For every $x^{-i} \in X^{-i}$, $f_k^i(\cdot, x^{-i})$ is a differentiable function of x^i, for all $k = 1, 2, \cdots, l_i$.*
2. *The system $f_k^i(x) = 0$, $k = 1, 2, \cdots, l_i$, has no solution.*

If Assumption 2 holds, the gradient of the payoff function of player i, defined by (3), is given by

$$\nabla_{x^i} u_i^{\alpha_i}(x^i, x^{-i}) = \left(J_{f^i(\cdot, x^{-i})}(x)\right)^T \mu_i + \frac{\left(J_{f^i(\cdot, x^{-i})}(x)\right)^T \Sigma_i f^i(x^i, x^{-i}) \phi_{Z_i}^{-1}(1-\alpha_i)}{||\Sigma_i^{1/2} f^i(x^i, x^{-i})||}, \tag{4}$$

where $J_{f^i(\cdot,x^{-i})}(x)$ is the Jacobian matrix of $f^i(\cdot, x^{-i})$. Define, a function $F : \mathbb{R}^m \to \mathbb{R}^m$, $F(x) = (F_1(x), F_2(x), \cdots, F_n(x))^T$, where for each $i \in I$

$$F_i(x) = -\left(J_{f^i(\cdot, x^{-i})}(x)\right)^T \mu_i - \frac{\left(J_{f^i(\cdot, x^{-i})}(x)\right)^T \Sigma_i f^i(x^i, x^{-i}) \phi_{Z_i}^{-1}(1-\alpha_i)}{||\Sigma_i^{1/2} f^i(x^i, x^{-i})||}. \tag{5}$$

Theorem 2 *Consider an n-player non-cooperative game with random payoffs. Let a random vector $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, where $\Sigma_i \succ 0$. Let Assumptions 1-2 hold. Then, for an $\alpha \in (0.5, 1]^n$, x^* is a Nash equilibrium of a CCG if and only if it is a solution of the VI(X, F).*

Proof Fix $\alpha \in (0.5, 1]^n$. Let $x^* \in X$ be a Nash equilibrium of a CCG at α. Then,

$$x^{i*} \in \arg\min_{x^i \in X^i} \{-u_i^{\alpha_i}(x^i, x^{-i*})\}, \ \forall\, i \in I.$$

For each $i \in I$, X^i is a convex set. Since, $\xi^i \sim Ellip(\mu_i, \Sigma_i)$ with $\Sigma_i \succ 0$, then, as claimed in Lemma 1, $-u_i^{\alpha_i}(\cdot, x^{-i*})$ given by (3) is a convex function of x_i. Then, from minimum principle, x^* is a Nash equilibrium if and only if for each $i \in I$

$$(y^i - x^{i*})^T \left(-\nabla_{x^i} u_i^{\alpha_i}(x^{i*}, x^{-i*})\right) \geq 0, \ \forall\, y^i \in X^i.$$

That is,

$$(y^i - x^{i*})^T F_i(x^*) \geq 0, \ \forall\, y^i \in X^i, i \in I. \tag{6}$$

By concatenating all the inequalities defined by (6), x^* is a solution of the VI(X, F).

Conversely, let $x^* \in X$ be a solution of the VI(X, F), then

$$(y - x^*)^T F(x^*) \geq 0, \ \forall\, y \in X.$$

For each $i \in I$, take $y = (x^{1*}, \cdots, y^i, \cdots, x^{*n})^T$. Hence, for each $i \in I$, we have

$$(y^i - x^{i*})^T \left(-\nabla_{x^i} u_i^{\alpha_i}(x^{i*}, x^{-i*})\right) \geq 0, \ \forall\, y^i \in X^i.$$

Again from minimum principle

$$x^{i*} \in \arg\min_{x^i \in X^i} \{-u_i^{\alpha_i}(x^i, x^{-i*})\}, \ \forall\, i \in I.$$

Hence, x^* is a Nash equilibrium of a CCG at α.

3 Generalized Nash equilibrium for chance-constrained game

Unlike in the previous section, we consider the case where the strategy set of each player depends on the strategies of other players. A GNE is an equilibrium concept for such games [12, 27]. Let $X^i(x^{-i}) \subset \mathbb{R}^{m_i}$, $i \in I$, be the strategy set of player i for a given strategy profile x^{-i} of other players. The set of all strategy profiles is defined by $X(x) = \times_{i=1}^n X^i(x^{-i})$. For a given $\alpha \in [0, 1]^n$, a strategy profile $x^* \in X(x^*)$ is said to be a GNE of a CCG if for each $i \in I$ the following inequality holds

$$u_i^{\alpha_i}(x^{i*}, x^{-i*}) \geq u_i^{\alpha_i}(x^i, x^{-i*}), \ \forall\, x^i \in X^i(x^{-i*}). \tag{7}$$

We assume that a convex and compact set $\mathcal{R} \subset \mathbb{R}^m$ is given, and for each x^{-i} the strategy set of player i is defined as,

$$X^i(x^{-i}) = \{x^i \in \mathbb{R}^{m_i} | (x^i, x^{-i}) \in \mathcal{R}\}. \tag{8}$$

This is an important case of GNE problem considered in the fundamental paper by Rosen [27]. It appears more often in practice, e.g., when all players share common resources.

