

Block Coordinate Descent Almost Surely Converges to a Stationary Point Satisfying the Second-order Necessary Condition

Enbin Song · Zhubin Shen · Qingjiang Shi

Abstract Given a non-convex twice continuously differentiable cost function with Lipschitz continuous gradient, we prove that all of the block coordinate gradient descent, block mirror descent and proximal block coordinate descent methods converge to stationary points satisfying the second-order necessary condition, almost surely with random initialization. All our results are ascribed to the center-stable manifold theorem and Ostrowski's lemma.

Keywords Block coordinate gradient descent · block mirror descent · proximal block coordinate descent · saddle points · local minimum · non-convex

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1 Introduction

Consider the following nonconvex optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \tag{1}$$

where f is a nonconvex and twice continuously differentiable function with Lipschitz continuous gradient. It is well-known that the nonconvex optimization appears in many applications of interest. A main source of difficulty in nonconvex optimization over continuous spaces is the proliferation of saddle points. There exist many instances where bad initializations could result in unfavorable saddle points for the basic gradient descent method [16, Section 1.2.3]. Despite the existence of such worst-case instances in theory, many simple algorithms including first-order algorithms and their variants, perform extremely well in terms of the quality of solutions of continuous optimization.

A well known family of algorithms for solving (1) is the block coordinate descent (BCD) type methods [2, 8, 9, 17, 21, 22, 26, 30–32], especially when the problem size is large. In this paper, we particularly prove that, with random initialization, the three popular block coordinate descent type first-order methods, including *the block coordinate gradient descent, block mirror descent and proximal block coordinate descent methods*, all can almost surely converge to stationary points

E. Song, Z. Shen
Department of Mathematics, Sichuan University, Chengdu, China
E-mail: e.b.song@163.com

Q. Shi
College of EIE, Nanjing University of Aeronautics and Astronautics, Nanjing, China
E-mail: qing.j.shi@gmail.com

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satisfying the second-order necessary condition, but neither access to second-order derivative information¹ nor randomness beyond initialization. Furthermore, we show that these results also hold true even for the cost functions with non-isolated critical points.

As in [12], our proof is built on the following basic facts. Suppose that $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an iterative mapping of an optimization method and the fixed point of g is the critical point of f as well. If the iterative mapping g possesses the following two properties, referred to as *stable mapping property* for convenience

- (i) g is a diffeomorphism;
- (ii) if x^* is a strict saddle point of f , then there exists at least one eigenvalue of the Jacobian $(Dg(x^*))^T$ ($Dg(x^*)$ denotes the gradient matrix [3] of g at x^*), whose magnitude is strictly greater than one,

then the corresponding optimization method almost surely converges to stationary points satisfying the second-order necessary condition. Hence, the main goal of our proof is to show that the iterative mappings of three BCD type algorithms satisfy the stable mapping property. However, answering these underlying questions for BCD type algorithms is not easy and the existing analysis methods are not applicable. In particular, the eigenvalue analysis of the Jacobians of the iterative mappings needs a nontrivial argument.

This paper presents a unified theoretical framework for analyzing the stable mapping property of BCD type algorithms. Specifically, Property (i) is proved by first decomposing the entire iterative mapping of each BCD type algorithm into multiple one-block mappings and then using the chain rule of diffeomorphisms. As to the proof of Property (ii), the main difficulty lies in that the Jacobian $(Dg(x^*))^T$ of one-block mapping at a strict saddle point is an asymmetric matrix and a complicated polynomial function of the original $\nabla^2 f(x^*)$ with degree p (the number of blocks of the decision variables). We overcome the above difficulty using two steps. The first step is to transform the original Jacobian $(Dg(x^*))^T$ into a more tractable form. Then, based on the simple form of $(Dg(x^*))^T$, the second step is to show that $(Dg(x^*))^T$ has at least one eigenvalue with magnitude strictly greater than one by resorting to Ostrowski's lemma (which essentially follows from Rouché's Theorem in complex analysis).

1.1 Related work

Recently, a milestone result of the gradient descent was established by [12]. The authors assume that a cost function satisfies the *strict saddle property*. Equivalently, each critical point x of f is either a local minimizer, or a "strict saddle", i.e., $\nabla^2 f(x)$ has at least one strictly negative eigenvalue. They demonstrated that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a twice continuously differentiable function whose gradient is Lipschitz continuous with constant L , then the gradient descent with a sufficiently small constant step-size α (i.e., $x_{k+1} = x_k - \alpha \nabla f(x_k)$ and $0 < \alpha < \frac{1}{L}$) converges to a local minimizer² almost surely with random initialization.

There is a followup work given by [19], which firstly proved that the results in [12] do hold true even for cost functions with non-isolated critical points. One key tool they used is that for every open cover there is a countable subcover in \mathbb{R}^n . Moreover, [19] has shown the globally Lipschitz assumption can be circumvented as long as the domain is convex and forward invariant with respect to gradient descent. In addition, they also provided an upper bound on the allowable step-size (such that those results hold true).

There are some prior works showing that first-order descent methods can indeed escape from strict saddle points with the assistance of near isotropic noise. Specifically, [20] established convergence of the Robbins-Monro stochastic approximation to local minimizers for strict saddle functions and [11] demonstrated that the perturbed versions of multiplicative weights algorithm can

¹ Certainly, some optimization methods, e.g., the trust region [15], could converge to stationary points satisfying the second-order necessary condition. However, they require the second-order derivative information.

² Actually, without further assumption it should be called stationary point satisfying the second-order necessary condition.

converge to local minima in generic potential games. In particular, [6] quantified the convergence rate of noise-added stochastic gradient descent to local minima. Note that the aforementioned methods generally require the assistance of isotropic noise, which can significantly slow down the convergence rate when the problem parameters (e.g., dimension) are large. In contrast, our setting is deterministic and corresponds to simpler implementations of block coordinate gradient descent (BCGD), block mirror descent (BMD) and proximal block coordinate descent (PBCD).

While we were preparing to submit our manuscript, we were informed of a relevant technical report [13] recently appearing in arXiv, which is closely related to our work, but with a totally different proof for the BCGD method. Moreover, [13] considers neither the BMD method nor the PBCD method.

1.2 Paper Organization

In Section 2, we introduce the basic setting and definitions used throughout the paper. Section 3 provides the main results for the BCGD method. The main results for BMD and PBCD are given in Section 4 and Section 5, respectively. Section 6 provides several important lemmas related to eigenvalue analysis. Finally, we conclude this paper in Section 7. The detailed proofs of some lemmas and propositions are presented in Section 8.

1.3 Notations

Denote a complex number z as $z = a + bi$, where a and b are real numbers and i is the imaginary unit with $i^2 = -1$. We also denote $a = \text{Re}(z)$ and $b = \text{Im}(z)$ as the real part and the imaginary part of z , respectively. For a matrix X , we denote $\text{eig}(X)$ as the set of eigenvalues of X , X^T as the transpose of X , X^H as the conjugate transpose or Hermitian transpose of X , $\rho(X)$ as the spectral radius of X (i.e., the maximum modulus of the eigenvalues of X), and $\|X\|$ as the spectral norm of X . When X is a real symmetric matrix, let $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ denote the maximum and minimum eigenvalues of X , respectively. Moreover, for two real symmetric matrices X_1 and X_2 , $X_1 \succ X_2$ (resp. $X_1 \succeq X_2$) means $X_1 - X_2$ is positive definite (resp. positive semi-definite). We use I_n to denote the identity matrix with dimension n , and we will simply use I when it is clear from context what the dimension is. For square matrices $X_s \in \mathbb{R}^{n_s \times n_s}$, $s = 1, 2, \dots, p$, we denote $\text{Diag}(X_1, X_2, \dots, X_p)$ as the block-diagonal matrix with X_s being the s -th diagonal block. For square matrices $X_s \in \mathbb{R}^{n \times n}$, $s = 1, \dots, p$, and $t, k \in \{1, 2, \dots, p\}$, we use $\prod_{s=t}^k X_s$ to denote the continued products $X_t \times X_{t+1} \times \dots \times X_{k-1} \times X_k$ if $t \leq k$ and $X_t \times X_{t-1} \times \dots \times X_{k+1} \times X_k$ if $t > k$. \mathbb{P}_ν denotes the probability with respect to a prior measure ν , which is assumed to be absolutely continuous with respect to Lebesgue measure.

2 Preliminaries

We make the following blanket assumption for problem (1).

Assumption 1 f is a twice continuously differentiable function whose gradient is Lipschitz continuous over \mathbb{R}^n , i.e., there exists a parameter $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad \text{for every } x, y \in \mathbb{R}^n. \quad (2)$$

Throughout this paper, in order to introduce the BCD type methods, we assume that the vector of decision variables x has the following partition:

$$x = \left((x_{(1)})^T, (x_{(2)})^T, \dots, (x_{(p)})^T \right)^T, \quad (3)$$

where $x^{(s)} \in \mathbb{R}^{n_s}$, and n_1, n_2, \dots, n_p are p positive integer numbers satisfying $\sum_{s=1}^p n_s = n$.

Moreover, we use the notations in [17] and define matrices $U_s \in \mathbb{R}^{n \times n_s}$, the s -th block-column of I_n , $s = 1, \dots, p$, such that

$$(U_1, U_2, \dots, U_p) = I_n. \quad (4)$$

Clearly, according to our notations, we have $x^{(s)} = U_s^T x$ for every $x \in \mathbb{R}^n$, $s = 1, \dots, p$. Consequently, $x = \sum_{s=1}^p U_s x^{(s)}$ and the derivative with respect to the variables in the vector $x^{(s)}$ can be expressed as

$$\nabla_s f(x) \equiv U_s^T \nabla f(x), \quad s = 1, \dots, p.$$

Below we give some necessary definitions as appeared in [12] and [19].

Definition 1

1. A point x^* is a critical point of f if $\nabla f(x^*) = \mathbf{0}$. We denote $C = \{x : \nabla f(x) = \mathbf{0}\}$ as the set of critical points (can be uncountably many).
2. A critical point x^* is isolated if there is a neighborhood U around x^* , and x^* is the only critical point in U^3 . Otherwise it is called non-isolated.
3. A critical point is a local minimum if there is a neighborhood U around x^* such that $f(x^*) \leq f(x)$ for all $x \in U$, and a local maximum if $f(x^*) \geq f(x)$.
4. A critical point is a saddle point if for all neighborhoods U around x^* , there are $x, y \in U$ such that $f(x) \leq f(x^*) \leq f(y)$.

Definition 2 (Strict Saddle) A critical point x^* of f is a strict saddle if $\lambda_{\min}(\nabla^2 f(x^*)) < 0$.

Definition 3 (Non-Strict Saddle) A critical point x^* of f is a non-strict saddle if $\lambda_{\min}(\nabla^2 f(x^*)) \geq 0$.

Definition 4 (Global Stable Set) The global stable set $W^s(x^*)$ of a critical point x^* is the set of initial points of an iterative mapping g of an optimization method that converge to x^* :

$$W^s(x^*) = \{x : \lim_k g^k(x) = x^*\},$$

where g^k denotes the composition of g with itself k times.

3 The BCGD method

In this section, we will prove that the BCGD method [2, 21, 30, 31] does not converge to strict saddle points under appropriate choices of step size, almost surely with random initialization. This lays the ground for the analyses of the BMD and PBCD methods. The section is organized as follows. We first describe the BCGD method. Then we prove that the iterative mapping of the BCGD method is a diffeomorphism. Next, we perform eigenvalue analysis for the Jacobian of the BCGD iterative mapping. Lastly, we conclude our results for the BCGD method.

3.1 The BCGD method description

For ease of later reference and also for the sake of clarity, we present a detailed description of the BCGD method [2] for problem (1) below.

³ If the critical points are isolated then they are countably many or finite.

Method 1 (BCGD)**Input:** $0 < \alpha < \frac{1}{L}$.**Initialization:** $x_0 \in \mathbb{R}^n$.**General Step** ($k = 0, 1, \dots$): Set $x_k^0 = x_k$ and define recursively

$$x_k^s = x_k^{s-1} - \alpha U_s \nabla_s f(x_k^{s-1}), \quad s = 1, \dots, p.$$

Set $x_{k+1} = x_k^p$.

For the BCGD method illustrated above, we use $g_{\alpha f}^s$ to denote the corresponding gradient mapping with respect to $x^{(s)}$, i.e.,

$$g_{\alpha f}^s(x) \triangleq x - \alpha U_s \nabla_s f(x), \quad s = 1, \dots, p. \quad (5)$$

Hence, the iteration of the BCGD method can be compactly written as

$$x_{k+1} = g_{\alpha f}(x_k), \quad (6)$$

where $g_{\alpha f}$ is a composite mapping defined by

$$g_{\alpha f}(x) \triangleq g_{\alpha f}^p \circ g_{\alpha f}^{p-1} \circ \dots \circ g_{\alpha f}^2 \circ g_{\alpha f}^1(x). \quad (7)$$

Moreover, by using the chain rule, we can obtain the gradient matrix of $g_{\alpha f}$ as follows

$$Dg_{\alpha f}(x) = Dg_{\alpha f}^1(y_1) \times Dg_{\alpha f}^2(y_2) \times \dots \times Dg_{\alpha f}^{p-1}(y_{p-1}) \times Dg_{\alpha f}^p(y_p), \quad (8)$$

where $y_1 = x$, $y_s = g_{\alpha f}^{s-1}(y_{s-1})$, $s = 2, \dots, p$, and $Dg_{\alpha f}^s(x)$ is the gradient matrix of $g_{\alpha f}^s$ given by

$$Dg_{\alpha f}^s(x) = I_n - \alpha \nabla^2 f(x) U_s U_s^T. \quad (9)$$

Now we are ready to show that the BCGD method can almost surely converge to non-strict saddle points. According to [12, Theorem 4], it suffices to show that the iterative mapping $g_{\alpha f}$ satisfies the stable mapping property, i.e.,

- (i) $g_{\alpha f}$ is a diffeomorphism; and
- (ii) if x^* is a strict saddle point of f , then there is at least one eigenvalue of the Jacobian $(Dg_{\alpha f}(x^*))^T$, whose magnitude is strictly greater than one.

Therefore, in what follows, our efforts are devoted to proving that $g_{\alpha f}$ is a stable mapping (i.e., a mapping that satisfies the stable mapping property).

3.2 The iterative mapping $g_{\alpha f}$ of BCGD is a diffeomorphism

Let us first study the property of $g_{\alpha f}^s$, $s = 1, \dots, p$.

Lemma 1 *If $\alpha < \frac{1}{L}$, then the mappings $g_{\alpha f}^s$ defined by (5), $s = 1, \dots, p$, are all diffeomorphisms.*

The proof of Lemma 1 is lengthy and thus has been relegated to Appendix.

Using Lemma 1 and the fact that the composition of two diffeomorphisms is also a diffeomorphism [14, Proposition 2.15], we immediately obtain the following result.

Proposition 1 *The mapping $g_{\alpha f}$ defined by (7) with step size $\alpha < \frac{1}{L}$ is a diffeomorphism.*

3.3 Eigenvalue analysis of the Jacobian of $g_{\alpha f}$ at a strict saddle point

In what follows, let us analyze the eigenvalues of the Jacobian of $g_{\alpha f}$ at a strict saddle point, and show that it has at least one eigenvalue with magnitude greater than one.

Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point. It follows that $g_{\alpha f}^s(x^*) = x^*$, $s = 1, 2, \dots, p$. As a result, using (8) and (9) we can express the Jacobian of $g_{\alpha f}$ at x^* as follows

$$\{Dg_{\alpha f}(x^*)\}^T = \left\{ \prod_{s=1}^p Dg_{\alpha f}^s(x^*) \right\}^T = \prod_{s=p}^1 (I_n - \alpha U_s U_s^T \nabla^2 f(x^*)). \quad (10)$$

To analyze the eigenvalues of the Jacobian $\{Dg_{\alpha f}(x^*)\}^T$, let us introduce matrix G defined by

$$G \triangleq \frac{1}{\alpha} \left[I_n - \{Dg_{\alpha f}(x^*)\}^T \right], \quad (11)$$

which plays a key role in our eigenvalue analysis by observing the relation

$$\lambda \in \text{eig}(G) \Leftrightarrow 1 - \alpha\lambda \in \text{eig}\left(\{Dg_{\alpha f}(x^*)\}^T\right). \quad (12)$$

Next, we express G in a more tractable and explicit form. To do so, let us first introduce the following notations:

$$A \triangleq \nabla^2 f(x^*) = (A_{st})_{1 \leq s, t \leq p}, \quad (13)$$

where A_{st} is the (s, t) -th block given by

$$A_{st} \triangleq \frac{\partial^2 f(x^*)}{\partial x^{*(s)} \partial x^{*(t)}}, \quad 1 \leq s, t \leq p. \quad (14)$$

Moreover, we denote by A_s the s -th block-row of A , i.e.,

$$A_s \triangleq (A_{st})_{1 \leq t \leq p}, \quad s = 1, \dots, p, \quad (15)$$

which can be further expressed as

$$A_s = U_s^T \nabla^2 f(x^*), \quad 1 \leq s \leq p. \quad (16)$$

Consequently, the matrix G can be shortly written as follows

$$G = \frac{1}{\alpha} \left[I_n - \prod_{s=p}^1 (I_n - \alpha U_s A_s) \right]. \quad (17)$$

Furthermore, by invoking Lemma 6 in Section 6, we obtain

$$G = (I_n + \alpha \check{A})^{-1} A, \quad (18)$$

where \check{A} is the strictly block lower triangular matrix of A (similar to the definition of \check{B} in (71) in Section 6).

