

Amenable cones: error bounds without constraint qualifications

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December 2017 (Revised: August 2019)

Abstract

We provide a framework for obtaining error bounds for linear conic problems without assuming constraint qualifications or regularity conditions. The key aspects of our approach are the notions of *amenable cones* and *facial residual functions*. For *amenable cones*, it is shown that error bounds can be expressed as a composition of facial residual functions. The number of compositions is related to the facial reduction technique and the singularity degree of the problem. In particular, we show that symmetric cones are amenable and compute facial residual functions. From that, we are able to furnish a new Hölderian error bound, thus extending and shedding new light on an earlier result by Sturm on semidefinite matrices. We also provide error bounds for the intersection of amenable cones, this will be used to prove error bounds for the doubly nonnegative cone. At the end, we list some open problems.

Keywords: error bounds, amenable cones, facial reduction, singularity degree, symmetric cones, feasibility problem.

1 Introduction

In this work, we are interested in proving error bounds for the following conic feasibility problem.

$$\text{find } x \in (\mathcal{L} + a) \cap \mathcal{K}, \tag{Feas}$$

where \mathcal{K} is a closed convex cone contained in a finite dimensional real vector space \mathcal{E} , $\mathcal{L} \subseteq \mathcal{E}$ is a subspace and $a \in \mathcal{E}$. We will write $(\mathcal{K}, \mathcal{L}, a)$ to denote the problem (Feas). We suppose that \mathcal{E} is equipped with some inner product $\langle \cdot, \cdot \rangle$ and that the norm is induced by $\langle \cdot, \cdot \rangle$, i.e., $\|x\| = \sqrt{\langle x, x \rangle}$. Given a set $C \subseteq \mathcal{E}$ and $x \in \mathcal{E}$, we define the distance between x and C as $\text{dist}(x, C) = \inf\{\|x - y\| \mid y \in C\}$.

Suppose that we are given some arbitrary $x \in \mathcal{E}$ and we wish to measure how far x is from $(\mathcal{L} + a) \cap \mathcal{K}$. Since $\mathcal{L} + a$ is an affine space, it is quite simple to compute $\text{dist}(x, \mathcal{L} + a)$. Also, in many cases, it is also straightforward to compute $\text{dist}(x, \mathcal{K})$. Naïvely, one might expect that if we combine $\text{dist}(x, \mathcal{L} + a)$ and $\text{dist}(x, \mathcal{K})$ in some appropriate fashion, we might get a reasonable estimate for $\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K})$. When \mathcal{K} is a polyhedral cone, this is indeed true. In fact, when \mathcal{K} is polyhedral, it follows from the celebrated Hoffman's Lemma that there is a constant $\kappa > 0$ such that

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa \text{dist}(x, \mathcal{L} + a) + \kappa \text{dist}(x, \mathcal{K}), \tag{1}$$

for every $x \in \mathcal{E}$. This is an example of an *error bound result*. As far as error bounds go, the polyhedral case is perhaps the best one could hope for. It is *global*, meaning that it holds for all $x \in \mathcal{E}$. No regularity assumptions are needed on the intersection $(\mathcal{L} + a) \cap \mathcal{K}$. It is also *Lipschitzian* meaning that there is a linear relation between the distances, so if we decrease the individual distances to \mathcal{K} and $\mathcal{L} + a$, the distance to $\mathcal{K} \cap (\mathcal{L} + a)$ will decrease at least by the same order of magnitude.

It is well known that when \mathcal{K} is not polyhedral, the situation can be quite unfavourable and we cannot expect a result as nice as (1) to hold. In order to obtain error bounds we need to sacrifice globality, the

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Lipschitzness or impose regularity conditions. The literature on error bounds is very rich and it is not possible to do it justice here. Instead, we refer to either the comprehensive survey by Pang [35] or to the chapter by Lewis and Pang [27]. We emphasize that many results for the nonpolyhedral case include some regularity assumption on the intersection $\mathcal{K} \cap (\mathcal{L} + c)$. For instance, compactness and the condition $(\text{ri } \mathcal{K}) \cap (\mathcal{L} + c) \neq \emptyset$ (i.e., Slater’s condition) might be required for some of the results to hold, see page 313 in [35]. Also, Baes and Lin recently proved Lipschitzian error bound results for the symmetric cone complementarity problem but they require Slater’s condition to hold [3]. For nonlinear semidefinite programs, Yamashita proved error bounds under a few regularity conditions [53].

Among the several error bounds results in the literature, the one proved by Sturm in [47] is, perhaps, one of the most extraordinary. Here, we provide a brief account. Let \mathcal{S}^n denote the space of $n \times n$ symmetric matrices and \mathcal{S}_+^n denote the cone of $n \times n$ symmetric positive semidefinite matrices. Given a symmetric matrix $x \in \mathcal{S}^n$, we will denote its minimum eigenvalue by $\lambda_{\min}(x)$. Combining Theorem 3.3 and Lemma 3.6 of [47], we have the following result by Sturm.

Theorem (Sturm’s Error Bound). *Let $\{x_\epsilon \mid 0 < \epsilon \leq 1\} \subseteq \mathcal{S}^n$ be a **bounded** set, with the property that $\text{dist}(x_\epsilon, \mathcal{L} + a) \leq \epsilon$ and $\lambda_{\min}(x_\epsilon) \geq -\epsilon$, for all $\epsilon \in (0, 1]$. Then, there exists constants $\kappa > 0$ and $\gamma \geq 0$ such that*

$$\text{dist}(x_\epsilon, (\mathcal{L} + a) \cap \mathcal{S}_+^n) \leq \kappa \epsilon^{(2^{-\gamma})},$$

where γ satisfies $\gamma \leq \min\{n - 1, \dim \mathcal{L}^\perp \cap \{a\}^\perp, \text{span}(\mathcal{L} + a)\}$.

There are several remarkable aspects of Sturm’s bound. First of all, no regularity condition is assumed on the intersection $\mathcal{S}_+^n \cap (\mathcal{L} + a)$. The drawback is that instead of “ ϵ ”, we get “ ϵ^λ ” at the right-hand-side, for some $\lambda \in (0, 1]$. Error bounds of this type are called “Hölderian”. We emphasize, however, that although the bound is Hölderian, we know that the exponent is not smaller than 2^{1-n} . Finally, Sturm also showed how γ can be *computed*, which is a significant advancement in comparison to earlier Hölderian error bounds where it is typically very hard to estimate the exponent, see the comments after Theorems 11 and 13 in [35]. It turns out that γ depends on the *singularity degree* of the system $(\mathcal{S}_+^n, \mathcal{L}, a)$. The singularity degree is currently understood as the minimum number of steps that the facial reduction algorithm (by Borwein and Wolkowicz) needs in order to fully regularize $(\mathcal{S}_+^n, \mathcal{L}, a)$. Sturm was also the first to link an error bound result to facial reduction.

The research on facial reduction [6, 52, 38, 14] has shown that problems that do not satisfy Slater’s condition are quite numerous. For those problems, results such as Sturm’s error bound are useful to derive convergence results. For a recent application see the paper by Drusvyatskiy, Li and Wolkowicz [12], where Sturm’s bound plays an important role in deriving a rate of convergence of the alternate projection method for semidefinite feasibility problems that do not satisfy Slater’s condition.

Sturm’s error bound was later extended to a mixed system of semidefinite and second order cone constraints, see the chapter by Luo and Sturm [32]. Apart from that, it seems that no other paper attempted to establish further links between error bounds and facial reduction. It is not known, for instance, for which convex cones a result similar to Sturm’s error bound holds. This paper is, hopefully, a step towards answering this question.

1.1 The contributions of this paper

Two concepts are introduced in this paper: *amenable cones* and *facial residual functions*. The main goal is to show that for *amenable cones*, a result analogous to Sturm’s error bound holds. This article has the following contributions.

1. We define amenable cones (Definition 8) and prove that polyhedral cones, projectionally exposed cones, symmetric cones and strictly convex cones are amenable (Propositions 9 and 33). Roughly speaking, a cone \mathcal{K} is amenable if for every face $\mathcal{F} \trianglelefteq \mathcal{K}$, we have that $\text{dist}(x, \mathcal{K})$ provides a reasonable upper bound to $\text{dist}(x, \mathcal{F})$, when $x \in \text{span } \mathcal{F}$.

Furthermore, we observe that amenable cones are nice (Proposition 13) and show that amenability is preserved by direct products and by taking injective linear images (Proposition 11).

2. We define facial residual functions (Definition 16). Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$, where \mathcal{F}^* is the dual cone of \mathcal{F} . A facial residual function provides way of estimating $\text{dist}(x, \mathcal{F} \cap \{z\}^\perp)$ by using other available information such as $\text{dist}(x, \mathcal{K})$, $\text{dist}(x, \text{span } \mathcal{F})$ and $\langle x, z \rangle$. We prove that symmetric cones admit facial residual functions of the form $\kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}$ (Theorem 35).

Furthermore, facial residual functions can be easily constructed for direct products of amenable cones, provided that facial residual functions are known for each individual cone. Similarly, facial residual functions are also easily constructed for injective linear images of convex cones. See Proposition 17.

3. For *amenable cones*, we prove a novel error bound result that does not require constraint qualifications. The error bound is expressed as a composition of facial residual functions. The number of function compositions is connected to facial reduction, see Theorem 23 and Proposition 24. We then use Theorem 23 to provide two Hölderian error bounds for symmetric cones, see Theorem 37 and Proposition 38.

We also study error bounds for the intersection of cones and derive a result for the doubly nonnegative cone, see Proposition 41.

This article is divided as follows. In Section 2 we review several necessary tools. If the reader already has experience with the material therein, we recommend skipping most of Section 2. In Section 3, we introduce amenable cones and facial residual functions. In Section 4, we derive error bound results. The case of symmetric cones is discussed in Section 4.1. In Section 5, we summarize this work and point out future research directions.