Theorem 3 *Consider an n-player non-cooperative game with random payoffs. Let a random vector $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, where $\Sigma_i \succ 0$. The strategy set of each player is given by* (8). *If condition 2 and condition 3 of Assumption 1 hold, there always exists a GNE of a CCG for all $\alpha \in (0.5, 1]^n$.*

Proof Fix $\alpha \in (0.5, 1]^n$. Since, condition 2 and 3 of Assumption 1 hold, and $\xi^i \sim Ellip(\mu_i, \Sigma_i)$ with $\Sigma_i \succ 0$, then, $u_i^{\alpha_i}(\cdot)$, $i \in I$, defined by (3), is a continuous function of x, and $u_i^{\alpha_i}(\cdot, x^{-i})$, $i \in I$, is a concave function of x^i. The set $\mathcal{R}$ is convex and compact. Then, the existence of a GNE for a CCG follows from Theorem 1 of [27].

Remark 3 Theorem 3 holds for $\alpha \in [0.5, 1]^n$, if for each $i \in I$, ξ^i has strictly positive density [14].

3.1 Variational inequality formulation

Facchinei et al. [12] proposed a variational inequality whose solution is a solution of the deterministic GNE problem considered by Rosen [27]. We characterize the generalized Nash equilibria of a CCG using the solution set of the VI($\mathcal{R}, F$).

Theorem 4 *Consider an n-player non-cooperative game with random payoffs. Let a random vector $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, where $\Sigma_i \succ 0$. The strategy set of each player is given by* (8)*. Let condition 2 and condition 3 of Assumption 1, and Assumption 2 hold. Then, for an $\alpha \in (0.5, 1]^n$, a solution of the VI($\mathcal{R}, F$) is a GNE of a CCG.*

Proof Fix $\alpha \in (0.5, 1]^n$. Let x^* be a solution of the VI($\mathcal{R}, F$) (which always exists because $\mathcal{R}$ is a compact set). Then,

$$(y - x^*)^T F(x^*) \geq 0, \ \forall \ y \in \mathcal{R}. \tag{9}$$

For each $i \in I$, consider a point $y^i \in X^i(x^{-i*})$. Then, $y = (x^{1*}, \cdots, y^i, \cdots, x^{n*})^T \in \mathcal{R}$. From (9), for each $i \in I$, we have

$$(y^i - x^{i*})^T \left(-\nabla_{x^i} u_i^{\alpha_i}(x^{i*}, x^{-i*})\right) \geq 0, \ \forall \ y^i \in X^i(x^{-i*}).$$

$X^i(x^{-i*})$ is a convex set because $\mathcal{R}$ is a convex set. Under the hypothesis of theorem, $-u_i^{\alpha_i}(\cdot, x^{-i*})$ is a convex function of x^i. Therefore, from minimum principle we have for each $i \in I$

$$x^{i*} \in \operatorname*{arg\,min}_{x^i \in X^i(x^{-i*})} \{-u_i^{\alpha_i}(x^i, x^{-i*})\}.$$

This implies that x^* is a GNE of a CCG.

Broadly, a GNE need not be a solution of a VI. In fact, a counter example is given in [12] where a GNE is not the solution of the corresponding VI. However, if $\mathcal{R}$ is defined by a finite number of convex and continuously differentiable constraints, there exists a set of generalized Nash equilibria which are solutions of a VI. Such a GNE is called a variational equilibrium [20]. We show the similar result for a CCG.

Let $\mathcal{R}$ be defined by a finite number of constraints as follows:

$$\mathcal{R} = \{x \in \mathbb{R}^m | \ g_k(x) \leq 0, \ \forall \ k = 1, 2, \cdots, K\}, \tag{10}$$

where all the constraints $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \cdots, K$, are convex and continuously differentiable. For an $\alpha \in (0.5, 1]^n$, let x be a GNE of a CCG corresponding to elliptically symmetric distributed random vector ξ^i, $i \in I$, and Assumptions 1-2 hold. Then, for each $i \in I$, x^i is a solution of the following convex optimization problem

$$\begin{aligned} &\min_{x^i} -u_i^{\alpha_i}(x^i, x^{-i}) \\ \text{s.t.} \\ &g_k(x^i, x^{-i}) \leq 0, \ \forall \ k = 1, 2, \cdots, K. \end{aligned} \tag{11}$$