Given the key expression of G in (18), we are able to provide a sufficient description of the distribution of the eigenvalues of G , which is summarized in Proposition 2.

Proposition 2 *Assume that G is defined by (11) for a strict saddle point $x^* \in \mathbb{R}^n$ with $\alpha \in (0, \frac{1}{L})$ and L being the gradient Lipschitz constant of f (cf. (2)). There exists at least one eigenvalue λ of G which lies in the closed left half complex plane excluding the origin, i.e., $\lambda \in \Omega$, where*

$$\Omega \triangleq \{a + bi \mid a, b \in \mathbb{R}, a \leq 0, (a, b) \neq (0, 0), i = \sqrt{-1}\}. \quad (19)$$

Proof Since $A = \nabla^2 f(x^*)$ and x^* is a strict saddle point, A has at least one negative eigenvalue. Moreover, according to [19, Lemma 7], we have $L \geq \rho(A)$ and thus $\alpha \in (0, \frac{1}{L}) \subseteq (0, \frac{1}{\rho(A)})$. Therefore, by noting (18) and invoking Lemma 10 in Section 6 with identifications $A \sim B$, $\tilde{A} \sim \tilde{B}$, $\alpha \sim \beta$ and $\rho(A) \sim \rho(B)$, we have $\lambda \in \Omega$. This completes the proof. \square

Based on Proposition 2 and the relation (12), we immediately obtain the key result in the following proposition.

Proposition 3 *Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point of problem (1). Then, for the BCGD iterative mapping $g_{\alpha f}$ defined by (7) with $\alpha \in (0, \frac{1}{L})$, the Jacobian $\{Dg_{\alpha f}(x^*)\}^T$ has at least one eigenvalue whose magnitude is strictly greater than one.*

Proof According to Proposition 2, G has an eigenvalue λ such $\lambda \in \Omega$ (see the definition of Ω in (19)). By noting that λ could be a complex number, let us express $\lambda = a + bi$. From the definition of Ω , we have $a \leq 0$ and $(a, b) \neq (0, 0)$. As a result, we obtain

$$|1 - \alpha(a + bi)| = \sqrt{1 - 2\alpha a + \alpha^2 a^2 + \alpha^2 b^2} \geq \sqrt{1 + \alpha^2(a^2 + b^2)} > 1.$$

This completes the proof due to the relation (12). \square

3.4 Main results of BCGD

So far, we have shown that the iterative mapping of BCGD satisfies the stable mapping property. In what follows, similar to the proof of Theorem 4 in [12], we establish the main results of BCGD by using the center-stable manifold theorem [7, 23, 24], which is a primary tool to give a local characterization of the stable set. For completeness, we rewrite it here as follows.

Theorem 1 [23, Theorem III. 7] *Let 0 be a fixed point for the C^r local diffeomorphism $\phi : U \rightarrow E$, where U is a neighborhood of zero in the Banach space E and $\infty > r \geq 1$. Let $E_{sc} \oplus E_u$ be the invariant splitting of \mathbb{R}^n into the generalized eigenspaces of $D\phi(0)$ corresponding to eigenvalues of absolute value less than or equal to one, and greater than one. Then there is a local ϕ invariant C^r embedded disc W_{loc}^{sc} tangent to E_{sc} at 0 and a ball B around zero in an adapted norm such that $\phi(W_{loc}^{sc}) \cap B \subset W_{loc}^{sc}$, and $\bigcap_{k=0}^{\infty} \phi^{-k}(B) \subset W_{loc}^{sc}$.*

Since the iterative mapping of BCGD satisfies the stable mapping property, based on Theorem 1 and similar to the proof of [12, Theorem 4], we can prove that, once the sequence generated by the BCGD method is convergent, its limit is almost surely a non-strict saddle point. The result is stated in the following theorem.

Theorem 2 *Let x^* be a strict saddle point of problem (1), and $W^s(x^*)$ be the global stable set at x^* of the BCGD iterative mapping with $0 < \alpha < \frac{1}{L}$. Then $W^s(x^*)$ is of (Lebesgue) measure zero.*

Further, by applying Theorem 2 and following the same arguments as those in the proofs of Theorem 2 and Corollary 12 in [19], we can immediately obtain Theorem 3 and Theorem 4 as follows. For brevity, we again omit the proofs.

Theorem 3 (Non-isolated) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice continuously differentiable function and $\sup_{x \in \mathbb{R}^n} \|\nabla^2 f(x)\|_2 \leq L < \infty$. The set of initial points $x \in \mathbb{R}^n$, from each of which the BCGD method with step size $0 < \alpha < \frac{1}{L}$ converges to a strict saddle point, is of (Lebesgue) measure zero, without the assumption that critical points are isolated.*

A straightforward result of Theorem 3 is that, once the sequence generated by the BCGD method is convergent, its limit is almost surely a stationary (critical) point satisfying the second-order necessary condition (i.e., a non-strict saddle point). This is our main result for BCGD summarized in the following theorem.

Theorem 4 *Let \mathcal{S}_{ns} denote the set of non-strict saddle points of problem (1), and the sequence $\{x_k\}$ be generated by the BCGD Method 1 with $0 < \alpha < \frac{1}{L}$. Suppose that $\{x_k\}$ is convergent and its limit is x^* , then we have $\mathbb{P}_\nu[x^* \in \mathcal{S}_{ns}] = 1$.*

4 The BMD method

In this section, we extend the above results to the BMD method in [5, 8, 21]. That is, the BMD method, which is based on Bregman's divergences, converges to non-strict saddle points as well, almost surely with random initialization. Note that, however, in what follows we only prove that the iterative mapping of BMD satisfies the stable mapping property, and omit for brevity the proof of our main claim (to avoid duplication of Subsection 3.4).

4.1 The BMD method description

In the BMD method, each time an approximate version of the objective is optimized with respect to one block variable while fixing the others. The approximate objective is obtained by introducing the Bregman's divergence (see [1, 4, 28] and references therein). Here, we focus on a class of BMD method with Bregman's divergences defined with respect to strongly convex and twice continuously differentiable functions.

To illustrate the BMD method, we introduce p functions $\varphi_t : \mathbb{R}^{n_t} \rightarrow \mathbb{R}$, $t = 1, 2, \dots, p$, and define the Bregman's divergence $B_{\varphi_t} : \mathbb{R}^{n_t} \times \mathbb{R}^{n_t} \rightarrow \mathbb{R}^+$ on them as follows

$$B_{\varphi_t}(x(t), y(t)) \triangleq \varphi_t(x(t)) - \varphi_t(y(t)) - \langle x(t) - y(t), \nabla \varphi_t(y(t)) \rangle, \quad t = 1, 2, \dots, p. \quad (20)$$

Furthermore, we make the following assumption on φ_t 's throughout this section.

Assumption 2 φ_t is a strongly convex and twice continuously differentiable function with parameter $\mu_t > 0$, i.e., for any $y(t)$ and $x(t) \in \mathbb{R}^{n_t}$,

$$\varphi_t(y(t)) \geq \varphi_t(x(t)) + \langle \nabla \varphi_t(x(t)), y(t) - x(t) \rangle + \frac{\mu_t}{2} \|y(t) - x(t)\|^2, \quad t = 1, 2, \dots, p. \quad (21)$$

Based on the Bregman's divergence defined above, the BMD method is described as Method 2 in the table, where $\mu \triangleq \min\{\mu_1, \mu_2, \dots, \mu_p\}$ and

$$x_k^s \triangleq \left((x_k^s(1))^T, (x_k^s(2))^T, \dots, (x_k^s(p))^T \right)^T, \quad s = 1, \dots, p; \quad k = 0, 1, \dots \quad (22)$$

Here x_k^s denotes the value of x after updating the first s block variables at the k -th outer iteration of the BMD method, and $x_k^s(t) \in \mathbb{R}^{n_t}$ denotes the t -th block variable of x_k^s .

Method 2 (BMD)	
Input: $\alpha < \frac{\mu}{L}$.	
Initialization: $x_0 \in \mathbb{R}^n$.	
General Step ($k = 0, 1, \dots$): Set $x_k^0 = x_k$ and define recursively for $s = 1, 2, \dots, p$:	
$t = 1, 2, \dots, p$,	
If $t = s$,	$x_k^s(t) = \arg \min_{x(t)} \langle x(t), \nabla_t f(x_k^{s-1}) \rangle + \frac{1}{\alpha} B_{\varphi_t}(x(t), x_k^{s-1}(t)). \quad (23)$
Else	$x_k^s(t) = x_k^{s-1}(t). \quad (24)$
End	
Set $x_{k+1} = x_k^p$.	

Next, let us derive the iterative mapping of BMD and its gradient matrix. First, note that φ_s is a strongly convex function. Hence, $B_{\varphi_s}(x(s), y(s))$ is a strongly convex function as well with respect to $x(s)$ if $y(s)$ is fixed. Let $x_k^s(s)$ be the unique solution of problem (23). The first-order optimality condition implies that

$$\mathbf{0} = \nabla_s f(x_k^{s-1}) + \frac{1}{\alpha} (\nabla \varphi_s(x_k^s(s)) - \nabla \varphi_s(x_k^{s-1}(s))), \quad (25)$$

equivalently,

$$\nabla\varphi_s(x_k^s(s)) = \nabla\varphi_s(x_k^{s-1}(s)) - \alpha\nabla_s f(x_k^{s-1}). \quad (26)$$

On the other hand, since φ_s is strongly convex, we infer that $\nabla\varphi_s$ is a diffeomorphism by invoking Lemma 14 in Appendix. This implies that the inverse mapping of $\nabla\varphi_s$ exists. Let $[\nabla\varphi_s]^{-1}$ denote its inverse. Then $x_k^s(s)$ can be expressed in terms of x_k^{s-1} as

$$x_k^s(s) = [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x_k^{s-1}(s)) - \alpha\nabla_s f(x_k^{s-1})). \quad (27)$$

Combining (27) and (24), we obtain

$$x_k^s = (I_n - U_s U_s^T) x_k^{s-1} + U_s [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x_k^{s-1}(s)) - \alpha\nabla_s f(x_k^{s-1})), \quad s = 1, 2, \dots, p, \quad (28)$$

where U_s is defined by (4).

Let us define $\psi_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$\psi_s(x) \triangleq (I_n - U_s U_s^T) x + U_s [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x(s)) - \alpha\nabla_s f(x)), \quad s = 1, 2, \dots, p. \quad (29)$$

Then the iteration of the BMD method can be compactly written as

$$x_{k+1} = \psi(x_k), \quad (30)$$

where $\psi(x)$ is a composite mapping defined by

$$\psi(x) \triangleq \psi_p \circ \psi_{p-1} \circ \dots \circ \psi_2 \circ \psi_1(x). \quad (31)$$

Moreover, by using the chain rule, we can obtain the gradient matrix of ψ as follows

$$D\psi(x) = D\psi_1(y_1) \times D\psi_2(y_2) \times \dots \times D\psi_{p-1}(y_{p-1}) \times D\psi_p(y_p), \quad (32)$$

where $y_1 = x$, $y_s = \psi_{s-1}(y_{s-1})$, $s = 2, \dots, p$, and $D\psi_s$ is the gradient matrix of ψ_s given by

$$\begin{aligned} D\psi_s(x) &= (I_n - U_s U_s^T) + D \left\{ [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x(s)) - \alpha\nabla_s f(x)) \right\} U_s^T \\ &= (I_n - U_s U_s^T) + D \left\{ \nabla\varphi_s(x(s)) - \alpha\nabla_s f(x) \right\} \left\{ \nabla^2\varphi_s \left\{ [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x(s)) - \alpha\nabla_s f(x)) \right\} \right\}^{-1} U_s^T \\ &= (I_n - U_s U_s^T) + \\ &\quad \left\{ U_s \nabla^2\varphi_s(x(s)) - \alpha\nabla^2 f(x) U_s \right\} \left\{ \nabla^2\varphi_s \left\{ [\nabla\varphi_s]^{-1}(\nabla\varphi_s(x(s)) - \alpha\nabla_s f(x)) \right\} \right\}^{-1} U_s^T, \end{aligned} \quad (33)$$

where the first equality is due to the chain rule, the second equality holds because of the chain rule and the inverse function theorem in [25], and the last equality follows from the definition of U_s .

4.2 The iterative mapping ψ of BMD is a diffeomorphism

Similar to Subsection 3.2, we show that ψ is a diffeomorphism as well. Again, we first state that ψ_s is a diffeomorphism in Lemma 2, whose proof is relegated to Appendix.

Lemma 2 *If the step size $\alpha < \frac{\mu}{L}$, then the mappings ψ_s defined by (29), $s = 1, \dots, p$, are diffeomorphisms.*

Based on Lemma 2 and using [14, Proposition 2.15], we have the following proposition.

Proposition 4 *The mapping ψ defined by (31) with step size $\alpha < \frac{\mu}{L}$ is a diffeomorphism.*

4.3 Eigenvalue analysis of the Jacobian of ψ at a strict saddle point

In what follows, we analyze the eigenvalues of the Jacobian of ψ at a strict saddle point and particularly show it has at least one eigenvalue with magnitude greater than one.

Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point. Then we have $\nabla f(x^*) = \mathbf{0}$. It follows that

$$\begin{aligned} D\psi_s(x^*) &= (I_n - U_s U_s^T) + \\ &\{U_s \nabla^2 \varphi_s(x^*(s)) - \alpha \nabla^2 f(x^*) U_s\} \left\{ \nabla^2 \varphi_s \left\{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s(x^*(s)) - \alpha \nabla_s f(x^*)) \right\} \right\}^{-1} U_s^T \\ &= (I_n - U_s U_s^T) + \{U_s \nabla^2 \varphi_s(x^*(s)) - \alpha \nabla^2 f(x^*) U_s\} \{ \nabla^2 \varphi_s(x^*(s)) \}^{-1} U_s^T \\ &= I_n - \alpha \nabla^2 f(x^*) U_s \{ \nabla^2 \varphi_s(x^*(s)) \}^{-1} U_s^T. \end{aligned} \quad (34)$$

By plugging (34) into (32), we obtain the Jacobian of ψ at x^* as follows

$$\{D\psi(x^*)\}^T = \prod_{s=p}^1 \left\{ I_n - \alpha U_s \{ \nabla^2 \varphi_s(x^*(s)) \}^{-1} U_s^T \nabla^2 f(x^*) \right\} = \prod_{s=p}^1 \{ I_n - \alpha U_s T_s \},$$

where in the last equality we have used the definition $T_s \triangleq \{ \nabla^2 \varphi_s(x^*(s)) \}^{-1} U_s^T \nabla^2 f(x^*) = \{ \nabla^2 \varphi_s(x^*(s)) \}^{-1} A_s$.

Similar to the case of BCGD, we define

$$\tilde{G} \triangleq \frac{1}{\alpha} \left[I_n - \{D\psi(x^*)\}^T \right] = \frac{1}{\alpha} \left[I_n - \prod_{s=p}^1 \{ I_n - \alpha U_s T_s \} \right]. \quad (35)$$

Then, by defining $T \triangleq [T_1^T \ T_2^T \ \dots \ T_p^T]^T$ and invoking Lemma 6 in Section 6, we obtain

$$\tilde{G} = (I_n + \alpha \tilde{T})^{-1} T, \quad (36)$$

where \tilde{T} is the strictly block lower triangular matrix of T . Although (36) has the same structure as (18), we cannot similarly obtain the eigenvalue distribution of \tilde{G} because the matrix T is not symmetric (while the matrix A in (18) is symmetric). Hence, let us introduce the matrix \bar{G} which has the same eigenvalue distribution as the matrix \tilde{G} , as shown in the following lemma.