2 Preliminaries

2.1 Basic definitions and assumptions

We recall our assumption that \mathcal{E} is equipped with some arbitrary inner product $\langle \cdot, \cdot \rangle$ and that the distance function $\text{dist}(\cdot, \cdot)$ is computed with respect the norm $\|\cdot\|$ induced by $\langle \cdot, \cdot \rangle$. For a direct product $\mathcal{E} = \mathcal{E}^1 \times \mathcal{E}^2$, we will assume that the inner product splits along the product so that

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle,$$

when $(x_1, x_2), (y_1, y_2) \in \mathcal{E}^1 \times \mathcal{E}^2$. By doing so, if $C^1 \subseteq \mathcal{E}^1$ and $C^2 \subseteq \mathcal{E}^2$, we have

$$\text{dist}((x_1, x_2), C^1 \times C^2) = \sqrt{\text{dist}(x_1, C^1)^2 + \text{dist}(x_2, C^2)^2}. \quad (2)$$

We remark that because all norms on a finite dimensional vector space are equivalent, our assumption that the norm is induced by the inner product is not very restrictive.

Let $C \subseteq \mathcal{E}$ be an arbitrary convex set. We will denote its relative interior, closure and linear span by $\text{ri } C$, $\text{cl } C$ and $\text{span } C$, respectively. We will write C^\perp for the orthogonal complement of C , which is defined as

$$C^\perp = \{x \in \mathcal{E} \mid \langle x, y \rangle = 0, \forall y \in C\}.$$

We recall that a set \mathcal{K} is a convex cone if for all nonnegative α, β and all $x, y \in \mathcal{K}$, we have $\alpha x + \beta y \in \mathcal{K}$. We will write \mathcal{K}^* for the dual cone of \mathcal{K} with respect the inner product $\langle \cdot, \cdot \rangle$. We have

$$\mathcal{K}^* = \{x \in \mathcal{E} \mid \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}.$$

We write $\text{lin } \mathcal{K}$ for the lineality space of \mathcal{K} , which is defined as

$$\text{lin } \mathcal{K} = \mathcal{K} \cap -\mathcal{K}.$$

A cone is said to be *pointed* if $\text{lin } \mathcal{K} = \{0\}$.

Let \mathcal{K} be a convex cone and $\mathcal{F} \subseteq \mathcal{K}$ be a convex cone contained in \mathcal{K} . \mathcal{F} is a *face* of \mathcal{K} if and only if the property below holds

$$x, y \in \mathcal{K}, x + y \in \mathcal{F} \Rightarrow x, y \in \mathcal{F}.$$

In this case, we write $\mathcal{F} \trianglelefteq \mathcal{K}$. If there exists $z \in \mathcal{K}^*$ such that $\mathcal{F} = \mathcal{K}^* \cap \{z\}^\perp$, then \mathcal{F} is said to be an *exposed face*. If all the faces of \mathcal{K} are exposed, then \mathcal{K} is said to be *facially exposed*.

We define the *conjugate face* of \mathcal{F} with respect to \mathcal{K} as

$$\mathcal{F}^\Delta = \mathcal{K}^* \cap \mathcal{F}^\perp.$$

Recall that if $\mathcal{F} \trianglelefteq \mathcal{K}$, then $\mathcal{F} = \mathcal{K} \cap \text{span } \mathcal{F}$. It follows that $\mathcal{F}^* = \text{cl}(\mathcal{K}^* + \mathcal{F}^\perp)$. A cone \mathcal{K} is said to be *nice* when the closure can be removed, that is, if the following property holds

$$\mathcal{F} \trianglelefteq \mathcal{K} \Rightarrow \mathcal{K}^* + \mathcal{F}^\perp \text{ is closed.}$$

Niceness plays an important role in the study of the facial structure of convex cones. It is also important in the context of optimality conditions, see, for example, Corollary 4.2 in the work of Borwein and Wolkowicz [8]. Regularization approaches such as facial reduction have very nice theoretical properties when the underlying cone is nice, see the works by Pataki [38, 37], related works by Tunçel and Wolkowicz [51], Roshchina [45] and by Roshchina and Tunçel [46].

In this work, we will need the following technical fact related to niceness.

Proposition 1. *Let \mathcal{K} be a closed convex cone such that \mathcal{K}^* is nice. Let $z \in \mathcal{K}^*$ and $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$. Then, $z \in \text{ri } \mathcal{F}^\Delta$.*

Proof. By definition of the conjugate face, we have $z \in \mathcal{F}^\Delta$. Suppose $z \notin \text{ri } \mathcal{F}^\Delta$. By invoking a separation theorem (e.g., Theorem 11.3 in [44]), we can find $x \in \mathcal{F}^{\Delta*}$ such that $\langle x, z \rangle = 0$ and $x \notin \mathcal{F}^{\Delta\perp}$. Then, the niceness of \mathcal{K}^* implies that

$$\mathcal{F}^{\Delta*} = \mathcal{K} + \mathcal{F}^{\Delta\perp}.$$

Therefore, $x = u + v$, where $u \in \mathcal{K}$ and $v \in \mathcal{F}^{\Delta\perp}$. Since $\langle x, z \rangle = 0$ and $z \in \mathcal{F}^\Delta$, we obtain that

$$\langle x, z \rangle = \langle u, z \rangle = 0,$$

that is, $u \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{F}^{\Delta\perp}$, we conclude that $x \in \mathcal{F}^{\Delta\perp}$, which is a contradiction. \square

If \mathcal{A} is a linear map, we will denote by \mathcal{A}^\top the corresponding adjoint map. The operator norm of \mathcal{A} will be denoted by $\|\mathcal{A}\| = \sup\{\|\mathcal{A}x\| \mid \|x\| \leq 1\}$. We will denote the set of nonnegative real numbers by \mathbb{R}_+ . We conclude this subsection with a reminder on our overall assumption on \mathcal{K} .

Assumption 1. *Throughout this paper, we assume that \mathcal{K} denotes a pointed closed convex cone.*

2.2 Hoffman's Lemma

Hoffman's Lemma can be stated in many different ways. For the sake of completeness we state below the format we will use throughout this article, which is a consequence of Hoffman's original result [21]. We recall that a set C is said to be *polyhedral* if it can be expressed as the solution set of a finite system of linear inequalities.

Theorem 2 (Hoffman's Lemma [21]). *Let $C_1, \dots, C_m \subseteq \mathcal{E}$ be polyhedral sets such that $\bigcap_{i=1}^m C_i \neq \emptyset$. There exists a positive constant κ such that*

$$\text{dist}(x, \bigcap_{i=1}^m C_i) \leq \kappa \sum_{i=1}^m \text{dist}(x, C_i), \quad \forall x \in \mathcal{E}.$$

2.3 Constraint qualifications

Although we will not assume that $(\mathcal{K}, \mathcal{L}, a)$ satisfies some constraint qualification, it is still necessary to discuss them. We say that $(\mathcal{K}, \mathcal{L}, a)$ satisfies *Slater's condition* if $(\text{ri } \mathcal{K}) \cap (\mathcal{L} + a) \neq \emptyset$. In this work, however, we will use a weaker constraint qualification called the *partial polyhedral Slater's (PPS) condition*, which is defined as follows.

Definition 3 (Partial Polyhedral Slater's condition). *Let $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$, where $\mathcal{K}^1, \mathcal{K}^2$ are closed convex cones such that \mathcal{K}^2 is polyhedral. We say that $(\mathcal{K}, \mathcal{L}, a)$ satisfies the Partial Polyhedral Slater's (PPS) condition if there exists $(x_1, x_2) \in \mathcal{L} + a$, such that $x_1 \in \text{ri } \mathcal{K}^1$ and $x_2 \in \mathcal{K}^2$.*

The PPS condition reflects the fact that we only care about having a relative interior point with respect to the part of the cone that we know that is not polyhedral. When a conic linear program satisfies the PPS condition, we get the same consequences of the usual Slater's condition: zero duality gap and, when the optimal value is finite, the dual problem is attained (e.g., Proposition 23 in [30]).

We will treat Slater's condition as a particular case of the PPS condition. In fact, if $(\mathcal{K}, \mathcal{L}, a)$ satisfies the Slater's condition, we can add an extra dummy coordinate, so that $(\mathcal{K} \times \{0\}, \mathcal{L} \times \{0\}, (a, 0))$ satisfies the PPS condition. Similarly, if \mathcal{K} is a polyhedral cone, we will also consider that the PPS condition holds, since we can also add an extra coordinate and take $\mathcal{K}^1 = \{0\}$.

2.4 Facial Reduction

The facial reduction algorithm originally appeared in [7] and was developed by Borwein and Wolkowicz as a way of dealing with conic convex programs that do not satisfy regularity conditions. More recently, Pataki [38] and Waki and Muramatsu [52] gave simplified descriptions of facial reduction for the special case of conic linear programs.

Suppose that $(\mathcal{K}, \mathcal{L}, a)$ is feasible. The basic idea is that there exists a unique face \mathcal{F}_{\min} of \mathcal{K} with the following properties:

- (a) $\mathcal{F}_{\min} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$,
- (b) $(\mathcal{F}_{\min}, \mathcal{L}, a)$ satisfies Slater's condition.

The first property means that the feasible region stays the same when we replace \mathcal{K} by \mathcal{F}_{\min} . It can be shown that properties (a) and (b) imply that \mathcal{F}_{\min} is the smallest face of \mathcal{K} containing $\mathcal{K} \cap (\mathcal{L} + a)$, see item (ii) of Proposition 2.2 in [36]. For this reason, \mathcal{F}_{\min} is called the *minimal face* of the problem $(\mathcal{K}, \mathcal{L}, a)$.