For a convex optimization problem, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for a global minimum if a suitable constraint qualification (CQ) holds. One such CQ is Abadie CQ. Other stronger CQs are linear independence CQ and Slater condition. We assume that one of these CQ holds. Then, a point $x \in \mathbb{R}^m$, where the KKT conditions of (11) corresponding to each player are satisfied simultaneously, is a GNE of a CCG. That is, x is a GNE of a CCG if and only if the following conditions hold:

$$\left.\begin{array}{r} -\nabla_{x^i} u_i^{\alpha_i}(x^i, x^{-i}) + \left(J_{g(\cdot, x^{-i})}(x)\right)^T \lambda^i = 0, \ \forall\, i \in I, \\ 0 \leq \lambda^i \perp g(x^i, x^{-i}) \leq 0, \ \forall\, i \in I, \end{array}\right\} \tag{12}$$

where $\lambda^i \in \mathbb{R}^K$ is a vector of Lagrange multipliers corresponding to player i, $\perp$ implies that at least one side of inequality is equality. If constraint g_k is inactive, the corresponding Lagrange multiplier is zero. However, for active constraints the Lagrange multipliers can be different for each player.

Now, suppose x is a solution of the VI($\mathcal{R}, F$), where $\mathcal{R}$ is defined by (10). Let x satisfies Abadie CQ. Then, the following KKT conditions are necessary and sufficient for the solution of the VI($\mathcal{R}, F$) [11]:

$$\left.\begin{array}{l} F(x) + \left(J_{g(\cdot)}(x)\right)^T \lambda = 0 \\ 0 \leq \lambda \perp g(x) \leq 0. \end{array}\right\}$$

Using the definition of $F(\cdot)$ from (5), the above KKT conditions can be written as

$$\left.\begin{array}{c} \begin{pmatrix} -\nabla_{x^1} u_1^{\alpha_1}(x^1, x^{-1}) \\ -\nabla_{x^2} u_2^{\alpha_2}(x^2, x^{-2}) \\ \vdots \\ -\nabla_{x^n} u_n^{\alpha_n}(x^n, x^{-n}) \end{pmatrix} + \begin{pmatrix} \left(J_{g(\cdot, x^{-1})}(x)\right)^T \\ \left(J_{g(\cdot, x^{-2})}(x)\right)^T \\ \vdots \\ \left(J_{g(\cdot, x^{-n})}(x)\right)^T \end{pmatrix} \lambda = 0, \\ 0 \leq \lambda \perp g(x) \leq 0. \end{array}\right\} \tag{13}$$

Theorem 5 *Consider an n-player non-cooperative game with random payoffs. Let a random vector $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, where $\Sigma_i \succ 0$. The strategy set of each player is defined using shared constraints' set given by* (10). *Let condition 2 and condition 3 of Assumption 1, and Assumption 2 hold, and $\alpha \in (0.5, 1]^n$. Then,*

(a) If x^* is a solution of the VI($\mathcal{R}, F$) such that (x^*, λ^*) satisfy KKT conditions (13)*, x^* is a GNE of a CCG at which KKT conditions* (12) *hold with $\lambda^{1*} = \lambda^{2*} = \cdots = \lambda^{n*} = \lambda^*$.*

(b) If x^* is a GNE of a CCG at which KKT conditions (12) *hold with $\lambda^{1*} = \lambda^{2*} = \cdots = \lambda^{n*}$, x^* is a solution of the VI($\mathcal{R}, F$).*

Proof (*a*) Let (13) holds at (x^*, λ^*). Then, it is easy to see that (12) holds at x^* with $\lambda^{1*} = \lambda^{2*} = \cdots = \lambda^{n*} = \lambda^*$. Since, $\xi^i \sim Ellip(\mu_i, \Sigma_i)$, $i \in I$, and condition 2 and condition 3 of Assumption 1, and Assumption 2 hold, then, KKT conditions (12) are sufficient to guarantee that x^* is a GNE of a CCG.

(b) Let x^* be a GNE of a CCG at which KKT conditions (12) hold with $\lambda^{1*} = \lambda^{2*} = \cdots = \lambda^{n*}$. Then, it is obvious that the KKT conditions (13) are satisfied by (x^*, λ^*), where $\lambda^* = \lambda^{1*}$. Under convexity of $\mathcal{R}$, (13) is sufficient to claim that x^* is a solution of the VI$(\mathcal{R}, F)$ [11].

4 Nash-Cournot competition in electricity market

In this section, we consider two examples of stochastic Nash-Cournot competition among electricity firms. We assume that the electricity firms are risk averse. We study these stochastic Nash games under chance-constrained game framework.