Lemma 3 *We have $\text{eig}(\tilde{G}) = \text{eig}(\bar{G})$, where \bar{G} is defined by*

$$\bar{G} \triangleq \left[I_n + \alpha \Psi^{-\frac{1}{2}} \check{A} \left(\Psi^{-\frac{1}{2}} \right)^T \right]^{-1} \Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T \quad \text{with} \quad (37)$$

$$\Psi \triangleq \text{Diag}(\nabla^2 \varphi_1(x^*(1)), \nabla^2 \varphi_2(x^*(2)), \dots, \nabla^2 \varphi_p(x^*(p))). \quad (38)$$

Proof Using the definitions of Ψ , A and T , we have the relations $T = \Psi^{-1} A$ and $\tilde{T} = \Psi^{-1} \check{A}$. By plugging these two relations into (36), we have

$$\begin{aligned} \tilde{G} &= (I_n + \alpha \Psi^{-1} \check{A})^{-1} \Psi^{-1} A = (\Psi + \alpha \check{A})^{-1} A = \left[\Psi^{\frac{1}{2}} \left(I_n + \alpha \Psi^{-\frac{1}{2}} \check{A} \Psi^{-\frac{1}{2}} \right) \Psi^{\frac{1}{2}} \right]^{-1} A \\ &= \Psi^{-\frac{1}{2}} \left(I_n + \alpha \Psi^{-\frac{1}{2}} \check{A} \Psi^{-\frac{1}{2}} \right)^{-1} \Psi^{-\frac{1}{2}} A = \left(\Psi^{-\frac{1}{2}} \right)^T \left[I_n + \alpha \Psi^{-\frac{1}{2}} \check{A} \left(\Psi^{-\frac{1}{2}} \right)^T \right]^{-1} \Psi^{-\frac{1}{2}} A, \end{aligned} \quad (39)$$

where $\Psi^{-\frac{1}{2}}$ exists because Ψ is positive definite. Clearly \bar{G} is obtained from \tilde{G} by moving the term $\left(\Psi^{-\frac{1}{2}} \right)^T$ in (39) to the end. Since it holds that $\text{eig}(XY) = \text{eig}(YX)$ for any two square matrices X and Y , we have $\text{eig}(\tilde{G}) = \text{eig}(\bar{G})$. This completes the proof. \square

According to the above lemma and the relation (35), it suffices to analyze the eigenvalues of \bar{G} in order to analyze the eigenvalues of the Jacobian. Now (37) has the same structure as (18) and particularly the matrix $E \triangleq \Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T$ is symmetric. On the other hand, since Ψ has a block diagonal structure, we have $\Psi^{-\frac{1}{2}} \check{A} \left(\Psi^{-\frac{1}{2}} \right)^T = \check{E}$ where \check{E} is the strictly block lower triangular matrix of E . Hence, we can shortly write

$$\bar{G} \triangleq [I_n + \alpha \check{E}]^{-1} E. \quad (40)$$

To study the eigenvalues of \bar{G} , let us first characterize the eigenvalue distribution of E in the following lemma.

Lemma 4 *Let $x^* \in \mathbb{R}^n$ be a strict saddle point. Assume that A and Ψ are defined by (13) and (38), respectively. Then,*

- (i) *the matrix $E \triangleq \Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T$ has at least one negative eigenvalue.*
- (ii) *The spectral radius of the symmetric matrix E is upper bounded by $\frac{L}{\mu}$.*

Proof (i) Since E is a congruent transformation of A , they have the same index of inertia. In addition, since x^* is a strict saddle point, the matrix $A = \nabla^2 f(x^*)$ has at least one negative eigenvalue. Hence, E has at least one negative eigenvalue as well.

(ii) We here prove that $\rho \left(\Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T \right) \leq \frac{L}{\mu}$. It suffices to show that $-\frac{L}{\mu} I_n \preceq \Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T \preceq \frac{L}{\mu} I_n$ holds. First, under Assumption 1, using [19, Lemma 7] we have

$$-L I_n \preceq \nabla^2 f(x^*) = A \preceq L I_n. \quad (41)$$

In addition, Eq. (21) implies that

$$\nabla^2 \varphi_s(x^*(s)) \succeq \mu_s I_{n_s} \succeq \mu I_{n_s}, \quad s = 1, 2, \dots, p, \quad (42)$$

and thus $\frac{1}{\mu} \Psi \succeq I_n$, which together with (41) further implies $-\frac{L}{\mu} \Psi \preceq A \preceq \frac{L}{\mu} \Psi$. Premultiplying the above inequalities by $\Psi^{-\frac{1}{2}}$ and postmultiplying by $\left(\Psi^{-\frac{1}{2}} \right)^T$, respectively, we arrive at

$$-\frac{L}{\mu} I_n \preceq \Psi^{-\frac{1}{2}} A \left(\Psi^{-\frac{1}{2}} \right)^T \preceq \frac{L}{\mu} I_n. \quad (43)$$

Thus, the proof is finished. \square

Based on Lemma 4 and Lemma 10 in Section 6, we are ready to show that there exists at least one eigenvalue of \bar{G} which belongs to Ω defined by (19). The result is stated in Proposition 5.

Proposition 5 *Assume that \bar{G} is defined by (40) for a strict saddle point $x^* \in \mathbb{R}^n$ with $\alpha \in (0, \frac{\mu}{L})$ and L being the gradient Lipschitz constant of f . There exists at least one eigenvalue λ of \bar{G} such $\lambda \in \Omega$.*

Furthermore, we have Proposition 6.

Proposition 6 *Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point of problem (1). Then, for the BMD iterative mapping ψ defined by (31) with $\alpha \in (0, \frac{\mu}{L})$, the Jacobian $\{D\psi(x^*)\}^T$ has at least one eigenvalue whose magnitude is strictly greater than one.*

Note that the proofs of the above two propositions are similar to those for Propositions 2 and 3. Thus we omit them for brevity.

5 The PBCD method

In this section, we will prove that the PBCD method in [8, 9, 32] converges to non-strict saddle points as well, almost surely with random initialization. Again, our effort is devoted to show that the iterative mapping of PBCD satisfies the stable mapping property.

5.1 The PBCD method description

In the PBCD method, each time an upper bound function of the objective is optimized with respect to one block variable while fixing the others (see [8, 9, 32] and references therein). Such an upper bound is obtained by adding a proximal term to the objective. The PBCD method is described as Method 3 in the following table, where x_k^s is defined in (22).

Method 3 (PBCD)	
Input: $\alpha < \frac{1}{L}$.	
Initialization: $x_0 \in \mathbb{R}^n$.	
General Step ($k = 0, 1, \dots$): Set $x_k^0 = x_k$ and define recursively for $s = 1, 2, \dots, p : t = 1, 2, \dots, p$,	
If $t = s$,	
$x_k^s(t) = \arg \min_{x(t)} \left\{ f(x_k^{s-1}(1), \dots, x_k^{s-1}(t-1), x(t), x_k^{s-1}(t+1), \dots, x_k^{s-1}(p)) + \frac{1}{2\alpha} \ x(t) - x_k^{s-1}(t)\ _2^2 \right\}.$	(44)
Else	
	$x_k^s(t) = x_k^{s-1}(t). \quad (45)$
End	
Set $x_{k+1} = x_k^p$.	

Next, let us derive the iterative mapping of PBCD and its gradient matrix. Because $\alpha < \frac{1}{L}$, the objective function of (44) is strongly convex with respect to the variable $x(s)$. Thus, the problem (44) has a unique solution. Let $x_k^s(s)$ be the unique minimizer of problem (44). Then by the first-order optimality condition, we have

$$\mathbf{0} = \nabla_s f(x_k^{s-1}(1), \dots, x_k^{s-1}(s-1), x_k^s(s), x_k^{s-1}(s+1), \dots, x_k^{s-1}(p)) + \frac{1}{\alpha} (x_k^s(s) - x_k^{s-1}(s)),$$

equivalently

$$x_k^{s-1}(s) = x_k^s(s) + \alpha \nabla_s f(x_k^{s-1}(1), \dots, x_k^{s-1}(s-1), x_k^s(s), x_k^{s-1}(s+1), \dots, x_k^{s-1}(p)). \quad (46)$$

Combining (45) and (46), we obtain

$$x_k^{s-1} = x_k^s + \alpha U_s \nabla_s f(x_k^s), \quad s = 1, \dots, p. \quad (47)$$

Recall that, for a given step size α with $0 < \alpha < \frac{1}{L}$, we have used $g_{\alpha f}^s$ (see its definition in (5)) to denote the gradient mapping of function f with respect to the variable $x(s)$, i.e.,

$$g_{\alpha f}^s(x) = x - \alpha U_s \nabla_s f(x), \quad s = 1, \dots, p. \quad (48)$$

By replacing f with $-f$ in (48), the gradient mapping of the function $-f$ with respect to the variable $x(s)$ can be expressed as

$$g_{\alpha(-f)}^s(x) = x + \alpha U_s \nabla_s f(x). \quad (49)$$

Using (49), we can write Eq. (47) shortly as follows

$$x_k^{s-1} = g_{\alpha(-f)}^s(x_k^s), \quad s = 1, \dots, p. \quad (50)$$

Similar to Lemma 1, we can show that the mapping $g_{\alpha(-f)}^s$ and its inverse $[g_{\alpha(-f)}^s]^{-1}$ are diffeomorphisms with $\alpha \in (0, \frac{1}{L})$. Using the inverse mapping, (50) can be shortly written as

$$x_k^s = [g_{\alpha(-f)}^s]^{-1}(x_k^{s-1}), \quad s = 1, \dots, p, \quad (51)$$

and thus the PBCD iteration can be described as follows

$$\begin{aligned} x_{k+1} &= [g_{\alpha(-f)}^p]^{-1} \circ [g_{\alpha(-f)}^{p-1}]^{-1} \circ \dots \circ [g_{\alpha(-f)}^2]^{-1} \circ [g_{\alpha(-f)}^1]^{-1}(x_k) \\ &= [g_{\alpha(-f)}^1 \circ g_{\alpha(-f)}^2 \circ \dots \circ g_{\alpha(-f)}^{p-1} \circ g_{\alpha(-f)}^p]^{-1}(x_k). \end{aligned} \quad (52)$$

For simplicity, let us define

$$g_{\alpha(-f)} \triangleq g_{\alpha(-f)}^1 \circ g_{\alpha(-f)}^2 \circ \dots \circ g_{\alpha(-f)}^{p-1} \circ g_{\alpha(-f)}^p. \quad (53)$$

Then we obtain

$$x_{k+1} = [g_{\alpha(-f)}]^{-1}(x_k). \quad (54)$$

Moreover, by using the chain rule, we can obtain the gradient matrix of the mapping $g_{\alpha(-f)}$ as follows

$$Dg_{\alpha(-f)}(x) = Dg_{\alpha(-f)}^p(y_p) \times Dg_{\alpha(-f)}^{p-1}(y_{p-1}) \times \dots \times Dg_{\alpha(-f)}^2(y_2) \times Dg_{\alpha(-f)}^1(y_1), \quad (55)$$

where $y_p = x$, $y_s = g_{\alpha(-f)}^{s+1}(y_{s+1})$, $s = p-1, \dots, 1$, and $Dg_{\alpha(-f)}^s$ is the gradient matrix of $g_{\alpha(-f)}^s$ given by

$$Dg_{\alpha(-f)}^s(x) = I_n + \alpha \nabla^2 f(x) U_s U_s^T. \quad (56)$$

5.2 The iterative mapping $[g_{\alpha(-f)}]^{-1}$ of PBCD is a diffeomorphism

In this subsection, we first show that $g_{\alpha(-f)}^s$, $s = 1, \dots, p$, are diffeomorphisms and then conclude that $[g_{\alpha(-f)}]^{-1}$ is a diffeomorphism as well.

Lemma 5 *If step size $\alpha < \frac{1}{L}$, then the mappings $g_{\alpha(-f)}^s$ defined by (49), $s = 1, \dots, p$, are diffeomorphisms.*

Proof Note that L is also the Lipschitz constant of gradient of $-f$. By replacing $-f$ with f , Lemma 1 in Subsection 3.2 implies that $g_{\alpha(-f)}^s$ is a diffeomorphism with $\alpha \in (0, \frac{1}{L})$. The proof is completed. \square

Based on the above lemma and using [14, Proposition 2.15], we can easily prove the following proposition.

Proposition 7 *The mapping $[g_{\alpha(-f)}]^{-1}$ determined by (54) with step size $\alpha < \frac{1}{L}$ is a diffeomorphism.*

5.3 Eigenvalue analysis of the Jacobian of $[g_{\alpha(-f)}]^{-1}$ at a strict saddle point

In this subsection, we analyze the eigenvalues of the Jacobian of $[g_{\alpha(-f)}]^{-1}$ at a strict saddle point, and show that it has at least one eigenvalue with magnitude greater than one.

In what follows, we first derive the Jacobian of $[g_{\alpha(-f)}]^{-1}$ at a strict saddle point x^* . Following the inverse function theorem in [25], we obtain

$$D[g_{\alpha(-f)}]^{-1}(x^*) = \left(Dg_{\alpha(-f)} \left([g_{\alpha(-f)}]^{-1}(x^*) \right) \right)^{-1}. \quad (57)$$

Since $x^* \in \mathbb{R}^n$ is a strict saddle point, we have $x^* = [g_{\alpha(-f)}]^{-1}(x^*)$. Plugging this into (57), we obtain

$$D[g_{\alpha(-f)}]^{-1}(x^*) = (Dg_{\alpha(-f)}(x^*))^{-1}. \quad (58)$$

According to (58), it suffices to perform eigenvalue analysis on $Dg_{\alpha(-f)}(x^*)$, which is the focus in what follows. First, using the chain rule (55) at x^* , we have

$$Dg_{\alpha(-f)}(x^*) = Dg_{\alpha(-f)}^p(x^*) \times Dg_{\alpha(-f)}^{p-1}(x^*) \times \cdots \times Dg_{\alpha(-f)}^2(x^*) \times Dg_{\alpha(-f)}^1(x^*). \quad (59)$$

It follows that

$$(Dg_{\alpha(-f)}(x^*))^T = \prod_{s=1}^p (Dg_{\alpha(-f)}^s(x^*))^T = \prod_{s=1}^p (I_n + \alpha U_s U_s^T \nabla^2 f(x^*)), \quad (60)$$

where the second equality is due to (56).

Next let us define

$$H \triangleq \frac{1}{\alpha} \left[I_n - (Dg_{\alpha(-f)}(x^*))^T \right] = \frac{1}{\alpha} \left[I_n - \prod_{s=1}^p (I_n - \alpha U_s (-A_s)) \right], \quad (61)$$

where the second equality is due to (60) and $A_s = U_s^T \nabla^2 f(x^*)$. Again, by invoking Lemma 6 in Section 6, we obtain

$$H = - \left(I_n - \alpha \hat{A} \right)^{-1} A, \quad (62)$$

where \hat{A} is the strictly block lower triangular matrix of A (similar to the definition of \hat{B} in (73) in Section 6).

Given (62), we can characterize the eigenvalue distribution of H in the following proposition.

Proposition 8 *Assume that H is defined by (61) for a strict saddle point $x^* \in \mathbb{R}^n$ with $\alpha \in (0, \frac{1}{L})$ and L being the gradient Lipschitz constant of $f(x)$. There is at least one nonzero eigenvalue λ of αH such that $\frac{1}{\lambda} \in \Xi(\alpha, A)$, where*

$$\Xi(\alpha, A) \triangleq \left\{ a + bi \mid a, b \in \mathbb{R}, \frac{1}{2} + \frac{1 - \alpha \rho(A)}{\alpha \rho(A)} \leq a, i = \sqrt{-1} \right\}. \quad (63)$$

Proof Since $A = \nabla^2 f(x^*)$ and x^* is a strict saddle point, $-A$ has at least one positive eigenvalue. Moreover, according to [19, Lemma 7], we have $L \geq \rho(A)$ and thus $\alpha \in (0, \frac{1}{L}) \subseteq (0, \frac{1}{\rho(A)})$. Therefore, (62) implies that $\alpha H = -\alpha \left(I_n - \alpha \hat{A} \right)^{-1} A$. Thus, by invoking Lemma 11 in Section 6 with identifications $-A \sim B$, $-\hat{A} \sim \hat{B}$, $\alpha \sim \beta$ and $\rho(-A) = \rho(A) \sim \rho(B)$, we have $\frac{1}{\lambda} \in \Xi(\alpha, A)$. This completes the proof. \square

Based on Proposition 8, we can characterize the eigenvalue distribution of $Dg_{\alpha(-f)}(x^*)$ through the relation (61). Then by observing the connection between $Dg_{\alpha(-f)}(x^*)$ and $D[g_{\alpha(-f)}]^{-1}(x^*)$ in (58), we obtain our key claim in the following proposition.