The classical facial reduction algorithm constructs a chain of faces as follows:

$$\mathcal{F}_{\min} = \mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K},$$

where $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ and $z_i \in \mathcal{F}_i^* \cap \mathcal{L} \cap \{a\}^\perp$, for $i = 1, \dots, \ell - 1$. The z_i are called *reducing directions* and computing them usually forms the bulk of the computational cost of facial reduction. There are quite a few recent works discussing how to compute those directions and how to do facial reduction efficiently and in a numerically stable manner [9, 29, 41, 19, 40, 34]. We regard finding each z_i as *one facial reduction step*.

2.4.1 Singularity degree and distance to polyhedrality

For a fixed $(\mathcal{K}, \mathcal{L}, a)$, we might need many facial reduction steps before \mathcal{F}_{\min} is reached. Motivated by that, we define the *singularity degree* of $(\mathcal{K}, \mathcal{L}, a)$ as the minimum number of facial reduction steps before \mathcal{F}_{\min} is reached. This definition of singularity degree is adopted, for example, in [28, 13] and in a recent survey [14]. However, the first usage of *singularity degree* in the context of facial reduction was due to Sturm in [47] and it had a slightly different meaning, see section 5.4 and footnote 3 in [30].

In particular, according to Sturm's definition, if $\mathcal{F}_{\min} = \{0\}$, then the singularity degree is zero. This makes perfect sense in the context of [47], since if $\mathcal{F}_{\min} = \{0\}$ then a Lipschitzian error bound holds for $(\mathcal{K}, \mathcal{L}, a)$, see page 1232 and Equation (2.5) therein. In this paper, we also make a similar observation in

Proposition 27. Nevertheless, it seems that most researchers are now inclined to define the singularity degree as in [28], so we shall also follow suit. In this case, if $\mathcal{F}_{\min} = \{0\}$, then the singularity degree should be at least one when $\dim \mathcal{K} \geq 1$.

We will denote the singularity degree of $(\mathcal{K}, \mathcal{L}, a)$ by $d_S(\mathcal{L}, a)$. Note that $d_S(\mathcal{L}, a)$ depends on \mathcal{K} , \mathcal{L} and a . However, it is possible to give a bound on the singularity degree that does not depend on \mathcal{L} nor a . In what follows, if we have a chain of faces $\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1$, the length of the chain is defined to be ℓ . Then, the *longest chain of faces of \mathcal{K}* is denoted by $\ell_{\mathcal{K}}$ and is defined as the length of the longest chain of face of \mathcal{K} such that all inclusions are strict. We have that $d_S(\mathcal{L}, a) \leq \ell_{\mathcal{K}}$.

Sometimes it is enough to find a face that satisfies a less strict constraint qualification. In particular, the *FRA-Poly* algorithm in [30] is divided in two phases. In the first phase, a face satisfying the PPS condition is found and in the second phase, \mathcal{F}_{\min} is computed. In many cases of interest, this two-phase strategy leads to better bounds on the singularity degree than the classical facial reduction algorithm, see for instance, Table 1 in [30]. We will recall here a few definitions and results from [30].

Definition 4. *The distance to polyhedrality $\ell_{\text{poly}}(\mathcal{K})$ is the length minus one of the longest strictly ascending chain of nonempty faces $\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1$ which satisfies:*

- (a) \mathcal{F}_ℓ is polyhedral;
- (b) \mathcal{F}_j is not polyhedral for $j < \ell$.

See Example 1 in [30] for the values of $\ell_{\text{poly}}(\mathcal{K})$ for some common cones. In particular, if \mathcal{K} is polyhedral, we have $\ell_{\text{poly}}(\mathcal{K}) = 0$. In this paper, we will compute a bound for $\ell_{\text{poly}}(\mathcal{K})$ when \mathcal{K} is a symmetric cone, see Remark 39. The next result gives an upper bound to the number of facial reduction steps that are necessary before a face satisfying the PPS condition is found.

Proposition 5. *Let $\mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^s$, where each \mathcal{K}^i is a pointed closed convex cone. Suppose $(\mathcal{K}, \mathcal{L}, a)$ is feasible. There is a chain of faces*

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

of length ℓ and vectors $(z_1, \dots, z_{\ell-1})$ satisfying the following properties.

- (i) $\ell - 1 \leq \sum_{i=1}^s \ell_{\text{poly}}(\mathcal{K}^i) \leq \dim \mathcal{K}$
- (ii) For all $i \in \{1, \dots, \ell - 1\}$, we have

$$\begin{aligned} z_i &\in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp, \\ \mathcal{F}_{i+1} &= \mathcal{F}_i \cap \{z_i\}^\perp. \end{aligned}$$

- (iii) $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ and $(\mathcal{F}_\ell, \mathcal{L}, a)$ satisfies the PPS condition.

Proof. As mentioned previously, *FRA-Poly* is divided in two phases [30]. In Phase 1, it computes the directions z_i as in item (ii). Then, it ends with a face satisfying the PPS condition, as in item (iii). The bound on the number of directions follows from item (i) of Proposition 8 in [30] and from the fact that $\ell_{\text{poly}}(\mathcal{K}^i) \leq \dim \mathcal{K}^i$ for every i .¹ \square

In this paper, we define the quantity $d_{\text{PPS}}(\mathcal{L}, a)$, which is the minimum number of reduction directions needed to find a face $\mathcal{F} \trianglelefteq \mathcal{K}$ that contains $\mathcal{K} \cap (\mathcal{L} + a)$ and such that $(\mathcal{F}, \mathcal{L}, a)$ satisfies the PPS condition.

¹Note that if $\mathcal{F} \trianglelefteq \mathcal{K}$ and $\mathcal{F} \subsetneq \mathcal{K}$, then $\dim \mathcal{F} < \dim \mathcal{K}$.

2.5 Distance functions and generalized eigenvalue functions

In this subsection, we will briefly discuss a generalization of the concept of eigenvalues introduced by Renegar in [43]. Let \mathcal{K} be a pointed closed convex cone and $d \in \text{ri}\mathcal{K}$, then the generalized eigenvalue function of \mathcal{K} with respect to d is

$$\lambda_{\mathcal{K}}^d(x) = \inf\{t \mid x - td \notin \mathcal{K}\}. \quad (3)$$

With that, we have $x - \lambda_{\mathcal{K}}^d(x)d \in \mathcal{K}$ for all $x \in \text{span}\mathcal{K}$. We also have

$$\begin{aligned} x \in \mathcal{K} &\iff \lambda_{\mathcal{K}}^d(x) \geq 0 \\ x \in \text{ri}\mathcal{K} &\iff \lambda_{\mathcal{K}}^d(x) > 0. \end{aligned}$$

We observe that if $\mathcal{K} = \mathcal{S}_+^n$ and d is the $n \times n$ identity matrix, then $\lambda_{\min}(x) = \lambda_{\mathcal{K}}^d(x)$, for all $x \in \mathcal{E}$. Renegar proved the following result in [43], see Proposition 2.1 therein.

Proposition 6. *Let \mathcal{K} be a closed pointed convex cone and $d \in \text{ri}\mathcal{K}$. Then, the function $\lambda_{\mathcal{K}}^d(x)$ is concave and Lipschitz continuous over $\text{span}\mathcal{K}$.*

The Lipschitz continuity of $\lambda_{\mathcal{K}}^d(\cdot)$ is important because it implies that it is reasonable to use $\lambda_{\mathcal{K}}^d(x)$ as an indirect way of measuring $\text{dist}(x, \mathcal{K})$. This idea is expressed in the next proposition.

Proposition 7. *Let $d \in \text{ri}\mathcal{K}$. There are positive constants κ_1 and κ_2 such that*

$$\kappa_1 \max(-\lambda_{\mathcal{K}}^d(x), 0) \leq \text{dist}(x, \mathcal{K}) \leq \kappa_2 \max(-\lambda_{\mathcal{K}}^d(x), 0), \quad \forall x \in \text{span}\mathcal{K}.$$

Proof. If $x \in \mathcal{K}$, then we have $\lambda_{\mathcal{K}}^d(x) \geq 0$ and $\text{dist}(x, \mathcal{K}) = 0$, so we are done. Suppose that $x \notin \mathcal{K}$. Since $x - \lambda_{\mathcal{K}}^d(x)d \in \mathcal{K}$, we have

$$\text{dist}(x, \mathcal{K}) \leq \|x - (x - \lambda_{\mathcal{K}}^d(x)d)\| = -\lambda_{\mathcal{K}}^d(x)\|d\|.$$

Let $v \in \mathcal{K}$ be such that $\text{dist}(x, \mathcal{K}) = \|x - v\|$. Since $x \notin \mathcal{K}$, v belongs to the relative boundary of \mathcal{K} , so that

$$\lambda_{\mathcal{K}}^d(v) = 0, \quad \lambda_{\mathcal{K}}^d(x) < 0.$$

Using the Lipschitz continuity of $\lambda_{\mathcal{K}}^d(\cdot)$, there is some $\tilde{\kappa}$ such that

$$-\lambda_{\mathcal{K}}^d(x) = |\lambda_{\mathcal{K}}^d(x) - \lambda_{\mathcal{K}}^d(v)| \leq \|x - v\|\tilde{\kappa} = \text{dist}(x, \mathcal{K})\tilde{\kappa},$$

for all $x \in \text{span}\mathcal{K}$. We conclude that the proposition holds with $\kappa_1 = 1/\tilde{\kappa}$ and $\kappa_2 = \|d\|$. \square

3 Amenable cones and facial residual functions

In this section, we introduce the two notions that are the cornerstones of this work: *amenable cones* and *facial residual functions*.