4.1 Cournot competition among electricity firms

We consider an electricity market comprises of n electricity generating firms. The firms compete in quantity as in Cournot model. Let $x^i \in X^i \subset \mathbb{R}_+$ denote the amount of electricity generated by firm i and $x = (x^1, x^2, \cdots, x^n)$ denote a vector of output level of all the firms. The market price is determined by the sum of the amount of electricity generated by all the firms. We assume that the market price is random due to various external factors and it is given by,

$$P(x, \omega) = a - b \cdot \sum_{i=1}^{n} x^i + \zeta(\omega),$$

where $\zeta : \Omega \to \mathbb{R}$ is a random variable, $a \in \mathbb{R}$, and $b \geq 0$. We assume that the quantile function of random variable ζ exists. Let $c_i(x^i)$ be the cost incurred by firm i for generating x^i amount of electricity. Then, for an $\omega \in \Omega$ the realization of the profit of firm i is given by,

$$r^i(x, \omega) = x^i \cdot P(x, \omega) - c_i(x^i). \tag{14}$$

The profit of firm i given by (14) is a random variable. Then, for a given output level vector x and confidence level vector α, the profit of firm i is given by,

$$u_i^{\alpha_i}(x) = \sup \left\{ \gamma \;\middle|\; P\left(\left\{ \omega \middle| \; x^i \left(a - b \cdot \sum_{j=1}^{n} x^j \right) + x^i \cdot \zeta(\omega) - c_i(x^i) \geq \gamma \right\} \right) \geq \alpha_i \right\}. \tag{15}$$

For each $i \in I$, if $x^i > 0$, from (15) we have,

$$\begin{aligned} u_i^{\alpha_i}(x) &= \sup\left\{\gamma \;\middle|\; P\left(\left\{\omega \middle| \, \zeta(\omega) \le \frac{\gamma - x^i\left(a - b\cdot\sum_{j=1}^n x^j\right) + c_i(x^i)}{x^i}\right\}\right) \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. \le 1-\alpha_i \right\} \\ &= \sup\left\{\gamma \;\middle|\; \gamma \le x^i\left(a - b\cdot\sum_{j=1}^n x^j\right) - c_i(x^i) + x^i\phi_\zeta^{-1}(1-\alpha_i)\right\} \\ &= x^i\left(a - b\cdot\sum_{j=1}^n x^j\right) - c_i(x^i) + x^i\phi_\zeta^{-1}(1-\alpha_i). \end{aligned}$$

For each $i \in I$, if $x^i = 0$, from (15) we have,

$$u_i^{\alpha_i}(x) = -c_i(x^i).$$

Therefore, for a given x and α, the payoff of firm i is given by,

$$u_i^{\alpha_i}(x) = x^i\left(a - b\cdot\sum_{j=1}^n x^j\right) - c_i(x^i) + x^i\phi_\zeta^{-1}(1-\alpha_i). \tag{16}$$

We assume that the cost function $c_i(\cdot)$, $i \in I$, is a differentiable and convex function of x^i. Then, the payoff function of player i, $i \in I$, is a differentiable and concave function of x^i. We consider the case where each firm i has a finite capacity C^i, i.e., $X^i = [0, C^i]$.

4.1.1 Equivalent nonlinear complementarity problem

In general, a CCG problem is equivalent to a VI problem as shown in Section 2.3. In this example the strategy set of each firm is defined by a closed interval. Using this property, we formulate the Nash equilibrium problem as an equivalent nonlinear complementarity problem (NCP) which is easier to solve than a VI. A best response of firm i, for a fixed output level x^{-i} of other firms, can be obtained by solving the following convex optimization problem,

$$\begin{aligned} &\min_{x^i} -x^i\left(a - b\cdot\sum_{j=1}^n x^j\right) + c_i(x^i) - x^i\phi_\zeta^{-1}(1-\alpha_i) \\ &s.t. \\ &\quad x^i - C^i \le 0, \\ &\quad x^i \ge 0. \end{aligned} \tag{17}$$

It is easy to see that Slater condition holds for (17). Then, the KKT conditions of the convex optimization problem (17) are necessary and sufficient conditions for the optimal solution. Hence, a best response of player i can be obtained by solving the following KKT conditions.

$$\left.\begin{aligned} &0 \leq x^i \perp -\left(a - b\sum_{j=1;j\neq i}^{n} x^j\right) + 2bx^i + \frac{dc_i(x^i)}{dx^i} - \phi_\zeta^{-1}(1-\alpha_i) + \lambda_i \geq 0,\\ &0 \leq \lambda_i \perp C^i - x^i \geq 0. \end{aligned}\right\} \tag{18}$$

Then, a Nash equilibrium of the CCG can be obtained by solving (18) for each i simultaneously. Consider the vectors $z = (x^1, x^2, \cdots, x^n, \lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^{2n}$ and $G(z) \in \mathbb{R}^{2n}$, where,

$$G_i(z) = \begin{cases} -\left(a - b\sum_{j=1;j\neq i}^{n} x^j\right) + 2bx^i + \frac{dc_i(x^i)}{dx^i} - \phi_\zeta^{-1}(1-\alpha_i) + \lambda_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ i = 1, 2, \cdots, n, \\ C^{i-n} - x^{i-n},\ \forall\ i = n+1, n+2, \cdots, 2n \end{cases}$$

Therefore, the strategy part x of z is a Nash equilibrium of a CCG at α if and only if z is a solution of the following NCP

$$0 \leq z \perp G(z) \geq 0. \tag{19}$$

The nonlinearity in $G(z)$ is present due to term $\frac{dc_i(x^i)}{dx^i}$, $i \in I$. Therefore, if the nonlinearity of cost function of each firm is at most quadratic, then (19) becomes a linear complementarity problem (LCP).