Proposition 9 *Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point of problem (1). Then, for the PBCD iterative mapping $[g_{\alpha(-f)}]^{-1}$ determined by (54) with $\alpha \in (0, \frac{1}{L})$, the Jacobian $\left\{ D[g_{\alpha(-f)}]^{-1}(x^*) \right\}^T$ has at least one eigenvalue whose magnitude is strictly greater than one.*

Proof From (61), we obtain

$$(Dg_{\alpha(-f)}(x^*))^T = I_n - \alpha H, \quad (64)$$

which together with (58) leads to

$$\text{eig} \left(\left\{ D[g_{\alpha(-f)}]^{-1}(x^*) \right\}^T \right) = \text{eig} \left(\left\{ (Dg_{\alpha(-f)}(x^*))^T \right\}^{-1} \right) = \text{eig} \left((I_n - \alpha H)^{-1} \right), \quad (65)$$

implying

$$\lambda \in \text{eig}(\alpha H) \Leftrightarrow \frac{1}{1-\lambda} \in \text{eig}\left(\left\{D[g_{\alpha(-f)}]^{-1}(x^*)\right\}^T\right). \quad (66)$$

Hence it suffices to show that there exists $\lambda \in \text{eig}(\alpha H)$ such $|\frac{1}{1-\lambda}| > 1$.

Obviously, any eigenvalue λ of αH can not equal 1, which, combined with Proposition 8, implies αH has at least one nonzero eigenvalue λ_0 with $\lambda_0 \neq 1$ such that $\frac{1}{\lambda_0} \in \Xi(\alpha, A)$, equivalently,

$$\text{Re}\left(\frac{1}{\lambda_0}\right) \geq \frac{1}{2} + \frac{1 - \alpha\rho(A)}{\alpha\rho(A)}. \quad (67)$$

On the other hand, according to [19, Lemma 7], we have $L \geq \rho(A)$ and thus $\alpha \in (0, \frac{1}{L}) \subseteq (0, \frac{1}{\rho(A)})$. It follows that

$$\frac{1}{2} + \frac{1 - \alpha\rho(A)}{\alpha\rho(A)} > \frac{1}{2}. \quad (68)$$

By combining (67) and (68), we have $\text{Re}\left(\frac{1}{\lambda_0}\right) > \frac{1}{2}$, which implies $|\frac{1}{1-\lambda_0}| > 1$ due to the following implication

$$\left|\frac{1}{1-\lambda_0}\right| > 1 \Leftrightarrow \left|\frac{\frac{1}{\lambda_0}}{\frac{1}{\lambda_0}-1}\right| > 1 \Leftrightarrow \left|\frac{1}{\lambda_0}\right| > \left|\frac{1}{\lambda_0}-1\right| \Leftrightarrow \frac{1}{2} < \text{Re}\left(\frac{1}{\lambda_0}\right). \quad (69)$$

This completes the proof. \square

6 Preliminary on Eigenvalue Analysis

To characterize the eigenvalue distributions of iterative mappings, we provide some basic important lemmas in what follows. Note that the proofs of some lemmas are lengthy and thus moved to the appendix.

First, let us introduce some notations. Unless otherwise specified, we assume that $B \in \mathbb{R}^{n \times n}$ is a symmetric matrix, which is partitioned into $p \times p$ blocks as follows

$$B \triangleq (B_{st})_{1 \leq s, t \leq p}, \quad (70)$$

where $B_{st} \in \mathbb{R}^{n_s \times n_t}$ is the (s, t) -th block, and n_1, n_2, \dots, n_p are p positive integer numbers satisfying $\sum_{s=1}^p n_s = n$.

In addition, we denote the strictly block lower triangular matrix of B as

$$\check{B} \triangleq (\check{B}_{st})_{1 \leq s, t \leq p} \quad (71)$$

with $p \times p$ blocks and its (s, t) -th block is given by

$$\check{B}_{st} = \begin{cases} B_{st}, & s > t, \\ \mathbf{0}, & s \leq t. \end{cases} \quad (72)$$

Similarly, we denote the strictly block upper triangular matrix of B as

$$\hat{B} \triangleq (\hat{B}_{st})_{1 \leq s, t \leq p} \quad (73)$$

with $p \times p$ blocks and its (s, t) -th block is given by

$$\hat{B}_{st} = \begin{cases} B_{st}, & s < t, \\ \mathbf{0}, & s \geq t. \end{cases} \quad (74)$$

With the above notations, we provide several lemmas below.

Lemma 6 Assume that $\beta > 0$, B defined by (70) (B could be asymmetric) has the following partition:

$$B = (B_1^T, B_2^T, \dots, B_p^T)^T,$$

where $B_s \in \mathbb{R}^{n_s \times n}$ denotes the s -th block-row of B and is defined as

$$B_s \triangleq (B_{st})_{1 \leq t \leq p}, \quad 1 \leq s \leq p. \quad (75)$$

Then

$$\frac{1}{\beta} \left[I_n - \prod_{s=p}^1 (I_n - \beta U_s B_s) \right] = (I_n + \beta \check{B})^{-1} B \quad (76)$$

and

$$\frac{1}{\beta} \left[I_n - \prod_{s=1}^p (I_n - \beta U_s B_s) \right] = (I_n + \beta \hat{B})^{-1} B, \quad (77)$$

where \check{B} and \hat{B} are defined by (71) and (73), respectively.

The proof is lengthy and has been relegated to Appendix.

Lemma 7 Assume that B , \check{B} and \hat{B} are defined by (70), (71) and (73), respectively. Then for an arbitrary n dimensional vector $\eta \in \mathbb{C}^n$ with $\|\eta\| = 1$, we have

$$-\rho(B) \leq \operatorname{Re}(\eta^H \check{B} \eta) \leq \rho(B), \quad (78)$$

and

$$-\rho(B) \leq \operatorname{Re}(\eta^H \hat{B} \eta) \leq \rho(B). \quad (79)$$

Proof By collecting the main diagonal blocks of B , we define the following block diagonal matrix:

$$\tilde{B} \triangleq \operatorname{Diag}(B_{11}, B_{22}, \dots, B_{pp}). \quad (80)$$

Thus, B can be expressed as

$$B = \check{B} + \tilde{B} + \check{B}^T. \quad (81)$$

By the definition of $\rho(B)$ and noting $\|\eta\| = 1$, we have

$$2\rho(B) \geq \eta^H (\rho(B)I_n + B) \eta.$$

By plugging (81) into the above inequality and then using the fact $\tilde{B} \geq -\rho(B)I_n$ [10, Theorem 4.3.15], we obtain

$$\begin{aligned} 2\rho(B) &\geq \eta^H (\rho(B)I_n + \tilde{B}) \eta + \eta^H (\check{B} + \check{B}^T) \eta \\ &= \eta^H (\rho(B)I_n + \tilde{B}) \eta + 2\operatorname{Re}(\eta^H \check{B} \eta) \geq 2\operatorname{Re}(\eta^H \check{B} \eta). \end{aligned} \quad (82)$$

Furthermore, by immediately using (82), we have

$$\rho(B) = \rho(-B) \geq \operatorname{Re}(\eta^H (-\check{B}) \eta) = -\operatorname{Re}(\eta^H \check{B} \eta),$$

which together with (82) implies (78). By using the same argument, we can prove (79) holds. \square

Lemma 8 Assume that B and \check{B} are defined by (70) and (71), respectively. Moreover, suppose B is invertible. For any $\beta \in \left(0, \frac{1}{\rho(\check{B})}\right)$ and $t \in [0, 1]$, if λ is an eigenvalue of $B^{-1}(I_n + t\beta\check{B})$, then $\operatorname{Re}(\lambda) \neq 0$.

Proof Let ξ be the unit-length eigenvector of $B^{-1}(I + t\beta\check{B})$ corresponding to λ . Then $B^{-1}(I_n + t\beta\check{B})\xi = \lambda\xi$, or equivalently,

$$(I_n + t\beta\check{B})\xi = \lambda B\xi. \quad (83)$$

Premultiplying both sides of the above equality by ξ^H , we arrive at

$$1 + t\beta\xi^H\check{B}\xi = \lambda\xi^HB\xi. \quad (84)$$

By noting that $0 < \beta < \frac{1}{\rho(B)}$ and $t \in [0, 1]$, Lemma 7 implies that $\text{Re}(1 + t\beta\xi^H\check{B}\xi) > 0$, i.e., the real part of $1 + t\beta\xi^H\check{B}\xi$ is a positive real number. Therefore, we infer from (84) that $\text{Re}(\lambda) \neq 0$. \square

Lemma 9 *Assume that B and \hat{B} are defined by (70) and (73), respectively. Moreover, suppose B is invertible. For any $\beta \in (0, \frac{1}{\rho(B)})$ and $t \in [0, 1]$, if λ is an eigenvalue of $(\beta B)^{-1}(I_n + t\beta\hat{B})$ and $\text{Re}(\lambda) > 0$, then $\text{Re}(\lambda) \geq \frac{1}{2} + \frac{1 - \beta\rho(B)}{\beta\rho(B)} > \frac{1}{2}$.*

The proof is lengthy and has been relegated to Appendix.

Lemma 10 *Assume that B and \check{B} are defined by (70) and (71), respectively. Furthermore if $\lambda_{\min}(B) < 0$, then, for an arbitrary $\beta \in (0, \frac{1}{\rho(B)})$, there is at least one eigenvalue λ of $(I_n + \beta\check{B})^{-1}B$ which lies in closed left half complex plane excluding the origin, i.e.,*

$$\forall \beta \in \left(0, \frac{1}{\rho(B)}\right) \Rightarrow \exists \lambda \in \left[\text{eig}\left((I_n + \beta\check{B})^{-1}B\right) \cap \Omega\right], \quad (85)$$

where

$$\Omega \triangleq \{a + bi \mid a, b \in \mathbb{R}, a \leq 0, (a, b) \neq (0, 0), i = \sqrt{-1}\}. \quad (86)$$

The proof is lengthy and has been relegated to Appendix.

Lemma 11 *Assume that B and \hat{B} are defined by (70) and (73), respectively. Furthermore if $\lambda_{\max}(B) > 0$, then, for an arbitrary $\beta \in (0, \frac{1}{\rho(B)})$, there is at least one nonzero eigenvalue λ of $\beta(I_n + \beta\hat{B})^{-1}B$ such that*

$$\frac{1}{\lambda} \in \Xi(\beta, B), \quad (87)$$

where

$$\Xi(\beta, B) \triangleq \left\{a + bi \mid a, b \in \mathbb{R}, \frac{1}{2} + \frac{1 - \beta\rho(B)}{\beta\rho(B)} \leq a, i = \sqrt{-1}\right\}. \quad (88)$$

The proof is lengthy and has been relegated to Appendix.

Lemma 12 *Assume that B , \check{B} and \hat{B} are defined by (70), (71) and (73), respectively. Then, for any $\beta \in (0, \frac{1}{\rho(B)})$, both $(I_n + \beta\check{B})^{-1}B$ and $(I_n + \beta\hat{B})^{-1}B$ have the same multiplicity of zero eigenvalue as B .*

The proof is lengthy and has been relegated to Appendix.

7 Conclusion

In this paper, given a non-convex twice continuously differentiable cost function with Lipschitz continuous gradient, we prove that all of BCGD, BMD and PBCD converge to stationary points satisfying the second-order necessary condition, almost surely with random initialization. Similar to [12], our analysis is also built on the center-stable manifold theorem. However, our main task is to show the Jacobians of the considered iterative mappings at a strict saddle point have at least one eigenvalue with magnitude strictly larger than one. This is done by using matrix analysis and particularly resorting to Ostrowski's lemma. We expect that the key philosophy behind our analysis can be extended to study the stable mapping property for other algorithms, e.g., BSUM [21] and ADMM [27]. This is left for our future investigation.

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8 Appendix

8.1 Proof of Lemma 1

Proof Without loss of generality, we focus on the first block and prove that $g_{\alpha f}^1$ with step size $\alpha < \frac{1}{L}$ is a diffeomorphism. The other blocks can be treated similarly.

The proof is divided into the following four steps.

(a) We prove that $g_{\alpha f}^1$ is injective from $\mathbb{R}^n \rightarrow \mathbb{R}^n$ for $\alpha < \frac{1}{L}$. Suppose that there exist x and y such that $g_{\alpha f}^1(x) = g_{\alpha f}^1(y)$. Then $x - y = \alpha U_1 (\nabla_1 f(x) - \nabla_1 f(y))$ and

$$\|x - y\| = \alpha \|U_1 (\nabla_1 f(x) - \nabla_1 f(y))\| = \alpha \|\nabla_1 f(x) - \nabla_1 f(y)\| \leq \alpha \|\nabla f(x) - \nabla f(y)\| \leq \alpha L \|x - y\|,$$

where the second equality is due to $U_1^T U_1 = I_{n_1}$. Since $\alpha L < 1$, the above inequality means $x = y$.

(b) To show $g_{\alpha f}^1$ is surjective, we construct an explicit inverse function. Given a point y in \mathbb{R}^n , suppose it has the following partition,

$$y = \left((y(1))^T, (y(2))^T, \dots, (y(p))^T \right)^T. \quad (89)$$

Then we define $n - n_1$ dimensional vector

$$y_{-(1)} \triangleq \left((y(2))^T, \dots, (y(p))^T \right)^T \quad (90)$$

and define a function $\hat{f}(\cdot; y_{-(1)}) : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$, $\hat{f}(x(1); y_{-(1)}) \triangleq f\left(\begin{pmatrix} x(1) \\ y_{-(1)} \end{pmatrix}\right)$, which is determined by function f and the remained block coordinate vector $y_{-(1)}$ of y . Thus, the proximal point mapping of $-\hat{f}(\cdot; y_{-(1)})$ centered at $y(1)$ is given by

$$x_{y(1)} = \arg \min_{x(1)} \frac{1}{2} \|x(1) - y(1)\|^2 - \alpha \hat{f}(x(1); y_{-(1)}). \quad (91)$$

For $\alpha < \frac{1}{L}$, the function above is strongly convex with respect to $x(1)$, so there is a unique minimizer. Let $x_{y(1)}$ be the unique minimizer, then by the first order optimality condition,

$$y(1) = x_{y(1)} - \alpha \nabla \hat{f}(x_{y(1)}; y_{-(1)}) = x_{y(1)} - \alpha \nabla_1 f\left(\begin{pmatrix} x_{y(1)} \\ y_{-(1)} \end{pmatrix}\right), \quad (92)$$

where the second equality is due to the definition of function $\hat{f}(\cdot; y_{-(1)})$. Let x_y be defined as

$$x_y \triangleq \begin{pmatrix} x_{y(1)} \\ y_{-(1)} \end{pmatrix}, \quad (93)$$

where $x_{y(1)}$ is defined by (91). Accordingly,

$$y = \begin{pmatrix} y(1) \\ y_{-(1)} \end{pmatrix} = \begin{pmatrix} x_{y(1)} - \alpha \nabla_1 f \left(\begin{pmatrix} x_{y(1)} \\ y_{-(1)} \end{pmatrix} \right) \\ y_{-(1)} \end{pmatrix} = x_y - \alpha U_1 \nabla_1 f(x_y) = g_{\alpha f}^1(x_y),$$

where the first equality is due to the definition of $y_{-(1)}$ (see (90)), the second equality thanks to (92), and the third equality holds because of the definitions of x_y (see (93)) and U_1 (see (4)).

Hence, x_y is mapped to y by the mapping $g_{\alpha f}^1$.

(c) We prove that $Dg_{\alpha f}^1(x)$ is invertible for $\alpha < \frac{1}{L}$. Note that the matrices $\nabla^2 f(x)U_1U_1^T \in \mathbb{R}^{n \times n}$ and $U_1^T \nabla^2 f(x)U_1 \in \mathbb{R}^{n_1 \times n_1}$ have different size. Following [10, Theorem 1.3.20], we obtain

$$\text{eig}(\nabla^2 f(x)U_1U_1^T) \subseteq \text{eig}(U_1^T \nabla^2 f(x)U_1) \cup \{0\}. \quad (94)$$

On the other hand, since the matrix $U_1^T \nabla^2 f(x)U_1 = \frac{\partial^2 f(x)}{\partial x(1) \partial x(1)}$ is the n_1 -by- n_1 principal submatrix of $\nabla^2 f(x)$, it follows from [10, Theorem 4.3.15] that

$$\text{eig}(U_1^T \nabla^2 f(x)U_1) \subseteq [\lambda_{\min}(\nabla^2 f(x)), \lambda_{\max}(\nabla^2 f(x))] \subseteq [-L, L], \quad (95)$$

where the last relation holds because of Eq. (2) and Lemma 7 in [19]. Since $\alpha < \frac{1}{L}$, Eqs. (94) and (95) imply $\text{eig}(\alpha \nabla^2 f(x)U_1U_1^T) \subseteq (-1, 1)$. Hence, we conclude that $Dg_{\alpha f}^1(x) = I - \alpha \nabla^2 f(x)U_1U_1^T$ is invertible for $\alpha < \frac{1}{L}$.