3.1 Amenable cones

Definition 8. *A closed convex cone \mathcal{K} is said to be amenable if for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there is a positive constant κ such that*

$$\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K}), \quad \forall x \in \text{span}\mathcal{F}. \quad (4)$$

Next, we prove that a few common cones are amenable. The proof that symmetric cones are amenable will be deferred to Proposition 33. We recall that a pointed cone \mathcal{K} is said to be *strictly convex* if the only faces besides \mathcal{K} and $\{0\}$ are extreme rays (i.e., one dimensional faces). Also, \mathcal{K} is said to be *projectionally exposed* if for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there is a projection (not necessarily orthogonal) \mathcal{P} such that $\mathcal{P}(\mathcal{K}) = \mathcal{F}$, see [49] by Sung and Tam. Here, we remind that a projection is a linear map $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{E}$ satisfying $\mathcal{P}^2 = \mathcal{P}$. If for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there is an *orthogonal* projection \mathcal{P} such that $\mathcal{P}(\mathcal{K}) = \mathcal{F}$, then \mathcal{K} is said to be *orthogonal projectionally exposed*.

Proposition 9. *The following cones are amenable.*

(i) *Projectionally exposed cones. In particular, if $\mathcal{F} \trianglelefteq \mathcal{K}$ and \mathcal{P} is a projection satisfying $\mathcal{P}(\mathcal{K}) = \mathcal{F}$, then (4) is satisfied with $\kappa = \|\mathcal{P}\|$.*

(ii) *Polyhedral cones.*

(iii) *Strictly convex cones.*

Proof. (i) Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{E}$ be a projection map such that $\mathcal{P}(\mathcal{K}) = \mathcal{F}$. Then, we have $\mathcal{P}(\text{span } \mathcal{K}) = \text{span } \mathcal{F}$. Furthermore, since $\mathcal{P}^2 = \mathcal{P}$, we have $\mathcal{P}(\text{span } \mathcal{F}) = \text{span } \mathcal{F}$.

Now, let $x \in \text{span } \mathcal{F}$ and let $y \in \mathcal{K}$ be such that $\text{dist}(x, \mathcal{K}) = \|x - y\|$. Then, since $\mathcal{P}(y) \in \mathcal{F}$ and $\mathcal{P}(x) = x$, we have

$$\text{dist}(x, \mathcal{F}) \leq \|x - \mathcal{P}(y)\| = \|\mathcal{P}(x) - \mathcal{P}(y)\| \leq \|\mathcal{P}\| \|x - y\|,$$

where $\|\mathcal{P}\|$ is the operator norm of \mathcal{P} . This shows that (4) is satisfied with $\kappa = \|\mathcal{P}\|$.

(ii) Let \mathcal{K} be a polyhedral cone and \mathcal{F} be a face of \mathcal{K} . Since \mathcal{K} is polyhedral, \mathcal{F} must be an exposed face (e.g., Corollary 2 in [50]), therefore there exists $z \in \mathcal{K}^*$ such that

$$\mathcal{F} = \mathcal{K} \cap \{z\}^\perp.$$

By Hoffman's Lemma (Theorem 2), there is a positive constant κ such that

$$\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K}) + \kappa \text{dist}(x, \{z\}^\perp).$$

We now observe that if $x \in \text{span } \mathcal{F}$ then $x \in \{z\}^\perp$. Therefore,

$$\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K}), \quad \forall x \in \text{span } \mathcal{F}.$$

(iii) Let \mathcal{K} be a strictly convex cone and let \mathcal{F} be a proper face of \mathcal{K} . If $\mathcal{F} = \{0\}$, since $\text{span } \mathcal{F} = \{0\}$, it is enough to take $\kappa = 1$.

We move on to the case where $\mathcal{F} = \{\alpha v \mid \alpha \geq 0\}$, for some nonzero $v \in \mathcal{K}$. We assume, without loss of generality, that $\|v\| = 1$. Then $\text{span } \mathcal{F} = \{\alpha v \mid \alpha \in \mathbb{R}\}$. Note that if $\alpha \geq 0$, we have

$$\text{dist}(\alpha v, \mathcal{F}) = \text{dist}(\alpha v, \mathcal{K}) = 0.$$

So suppose that $\alpha < 0$. Let $u \in \mathcal{K}$ be such that $\text{dist}(-v, \mathcal{K}) = \|u + v\|$. We have

$$\text{dist}(\alpha v, \mathcal{F}) = -\alpha, \quad \text{dist}(\alpha v, \mathcal{K}) = -\alpha \|u + v\|.$$

It follows that for every $x \in \text{span } \mathcal{F}$, we have

$$\text{dist}(x, \mathcal{F}) \leq \frac{1}{\|u + v\|} \text{dist}(x, \mathcal{K}).$$

We remark that, since \mathcal{K} is pointed, $-v \notin \mathcal{K}$, so $\|u + v\| > 0$. □

Remark 10. *In Definition 8, the constant κ may depend on \mathcal{F} . Nevertheless, there are cones that admit a finite “universal” constant $\kappa_{\mathcal{K}}$ depending only on \mathcal{K} and such that (4) holds for all faces. For example, we will see in Proposition 33 that $\kappa_{\mathcal{K}} = 1$ is enough for symmetric cones. Also, if \mathcal{K} is polyhedral, since the number of faces is finite, we may pick a constant $\kappa_{\mathcal{F}}$ for each face and let $\kappa_{\mathcal{K}}$ be the maximum among the $\kappa_{\mathcal{F}}$. Finally, if \mathcal{K} is a pointed strictly convex cone, the proof of item (iii) of Proposition 9 shows that we may take*

$$\kappa_{\mathcal{K}} = \sup_{v \in \mathcal{K}, \|v\|=1} \text{dist}(-v, \mathcal{K})^{-1}$$

as a universal constant for \mathcal{K} . Since $\text{dist}(\cdot, \mathcal{K})$ is a continuous function, $\kappa_{\mathcal{K}}$ must be finite.

However, if \mathcal{K} is a projectionally exposed cone, the proof of item (i) only shows that the constant κ can be taken to be the operator norm of the projection \mathcal{P} . Because \mathcal{P} is not necessarily orthogonal, $\|\mathcal{P}\|$ can be quite large and we do not know whether a finite universal constant exists. More generally, it is not known whether arbitrary amenable cones admit finite universal constants.

We note that amenability is preserved by simple operations, see Appendix A for proofs.

Proposition 11 (Preservation of amenability). *The following hold.*

- (i) *If \mathcal{K}^1 and \mathcal{K}^2 are amenable cones then $\mathcal{K}^1 \times \mathcal{K}^2$ is amenable.*
- (ii) *If \mathcal{K} is amenable and \mathcal{A} is an injective linear map, then $\mathcal{A}(\mathcal{K})$ is amenable.*

Next, we examine the connection between amenability and related concepts. Two sets S_1, S_2 are said to have *subtransversal intersection* at $\bar{x} \in S_1 \cap S_2$ if there exists a positive κ and a neighbourhood V of \bar{x} such that

$$\text{dist}(x, S_1 \cap S_2) \leq \kappa(\text{dist}(x, S_1) + \text{dist}(x, S_2)), \quad \forall x \in V. \quad (5)$$

Subtransversality is discussed extensively in Ioffe's book [22], see Chapter 7 and Definition 7.5 therein. Another related concept is *bounded linear regularity*. The sets S_1, S_2 are said to be boundedly linearly regular, if $S_1 \cap S_2 \neq \emptyset$ and for every bounded set $B \subseteq \mathcal{E}$ there exists κ_B such that

$$\text{dist}(x, S_1 \cap S_2) \leq \kappa_B \max(\text{dist}(x, S_1), \text{dist}(x, S_2)), \quad \forall x \in B.$$

See, for example, the work by Bauschke, Borwein and Li [5]. Now, recalling that $\mathcal{F} = \mathcal{K} \cap \text{span } \mathcal{F}$ holds for every face $\mathcal{F} \trianglelefteq \mathcal{K}$, we obtain the following proposition, see Appendix A for the proof.

Proposition 12. *Let \mathcal{K} be a closed convex cone and $\mathcal{F} \trianglelefteq \mathcal{K}$. The following are equivalent.*

- (i) *There exists a positive constant κ such that $\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K})$ holds for every $x \in \text{span } \mathcal{F}$.*
- (ii) *\mathcal{K} and $\text{span } \mathcal{F}$ intersect at 0 subtransversally.*
- (iii) *\mathcal{K} and $\text{span } \mathcal{F}$ are boundedly linearly regular.*

The overall conclusion is that \mathcal{K} is amenable if and only if every face $\mathcal{F} \trianglelefteq \mathcal{K}$ is such that \mathcal{K} and $\text{span } \mathcal{F}$ are boundedly linearly regular or intersect subtransversally at the origin.

This is good news because, thanks to Theorem 10 in [5], it turns out that two (not necessarily pointed) convex cones $\mathcal{K}_1, \mathcal{K}_2$ are boundedly linearly regular if and only if $-\mathcal{K}_1^* - \mathcal{K}_2^*$ is closed and there is $\alpha > 0$ such that

$$U \cap (-\mathcal{K}_1^* - \mathcal{K}_2^*) \subseteq \alpha((-\mathcal{K}_1^* \cap U) + (-\mathcal{K}_2^* \cap U)), \quad (6)$$

where $U = \{x \in \mathcal{E} \mid \|x\| \leq 1\}$ is the unit ball. This leads us to the following result.

Proposition 13. *Amenable cones are nice and, in particular, are facially exposed.*

Proof. Let $\mathcal{F} \trianglelefteq \mathcal{K}$ be a face of an amenable cone \mathcal{K} . By the preceding discussion, \mathcal{K} and $\text{span } \mathcal{F}$ are boundedly linearly regular, so Theorem 10 in [5] implies that $-\mathcal{K}^* + \mathcal{F}^\perp$ is closed. Therefore, $\mathcal{K}^* + \mathcal{F}^\perp$ is closed and \mathcal{K} must be nice. To conclude, we recall that Pataki proved in Theorem 3 of [37] that all nice cones are facially exposed. \square

Remark 14. *Propositions 9 and 13 together imply that projectionally exposed cones are nice. This has been proved earlier by Permenter [39].*

We do not know whether nice cones must necessarily be amenable. At this moment, this seems unlikely because it would imply that the condition given in (6) (also called property (G) in [5]) is somehow superfluous when $\mathcal{K}_1 = \mathcal{K}$, $\mathcal{K}_2 = \text{span } \mathcal{F}$ and $\mathcal{F} \trianglelefteq \mathcal{K}$. Nevertheless, a nice but not amenable cone remains to be found.