4.2 Cournot competition among electricity firms over a network

We consider an electricity market where firms compete over an electricity network comprises of a set of nodes. Let N denote the set of nodes and N_i denote the subset of nodes where firm i has installed its generation facilities. Let I_k denote the set of firms who owns generation facilities at node k. Let x_k^i be the generation quantity for firm i at node k and $c_{ik}(x_k^i)$ be its cost of generation. For each i and k, we assume $c_{ik}(\cdot)$ to be a differentiable convex function of x_k^i. Denote a generation level vector of firm i by $x^i = (x_k^i)_{k\in N_i}$, and a generation level vector at node k by $\hat{x}_k = (x_k^i)_{i\in I_k}$. The price at each node k depends only on the generation quantities of the firms whose generation facilities are installed at node k and it is determined by the sum of the generation quantities. The case where prices are deterministic is considered in [21]. We consider the case where prices are random. The random price at node k is given by,

$$P_k(\hat{x}_k, \omega) = a_k - b_k \sum_{j\in I_k} x_k^j + \zeta_k(\omega),\ k \in N, \tag{20}$$

where ζ_k is a random variable, and $b_k \geq 0$ for all $k \in N$. Therefore, for an $\omega \in \Omega$, the realization of random profit of firm i is given by,

$$r^i(x,\omega) = \sum_{k\in N_i} \left(x_k^i \left(a_k - b_k \sum_{j\in I_k} x_k^j + \zeta_k(\omega) \right) - c_{ik}(x_k^i) \right).$$

We assume that $\{\zeta_k\}_{k\in N}$ are independent normal random variables, where the mean and variance of ζ_k are μ_k and σ_k^2 respectively. Then, for a given output level vector x and confidence level vector α, the profit of firm i is given by,

$$\begin{aligned} u_i^{\alpha_i}(x) = \sup \Bigg\{ \gamma \,\Big|\, P\Bigg(\Bigg\{ \omega \,\Big|\, \sum_{k\in N_i} \Bigg(x_k^i \Bigg(a_k - b_k \sum_{j\in I_k} x_k^j \Bigg) - c_{ik}(x_k^i) \Bigg) \\ + \sum_{k\in N_i} x_k^i \zeta_k(\omega) \geq \gamma \Bigg\} \Bigg) \geq \alpha_i \Bigg\} \\ = \sum_{k\in N_i} \left[x_k^i \left(a_k - b_k \sum_{j\in I_k} x_k^j \right) - c_{ik}(x_k^i) \right] + \sum_{k\in N_i} \mu_k x_k^i \\ + \left(\sum_{k\in N_i} \sigma_k^2 (x_k^i)^2 \right)^{1/2} \phi_{Z_i^N}^{-1}(1-\alpha_i), \end{aligned}$$

where $\phi_{Z_i^N}^{-1}(1-\alpha_i)$ is a quantile function of a standard normal random variable Z_i^N. Hence,

$$\begin{aligned} u_i^{\alpha_i}(x) = \sum_{k\in N_i} \left[x_k^i \left(a_k - b_k \sum_{j\in I_k} x_k^j \right) - c_{ik}(x_k^i) \right] + \sum_{k\in N_i} \mu_k x_k^i \\ + \left(\sum_{k\in N_i} \sigma_k^2 (x_k^i)^2 \right)^{1/2} \phi_{Z_i^N}^{-1}(1-\alpha_i). \end{aligned} \tag{21}$$

We assume that the production level of each firm at each node is strictly positive and all the generation facilities have finite capacity. That is, for each i and k, let $x_k^i \in [\beta_k^i, C_k^i]$, where $\beta_k^i > 0$. We consider the case where each node has a certain demand. Let d_k be the demand at node k. The total electricity generation has to be sufficient to meet the demand at each node. The shared constraint set in this case is defined by

$$\mathcal{R} = \Big\{ x | x_k^i \in [\beta_k^i, C_k^i],\ \forall\, i \in I_k, k \in N, \text{ and } \sum_{i\in I_k} x_k^i \geq d_k,\ \forall\, k \in N \Big\}. \tag{22}$$

Then, the objective of each firm is to maximize its payoff given by (21) subject to its constraints given by (22). It is easy to check that for each $i \in I$, $u_i^{\alpha_i}(\cdot, x^{-i})$ is a concave function of x^i for all $\alpha_i \in [0.5, 1]$, $u_i^{\alpha_i}(\cdot)$ is a continuous function of x, and $\mathcal{R}$ is a compact set. Therefore, a generalized Nash equilibrium exists for this game. The payoff function $u_i^{\alpha_i}(\cdot, x^{-i})$ is also differentiable on an open set containing X^i because $x_k^i \geq \beta_k^i > 0$. A solution of VI($\mathcal{R}, F$) exists because $\mathcal{R}$ is a compact set. It is easy to check that the Slater condition holds for suitable $(d_k)_{k\in N}$ and β_k^i, C_k^i, $k \in N$, $i \in I_k$. Therefore, a generalized Nash equilibrium can be computed by solving KKT conditions (13).