(d) Note that we have shown that $g_{\alpha f}^1$ is bijective, and continuously differentiable. Since $Dg_{\alpha f}^1(x)$ is invertible for $\alpha < \frac{1}{L}$, the inverse function theorem guarantees $[g_{\alpha f}^1]^{-1}$ is continuously differentiable. Thus, $g_{\alpha f}^1$ is a diffeomorphism. \square

8.2 Proof of Lemma 6

Proof Since (77) can be treated similarly, we here only prove (76). To do so, we define recursively

$$J[s] \triangleq \frac{1}{\beta} \left[I_n - \prod_{t=s}^1 (I_n - \beta U_t B_t) \right], \quad 1 \leq s \leq p. \quad (96)$$

Hence, it suffices to show

$$J[p] = (I_n + \beta \check{B})^{-1} B, \quad (97)$$

or equivalently, $J[p] = B - \beta \check{B} J[p]$.

In what follows, let us first prove

$$U_s^T J[p] = B_s - \beta \sum_{t=1}^{s-1} B_{st} U_t^T J[p], \quad s = 1, \dots, p. \quad (98)$$

It is easily seen that

$$U_s^T (I_n - \beta U_t B_t) = U_s^T - \beta U_s^T U_t B_t = U_s^T, \quad \forall s \neq t. \quad (99)$$

Eq. (99) further implies

$$U_s^T J[k] = \frac{1}{\beta} U_s^T \left[I_n - \prod_{t=k}^1 (I_n - \beta U_t B_t) \right] = \frac{1}{\beta} \left[U_s^T - U_s^T \prod_{t=k}^1 (I_n - \beta U_t B_t) \right] = \mathbf{0}, \quad \forall k < s. \quad (100)$$

On the other hand, for any q such $1 \leq s < q \leq p$, we have

$$\begin{aligned} U_s^T (\beta J[q]) &= U_s^T \left[I_n - \prod_{t=q}^1 (I_n - \beta U_t B_t) \right] = U_s^T I_n - U_s^T \prod_{t=q}^{s+1} (I_n - \beta U_t B_t) \prod_{t=s}^1 (I_n - \beta U_t B_t) \\ &= U_s^T I_n - U_s^T \prod_{t=s}^1 (I_n - \beta U_t B_t) = U_s^T \left[I_n - \prod_{t=s}^1 (I_n - \beta U_t B_t) \right] = U_s^T \beta J[s], \end{aligned} \quad (101)$$

where the third equality is due to (99) and the last equality uses the definition of $J[s]$ in (96). Particularly, when $q = p$, the above equation (101) reduces to

$$U_s^T \beta J[s] = U_s^T \beta J[p]. \quad (102)$$

From Eq. (102), we further have

$$\begin{aligned} U_s^T (\beta J[p]) &= U_s^T [I_n - (I_n - \beta J[s])] \stackrel{(a)}{=} U_s^T [I_n - (I_n - \beta U_s B_s) (I_n - \beta J[s-1])] \\ &= \beta U_s^T J[s-1] + \beta B_s - \beta^2 B_s J[s-1] \stackrel{(b)}{=} \beta B_s - \beta^2 B_s J[s-1] \stackrel{(c)}{=} \beta B_s - \beta^2 B_s \sum_{t=1}^p U_t U_t^T J[s-1] \\ &\stackrel{(d)}{=} \beta B_s - \beta^2 B_s \sum_{t=1}^{s-1} U_t U_t^T J[s-1] \stackrel{(e)}{=} \beta B_s - \beta^2 B_s \sum_{t=1}^{s-1} U_t U_t^T J[t] \stackrel{(f)}{=} \beta B_s - \beta^2 B_s \sum_{t=1}^{s-1} U_t U_t^T J[p] \\ &= \beta B_s - \beta^2 \sum_{t=1}^{s-1} B_s U_t U_t^T J[p] = \beta B_s - \beta^2 \sum_{t=1}^{s-1} B_{st} U_t^T J[p], \end{aligned}$$

where (a) uses the definitions of $J[s]$ in (96); (b) is due to (100); (c) thanks to the definition (4) of U_t ; (d) uses (100) again; (e) and (f) are due to (101) and (102), respectively.

Dividing both sides of the above equation by β , we have (98), equivalently $J[p] = B - \beta \check{B} J[p]$, or $(I_n + \beta \check{B}) J[p] = B$. Further, by noting that $I_n + \beta \check{B}$ is an invertible matrix because \check{B} is a block strictly lower triangular matrix, we arrive at (76). This completes the proof. \square

8.3 Proof of Lemma 10

Proof We divide the proof into two cases.

Case 1: when B is an invertible matrix. In this case, we clearly have $\left((I_n + \beta \check{B})^{-1} B \right)^{-1} = B^{-1} (I_n + \beta \check{B})$. In what follows, we prove that (85) is true by using Lemma 13 in Appendix.

Firstly, we define an analytic function with t as a parameter:

$$\mathcal{X}_t(z) \triangleq \det \{ z I_n - [(1-t)B^{-1} + tB^{-1}(I_n + \beta \check{B})] \} = \det \{ z I_n - B^{-1}(I_n + t\beta \check{B}) \}, \quad 0 \leq t \leq 1. \quad (103)$$

Furthermore, let us define a closed rectangle in the complex plane as follows

$$\mathcal{D} \triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \quad -2\nu \leq b \leq 2\nu \}, \quad (104)$$

with ν defined by

$$\nu \triangleq \|B^{-1}\| + \frac{1}{\rho(B)} \|B^{-1}\check{B}\| \geq \|B^{-1}\| + t\beta \|B^{-1}\check{B}\| \geq \|B^{-1}(I_n + t\beta \check{B})\|, \quad \forall t \in [0, 1], \quad (105)$$

where the first inequality holds because of $\beta \in \left(0, \frac{1}{\rho(B)}\right)$ and $t \in [0, 1]$. Note that the boundary $\partial \mathcal{D}$ of \mathcal{D} consists of a finite number of smooth curves. Specifically, define

$$\begin{aligned} \gamma_1 &\triangleq \{ a + bi \mid a = 0, \quad -2\nu \leq b \leq 2\nu \}, \quad \gamma_2 \triangleq \{ a + bi \mid a = -2\nu, \quad -2\nu \leq b \leq 2\nu \}, \\ \gamma_3 &\triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \quad b = 2\nu \}, \quad \gamma_4 \triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \quad b = -2\nu \}, \end{aligned} \quad (106)$$

then $\partial \mathcal{D} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. In order to apply Lemma 13, we will show that

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \partial \mathcal{D}. \quad (107)$$

On one hand, since the spectral norm of a matrix is larger than or equal to its spectral radius, the above inequality (105) yields that, for any $t \in [0, 1]$, the magnitude of any eigenvalue of

$B^{-1}(I + t\beta\check{B})$ is less than ν . Note that for an arbitrary $z \in \gamma_2 \cup \gamma_3 \cup \gamma_4$, we have $|z| \geq 2\nu > \nu$. Consequently,

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \gamma_2 \cup \gamma_3 \cup \gamma_4. \quad (108)$$

On the other hand, it follows from Lemma 8 that, for any $t \in [0, 1]$, if λ is an eigenvalue of $B^{-1}(I_n + t\beta\check{B})$, then $\text{Re}(\lambda) \neq 0$. Hence, $\lambda \notin \gamma_1$, implying

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \gamma_1. \quad (109)$$

Combining (108) and (109), we obtain (107).

Based on (107) and using Lemma 13 in Appendix, we infer that $\mathcal{X}_0(z) = \det\{zI_n - B^{-1}\}$ and $\mathcal{X}_1(z) = \det\{zI_n - B^{-1}(I_n + \beta\check{B})\}$ have the same number of zeros in \mathcal{D} . Note that $\lambda_{\min}(B) < 0$ implies that there is at least one negative eigenvalue $\frac{1}{\lambda_{\min}(B)}$ of B^{-1} . Recalling the definition (105) of ν , we know $\left|\frac{1}{\lambda_{\min}(B)}\right| \leq \nu$. Thus $\frac{1}{\lambda_{\min}(B)}$ must lie inside \mathcal{D} , i.e., there exists at least one zero of $\mathcal{X}_0(z)$ inside \mathcal{D} , which in turn shows that $\mathcal{X}_1(z)$ has at least one zero inside \mathcal{D} as well. Thus, there must exist at least one eigenvalue of $B^{-1}(I_n + \beta\check{B})$ lying inside \mathcal{D} . We denote it as $x + yi$, then $-2\nu < x < 0$ and $-2\nu < y < 2\nu$. Consequently, $\frac{1}{x+yi} = \frac{x-yi}{x^2+y^2}$ is an eigenvalue of $(I_n + \beta\check{B})^{-1}B$ with real part $\frac{x}{x^2+y^2} < 0$. Hence, $\frac{1}{x+yi}$ lies in Ω defined by (86) and the proof is finished in this case.

Case 2: when B is a singular matrix. In this case, based on the results in **Case 1**, we use the perturbation theorem to prove (85).

Suppose the multiplicity of zero eigenvalue of B is m and B has an eigenvalue decomposition in the form of

$$B = V \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^T = V_1 \Theta V_1^T, \quad (110)$$

where $\Theta = \text{Diag}(\theta_1, \theta_2, \dots, \theta_{n-m})$, θ_s , $s = 1, \dots, n-m$, are the nonzero eigenvalues of B and

$$V = (V_1 \ V_2) \quad (111)$$

is an orthogonal matrix and V_1 consists of the first $(n-m)$ columns of V .

Denote

$$\delta \triangleq \min\{|\theta_1|, |\theta_2|, \dots, |\theta_{n-m}|\}. \quad (112)$$

For any $\epsilon \in (0, \delta)$, we define

$$B(\epsilon) \triangleq B + \epsilon I_n, \quad (113)$$

then,

$$\text{eig}(B(\epsilon)) = \{\theta_1 + \epsilon, \theta_2 + \epsilon, \dots, \theta_{n-m} + \epsilon, \epsilon\} \not\equiv 0, \quad \forall \epsilon \in (0, \delta), \quad (114)$$

and

$$\lambda_{\min}(B(\epsilon)) = \lambda_{\min}(B) + \epsilon \leq -\delta + \epsilon < 0, \quad \forall \epsilon \in (0, \delta), \quad (115)$$

where the first inequality is due to the definition of δ and $\min\{\theta_1, \theta_2, \dots, \theta_{n-m}\} = \lambda_{\min}(B) < 0$.

Since B is defined by (70), $B(\epsilon)$ has $p \times p$ blocks form as well. Specifically,

$$B(\epsilon) = (B(\epsilon)_{st})_{1 \leq s, t \leq p}, \quad (116)$$

and its (s, t) -th block is given by

$$B(\epsilon)_{st} = \begin{cases} B_{st} + \epsilon I_{n_s}, & s = t, \\ B_{st}, & s \neq t, \end{cases} \quad (117)$$

where n_1, n_2, \dots, n_p are p positive integer numbers satisfying $\sum_{s=1}^p n_s = n$. Similar to the definition (71) of \check{B} , we denote the strictly block lower triangular matrix of $B(\epsilon)$ as

$$\check{B}(\epsilon) \triangleq (\check{B}(\epsilon)_{st})_{1 \leq s, t \leq p} \quad (118)$$

with $p \times p$ blocks and its (s, t) -th block is given by

$$\check{B}(\epsilon)_{st} = \begin{cases} B(\epsilon)_{st}, & s > t, \\ \mathbf{0}, & s \leq t, \end{cases} = \begin{cases} B_{st}, & s > t, \\ \mathbf{0}, & s \leq t, \end{cases} = \check{B}_{st}, \quad (119)$$

where the second equality holds because of (117) and the last equality is due to (72). It follows easily from Eqs. (71), (72) (118) and (119) that

$$\check{B}(\epsilon) = \check{B}. \quad (120)$$

Consequently,

$$(I_n + \beta \check{B}(\epsilon))^{-1} B(\epsilon) = (I_n + \beta \check{B})^{-1} B(\epsilon) = (I_n + \beta \check{B})^{-1} (B + \epsilon I_n), \quad (121)$$

where the first equality is due to (120) and the second equality holds because of (113). For simplicity, let

$$\lambda_s^\beta(\epsilon), \quad s = 1, \dots, n, \quad (122)$$

be the eigenvalues of $(I_n + \beta \check{B})^{-1} (B + \epsilon I_n)$.

Note that for any $\epsilon \in (0, \delta)$, $B(\epsilon)$ is invertible and $\lambda_{\min}(B(\epsilon)) < 0$ (see (115)). According to the definitions of $B(\epsilon)$ and $\check{B}(\epsilon)$, a similar argument in **Case 1** can be applied with the identifications $B(\epsilon) \sim B$, $\check{B}(\epsilon) \sim \check{B}$, $\beta \sim \beta$ and $\rho(B(\epsilon)) \sim \rho(B)$, to prove that, for any $\beta \in \left(0, \frac{1}{\rho(B(\epsilon))}\right)$, there must exist at least one eigenvalue of $(I_n + \beta \check{B}(\epsilon))^{-1} B(\epsilon)$ which lies in Ω defined by (86). Taking into account the definition (113), we have $\rho(B(\epsilon)) \leq \rho(B) + \epsilon$. Hence, for any $\epsilon \in (0, \delta)$ and $\beta \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right)$, there exists at least one index denoted as $s(\epsilon) \in \{1, 2, \dots, n\}$ such that

$$\lambda_{s(\epsilon)}^\beta(\epsilon) \in \Omega. \quad (123)$$

Furthermore, it is well known that the eigenvalues of a matrix M are continuous functions of the entries of M . Therefore, for any $\beta \in \left(0, \frac{1}{\rho(B)}\right)$, $\lambda_s^\beta(\epsilon)$ is a continuous function of ϵ and

$$\lim_{\epsilon \rightarrow 0^+} \lambda_s^\beta(\epsilon) = \lambda_s^\beta(0), \quad s = 1, \dots, n, \quad (124)$$

where $\lambda_s^\beta(0)$ is the eigenvalue of $(I_n + \beta \check{B})^{-1} B$.

Now we are ready to prove (85) by contradiction. Suppose for sake of contradiction that, there exists a $\beta^* \in \left(0, \frac{1}{\rho(B)}\right)$ such that, for any $s \in \{1, \dots, n\}$, we have

$$\lambda_s^{\beta^*}(0) \notin \Omega, \quad (125)$$

where $\lambda_s^{\beta^*}(0)$ is the eigenvalue of $(I_n + \beta^* \check{B})^{-1} B$.

According to Lemma 12 in Section 6 and the assumption that the multiplicity of zero eigenvalue of B is m , we know that the multiplicity of eigenvalue 0 of $(I_n + \beta^* \check{B})^{-1} B$ is m as well. Then there are exactly m eigenvalue functions of ϵ whose limits are 0 as ϵ approaches zero from above. Without loss of generality, we assume

$$\lim_{\epsilon \rightarrow 0^+} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) = 0, \quad s = 1, \dots, m, \quad (126)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) \neq 0, \quad s = m + 1, \dots, n. \quad (127)$$

In the sequel, under the assumption (125), we will prove that there exists a $\delta^* > 0$ with $\delta^* < \delta$ such that, for any $\epsilon \in (0, \delta^*)$, then $\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right)$ and there exists no $s \in \{1, \dots, n\}$ such that $\lambda_s^{\beta^*}(\epsilon)$ belongs to Ω . This would contradict (123). The proof is given by the following three steps.

Step (a): Under the assumption (125), we first prove that there exists a $\delta_1^* > 0$ with $\delta_1^* < \delta$ such that, for any $\epsilon \in (0, \delta_1^*)$, there does not exist any $s \in \{m+1, \dots, n\}$ such that $\lambda_s^{\beta^*}(\epsilon)$ belongs to Ω .