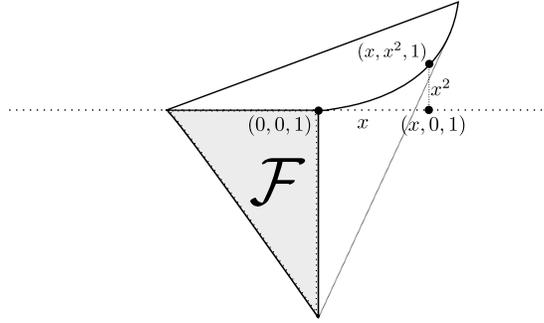


Figure 1: A cone that is not amenable.

Recently, Roshchina and Tunçel introduced the concept of *tangentially exposed cones* in [46] and they showed that nice cones are always tangentially exposed, although the converse does not hold in general, see Example 2 in [46]. As amenable cones are nice, they must be tangentially exposed as well.

We conclude this section by showing how amenability might break down.

Example 15 (A non-amenable cone, Figure 1). *Let $C \subseteq \mathbb{R}^2$ be the smallest closed convex set containing*

$$\{(x, x^2) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(x, 0) \in \mathbb{R}^2 \mid -1 \leq x \leq 0\}.$$

Let $\mathcal{K} \subseteq \mathbb{R}^3$ be the smallest closed convex cone containing $C \times \{1\}$. As seen in Figure 1, \mathcal{K} is not facially exposed, so \mathcal{K} cannot be amenable, because of Proposition 13. Nevertheless, we will check precisely why the amenability condition fails.

Let $\hat{C} = \{(x, 0) \mid -1 \leq x \leq 0\}$ and \mathcal{F} be the smallest closed convex cone containing $\hat{C} \times \{1\}$. Since \hat{C} is a face of C , \mathcal{F} is a face of \mathcal{K} . We have

$$\mathcal{F} = \{(x, 0, z) \in \mathbb{R}^3 \mid 0 \leq -x \leq z\}$$

Now, let $x \in (0, 1]$. We consider the point $(x, 0, 1) \in \text{span } \mathcal{F}$. The projection of $(x, 0, 1)$ on \mathcal{F} is $(0, 0, 1)$. Therefore, $\text{dist}((x, 0, 1), \mathcal{F}) = x$. However,

$$\text{dist}((x, 0, 1), \mathcal{K}) \leq \|(x, 0, 1) - (x, x^2, 1)\| = x^2.$$

Therefore, the quotient $\text{dist}((x, 0, 1), \mathcal{F})/\text{dist}((x, 0, 1), \mathcal{K})$ gets unbounded as x goes to zero, thus showing that Definition 8 can never be satisfied for any positive constant κ .

3.2 Facial residual functions

Let \mathcal{F} be a face of \mathcal{K} , $z \in \mathcal{F}^*$ and $\hat{\mathcal{F}} = \mathcal{F} \cap \{z\}^\perp$. The motivation for the definition of *facial residual functions* comes from the fact that if for some x we have

$$\text{dist}(x, \mathcal{K}) = \langle x, z \rangle = \text{dist}(x, \text{span } \mathcal{F}) = 0$$

then it must be the case that $x \in \hat{\mathcal{F}}$. This is because for any face $\mathcal{F} \trianglelefteq \mathcal{K}$ we have $\mathcal{F} = \mathcal{K} \cap \text{span } \mathcal{F}$. If x almost satisfies the equations above, we would hope that the distance between x and $\hat{\mathcal{F}}$ would also be small. Unfortunately, that is not what happens in general and we usually have to take into account the norm of x . Accordingly, we settle for the less ambitious goal that $\text{dist}(x, \mathcal{K})$ should be bounded by some function $\psi_{\mathcal{F}, z}$ that also depends on the norm of x . However, this dependency is not completely arbitrary and we require $\psi_{\mathcal{F}, z}$ to be zero if x belongs to $\hat{\mathcal{F}}$.

Definition 16 (Facial residual functions). *Let \mathcal{K} be a closed convex cone and \mathcal{F} a face of \mathcal{K} . Let $z \in \mathcal{F}^*$ and $\hat{\mathcal{F}} = \mathcal{F} \cap \{z\}^\perp$. Suppose that $\psi_{\mathcal{F}, z} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the following properties:*

(i) $\psi_{\mathcal{F},z}$ is nonnegative, monotone nondecreasing in each argument and $\psi(0, \alpha) = 0$ for every $\alpha \in \mathbb{R}_+$.

(ii) whenever $x \in \text{span } \mathcal{K}$ satisfies the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$$

we have:

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F},z}(\epsilon, \|x\|).$$

Then, $\psi_{\mathcal{F},z}$ is said to be facial residual function (FRF) for \mathcal{F} and z .

It not obvious whether facial residual functions always exist, so will now take a look at this issue. Let $\bar{\psi}_{\mathcal{F},z}(\epsilon, \|x\|)$ be the optimal value of the following problem.

$$\begin{aligned} & \sup_{v \in \text{span } \mathcal{K}} \text{dist}(v, \hat{\mathcal{F}}) & (\text{P}) \\ \text{subject to} & \text{dist}(v, \mathcal{K}) \leq \epsilon \\ & \text{dist}(v, \text{span } \mathcal{F}) \leq \epsilon \\ & \langle v, z \rangle \leq \epsilon \\ & \|v\| \leq \|x\| \end{aligned}$$

The functions $\text{dist}(\cdot, \mathcal{K})$ and $\text{dist}(\cdot, \text{span } \mathcal{F})$ are continuous convex functions. Since x is fixed in (P), the feasible region of (P) is a compact convex set, due to the presence of the constraint “ $\|v\| \leq \|x\|$ ”. In particular, $\bar{\psi}_{\mathcal{F},z}(\epsilon, \|x\|)$ is finite and nonnegative. Furthermore, increasing either ϵ or $\|x\|$ enlarges the feasible region, so that $\bar{\psi}_{\mathcal{F},z}(\cdot, \cdot)$ is monotone nondecreasing in each argument. If $\epsilon = 0$ and v is feasible for (P) it must be the case that $v \in \hat{\mathcal{F}}$, so $\text{dist}(v, \hat{\mathcal{F}}) = 0$. Therefore, $\epsilon = 0$ implies $\bar{\psi}_{\mathcal{F},z}(0, \alpha) = 0$ for every $\alpha \in \mathbb{R}_+$. This shows that $\bar{\psi}_{\mathcal{F},z}(\cdot, \cdot)$ is indeed a facial residual function and we will call $\bar{\psi}_{\mathcal{F},z}$ the *canonical facial residual function* for \mathcal{F} and z . It is the best possible, since, by definition, $\bar{\psi}_{\mathcal{F},z}(\epsilon, \|x\|) \leq \psi_{\mathcal{F},z}(\epsilon, \|x\|)$, if $\psi_{\mathcal{F},z}$ is another facial residual function.

The existence of canonical facial residual function shows that, in principle, error bounds for amenable cones can always be established, see Theorem 23. Unfortunately, computing $\bar{\psi}_{\mathcal{F},z}$ is complicated since it boils down to *maximization* of a convex function over a convex set. It is also likely that $\bar{\psi}_{\mathcal{F},z}$ will have no easy formula as a function of ϵ and $\|x\|$.

In face of these difficulties, one of the goals in this paper is to show that many useful cones admit simpler facial residual functions. For example, we will show in Theorem 35 that for symmetric cones, we can use $\kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}$ as a facial residual function, where κ is a positive constant.

We say that a function $\tilde{\psi}_{\mathcal{F},z}$ is a *positive rescaling* of $\psi_{\mathcal{F},z}$ if there are positive constants M_1, M_2, M_3 such that

$$\tilde{\psi}_{\mathcal{F},z}(\epsilon, \|x\|) = M_3\psi_{\mathcal{F},z}(M_1\epsilon, M_2\|x\|).$$

Two functions ψ_1, ψ_2 are the same *up to positive rescaling* if ψ_1 is equal to a positive rescaling of ψ_2 . It is possible that $\psi_{\mathcal{F},z}$ is different for each choice of \mathcal{F} and z . However, in a few cases of interest such as symmetric cones, $\psi_{\mathcal{F},z}$ can be taken to be a positive rescaling of the same fixed facial residual function, see Theorem 35.

In the next proposition, we will see that when we perform a simple operation on a cone \mathcal{K} , we may still use the same facial residual functions for \mathcal{K} if we positive rescale them. As the proof is long but routine, it is deferred to Appendix A.

Proposition 17. *The following hold.*

(i) Let $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$, where $\mathcal{K}^1 \subseteq \mathcal{E}^1, \mathcal{K}^2 \subseteq \mathcal{E}^2$ are amenable cones.

Let $\mathcal{F} \trianglelefteq \mathcal{K}$ and $z \in \mathcal{F}^*$. Write $\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^2$ where $\mathcal{F}_1 \trianglelefteq \mathcal{K}^1, \mathcal{F}_2 \trianglelefteq \mathcal{K}^2$. Write $z = (z_1, z_2)$ with $z_1 \in (\mathcal{F}^1)^*$ and $z_2 \in (\mathcal{F}^2)^*$.

Let $\psi_{\mathcal{F}_1, z_1}, \psi_{\mathcal{F}_2, z_2}$ be facial residual functions for \mathcal{F}_1, z_1 and \mathcal{F}_2, z_2 , respectively.

Then, there is a positive rescaling of $\psi_{\mathcal{F}_1, z_1} + \psi_{\mathcal{F}_2, z_2}$ that is also a facial residual function for \mathcal{F}, z .

(ii) Let \mathcal{A} be an injective linear map.