5 Numerical results

We consider two instances of chance-constrained Nash-Cournot model considered in Section 4. We compute the Nash equilibria by solving equivalent complementarity problems using MATLAB. The numerical experiments are carried out on an Intel(R) 32-bit core(TM) i3-3110M CPU @ 2.40GHz×4 and 3.8 GiB of RAM machine.

Example 1 We consider an example of the stochastic Nash-Cournot model given in Section 4.1. We consider the case of 5 electricity firms where the cost function of firm i is $c_i(x^i) = (x^i)^2$. The random variable ζ follows a normal random variable with mean μ and variance σ^2.

Since, the cost function of each firm is quadratic, then, the Nash equilibria of the CCG corresponding to stochastic Nash game given in Example 1 can be computed by solving the following equivalent linear complementarity problem:

$$0 \leq z \perp Mz + q \geq 0, \tag{23}$$

where, $z = \left(x^1, x^2, x^3, x^4, x^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right)^T$,

$$M = \begin{pmatrix} 2b+2 & b & b & b & b & 1 & 0 & 0 & 0 & 0 \\ b & 2b+2 & b & b & b & 0 & 1 & 0 & 0 & 0 \\ b & b & 2b+2 & b & b & 0 & 0 & 1 & 0 & 0 \\ b & b & b & 2b+2 & b & 0 & 0 & 0 & 1 & 0 \\ b & b & b & b & 2b+2 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} -a - \phi_\zeta^{-1}(1-\alpha_1) \\ -a - \phi_\zeta^{-1}(1-\alpha_2) \\ -a - \phi_\zeta^{-1}(1-\alpha_3) \\ -a - \phi_\zeta^{-1}(1-\alpha_4) \\ -a - \phi_\zeta^{-1}(1-\alpha_5) \\ C^1 \\ C^2 \\ C^3 \\ C^4 \\ C^5 \end{pmatrix}.$$

We solve LCP 23 by using available LCP solver in MATLAB. The average CPU time to compute a Nash equilibrium is within 1 second. Table 1 summarizes the Nash equilibria of the CCG. The column 1 of Table 1 represent the values of parameters a and b that define the market price. The production capacity of each firm is given in column 2. The values of mean and variance of normally distributed random variable ζ are given in column 3, and column 4 contains the values of risk level vector α. The Nash equilibrium of the game is given in column 5.

Table 1: Nash equilibria of chance-constrained game

a, b	$(C^i)_{i=1}^5$	μ, σ^2	$\alpha = (\alpha_i)_{i=1}^5$	Nash Equilibrium x^*
$a = 1$ $b = 2$	$C^1 = 5$ $C^2 = 5$ $C^3 = 5$ $C^4 = 5$ $C^5 = 5$	$\mu = 1$ $\sigma^2 = 2$	(0.6, 0.6, 0.6, 0.6, 0.6)	(0.1173, 0.1173, 0.1173, 0.1173, 0.1173)
			(0.6, 0.6, 0.7, 0.7, 0.7)	(0.1583, 0.1583, 0.0625, 0.0625, 0.0625)
			(0.55, 0.65, 0.7, 0.75, 0.75)	(0.2189, 0.1271, 0.0779, 0.0248, 0.0248)
$a = 2$ $b = 5$	$C^1 = 7$ $C^2 = 7$ $C^3 = 7$ $C^4 = 7$ $C^5 = 7$	$\mu = 1$ $\sigma^2 = 3$	(0.6, 0.6, 0.6, 0.6, 0.6)	(0.0800, 0.0800, 0.0800, 0.0800, 0.0800)
			(0.6, 0.6, 0.7, 0.7, 0.7)	(0.1115, 0.1115, 0.0444, 0.0444, 0.0444)
			(0.55, 0.65, 0.7, 0.75, 0.75)	(0.1548, 0.0906, 0.0562, 0.0190, 0.0190)
$a = 3$ $b = 7$	$C^1 = 10$ $C^2 = 10$ $C^3 = 10$ $C^4 = 10$ $C^5 = 10$	$\mu = 2$ $\sigma^2 = 5$	(0.6, 0.6, 0.6, 0.6, 0.6)	(0.1008, 0.1008, 0.1008, 0.1008, 0.1008)
			(0.6, 0.6, 0.7, 0.7, 0.7)	(0.1329, 0.1329, 0.0656, 0.0656, 0.0656)
			(0.55, 0.65, 0.7, 0.75, 0.75)	(0.1767, 0.1121, 0.0776, 0.0403, 0.0403)

Example 2 We consider an example of the stochastic Nash-Cournot model given in Section 4.2. We consider the case of 5 electricity firms and 3 nodes. The cost of firm i at node k is $c_{ik}(x_k^i) = \left(x_k^i\right)^2$. The sets N_i and I_k for different values of i and k are as follows: $N_1 = \{1,2,3\}$, $N_2 = \{1,2\}$, $N_3 = \{1,3\}$, $N_4 = \{1,2\}$, $N_5 = \{1,2,3\}$, and $I_1 = \{1,2,3,4,5\}$, $I_2 = \{1,2,4,5\}$, $I_3 = \{1,3,5\}$.