Taking into account the definition of Ω and Eq. (125) we have

$$\operatorname{Re} \left(\lambda_s^{\beta^*}(0) \right) > 0, \quad \forall s \in \{m+1, \dots, n\}. \quad (128)$$

Moreover, note that $\lambda_s^{\beta^*}(\epsilon)$ is a continuous function of ϵ and (127) holds. Combining these with the above inequalities, we know that there exists a $\delta_1^* > 0$ with $\delta_1^* < \delta$ such that

$$\left| \lambda_s^{\beta^*}(\epsilon) - \lambda_s^{\beta^*}(0) \right| < \frac{1}{3} \operatorname{Re} \left(\lambda_s^{\beta^*}(0) \right), \quad \forall s \in \{m+1, \dots, n\}, \forall \epsilon \in [0, \delta_1^*], \quad (129)$$

which further means that

$$0 < \frac{2}{3} \operatorname{Re} \left(\lambda_s^{\beta^*}(0) \right) < \operatorname{Re} \left(\lambda_s^{\beta^*}(\epsilon) \right), \quad \forall s \in \{m+1, \dots, n\}, \forall \epsilon \in [0, \delta_1^*]. \quad (130)$$

Hence, we arrive at

$$\lambda_s^{\beta^*}(\epsilon) \notin \Omega, \quad \forall s \in \{m+1, \dots, n\}, \forall \epsilon \in [0, \delta_1^*]. \quad (131)$$

Step (b): In this step, we prove that there exists a $\delta_2^* > 0$ with $\delta_2^* < \delta$ such that, for any $\epsilon \in (0, \delta_2^*]$ and $s \in \{1, \dots, m\}$,

$$\operatorname{Re} \left(\lambda_s^{\beta^*}(\epsilon) \right) > 0, \quad (132)$$

equivalently,

$$\lambda_s^{\beta^*}(\epsilon) \notin \Omega, \quad \forall s \in \{1, \dots, m\}, \forall \epsilon \in (0, \delta_2^*]. \quad (133)$$

To do so, let

$$\check{C}_{ij} \triangleq V_i^T \check{B} V_j, \quad 1 \leq i, j \leq 2, \quad (134)$$

where V_1 and V_2 are given by (111). Then we consider $\lambda_s^{\beta^*}(\epsilon)$ where s is an arbitrary number in $\{1, \dots, m\}$. Since we assume that $\lambda_s^{\beta^*}(\epsilon)$ is the eigenvalue of $(I_n + \beta^* \check{B})^{-1} (B + \epsilon I_n)$, then, for any $\epsilon \in (0, \delta)$,

$$\det \left\{ (I_n + \beta^* \check{B})^{-1} (B + \epsilon I_n) - \lambda_s^{\beta^*}(\epsilon) I_n \right\} = 0,$$

which is clearly equivalent to

$$\det \left\{ (I_n + \beta^* \check{B}) \right\} \det \left\{ (I_n + \beta^* \check{B})^{-1} (B + \epsilon I_n) - \lambda_s^{\beta^*}(\epsilon) I_n \right\} = 0, \forall \epsilon \in (0, \delta). \quad (135)$$

It is easily seen from the above Eq. (135) that, for any $\epsilon \in (0, \delta)$,

$$\begin{aligned} 0 &= \det \left\{ (B + \epsilon I_n) - \lambda_s^{\beta^*}(\epsilon) (I_n + \beta^* \check{B}) \right\} = \det \left\{ \left(V \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^T + \epsilon I_n \right) - \lambda_s^{\beta^*}(\epsilon) (I_n + \beta^* \check{B}) \right\} \\ &= \det \left\{ \begin{pmatrix} \Theta + \epsilon I_{n-m} & \mathbf{0} \\ \mathbf{0} & \epsilon I_m \end{pmatrix} - \lambda_s^{\beta^*}(\epsilon) (I_n + \beta^* V^T \check{B} V) \right\} \\ &= \det \left\{ \begin{pmatrix} \Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) & -\lambda_s^{\beta^*}(\epsilon) \beta^* \check{C}_{12} \\ -\lambda_s^{\beta^*}(\epsilon) \beta^* \check{C}_{21} & \epsilon I_m - \lambda_s^{\beta^*}(\epsilon) (I_m + \beta^* \check{C}_{22}) \end{pmatrix} \right\}, \end{aligned} \quad (136)$$

where (110) leads to the second equality and the last equality holds because of Eqs. (111) and (134).

Besides, recalling

$$\lim_{\epsilon \rightarrow 0^+} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) = 0, \quad (137)$$

we have

$$\lim_{\epsilon \rightarrow 0^+} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right] = \Theta, \quad (138)$$

which is an invertible matrix because Θ is given by (110). Clearly, the above (138) further implies that, there exists a $\delta_1 > 0$ with $\delta_1 < \delta$ such that, for any $\epsilon \in [0, \delta_1]$,

$$\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \quad (139)$$

is an invertible matrix as well, and

$$\lim_{\epsilon \rightarrow 0^+} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} = \Theta^{-1}. \quad (140)$$

Since the inverse of a matrix M , M^{-1} , is a continuous function of the elements of M , there exists a $\delta_2 > 0$ with $\delta_2 \leq \delta_1$, such that, for any $\epsilon \in [0, \delta_2]$,

$$\left\| \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \right\| \leq 2 \|\Theta^{-1}\|. \quad (141)$$

Consequently, for any $\epsilon \in [0, \delta_2]$, it follows easily from (136) and the fact that $\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* V_1^T \check{B} V_1)$ is an invertible matrix (see (139)) that

$$\begin{aligned} 0 &= \det \left\{ \Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right\} \times \\ &\det \left\{ \epsilon I_m - \lambda_s^{\beta^*}(\epsilon) (I_m + \beta^* \check{C}_{22}) - \left(\beta^* \lambda_s^{\beta^*}(\epsilon) \right)^2 \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12} \right\}, \end{aligned}$$

showing that

$$\epsilon I_m - \lambda_s^{\beta^*}(\epsilon) (I_m + \beta^* \check{C}_{22}) - \left(\beta^* \lambda_s^{\beta^*}(\epsilon) \right)^2 \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12}$$

is a singular matrix. Therefore, there exists one nonzero vector $v(\epsilon) \in \mathbb{C}^m$ with $\|v(\epsilon)\| = 1$ satisfying

$$\begin{aligned} &\left\{ \epsilon I_m - \lambda_s^{\beta^*}(\epsilon) (I_m + \beta^* \check{C}_{22}) - \left(\beta^* \lambda_s^{\beta^*}(\epsilon) \right)^2 \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12} \right\} v(\epsilon) \\ &= \mathbf{0}. \end{aligned}$$

Moreover, for any $\epsilon \in [0, \delta_2]$, premultiplying both sides of the above equality by $(v(\epsilon))^H$, we have

$$\begin{aligned} &(v(\epsilon))^H \left\{ \epsilon I_m - \lambda_s^{\beta^*}(\epsilon) (I_m + \beta^* \check{C}_{22}) \right. \\ &\quad \left. - \left(\beta^* \lambda_s^{\beta^*}(\epsilon) \right)^2 \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12} \right\} v(\epsilon) = 0, \end{aligned} \quad (142)$$

or equivalently,

$$\begin{aligned} &\epsilon - \lambda_s^{\beta^*}(\epsilon) \left(1 + \beta^* (v(\epsilon))^H \check{C}_{22} v(\epsilon) \right) \\ &= (\beta^*)^2 \left(\lambda_s^{\beta^*}(\epsilon) \right)^2 (v(\epsilon))^H \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12} v(\epsilon). \end{aligned} \quad (143)$$

Dividing both sides of the above equation by $\lambda_s^{\beta^*}(\epsilon)$, we have, for any $\epsilon \in (0, \delta_2]$,

$$\begin{aligned} 0 &\leq \left| \frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} - \left(1 + \beta^* (v(\epsilon))^H \check{C}_{22} v(\epsilon) \right) \right| \\ &= \left| (\beta^*)^2 \lambda_s^{\beta^*}(\epsilon) (v(\epsilon))^H \check{C}_{21} \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \check{C}_{12} v(\epsilon) \right| \\ &\leq (\beta^*)^2 \left| \lambda_s^{\beta^*}(\epsilon) \right| \|\check{C}_{21}\| \left\| \left[\Theta + \epsilon I_{n-m} - \lambda_s^{\beta^*}(\epsilon) (I_{n-m} + \beta^* \check{C}_{11}) \right]^{-1} \right\| \|\check{C}_{12}\| \\ &\leq 2 (\beta^*)^2 \left| \lambda_s^{\beta^*}(\epsilon) \right| \|\check{C}_{21}\| \|\Theta^{-1}\| \|\check{C}_{12}\|, \end{aligned} \quad (144)$$

where the second inequality is due to $\|v(\epsilon)\| = 1$ and Cauchy–Schwartz inequality, and the last inequality follows from (141). As $\lim_{\epsilon \rightarrow 0^+} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) = 0$, the right-hand limit in the above expression is equal to zero. Hence, we have

$$\lim_{\epsilon \rightarrow 0^+} \left| \frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} - \left(1 + \beta^*(v(\epsilon))^H \check{C}_{22} v(\epsilon)\right) \right| = 0. \quad (145)$$

Recall that

$$\begin{aligned} \operatorname{Re} \left(1 + \beta^*(v(\epsilon))^H \check{C}_{22} v(\epsilon)\right) &= \operatorname{Re} \left(1 + \beta^*(v(\epsilon))^H V_2^T \check{B} V_2 v(\epsilon)\right) \\ &= \operatorname{Re} \left(1 + \beta^*(v(\epsilon))^H V_2^H \check{B} V_2 v(\epsilon)\right) = \operatorname{Re} \left(1 + \beta^*(V_2 v(\epsilon))^H \check{B} V_2 v(\epsilon)\right) \geq 1 - \beta^* \rho(B) > 0, \end{aligned} \quad (146)$$

where the first equality follows from (134); the second equality holds because V_2 is a real matrix (see Eqs. (110) and (111)); the first inequality follows easily from Lemma 7 and $\|V_2 v(\epsilon)\|^2 = (v(\epsilon))^H V_2^H V_2 v(\epsilon) = (v(\epsilon))^H v(\epsilon) = 1$; and the last inequality thanks to $\beta^* \in \left(0, \frac{1}{\rho(B)}\right)$. Consequently, there exists a δ_3 with $\delta_3 \leq \delta_2$ such that, for any $\epsilon \in (0, \delta_3]$,

$$\begin{aligned} \frac{1}{3} (1 - \beta^* \rho(B)) &\geq \left| \frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} - \left(1 + \beta^*(v(\epsilon))^H \check{C}_{22} v(\epsilon)\right) \right| \\ &\geq \left| \operatorname{Re} \left(\frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} - \left(1 + \beta^*(v(\epsilon))^H \check{C}_{22} v(\epsilon)\right) \right) \right| = \left| \operatorname{Re} \left(\frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} \right) - \operatorname{Re} \left(1 + \beta^*(v(\epsilon))^H \check{C}_{22} v(\epsilon)\right) \right|. \end{aligned} \quad (147)$$

The above inequalities (146) and (147) imply that, for any $\epsilon \in (0, \delta_3]$,

$$0 < \frac{2}{3} (1 - \beta^* \rho(B)) \leq \operatorname{Re} \left(\frac{\epsilon}{\lambda_s^{\beta^*}(\epsilon)} \right) = \frac{\epsilon}{|\lambda_s^{\beta^*}(\epsilon)|^2} \operatorname{Re} \left(\lambda_s^{\beta^*}(\epsilon) \right). \quad (148)$$

Since the above argument is applied to any $s \in \{1, \dots, m\}$, there exists a $\delta_2^* > 0$ with $\delta_2^* \leq \delta_3$ such that,

$$0 < \operatorname{Re} \left(\lambda_s^{\beta^*}(\epsilon) \right), \quad \forall s \in \{1, \dots, m\}, \forall \epsilon \in (0, \delta_2^*], \quad (149)$$

which further implies that

$$\lambda_s^{\beta^*}(\epsilon) \notin \Omega, \quad \forall s \in \{1, \dots, m\}, \forall \epsilon \in (0, \delta_2^*]. \quad (150)$$

Step (c): According to $\beta^* \in \left(0, \frac{1}{\rho(B)}\right)$, there exists a $\bar{\delta} > 0$ with $\bar{\delta} < \delta$ such that, for any $\epsilon \in (0, \bar{\delta})$,

$$\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right). \quad (151)$$

Let

$$\delta^* = \min \{\delta_1^*, \delta_2^*, \bar{\delta}\}. \quad (152)$$

Then, for any $\epsilon \in (0, \delta^*)$, we have

$$\epsilon \in (0, \delta), \quad (153)$$

$$\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right) \quad (154)$$

and

$$\lambda_s^{\beta^*}(\epsilon) \notin \Omega, \quad \forall s \in \{1, \dots, n\}, \quad (155)$$

where (153) uses the definition (152) of δ^* (i.e., $\delta^* \leq \bar{\delta} < \delta$); (154) is due to the definition (152) of δ^* (i.e., $\delta^* \leq \bar{\delta}$) and (151); and (155) thanks to (131), (150) and the definition (152) of δ^* . Clearly, this contradicts (123).

Hence, we conclude that (85) holds true. \square

8.4 Proof of Lemma 2

Proof Without loss of generality, we focus on the first block and prove that ψ_1 with step size $\alpha < \frac{\mu}{L}$ is a diffeomorphism. The other blocks can be treated similarly.

The proof is divided into the following four steps.

(a) We first prove that ψ_1 is injective from $\mathbb{R}^n \rightarrow \mathbb{R}^n$ for $\alpha < \frac{\mu}{L}$. Suppose that there exist x and y such that $\psi_1(x) = \psi_1(y)$, which implies that

$$\begin{cases} [\nabla\varphi_1]^{-1}(\nabla\varphi_1(x(t)) - \alpha\nabla_1 f(x)) = [\nabla\varphi_1]^{-1}(\nabla\varphi_1(y(t)) - \alpha\nabla_1 f(y)), & t = 1, \\ x(t) = y(t), & t = 2, 3, \dots, p. \end{cases} \quad (156)$$

Since Lemma 14 in Appendix asserts that $\nabla\varphi_1$ is a diffeomorphism, then $[\nabla\varphi_1]^{-1}$ is a diffeomorphism as well. Hence, the above equality (156) is equivalent to

$$\begin{cases} \nabla\varphi_1(x(t)) - \alpha\nabla_1 f(x) = \nabla\varphi_1(y(t)) - \alpha\nabla_1 f(y), & t = 1, \\ x(t) = y(t), & t = 2, 3, \dots, p. \end{cases} \quad (157)$$

In particular, $\nabla\varphi_1(x(1)) - \alpha\nabla_1 f(x) = \nabla\varphi_1(y(1)) - \alpha\nabla_1 f(y)$ further implies that

$$\begin{aligned} \|x(1) - y(1)\| &\leq \frac{1}{\mu} \|\nabla\varphi_1(x(1)) - \nabla\varphi_1(y(1))\| = \frac{\alpha}{\mu} \|\nabla_1 f(x) - \nabla_1 f(y)\| \\ &\leq \frac{\alpha}{\mu} \|\nabla f(x) - \nabla f(y)\| \leq \frac{\alpha L}{\mu} \|x - y\| = \frac{\alpha L}{\mu} \|x(1) - y(1)\|, \end{aligned} \quad (158)$$

where the first inequality is due to the strong convexity (see (21) and $\mu \triangleq \min\{\mu_1, \mu_2, \dots, \mu_p\}$), the third inequality thanks to (2), and the last equality holds because of (157). Since $\frac{\alpha L}{\mu} < 1$, (158) means $x(1) = y(1)$. Combining with (157), we have $x = y$.

(b) To show ψ_1 is surjective, we construct an explicit inverse function. Given a point y in \mathbb{R}^n , we use the same partition for y and the same definitions for $y_{-(1)}$ and $\hat{f}(x(1); y_{-(1)})$ as in Lemma 1. Consider the following problem,

$$\min_{x(1)} B_{\varphi_1}(x(1), y(1)) - \alpha\hat{f}(x(1); y_{-(1)}) \quad (159)$$

When $\alpha < \frac{\mu}{L}$, the function above is strongly convex with respect to $x(1)$, so problem (159) has a unique minimizer. Let $x_{y(1)}$ be the unique minimizer, then by the first-order optimality condition, we have $\nabla\varphi_1(y(1)) = \nabla\varphi_1(x_{y(1)}) - \alpha\nabla\hat{f}(x_{y(1)}; y_{-(1)})$, which is equivalent to

$$y(1) = [\nabla\varphi_1]^{-1} \left(\nabla\varphi_1(x_{y(1)}) - \alpha\nabla_1 f \left(\begin{pmatrix} x_{y(1)} \\ y_{-(1)} \end{pmatrix} \right) \right). \quad (160)$$

Let x_y be defined as in (93). Then we have

$$\begin{aligned} y &= \begin{pmatrix} y(1) \\ y_{-(1)} \end{pmatrix} = \begin{pmatrix} [\nabla\varphi_1]^{-1} \left(\nabla\varphi_1(x_{y(1)}) - \alpha\nabla_1 f \left(\begin{pmatrix} x_{y(1)} \\ y_{-(1)} \end{pmatrix} \right) \right) \\ y_{-(1)} \end{pmatrix} \\ &= (I_n - U_1 U_1^T) x_y + U_1 [\nabla\varphi_1]^{-1} (\nabla\varphi_1(x_{y(1)}) - \alpha\nabla_1 f(x_y)) = \psi_1(x_y), \end{aligned} \quad (161)$$

where the second equality thanks to (160), the third equality holds because of the definition of U_1 (see (4)), and the last equality is due to (29).