Let $\mathcal{A}(\mathcal{F}) \trianglelefteq \mathcal{A}(\mathcal{K})$, where $\mathcal{F} \trianglelefteq \mathcal{K}$. Let $z \in (\mathcal{A}(\mathcal{F}))^*$.

Let $\psi_{\mathcal{F}, \mathcal{A}^\top z}$ be a facial residual function for $\mathcal{F}, \mathcal{A}^\top z$.

Then, there is a positive rescaling of $\psi_{\mathcal{F}, \mathcal{A}^\top z}$ that is a facial residual function for $\mathcal{A}(\mathcal{F}), z$.

We will now show that polyhedral cones admit facial residual functions that are linear in ϵ and do not depend on $\|x\|$.

Proposition 18. *Let \mathcal{K} be a polyhedral cone and \mathcal{F} a face of \mathcal{K} . Let $z \in \mathcal{F}^*$ and $\hat{\mathcal{F}} = \mathcal{F} \cap \{z\}^\perp$. Then, there is a positive constant κ (depending on $\mathcal{K}, \mathcal{F}, z$) such that whenever x satisfies the inequalities*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$$

we have:

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \kappa \epsilon.$$

That is, we can take $\psi_{\mathcal{F}, z}(\epsilon, \|x\|) = \kappa \epsilon$ as a facial residual function for \mathcal{F} and z .

Proof. Suppose x satisfies $\text{dist}(x, \mathcal{K}) \leq \epsilon, \langle x, z \rangle \leq \epsilon$ and $\text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$. The face \mathcal{F} can be written as the nonempty intersection of two polyhedral sets

$$\mathcal{F} = \mathcal{K} \cap \text{span } \mathcal{F}.$$

Therefore, from Hoffman's Lemma (Theorem 2), there exists κ_1 (not depending on x) such that

$$\text{dist}(x, \mathcal{F}) \leq \kappa_1 (\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F})) \leq 2\epsilon \kappa_1.$$

Therefore, there exists v such that $\|v\| \leq 2\epsilon \kappa_1$ such that $x + v \in \mathcal{F}$. Since $\langle x, z \rangle \leq \epsilon$ and $\langle x + v, z \rangle \geq 0$, we obtain

$$-2\epsilon \kappa_1 \|z\| \leq \langle x, z \rangle \leq \epsilon(1 + 2\epsilon \kappa_1 \|z\|).$$

Therefore,

$$|\langle x, z \rangle| \leq \epsilon(1 + 2\kappa_1 \|z\|) \tag{7}$$

The face $\hat{\mathcal{F}}$ can be written as the nonempty intersection of three polyhedral sets

$$\hat{\mathcal{F}} = \mathcal{K} \cap \text{span } \mathcal{F} \cap \{z\}^\perp.$$

From Hoffman's Lemma, there is $\kappa_2 > 0$ such that

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \kappa_2 (\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F}) + \text{dist}(x, \{z\}^\perp)).$$

Note that $\text{dist}(x, \{z\}^\perp) = |\langle x, z \rangle| / \|z\|$. From (7), we obtain

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \epsilon \kappa_2 \left(2 + \frac{1 + 2\kappa_1 \|z\|}{\|z\|} \right).$$

We then take $\kappa = \kappa_2 (2 + \frac{1 + 2\kappa_1 \|z\|}{\|z\|})$ to conclude the proof. \square

Proposition 18 is not useful by itself, since we can readily obtain error bounds directly from Hoffman's Lemma. However, there are cases where we have to deal with the direct product of polyhedral cones and nonpolyhedral cones. Then, since we can take as FRFs the sum of the individual FRFs (item (i) of Proposition 17), it becomes clear that the polyhedral cones only give linear contributions to the overall sum. This means that all source of non-Lipschitzness and nastiness in the error bounds must come from the nonpolyhedral parts, which is unsurprising but serves as a sanity check for the theory developed here.

4 Error bounds

We recall that our goal is to obtain error bounds for $(\mathcal{K}, \mathcal{L}, a)$ without assuming regularity conditions. Namely, given some arbitrary x we would like to bound $\text{dist}(x, \mathcal{K} \cap (\mathcal{L} + a))$ by some quantity involving $\text{dist}(x, \mathcal{K})$ and $\text{dist}(x, \mathcal{L} + a)$.

Our first result is an error bound that is useful in situations where, for some reason, we know a face \mathcal{F} of \mathcal{K} that contains the feasible region of $(\mathcal{K}, \mathcal{L}, a)$ and such that $(\mathcal{F}, \mathcal{L}, a)$ satisfies the PPS condition. In particular, this covers the case where we know \mathcal{F}_{\min} , which is the minimal face of \mathcal{K} that contains $\mathcal{K} \cap (\mathcal{L} + a)$.

Proposition 19 (Error bound for when a face satisfying the PPS condition is known). *Let \mathcal{K} be a closed convex amenable cone and let \mathcal{F} denote a face of \mathcal{K} containing $(\mathcal{L} + a) \cap \mathcal{K}$ and such that the PPS condition is satisfied.*

Then, there is a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a, \mathcal{F}$) such that whenever $x \in \text{span } \mathcal{K}$ and ϵ satisfy the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa \|x\| \epsilon + \kappa \epsilon.$$

Proof. Since the PPS condition is satisfied for $(\mathcal{F}, \mathcal{L}, a)$, at least one of the statements below must be true (see Section 2.3).

1. \mathcal{F} is polyhedral.
2. $(\text{ri } \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$.
3. $\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^2$ where \mathcal{F}^1 and \mathcal{F}^2 are closed convex cones such that \mathcal{F}^2 is polyhedral and

$$((\text{ri } \mathcal{F}^1) \times \mathcal{F}^2) \cap (\mathcal{L} + a) \neq \emptyset.$$

Recall that cases 1. and 2. can be seen as special cases of 3. if we add extra dummy coordinates. Therefore, without loss of generality, we assume that 3. holds.

Due to the amenability of \mathcal{K} , there is κ_1 such that

$$\text{dist}(z, \mathcal{F}) \leq \kappa_1 \text{dist}(z, \mathcal{K}), \quad \forall z \in \text{span } \mathcal{F}. \quad (8)$$

Now, let u be such that $\|u\| \leq \epsilon$ and $x + u \in \text{span } \mathcal{F}$. We have

$$\text{dist}(x + u, \mathcal{F}) \leq \kappa_1 \text{dist}(x + u, \mathcal{K}) \leq 2\kappa_1 \epsilon.$$

Then, observing that $\text{dist}(x, \mathcal{F}) \leq \text{dist}(-u, \mathcal{F}) + \text{dist}(x + u, \mathcal{F})$, we obtain that

$$\text{dist}(x, \mathcal{F}) \leq (1 + 2\kappa_1)\epsilon.$$

Next, since $\text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$ and $\mathcal{F} \subseteq (\text{span } \mathcal{F}^1) \times \mathcal{F}^2$, we conclude that

$$\text{dist}(x, (\text{span } \mathcal{F}^1) \times \mathcal{F}^2) \leq \text{dist}(x, \mathcal{F}) \leq (1 + 2\kappa_1)\epsilon.$$

Let $\hat{\kappa}_1 = (1 + 2\kappa_1)$. Since \mathcal{F}^2 is a polyhedral cone and $(\mathcal{L} + a) \cap ((\text{span } \mathcal{F}^1) \times \mathcal{F}^2) \neq \emptyset$, we can invoke Hoffman's Lemma (Theorem 2) which tells us that there exists a constant κ_2 such that whenever $x \in \text{span } \mathcal{K}$ satisfies

$$\text{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \text{dist}(x, (\text{span } \mathcal{F}^1) \times \mathcal{F}^2) \leq \hat{\kappa}_1 \epsilon$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap ((\text{span } \mathcal{F}^1) \times \mathcal{F}^2)) \leq \epsilon \kappa_2.$$

Therefore, there is y such that $\|y\| \leq \epsilon \kappa_2$ and

$$x + y \in (\mathcal{L} + a) \cap ((\text{span } \mathcal{F}^1) \times \mathcal{F}^2).$$

Since $x + y \in (\text{span } \mathcal{F}^1) \times \mathcal{F}^2$, we can write $x + y = (z_1, z_2)$, with $z_1 \in \text{span } \mathcal{F}^1$ and $z_2 \in \mathcal{F}^2$. By (8) and since $x + y$ lies in $\text{span } \mathcal{F}$, we have

$$\text{dist}(z_1, \mathcal{F}^1) \leq \text{dist}(x + y, \mathcal{F}) \leq \kappa_1 \text{dist}(x + y, \mathcal{K}) \leq \epsilon(\kappa_1 + \kappa_1 \kappa_2). \quad (9)$$

Since the PPS condition is satisfied, there exists

$$d = (d_1, d_2) \in ((\text{ri } \mathcal{F}^1) \times \mathcal{F}^2) \cap (\mathcal{L} + a).$$

By Proposition 7, there is $\kappa_3 > 0$ such that

$$-\lambda_{\mathcal{F}^1}^{d_1}(z_1)\kappa_3 \leq \text{dist}(z_1, \mathcal{F}^1). \quad (10)$$

Let $t_\epsilon = \epsilon(\kappa_1 + \kappa_1 \kappa_2)/\kappa_3$. It follows from (9) and (10) that $z_1 + t_\epsilon d_1 \in \mathcal{F}^1$. As $d_2, z_2 \in \mathcal{F}^2$, we conclude that

$$x + y + t_\epsilon d \in \mathcal{F}. \quad (11)$$

We have

$$x + y + t_\epsilon d = (x + y - a) + t_\epsilon(d - a) + a(1 + t_\epsilon).$$

Furthermore, since $x + y \in \mathcal{L} + a$ and $d \in \mathcal{L} + a$, the first two terms of the right hand side belong to \mathcal{L} . Therefore, if we divide the whole expression by $(1 + t_\epsilon)$ we get

$$\frac{x + y + t_\epsilon d}{1 + t_\epsilon} \in (\mathcal{L} + a) \cap \mathcal{F}.$$

We conclude that

$$\begin{aligned} \text{dist}(x, (\mathcal{L} + a) \cap \mathcal{F}) &\leq \left\| x - \left(\frac{x + y + t_\epsilon d}{1 + t_\epsilon} \right) \right\| \\ &\leq \|x\| \frac{t_\epsilon}{1 + t_\epsilon} + \frac{\epsilon \kappa_2}{1 + t_\epsilon} + \|d\| \frac{t_\epsilon}{1 + t_\epsilon} \\ &\leq \|x\| t_\epsilon + \epsilon \kappa_2 + \|d\| t_\epsilon \\ &\leq \kappa \|x\| \epsilon + \kappa \epsilon, \end{aligned}$$

where $\kappa = \max\{(\kappa_1 + \kappa_1 \kappa_2)/\kappa_3, \kappa_2 + \|d\|(\kappa_1 + \kappa_1 \kappa_2)/\kappa_3\}$. \square

Proposition 19 has the following immediate corollary, where $\text{dist}(x, \text{span } \mathcal{F})$ is embedded directly into the error bound.