For ease in computation we consider the case where only the price at node 1 is random and other prices are deterministic. That is, $\zeta_2 = \zeta_3 = 0$, therefore $\mu_2 = \mu_3 = 0$, and $\sigma_2^2 = \sigma_3^2 = 0$. Denote, $\zeta_1 = \zeta$, and $\mu_1 = \mu$, $\sigma_1^2 = \sigma^2$. By putting these values in (21), the payoff function of player i is given by

$$u_i^{\alpha_i}(x) = \sum_{k\in N_i}\left[x_k^i\left(a_k - b_k\sum_{j\in I_k} x_k^j\right) - c_{ik}(x_k^i)\right] + x_1^i\phi_\zeta^{-1}(1-\alpha_i),\ \forall\, i = 1,2,3,4,5, \tag{24}$$

where $\phi_\zeta^{-1}(1-\alpha_i) = \mu + \sigma\phi_{Z_i^N}^{-1}(1-\alpha_i)$ is a quantile function of normal random variable ζ with mean μ and variance σ^2. The payoff function of firm i given by (24) is differentiable for all x^i because second term of (24) is linear in x^i. Therefore, we consider the case where $x_k^i \in [0, C_k^i]$ for all i and k. Since, the cost function $c_{ik}(\cdot)$ for each i and k is quadratic, then the KKT conditions (13) can be reformulated as an LCP. Therefore, to compute a generalized Nash equilibrium of the CCG corresponding to the stochastic Nash game given in Example 2, we have the following equivalent LCP:

$$0 \leq z \perp Mz + q \geq 0, \tag{25}$$

where, $M = \begin{pmatrix} M_1 & J \\ C & \mathbf{0} \end{pmatrix}$, and $z = \begin{pmatrix} x \\ \lambda \end{pmatrix}$, $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ are such that

$$x = \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_1^2 \\ x_2^2 \\ x_1^3 \\ x_3^3 \\ x_1^4 \\ x_2^4 \\ x_1^5 \\ x_2^5 \\ x_3^5 \end{pmatrix},\ \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{15} \end{pmatrix},\ q_1 = \begin{pmatrix} -a_1 - \phi_\zeta^{-1}(1-\alpha_1) \\ -a_2 \\ -a_3 \\ -a_1 - \phi_\zeta^{-1}(1-\alpha_2) \\ -a_2 \\ -a_1 - \phi_\zeta^{-1}(1-\alpha_3) \\ -a_3 \\ -a_1 - \phi_\zeta^{-1}(1-\alpha_4) \\ -a_2 \\ -a_1 - \phi_\zeta^{-1}(1-\alpha_5) \\ -a_2 \\ -a_3 \end{pmatrix},\ q_2 = \begin{pmatrix} -d_1 \\ -d_2 \\ -d_3 \\ C_1^1 \\ C_2^1 \\ C_3^1 \\ C_1^2 \\ C_2^2 \\ C_1^3 \\ C_3^3 \\ C_1^4 \\ C_2^4 \\ C_1^5 \\ C_2^5 \\ C_3^5 \end{pmatrix}.$$

Denote $\delta_k = 2b_k + 2$, $k = 1, 2, 3$. Then, the matrices defining M are as follows:

$$M_1 = \begin{pmatrix} \delta_1 & 0 & 0 & b_1 & 0 & b_1 & 0 & b_1 & 0 & b_1 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 & b_2 & 0 & 0 & 0 & b_2 & 0 & b_2 & 0 \\ 0 & 0 & \delta_3 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & 0 & b_3 \\ b_1 & 0 & 0 & \delta_1 & 0 & b_1 & 0 & b_1 & 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & \delta_2 & 0 & 0 & 0 & b_2 & 0 & b_2 & 0 \\ b_1 & 0 & 0 & b_1 & 0 & \delta_1 & 0 & b_1 & 0 & b_1 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 & \delta_3 & 0 & 0 & 0 & 0 & b_3 \\ b_1 & 0 & 0 & b_1 & 0 & b_1 & 0 & \delta_1 & 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & b_2 & 0 & 0 & 0 & \delta_2 & 0 & b_2 & 0 \\ b_1 & 0 & 0 & b_1 & 0 & b_1 & 0 & b_1 & 0 & \delta_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & b_2 & 0 & 0 & 0 & b_2 & 0 & \delta_2 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & 0 & \delta_3 \end{pmatrix},$$

$$J = \begin{pmatrix} -\mathbb{L}^T & \mathbb{E}_{12} \end{pmatrix}, \; C = \begin{pmatrix} \mathbb{L} \\ -\mathbb{E}_{12} \end{pmatrix},$$

where $\mathbb{E}_{12}$ is a 12×12 identity matrix, and

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\mathbf{0}$ is a 15×15 zero matrix. Table 2 summarizes the Nash equilibria of the CCG. The average CPU time to compute a Nash equilibrium is within 1 second. The column 1 represents the parameters defining the market prices at different nodes. The production capacities of the generation facilities and the demand at different nodes are given in column 2. The mean and variance of normal random variable ζ, and the risk level vector α is given in column 3. The column 4 contains the Nash equilibrium of the game.