Hence, x_y is mapped to y by the mapping ψ_1 .

(c) We prove that the matrix $D\psi_1(x)$ is invertible. For notational convenience, we define $F_1 \triangleq \nabla^2 \varphi_1(x(1)) - \alpha A_{11}$ and $F_2 \triangleq \left\{ \nabla^2 \varphi_1 \left\{ [\nabla \varphi_1]^{-1} (\nabla \varphi_1(x(1)) - \alpha \nabla_1 f(x)) \right\} \right\}^{-1}$. Recalling (33), we have

$$\begin{aligned} D\psi_1(x) &= (I_n - U_1 U_1^T) + \{U_1 \nabla^2 \varphi_1(x(1)) - \alpha \nabla^2 f(x) U_1\} F_2 U_1^T \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I_{n_p} \end{pmatrix} + \begin{pmatrix} F_1 F_2 & \mathbf{0} & \cdots & \mathbf{0} \\ -\alpha A_{21} F_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha A_{p1} F_2 & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} = \begin{pmatrix} F_1 F_2 & \mathbf{0} & \cdots & \mathbf{0} \\ -\alpha A_{21} F_2 & I_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha A_{p1} F_2 & \mathbf{0} & \cdots & I_{n_p} \end{pmatrix}, \end{aligned}$$

where since F_1 and F_2 are defined above, the second equality follows from the definitions of U_1 and A_{s1} which are given by (4) and (14), respectively. The above equality means that $\text{eig}(D\psi_1(x)) = \{1\} \cup \text{eig}(F_1 F_2)$. Moreover, $F_1 \succeq \nabla^2 \varphi_1(x(1)) - \alpha L I_{n_1} \succ \nabla^2 \varphi_1(x(1)) - \frac{\mu}{L} L I_{n_1} = \nabla^2 \varphi_1(x(1)) - \mu I_{n_1} \succeq \mathbf{0}$, where the first inequality holds because of $A_{11} = \nabla_{11}^2 f(x)$, (2) and Lemma 7 in [19]; the second inequality is due to $\alpha < \frac{\mu}{L}$; the last inequality thanks to (21) and $\mu_1 \geq \mu$. Hence, F_1 is an invertible matrix. Consequently, $D\psi_1(x)$ is an invertible matrix as well.

(d) We have shown that ψ_1 is bijective and $D\psi_1(x)$ is invertible for $\alpha < \frac{\mu}{L}$. Meanwhile, under Assumptions 1 and 2, we infer that ψ_1 is continuously differentiable. Therefore, using the inverse function theorem, we conclude that $[\psi_1]^{-1}$ is continuously differentiable. Thus, ψ_1 is a diffeomorphism. \square

8.5 Proof of Lemma 9

Proof Let λ be an eigenvalue of $(\beta B)^{-1} (I + t\beta \hat{B})$ and ξ be the corresponding eigenvector of unit length. Thus, we have $(I_n + t\beta \hat{B}) \xi = \lambda (\beta B) \xi$. It follows that $\lambda = \frac{1+t\beta \xi^H \hat{B} \xi}{\beta \xi^H B \xi}$. Since $0 < \beta < \frac{1}{\rho(B)}$ and $t \in [0, 1]$, Lemma 7 implies $0 < \text{Re}(1 + t\beta \xi^H \hat{B} \xi)$. Combining this with the fact that $\text{Re}(\lambda) > 0$ (by the assumption) and B is a symmetric matrix, we have $\beta \xi^H B \xi > 0$.

On the other hand, the symmetric matrix B can be expressed as $B = \hat{B} + \tilde{B} + \hat{B}^T$, where \hat{B} and \tilde{B} are defined by (73) and (80), respectively. By invoking [10, Theorem 4.3.15], we have $\xi^H \tilde{B} \xi \leq \rho(B)$.

Now we are ready to show that our main claim is true. The proof is divided into two cases.

1) If $t \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} \text{Re}(\lambda) - \frac{1}{2} &= \frac{\text{Re}(1 + t\beta \xi^H \hat{B} \xi)}{\beta \xi^H B \xi} - \frac{1}{2} = \frac{2\text{Re}(1 + t\beta \xi^H \hat{B} \xi) - \beta \xi^H B \xi}{2\beta \xi^H B \xi} \\ &= \frac{2\text{Re}(1 + t\beta \xi^H \hat{B} \xi) - \beta \xi^H (\hat{B} + \tilde{B} + \hat{B}^T) \xi}{2\beta \xi^H B \xi} = \frac{2 + 2t\beta \text{Re}(\xi^H \hat{B} \xi) - 2\beta \text{Re}(\xi^H \hat{B} \xi) - \beta \xi^H \tilde{B} \xi}{2\beta \xi^H B \xi} \\ &= \frac{2 + 2(t-1)\beta \text{Re}(\xi^H \hat{B} \xi) - \beta \xi^H \tilde{B} \xi}{2\beta \xi^H B \xi} \geq \frac{2 + 2(t-1)\beta \rho(B) - \beta \rho(B)}{2\beta \xi^H B \xi} \geq \frac{1 - \beta \rho(B)}{\beta \rho(B)} > 0, \end{aligned}$$

where the first inequality thanks to $\text{Re}(\xi^H \hat{B} \xi) \leq \rho(B)$ (cf. Eq. (79)), $\xi^H \tilde{B} \xi \leq \rho(B)$ and $\beta \xi^H B \xi > 0$, and the last two inequalities hold because of $t \in [\frac{1}{2}, 1]$ and $\beta \in (0, \frac{1}{\rho(B)})$, respectively.

2) If $t \in [0, \frac{1}{2}]$, then

$$\begin{aligned} \operatorname{Re}(\lambda) - \frac{1}{2} &= \frac{\operatorname{Re}\left(1 + t\beta\xi^H \hat{B}\xi\right)}{\beta\xi^H B\xi} - \frac{1}{2} = \frac{2\operatorname{Re}\left(1 + t\beta\xi^H \hat{B}\xi\right) - \beta\xi^H B\xi}{2\beta\xi^H B\xi} \\ &= \frac{2 + 2t\beta\operatorname{Re}\left(\xi^H \hat{B}\xi\right) - \beta\xi^H B\xi}{2\beta\xi^H B\xi} \geq \frac{2 - 2t\beta\rho(B) - \beta\rho(B)}{2\beta\rho(B)} \geq \frac{1 - \beta\rho(B)}{\beta\rho(B)} > 0, \end{aligned}$$

where the first inequality is due to $\operatorname{Re}\left(\xi^H \hat{B}\xi\right) \geq -\rho(B)$ (cf. Eq. (79)) and $0 < \xi^H B\xi \leq \rho(B)$ and the last two inequalities hold because of $t \in [0, \frac{1}{2}]$ and $\beta \in \left(0, \frac{1}{\rho(B)}\right)$.

Thus the proof is finished. \square

8.6 Proof of Lemma 11

Proof We divide the proof into two cases.

Case 1: when B is an invertible matrix. In this case, we have $\left(\beta\left(I_n + \beta\hat{B}\right)^{-1}B\right)^{-1} = (\beta B)^{-1}\left(I_n + \beta\hat{B}\right)$. Hence, it suffices to show that, for an arbitrary $\beta \in \left(0, \frac{1}{\rho(B)}\right)$, there is at least one nonzero eigenvalue σ of $(\beta B)^{-1}\left(I_n + \beta\hat{B}\right)$ such $\sigma \in \Xi(\beta, B)$. In what follows, we prove it by using Lemma 13 in Appendix.

Let us define an analytic function with t as parameter:

$$\begin{aligned} \mathcal{X}_t(z) &\triangleq \det\left\{zI_n - \left[(1-t)(\beta B)^{-1} + t(\beta B)^{-1}\left(I_n + \beta\hat{B}\right)\right]\right\} \\ &= \det\left\{zI_n - (\beta B)^{-1}\left(I_n + t\beta\hat{B}\right)\right\}, \quad 0 \leq t \leq 1, \end{aligned} \quad (162)$$

and

$$\begin{aligned} \nu &\triangleq \left\|(\beta B)^{-1}\right\| + \frac{1}{\rho(B)}\left\|(\beta B)^{-1}\hat{B}\right\| \geq \left\|(\beta B)^{-1}\right\| + t\beta\left\|(\beta B)^{-1}\hat{B}\right\| \\ &\geq \left\|(\beta B)^{-1}\left(I_n + t\beta\hat{B}\right)\right\|, \quad \forall t \in [0, 1], \end{aligned} \quad (163)$$

where the first inequality holds because of $\beta \in \left(0, \frac{1}{\rho(B)}\right)$ and $t \in [0, 1]$. By setting $t = 0$ in (163), we can obtain

$$\nu \geq \left\|(\beta B)^{-1}\right\| \geq \rho\left((\beta B)^{-1}\right) \geq \frac{1}{\beta\rho(B)} > \frac{1}{\beta\rho(B)} - \frac{1}{2} = \frac{1}{2} + \frac{1 - \beta\rho(B)}{\beta\rho(B)} > \frac{1}{2}, \quad (164)$$

where the second inequality is due to the definitions of spectral norm and spectral radius, the third inequality thanks to property of spectral radius, and the last inequality holds because of $\beta \in \left(0, \frac{1}{\rho(B)}\right)$.

Given ν satisfying (164), we can define a closed rectangle as follows

$$\mathcal{D} \triangleq \left\{a + bi \mid \frac{1}{2} \leq a \leq 2\nu, \quad -2\nu \leq b \leq 2\nu\right\}, \quad (165)$$

which is a closed region in the complex plane. Note that its boundary $\partial\mathcal{D}$ consists of a finite number of smooth curves. Specifically, define

$$\begin{aligned} \gamma_1 &\triangleq \left\{a + bi \mid a = \frac{1}{2}, \quad -2\nu \leq b \leq 2\nu\right\}, \quad \gamma_2 \triangleq \left\{a + bi \mid a = 2\nu, \quad -2\nu \leq b \leq 2\nu\right\}, \\ \gamma_3 &\triangleq \left\{a + bi \mid \frac{1}{2} \leq a \leq 2\nu, \quad b = 2\nu\right\}, \quad \gamma_4 \triangleq \left\{a + bi \mid \frac{1}{2} \leq a \leq 2\nu, \quad b = -2\nu\right\}, \end{aligned} \quad (166)$$

then $\partial\mathcal{D} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.

In order to apply Lemma 13, we first prove

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \partial\mathcal{D} \quad (167)$$

in the sequel. On one hand, since the spectral norm of a matrix is larger than or equal to its spectral radius, the above inequality (163) yields that, for any $t \in [0, 1]$, every eigenvalue of $(\beta B)^{-1} (I + t\beta\hat{B})$ has a magnitude less than ν . Note that for an arbitrary $z \in \gamma_2 \cup \gamma_3 \cup \gamma_4$, we have $|z| \geq 2\nu$. Consequently,

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \gamma_2 \cup \gamma_3 \cup \gamma_4. \quad (168)$$

On the other hand, if σ_t is an eigenvalue of $(\beta B)^{-1} (I_n + t\beta\hat{B})$ with any $t \in [0, 1]$, and $\text{Re}(\sigma_t) > 0$, then Lemma 9 implies $\text{Re}(\sigma_t) > \frac{1}{2}$, equivalently, $\sigma_t \notin \gamma_1$. Hence, we have

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \gamma_1. \quad (169)$$

Combining (168) and (169), we obtain (167).

Based on (167) and using Lemma 13 in Appendix, we infer that $\mathcal{X}_0(z) = \det \{zI_n - (\beta B)^{-1}\}$ and $\mathcal{X}_1(z) = \det \{zI_n - (\beta B)^{-1} (I_n + \beta\hat{B})\}$ have the same number of zeros in \mathcal{D} . Note that $\lambda_{\max}(B) > 0$ implies that the matrix $(\beta B)^{-1}$ has a positive eigenvalue $\frac{1}{\beta\lambda_{\max}(B)}$, which satisfies $\frac{1}{\beta\lambda_{\max}(B)} \geq \frac{1}{\beta\rho(B)} > 1$. Recalling the definition (163) of ν , we know $\frac{1}{\beta\lambda_{\max}(B)} \leq \nu$. Thus $\frac{1}{\beta\lambda_{\max}(B)}$ must lie inside \mathcal{D} , i.e., there exists at least one zero of $\mathcal{X}_0(z)$ inside \mathcal{D} , which in turn shows that $\mathcal{X}_1(z)$ has at least one zero inside \mathcal{D} as well. Thus, there must exist at least one eigenvalue of $(\beta B)^{-1} (I_n + \beta\hat{B})$ lying inside \mathcal{D} . We denote it as σ , then $\text{Re}(\sigma) > \frac{1}{2}$. Moreover, Lemma 9 implies $\text{Re}(\sigma) \geq \frac{1}{2} + \frac{1-\beta\rho(B)}{\beta\rho(B)}$. Hence, σ lies in $\Xi(\beta, B)$ defined by (88) and the proof is finished in this case.

Case 2: when B is a singular matrix. In this case, based on the results in **Case 1** we apply the perturbation theorem to prove (87).

Suppose the multiplicity of zero eigenvalue of B is m . For clarity of notation, we rewrite the eigenvalue decomposition (110) of B below:

$$B = V \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^T = V_1 \Theta V_1^T, \quad (170)$$

where $\Theta = \text{Diag}(\theta_1, \theta_2, \dots, \theta_{n-m})$, θ_s , $s = 1, \dots, n-m$, are the nonzero eigenvalues of B and

$$V = (V_1 \ V_2) \quad (171)$$

is an orthogonal matrix and V_1 consists of the first $(n-m)$ columns of V .

Denote

$$\delta \triangleq \min \{|\theta_1|, |\theta_2|, \dots, |\theta_{n-m}|\}. \quad (172)$$

For any $\epsilon \in (-\delta, 0)$, we define

$$B(\epsilon) \triangleq B + \epsilon I_n. \quad (173)$$

Then,

$$\text{eig}(B(\epsilon)) = \{\theta_1 + \epsilon, \theta_2 + \epsilon, \dots, \theta_{n-m} + \epsilon, \epsilon\} \not\equiv 0, \quad \forall \epsilon \in (-\delta, 0), \quad (174)$$

and

$$\lambda_{\max}(B(\epsilon)) = \lambda_{\max}(B) + \epsilon \geq \delta + \epsilon > 0, \quad \forall \epsilon \in (-\delta, 0), \quad (175)$$

where the first inequality is due to the definition of δ and $\max\{\theta_1, \theta_2, \dots, \theta_{n-m}\} = \lambda_{\max}(B) > 0$.

Since B is defined by (70), $B(\epsilon)$ has $p \times p$ blocks form as well. Specifically,

$$B(\epsilon) = (B(\epsilon)_{st})_{1 \leq s, t \leq p}, \quad (176)$$

and its (s, t) -th block is given by

$$B(\epsilon)_{st} = \begin{cases} B_{st} + \epsilon I_{n_s}, & s = t, \\ B_{st}, & s \neq t, \end{cases} \quad (177)$$

where n_1, n_2, \dots, n_p are p positive integer numbers satisfying $\sum_{s=1}^p n_s = n$. Similar to definition (73), we denote the strictly block upper triangular matrix of $B(\epsilon)$ as

$$\hat{B}(\epsilon) \triangleq \left(\hat{B}(\epsilon)_{st} \right)_{1 \leq s, t \leq p} \quad (178)$$

with $p \times p$ blocks and its (s, t) -th block is given by

$$\hat{B}(\epsilon)_{st} = \begin{cases} B(\epsilon)_{st}, & s < t, \\ \mathbf{0}, & s \geq t, \end{cases} = \begin{cases} B_{st}, & s < t, \\ \mathbf{0}, & s \geq t, \end{cases} = \hat{B}_{st}, \quad (179)$$

where the second equality holds because of (177); the last equality is due to (74).

It follows easily from Eqs. (73), (74) (178) and (179) that

$$\hat{B}(\epsilon) = \hat{B}. \quad (180)$$

Consequently,

$$\beta \left(I_n + \beta \hat{B}(\epsilon) \right)^{-1} B(\epsilon) = \beta \left(I_n + \beta \hat{B} \right)^{-1} B(\epsilon) = \beta \left(I_n + \beta \hat{B} \right)^{-1} (B + \epsilon I_n),$$

where the first equality is due to (180) and the second equality holds because of (173). For simplicity, let $\lambda_s^\beta(\epsilon)$, $s = 1, \dots, n$, be the eigenvalues of $\beta \left(I_n + \beta \hat{B} \right)^{-1} (B + \epsilon I_n)$.