Corollary 20. *Let \mathcal{K} be a closed convex amenable cone and $\mathcal{F} \trianglelefteq \mathcal{K}$ a face of \mathcal{K} containing $(\mathcal{L} + a) \cap \mathcal{K}$ and such that $(\mathcal{F}, \mathcal{L}, a)$ satisfies the PPS condition.*

Then, there is a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a, \mathcal{F}$) such that whenever $x \in \text{span } \mathcal{K}$ and ϵ satisfy the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa \|x\| + \kappa)(\epsilon + \text{dist}(x, \text{span } \mathcal{F})).$$

Proof. We apply the previous proposition by taking $\hat{\epsilon} = \text{dist}(x, \mathcal{K}) + \text{dist}(x, \mathcal{L} + a) + \text{dist}(x, \text{span } \mathcal{F})$, which tells us that

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa \|x\| + \kappa)(\text{dist}(x, \mathcal{K}) + \text{dist}(x, \mathcal{L} + a) + \text{dist}(x, \text{span } \mathcal{F})).$$

Adjusting the constant κ , we get that whenever $x \in \text{span } \mathcal{K}$ satisfies

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa \|x\| + \kappa)(\epsilon + \text{dist}(x, \text{span } \mathcal{F})). \quad \square$$

From Proposition 19 and Corollary 20, it becomes clear that the key to general error bounds for $(\mathcal{K}, \mathcal{L}, a)$ is to know some face $\hat{\mathcal{F}}$ of \mathcal{K} for which the PPS condition is satisfied *and* we should also know some bound on $\text{dist}(x, \text{span } \hat{\mathcal{F}})$.

This is where we will use facial reduction (Section 2.4). If $(\mathcal{K}, \mathcal{L}, a)$ is feasible, but the PPS condition is not satisfied, then there exists $z_1 \in \mathcal{K}^* \cap \mathcal{L} \cap \{a\}^\perp$ with $z_1 \notin \mathcal{K}^\perp$, e.g., Theorem 4 in [30]. In particular, $\mathcal{F}_1 := \mathcal{K} \cap \{z_1\}^\perp$ is a proper face of \mathcal{K} that contains the feasible region of $(\mathcal{K}, \mathcal{L}, a)$. Again, if $(\mathcal{K} \cap \{z_1\}^\perp, \mathcal{L}, a)$ still does not satisfy the PPS condition, we use the same principle to obtain a new z_2 together with the face $\mathcal{F}_2 := \mathcal{K} \cap \{z_1\}^\perp \cap \{z_2\}^\perp$. Then, we proceed until a face satisfying the PPS condition is found. At each step, we will use a facial residual function to keep track of the distance between x and the face \mathcal{F}_i . The next proposition is the first step towards this idea.

Proposition 21. *Let $(\mathcal{K}, \mathcal{L}, a)$ be feasible. Let \mathcal{F} be a face of \mathcal{K} ,*

$$\begin{aligned} z &\in \mathcal{F}^* \cap \mathcal{L}^\perp \cap \{a\}^\perp \\ \hat{\mathcal{F}} &= \mathcal{F} \cap \{z\}^\perp, \end{aligned}$$

with $z \neq 0$. Let $\psi_{\mathcal{F}, z}$ be a facial residual function for \mathcal{F} and z . Then, there is a positive rescaling of $\psi_{\mathcal{F}, z}$ such that whenever $x \in \text{span } \mathcal{K}$ satisfies the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have:

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, z}(\epsilon + \text{dist}(x, \text{span } \mathcal{F}), \|x\|).$$

Proof. Positive rescaling $\psi_{\mathcal{F}, z}$ if necessary, we may assume that $\psi_{\mathcal{F}, z}$ is such that whenever $x \in \text{span } \mathcal{K}$ and $\tilde{\epsilon}$ satisfy the inequalities

$$\text{dist}(x, \mathcal{K}) \leq 2\tilde{\epsilon}, \quad \langle x, z \rangle \leq 2\tilde{\epsilon}\|z\|, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \tilde{\epsilon}$$

we have

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, z}(\tilde{\epsilon}, \|x\|).$$

Let

$$\tilde{\epsilon} = \frac{\text{dist}(x, \mathcal{K})}{2} + \frac{|\langle x, z \rangle|}{2\|z\|} + \text{dist}(x, \text{span } \mathcal{F}).$$

Then, the following inequality holds for every $x \in \text{span } \mathcal{K}$.

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, z} \left(\frac{\text{dist}(x, \mathcal{K})}{2} + \frac{|\langle x, z \rangle|}{2\|z\|} + \text{dist}(x, \text{span } \mathcal{F}), \|x\| \right). \quad (12)$$

Now, suppose that $x \in \text{span } \mathcal{K}$ satisfies

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon.$$

Since $\text{dist}(x, \mathcal{L} + a) \leq \epsilon$, there exists u such that $\|u\| \leq \epsilon$ and $x + u \in \mathcal{L} + a$. Because z is orthogonal to $\mathcal{L} + a$, it follows that $\langle x + u, z \rangle = 0$ and that

$$|\langle x, z \rangle| \leq \|z\|\epsilon. \quad (13)$$

Finally, from (12), (13), $\text{dist}(x, \mathcal{K}) \leq \epsilon$ and the monotonicity of $\psi_{\mathcal{F}, z}$, we obtain that $\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, z}(\epsilon + \text{dist}(x, \text{span } \mathcal{F}), \|x\|)$. \square

For what follows, we introduce a special notation for function composition. Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be real functions. We define $f \diamond g$ to be the function satisfying

$$(f \diamond g)(a, b) = f(a + g(a, b), b),$$

for every $a, b \in \mathbb{R}$. Note that if f and g are monotone nondecreasing on each argument, then the same is true for $f \diamond g$.

Lemma 22. *Let $\mathcal{L} \subseteq \mathcal{E}$ be a subspace and $a \in \mathcal{E}$. Let*

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of \mathcal{K} together with $z_i \in \mathcal{F}_i^ \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$, for $i = 1, \dots, \ell - 1$. For those i , let ψ_i be a facial residual function for \mathcal{F}_i, z_i . Then, there is a positive rescaling of the ψ_i such that if $x \in \text{span } \mathcal{K}$ satisfies the inequalities*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have:

$$\text{dist}(x, \mathcal{F}_\ell) \leq \varphi(\epsilon, \|x\|),$$

where $\varphi = \psi_{\ell-1} \diamond \cdots \diamond \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

Proof. The case $\ell = 1$ is straightforward. For the case $\ell \geq 2$, we proceed by induction. When $\ell = 2$, we apply Proposition 21 to $\mathcal{K}, \mathcal{F}_1, z_1$ and ψ_1 . Therefore, after positive rescaling ψ_1 appropriately, whenever $x \in \text{span } \mathcal{K}$ satisfies the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have:

$$\text{dist}(x, \mathcal{F}_2) \leq \psi_1(\epsilon + \text{dist}(x, \text{span } \mathcal{F}_1), \|x\|).$$

In this case, since $x \in \text{span } \mathcal{K}$ and $\mathcal{F}_1 = \mathcal{K}$, we have $\text{dist}(x, \text{span } \mathcal{F}_1) = 0$.

We now suppose that the lemma holds for chains of length $\hat{\ell}$ and will show that it must hold when the length is $\hat{\ell} + 1$. By the inductive hypothesis, we have that whenever

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have:

$$\text{dist}(x, \mathcal{F}_{\hat{\ell}}) \leq (\psi_{\hat{\ell}-1} \diamond \cdots \diamond \psi_1)(\epsilon, \|x\|).$$

From the the definition of $\psi_{\hat{\ell}}$ and its monotonicity in the first argument we get

$$\begin{aligned} \text{dist}(x, \mathcal{F}_{\hat{\ell}+1}) &\leq \psi_{\hat{\ell}}(\epsilon + \text{dist}(x, \text{span } \mathcal{F}_{\hat{\ell}}), \|x\|) \\ &\leq \psi_{\hat{\ell}}(\epsilon + (\psi_{\hat{\ell}-1} \diamond \cdots \diamond \psi_1)(\epsilon, \|x\|), \|x\|) \\ &\leq (\psi_{\hat{\ell}} \diamond \cdots \diamond \psi_1)(\epsilon, \|x\|), \end{aligned}$$

where we used the fact that $\text{dist}(x, \text{span } \mathcal{F}_{\hat{\ell}}) \leq \text{dist}(x, \mathcal{F}_{\hat{\ell}})$ to obtain the second inequality. \square

Using Lemma 22, we obtain one of the main results of this paper.