6 Conclusions

In this paper, we show the existence of a Nash equilibrium of a CCG for elliptically symmetric distributed random payoffs and continuous strategy sets. We characterize the set of Nash equilibria of a CCG using the solution set of a VI problem. For the case of shared constraints, we show the existence of a GNE and give a characterization of the set of a certain types of generalized Nash equilibria using the solution set of a VI problem. As an application of these games, we study two examples from electricity markets. We illustrate our theoretical results by taking few instances of these games.

Table 2: Nash equilibria of chance-constrained game

$(a_k)_{k=1}^3$, $(b_k)_{k=1}^3$	$((C_k^i)_{k=1}^3)_{i=1}^5$, $(d_k)_{k=1}^3$	μ, σ^2, $(\alpha_i)_{i=1}^5$	Nash Equilibrium x^*
$a_1 = 1$ $b_1 = 2$ $a_2 = 2$ $b_2 = 3$ $a_3 = 1$ $b_3 = 3$	$C_1^1 = 5$, $C_2^1 = 5$ $C_3^1 = 5$, $C_1^2 = 5$ $C_2^2 = 5$, $C_1^3 = 5$ $C_3^3 = 5$, $C_1^4 = 5$ $C_2^4 = 5$, $C_1^5 = 5$ $C_2^5 = 5$, $C_3^5 = 5$ $d_1 = 8$ $d_2 = 5$ $d_3 = 3$	$\mu = 2$ $\sigma^2 = 5$ $\alpha_1 = 0.6$ $\alpha_2 = 0.6$ $\alpha_3 = 0.7$ $\alpha_4 = 0.7$ $\alpha_5 = 0.8$	$\begin{pmatrix} 1.7264 \\ 1.25 \\ 1.0588 \\ 1.7264 \\ 1.25 \\ 1.5749 \\ 1.0588 \\ 1.5749 \\ 1.25 \\ 1.3975 \\ 1.25 \\ 0.8824 \end{pmatrix}$
$a_1 = 2$ $b_1 = 5$ $a_2 = 3$ $b_2 = 4$ $a_3 = 1$ $b_3 = 5$	$C_1^1 = 7$, $C_2^1 = 7$ $C_3^1 = 7$, $C_1^2 = 7$ $C_2^2 = 7$, $C_1^3 = 7$ $C_3^3 = 7$, $C_1^4 = 7$ $C_2^4 = 7$, $C_1^5 = 7$ $C_2^5 = 7$, $C_3^5 = 7$ $d_1 = 10$ $d_2 = 6$ $d_3 = 4$	$\mu = 1$ $\sigma^2 = 3$ $\alpha_1 = 0.7$ $\alpha_2 = 0.7$ $\alpha_3 = 0.6$ $\alpha_4 = 0.8$ $\alpha_5 = 0.7$	$\begin{pmatrix} 2.0023 \\ 1.5 \\ 1.3913 \\ 2.0023 \\ 1.5 \\ 2.0694 \\ 1.3913 \\ 1.9238 \\ 1.5 \\ 2.0023 \\ 1.5 \\ 1.2174 \end{pmatrix}$
$a_1 = 3$ $b_1 = 4$ $a_2 = 2$ $b_2 = 4$ $a_3 = 3$ $b_3 = 7$	$C_1^1 = 10$, $C_2^1 = 10$ $C_3^1 = 10$, $C_1^2 = 10$ $C_2^2 = 10$, $C_1^3 = 10$ $C_3^3 = 10$, $C_1^4 = 10$ $C_2^4 = 10$, $C_1^5 = 10$ $C_2^5 = 10$, $C_3^5 = 10$ $d_1 = 12$ $d_2 = 9$ $d_3 = 5$	$\mu = 3$ $\sigma^2 = 7$ $\alpha_1 = 0.8$ $\alpha_2 = 0.8$ $\alpha_3 = 0.8$ $\alpha_4 = 0.7$ $\alpha_5 = 0.6$	$\begin{pmatrix} 2.3201 \\ 2.25 \\ 1.7241 \\ 2.3201 \\ 2.25 \\ 2.3201 \\ 1.7241 \\ 2.46 \\ 2.25 \\ 2.5795 \\ 2.25 \\ 1.5517 \end{pmatrix}$

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