Note that for any $\epsilon \in (-\delta, 0)$, $B(\epsilon)$ is invertible and $\lambda_{\max}(B(\epsilon)) > 0$ (see (175)). According to the definitions of $B(\epsilon)$ and $\hat{B}(\epsilon)$, a similar argument in **Case 1** can be applied with the identifications $B(\epsilon) \sim B$, $\hat{B}(\epsilon) \sim \hat{B}$, $\beta \sim \beta$ and $\rho(B(\epsilon)) \sim \rho(B)$, to prove that, for any $\beta \in \left(0, \frac{1}{\rho(B(\epsilon))} \right)$, there must exist at least one eigenvalue of $\beta \left(I_n + \beta \hat{B}(\epsilon) \right)^{-1} B(\epsilon)$ which lies in $\Xi(\beta, B(\epsilon))$ defined by (88). Taking into account definition (173), we have $\rho(B(\epsilon)) \leq \rho(B) - \epsilon$. Hence, for any $\epsilon \in (-\delta, 0)$ and $\beta \in \left(0, \frac{1}{\rho(B) - \epsilon} \right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))} \right)$, there exists at least one index denoted as $s(\epsilon) \in \{1, 2, \dots, n\}$ such that

$$\frac{1}{\lambda_{s(\epsilon)}^\beta(\epsilon)} \in \Xi(\beta, B(\epsilon)), \quad (181)$$

where $\Xi(\beta, B(\epsilon))$ is defined in (88). Furthermore, it is well known that the eigenvalues of a matrix M are continuous functions of the entries of M . Therefore, for any $\beta \in \left(0, \frac{1}{\rho(B)} \right)$, $\lambda_s^\beta(\epsilon)$ is a continuous function of ϵ and

$$\lim_{\epsilon \rightarrow 0^-} \lambda_s^\beta(\epsilon) = \lambda_s^\beta(0), \quad s = 1, \dots, n, \quad (182)$$

where $\lambda_s^\beta(0)$ is the eigenvalue of $\beta \left(I_n + \beta \hat{B} \right)^{-1} B$.

In what follows, we will prove that (87) holds true by contradiction.

Suppose for sake of contradiction that, there exists a $\beta^* \in \left(0, \frac{1}{\rho(B)} \right)$ such that, for any $s \in \{1, \dots, n\}$, if $\lambda_s^{\beta^*}(0) \neq 0$, then

$$\frac{1}{\lambda_s^{\beta^*}(0)} \notin \Xi(\beta^*, B(0)) = \Xi(\beta^*, B), \quad (183)$$

where $\lambda_s^{\beta^*}(0)$ is the eigenvalue of $\beta^* \left(I_n + \beta^* \hat{B} \right)^{-1} B$.

According to Lemma 12 in Section 6 and the assumption that the multiplicity of zero eigenvalue of B is m , we know that the multiplicity of eigenvalue 0 of $\beta \left(I_n + \beta^* \hat{B} \right)^{-1} B$ is m as well. Then there are exactly m eigenvalue functions of ϵ whose limits are 0 as ϵ approaches zero from below. Without loss of generality, we assume

$$\lim_{\epsilon \rightarrow 0^-} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) = 0, \quad s = 1, \dots, m, \quad (184)$$

and

$$\lim_{\epsilon \rightarrow 0^-} \lambda_s^{\beta^*}(\epsilon) = \lambda_s^{\beta^*}(0) \neq 0, \quad s = m+1, \dots, n. \quad (185)$$

Subsequently, under the assumption (183), we will prove that there exists a $\delta^* > 0$ with $\delta^* < \delta$ such that, for any $\epsilon \in (-\delta^*, 0)$, then $\beta^* \in \left(0, \frac{1}{\rho(B) - \epsilon} \right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))} \right)$, and there does not exist any $s \in \{1, \dots, n\}$ such that $\frac{1}{\lambda_s^{\beta^*}(\epsilon)}$ belongs to $\Xi(\beta^*, B(\epsilon))$. This would contradict (181). The proof is given by the following three steps.

Step (a): Under the assumption (183), we first prove that there exists a $\delta_1^* > 0$ such that, for any $\epsilon \in (-\delta_1^*, 0)$, there does not exist any $s \in \{m+1, \dots, n\}$ such that $\frac{1}{\lambda_s^{\beta^*}(\epsilon)}$ belongs to $\Xi(\beta^*, B(\epsilon))$.

Note that $\rho(B(\epsilon))$ is a continuous function of ϵ . It follows from the definition of $\Xi(\beta^*, B(\epsilon))$ and Eq. (183) that, there exists a $\delta_1 > 0$ with $\delta_1 < \delta$ such that, for any $\epsilon \in (-\delta_1, 0)$,

$$\operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \right) < \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))}, \quad \forall s \in \{m+1, \dots, n\}. \quad (186)$$

Moreover, note that $\frac{1}{\lambda_s^{\beta^*}(\epsilon)}$ is a continuous function of ϵ and (185) holds. Combining these with the above inequalities, we know that there exists a $\delta_1^* > 0$ with $\delta_1^* \leq \delta_1$ such that

$$\left| \frac{1}{\lambda_s^{\beta^*}(\epsilon)} - \frac{1}{\lambda_s^{\beta^*}(0)} \right| < \frac{1}{3} \left[\frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(0)} \right) \right], \quad \forall s \in \{m+1, \dots, n\}, \forall \epsilon \in (-\delta_1^*, 0),$$

which further means that, for any $s \in \{m+1, \dots, n\}$ and $\epsilon \in (-\delta_1^*, 0)$,

$$\begin{aligned} \operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \right) &< \operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(0)} \right) + \frac{1}{3} \left[\frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(0)} \right) \right] \\ &= \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \frac{2}{3} \left[\frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(0)} \right) \right] < \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))}. \end{aligned}$$

Hence, we arrive at

$$\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \notin \Xi(\beta^*, B(\epsilon)), \quad \forall s \in \{m+1, \dots, n\}, \forall \epsilon \in (-\delta_1^*, 0). \quad (187)$$

Step (b): As in **Step (b)** in the proof of Lemma 10, we are able to apply the similar arguments to prove, that there exists a $\delta_2^* > 0$ such that, for any $\epsilon \in (-\delta_2^*, 0)$ and $s \in \{1, \dots, m\}$, $\operatorname{Re}(\lambda_s^{\beta^*}(\epsilon)) < 0$, or equivalently, $\operatorname{Re} \left(\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \right) < 0$, which immediately implies that

$$\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \notin \Xi(\beta^*, B(\epsilon)), \quad \forall s \in \{1, \dots, m\}, \forall \epsilon \in (-\delta_2^*, 0). \quad (188)$$

Step (c): According to $\beta^* \in \left(0, \frac{1}{\rho(B)} \right)$, there exists a $\bar{\delta} > 0$ with $\bar{\delta} < \delta$ such that, for any $\epsilon \in (-\bar{\delta}, 0)$,

$$\beta^* \in \left(0, \frac{1}{\rho(B) - \epsilon} \right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))} \right). \quad (189)$$

Let

$$\delta^* = \min \{ \delta_1^*, \delta_2^*, \bar{\delta} \}. \quad (190)$$

Then, for any $\epsilon \in (-\delta^*, 0)$, we have

$$\epsilon \in (-\delta, 0), \quad (191)$$

$$\beta^* \in \left(0, \frac{1}{\rho(B) - \epsilon} \right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))} \right) \quad (192)$$

and

$$\frac{1}{\lambda_s^{\beta^*}(\epsilon)} \notin \Xi(\beta^*, B(\epsilon)), \quad \forall s \in \{1, \dots, n\}, \quad (193)$$

where (191) uses the definition (190) of δ^* (i.e., $\delta^* \leq \bar{\delta} < \delta$); (192) is due to the definition (190) of δ^* (i.e., $\delta^* \leq \bar{\delta}$) and (189); and (193) thanks to (187), (188) and the definition (190) of δ^* . Clearly, this contradicts (181).

Hence, we conclude that Eq. (87) holds true. \square

8.7 Proof of Lemma 12

Proof Without loss of generality, suppose the multiplicity of zero eigenvalue of B is m and B has an eigenvalue decomposition in the form of (110). Let us consider the matrix $(I_n + \beta\check{B})^{-1}B$. Its characteristic polynomial is given below:

$$\begin{aligned} \det \left\{ (I_n + \beta\check{B})^{-1}B - \lambda I_n \right\} &= \det \left\{ B - \lambda (I_n + \beta\check{B}) \right\} = \det \left\{ V \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^T - \lambda (I_n + \beta\check{B}) \right\} \\ &= \det \left\{ \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} - \lambda (I_n + \beta V^T \check{B} V) \right\} = \det \left\{ \begin{pmatrix} \Theta - \lambda (I_{n-m} + \beta\check{C}_{11}) & -\lambda\beta\check{C}_{12} \\ -\lambda\beta\check{C}_{21} & -\lambda (I_m + \beta\check{C}_{22}) \end{pmatrix} \right\} \\ &= \det \left\{ \Theta - \lambda (I_{n-m} + \beta\check{C}_{11}) \right\} \det \left\{ -\lambda (I_m + \beta\check{C}_{22}) - \lambda^2 \beta^2 \check{C}_{21} [\Theta - \lambda (I_{n-m} + \beta\check{C}_{11})]^{-1} \check{C}_{12} \right\} \\ &= \lambda^m \det \left\{ \Theta - \lambda (I_{n-m} + \beta\check{C}_{11}) \right\} \det \left\{ - (I_m + \beta\check{C}_{22}) - \lambda \beta^2 \check{C}_{21} [\Theta - \lambda (I_{n-m} + \beta\check{C}_{11})]^{-1} \check{C}_{12} \right\}, \end{aligned} \quad (194)$$

where the second equality holds because of the eigenvalue decomposition form (110) of B and the fourth equality is due to the definitions (134) of \check{C}_{ij} , $i, j = 1, 2$. Since Θ is invertible, we must have $\det \left\{ \Theta - \lambda (I_{n-m} + \beta\check{C}_{11}) \right\} \neq 0$ at $\lambda = 0$. Hence, it suffices to show that at $\lambda = 0$

$$\det \left\{ - (I_m + \beta\check{C}_{22}) - \lambda \beta^2 \check{C}_{21} [\Theta - \lambda (I_{n-m} + \beta\check{C}_{11})]^{-1} \check{C}_{12} \right\} \neq 0, \quad (195)$$

equivalently, to show $\det[(I_m + \beta\check{C}_{22})] \neq 0$.

Let σ be an arbitrary eigenvalue of $I_m + \beta\check{C}_{22}$ and $v \in \mathbb{C}^m$ be the corresponding eigenvector of length one. Then we have

$$\begin{aligned} \operatorname{Re}(\sigma) &= \operatorname{Re}(v^H (I_m + \beta\check{C}_{22}) v) = \operatorname{Re}(1 + \beta v^H \check{C}_{22} v) = \operatorname{Re}(1 + \beta v^H V_2^T \check{B} V_2 v) \\ &= \operatorname{Re}(1 + \beta v^H V_2^H \check{B} V_2 v) = \operatorname{Re}(1 + \beta (V_2 v)^H \check{B} V_2 v) \geq 1 - \beta \rho(B) > 0, \end{aligned} \quad (196)$$

where the third equality follows from (134); the fourth equality holds because V_2 is a real matrix according to Eqs. (110) and (111); the first inequality follows from (78) in Lemma 7 and $\|V_2 v\|^2 = (V_2 v)^H V_2 v = v^H v = 1$. The above inequality shows that the real parts of the eigenvalues of $I_m + \beta\check{C}_{22}$ are not zero. Thus it holds that $\det[(I_m + \beta\check{C}_{22})] \neq 0$. As a result, we conclude that the multiplicity of zero eigenvalue of $(I_n + \beta\check{B})^{-1}B$ is exactly m . Similar arguments apply to $(I_n + \beta\hat{B})^{-1}B$ as well. This completes the proof. \square

8.8 Lemma 13 and its proof

This lemma is known as Ostrowski's lemma [18]. For convenience, we duplicate it here.

Lemma 13 (*[18, Ostrowski's lemma]*) Assume $M, N \in \mathbb{C}^{n \times n}$ and define

$$\mathcal{X}_t(z) \triangleq \det \{zI_n - [(1-t)M + tN]\}, \quad 0 \leq t \leq 1. \quad (197)$$

Moreover, suppose that \mathcal{D} is a closed region in \mathbb{C} and its boundary $\partial\mathcal{D}$ consists of a finite number of smooth curves. If

$$\mathcal{X}_t(z) \neq 0, \quad \forall t \in [0, 1], \quad \forall z \in \partial\mathcal{D}, \quad (198)$$

then, for any $t_1, t_2 \in [0, 1]$, $\mathcal{X}_{t_1}(z)$ and $\mathcal{X}_{t_2}(z)$ have the same number of zeros in \mathcal{D} counted according to their multiplicities.

Proof For any $t_0 \in [0, 1]$, define $u(z) \triangleq \mathcal{X}_{t_0}(z)$ and $\Gamma \triangleq \min_{z \in \partial\mathcal{D}} |\mathcal{X}_{t_0}(z)| > 0$. Besides, for any $t_0 + \epsilon \in [0, 1]$, let $v(z) \triangleq \mathcal{X}_{t_0+\epsilon}(z) - \mathcal{X}_{t_0}(z)$. Since $\mathcal{X}_t(z)$ is a continuous function on the closed set $[0, 1] \times \partial\mathcal{D}$, for sufficiently small $|\epsilon|$, we have $\max_{z \in \partial\mathcal{D}} |v(z)| < \Gamma$. Then, two single-valued analytic functions $u(z)$ and $v(z)$ on region \mathcal{D} satisfy $|u(z)| > |v(z)|, \forall z \in \partial\mathcal{D}$. According to Rouché's theorem [29], $u(z) + v(z) = \mathcal{X}_{t_0+\epsilon}(z)$ and $u(z) = \mathcal{X}_{t_0}(z)$ have the same number of zeros in \mathcal{D} . Hence, for t in the $|\epsilon|$ neighborhood of t_0 , the number of zeros of $\mathcal{X}_t(z)$ in the region \mathcal{D} , denoted as $N(t)$, is a constant. Thereby, $N(t)$ is a continuous function defined on $[0, 1]$. However, the function $N(t)$ can only be nonnegative integers. Thus, $N(t)$ is a constant function on $[0, 1]$. \square

8.9 Lemma 14 and its proof

Lemma 14 Suppose that $\phi(x)$ is a strongly convex twice continuously differentiable function with parameter $\sigma > 0$, i.e., for any y and $x \in \mathbb{R}^n$,

$$\phi(x) \geq \phi(y) + \langle x - y, \nabla\phi(y) \rangle + \frac{\sigma}{2} \|x - y\|^2, \quad (199)$$

then its gradient mapping $\nabla\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism.

Proof Since ϕ is a σ -strongly convex function, it follows easily from (199) that

$$\langle \nabla\phi(y) - \nabla\phi(x), y - x \rangle \geq \sigma \|y - x\|^2. \quad (200)$$

We first check that $\nabla\phi$ is injective from $\mathbb{R}^n \rightarrow \mathbb{R}^n$. Suppose that there exist x and y such that $\nabla\phi(x) = \nabla\phi(y)$. Then we have $\|y - x\|^2 \leq \frac{1}{\sigma} \langle \nabla\phi(y) - \nabla\phi(x), y - x \rangle = 0$, which means $x = y$. Thus, $\nabla\phi$ is injective.

To show that the gradient mapping $\nabla\phi$ is surjective, we construct an explicit inverse function. Let us consider the optimization problem $\min_x \phi(x) - \langle y, x \rangle$. Since ϕ is a σ -strongly convex function, the problem has a unique minimizer, denoted as x_y . By the first-order optimality condition, we have $\nabla\phi(x_y) = y$. Thus, x_y is mapped to y by the mapping $\nabla\phi$. Consequently, $\nabla\phi$ is bijective.

The assumption also means $\nabla\phi$ is continuously differentiable, and moreover $D\nabla\phi(x) = \nabla^2\phi(x) \succeq \sigma I_n \succ \mathbf{0}$. It follows that $D\nabla\phi(x)$ is invertible. Consequently, the inverse function theorem guarantees that $(\nabla\phi)^{-1}$ is continuously differentiable. Finally, we conclude that $\nabla\phi$ is a diffeomorphism. \square

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