Theorem 23 (Error bound for amenable cones). *Let $\mathcal{L} \subseteq \mathcal{E}$ be a subspace and $a \in \mathcal{E}$. Let \mathcal{K} be a closed convex amenable cone and let*

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of \mathcal{K} together with $z_i \in \mathcal{F}_i^ \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that $(\mathcal{F}_\ell, \mathcal{L}, a)$ satisfies the PPS condition and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i . For $i = 1, \dots, \ell - 1$, let ψ_i be a facial residual function for \mathcal{F}_i, z_i . Then, after positive rescaling the ψ_i , there is a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a, \mathcal{F}_\ell$) such that if $x \in \text{span } \mathcal{K}$ satisfies the inequalities*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa \|x\| + \kappa)(\epsilon + \varphi(\epsilon, \|x\|)),$$

where $\varphi = \psi_{\ell-1} \diamond \cdots \diamond \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

Proof. The case $\ell = 1$ follows from Proposition 19, by taking $\mathcal{F} = \mathcal{F}_1$. Now, suppose $\ell \geq 2$. We apply Lemma 22, which tells us that, after positive rescaling the ψ_i , if $x \in \text{span } \mathcal{K}$ satisfies

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

we have:

$$\text{dist}(x, \mathcal{F}_\ell) \leq \varphi(\epsilon, \|x\|),$$

where $\varphi = \psi_{\ell-1} \diamond \dots \diamond \psi_1$. Since \mathcal{K} is amenable and $(\mathcal{F}_\ell, \mathcal{L}, a)$ satisfies the PPS condition, we invoke Corollary 20 which implies that

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa\|x\| + \kappa)(\epsilon + \varphi(\epsilon, \|x\|)),$$

for a positive constant κ depending on $\mathcal{K}, \mathcal{L}, a, \mathcal{F}_\ell$. \square

We now clarify a few aspects of Theorem 23. First of all, Theorem 23 assumes that there is a chain of faces ending in a face \mathcal{F}_ℓ such that $(\mathcal{F}_\ell, \mathcal{K}, a)$ satisfies the PPS condition. The existence of such a chain is a nontrivial consequence of facial reduction theory. In particular, its existence follows from Proposition 5. It also follows from Theorem 3.2 in [52] or from Theorem 1 in [38].

Now, that the question of existence of a chain satisfying the requirements of Theorem 23 is settled, we will take a look at efficiency issues. If we fix $(\mathcal{K}, \mathcal{L}, a)$ there could be several chains of faces that meet the criteria in Theorem 23. Since it is desirable to have an error bound with ℓ as small as possible, we will use facial reduction theory to give bounds on ℓ . Here, we recall that $d_{\text{PPS}}(\mathcal{L}, a)$ is the minimal number of reducing directions needed to find a face that satisfies the PPS condition and $d_S(\mathcal{L}, a)$ is the singularity degree, see Section 2.4.1.

Proposition 24 (Efficiency of the error bound). *Let $\mathcal{K} = \mathcal{K}^1 \times \dots \times \mathcal{K}^s$, where each \mathcal{K}^i is a closed convex cone. Suppose $(\mathcal{K}, \mathcal{L}, a)$ is feasible. Then there is a chain of faces of \mathcal{K}*

$$\mathcal{F}_{d_{\text{PPS}}(\mathcal{L}, a)+1} \subsetneq \dots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

satisfying the requirements of Theorem 23 such that the following bounds are satisfied

- (i) $d_{\text{PPS}}(\mathcal{L}, a) \leq \sum_{i=1}^s \ell_{\text{poly}}(\mathcal{K}^i)$
- (ii) $d_{\text{PPS}}(\mathcal{L}, a) \leq \dim(\mathcal{L}^\perp \cap \{a\}^\perp)$
- (iii) $d_{\text{PPS}}(\mathcal{L}, a) \leq d_S(\mathcal{L}, a)$.

Proof. By definition, there exists at least one chain of length $d_{\text{PPS}}(\mathcal{L}, a) + 1$ satisfying the requirements of Theorem 23. The bound in item (i) follows from Proposition 5. We will now prove item (ii). Let

$$\mathcal{F}_{d_{\text{PPS}}(\mathcal{L}, a)+1} \subsetneq \dots \subsetneq \mathcal{F}_1 = \mathcal{K} \tag{14}$$

be a chain of faces of \mathcal{K} together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i . The inclusions in (14) must be strict, otherwise we would be able to remove some faces of the chain, shrink it and contradict the minimality of $d_{\text{PPS}}(\mathcal{L}, a)$. Finally, we note that for $i > 1$, if z_i belongs to the space spanned by $\{z_1, \dots, z_{i-1}\}$, then we would have $\mathcal{F}_{i+1} = \mathcal{F}_i$. Therefore, $\{z_1, \dots, z_{d_{\text{PPS}}(\mathcal{L}, a)}\}$ is a linear independent set contained in $\mathcal{L}^\perp \cap \{a\}^\perp$.

Item (iii) holds because the PPS condition is less strict than Slater's condition, so a chain of faces ending with a face for which Slater's condition holds will also satisfy the requirements of Theorem 23. \square

In particular, Proposition 24 shows that the number of function compositions appearing in Theorem 23 can be taken to be no more than the singularity degree of $(\mathcal{K}, \mathcal{L}, a)$.

Remark 25. Let $d \in \text{ri}\mathcal{K}$ and consider the generalized eigenvalue function $\lambda_{\mathcal{K}}^d(\cdot)$ defined in Section 2.5. From Proposition 7, there is a constant $\kappa' > 0$ depending on d such that

$$\lambda_{\mathcal{K}}^d(x) \geq -\epsilon \quad \Rightarrow \quad \text{dist}(x, \mathcal{K}) \leq \kappa'\epsilon,$$

for all $x \in \text{span}\mathcal{K}$. Therefore, under the setting of Theorem 23, we get that the inequalities

$$\lambda_{\mathcal{K}}^d(x) \geq -\epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon$$

imply

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa\|x\| + \kappa)((\kappa' + 1)\epsilon + \varphi((\kappa' + 1)\epsilon, \|x\|)),$$

where κ is some positive constant. Noting that $\varphi((\kappa' + 1)\epsilon, \|x\|)$ is a positive rescaling of $\varphi(\epsilon, \|x\|)$, we see that Theorem 23 is still valid if we replace “ $\text{dist}(x, \mathcal{K}) \leq \epsilon$ ” by “ $\lambda_{\mathcal{K}}^d(\cdot) \geq -\epsilon$ ”.

Similarly, if $\mathcal{L} + a$ is described as the solution set of some system of linear equalities “ $\mathcal{A}x = b$ ”, we can substitute “ $\text{dist}(x, \mathcal{K}) \leq \epsilon$ ” by some quantity measuring the error with respect to that system. For instance, we could use “ $\sum_{i=1}^m |b_i - \mathcal{A}_i(x)| \leq \epsilon$ ”, where the \mathcal{A}_i are such that $\mathcal{A}(x) = (\mathcal{A}_1(x), \dots, \mathcal{A}_m(x))$.

Next, we will make a brief detour and generalize an observation made by Sturm in [47]. He noticed that if $(\mathcal{S}_+^n, \mathcal{L}, a)$ is such that $\mathcal{F}_{\min} = \{0\}$, then a Lipschitzian error bound holds, see (2.5) in [47]. First, we need the following auxiliary result.

Lemma 26. Let $z \in \text{ri}\mathcal{K}^*$. Then, there is a positive constant κ such that

$$\|x\| \leq \kappa\langle x, z \rangle, \quad \forall x \in \mathcal{K}.$$

Proof. Let $C = \{x \in \mathcal{K} \mid \langle x, z \rangle = 1\}$. The recession cone of C is the set

$$\text{rec } C = \{x \in \mathcal{K} \mid \langle x, z \rangle = 0\}.$$

If $x \in \mathcal{K}$, $x \notin (\mathcal{K}^*)^\perp$ and $\langle x, z \rangle = 0$, then $\{x\}^\perp$ is a hyperplane that properly separates z from \mathcal{K}^* . Such a hyperplane exists if and only if $z \notin \text{ri}\mathcal{K}^*$, see Theorem 20.2 in [44]. We conclude that $\text{rec } C \subseteq (\mathcal{K}^*)^\perp$.

Since $\text{lin } \mathcal{K} = (\mathcal{K}^*)^\perp$ and \mathcal{K} is pointed (Assumption 1), we have $\text{rec } C = \{0\}$. Therefore, C must be compact. Let $\kappa = \sup_{u \in C} \|u\|$. Then for nonzero $x \in \mathcal{K}$, we have

$$\frac{\|x\|}{\langle x, z \rangle} \leq \kappa.$$

□

Proposition 27 (Error bound for trivial intersections). Suppose that $(\mathcal{K}, \mathcal{L}, a)$ is such that

$$(\mathcal{L} + a) \cap \mathcal{K} = \{0\}.$$

Then, there exists a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a$) such that

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon \quad \Rightarrow \quad \|x\| \leq \kappa\epsilon.$$

Proof. Since $(\mathcal{L} + a) \cap \mathcal{K} = \{0\}$ holds, we have, in particular, $0 \in \mathcal{L} + a$. Therefore, $\mathcal{L} + a = \mathcal{L}$. We conclude that $\mathcal{L} \cap \mathcal{K} = \{0\}$. By the Gordan-Stiemke’s Theorem (see Corollary 2 in Luo, Sturm and Zhang [33]), $\mathcal{L} \cap \mathcal{K} = \{0\}$ holds if and only if there exists $z \in (\text{ri}\mathcal{K}^*) \cap \mathcal{L}^\perp$.

Since $\text{dist}(x, \mathcal{L}) \leq \epsilon$, there exists u such that $\|u\| \leq \epsilon$ and $x + u \in \mathcal{L}$. Since $\langle x + u, z \rangle = 0$, we conclude that

$$\langle x, z \rangle \leq \epsilon\|z\|. \tag{15}$$

Since $\text{dist}(x, \mathcal{K}) \leq \epsilon$, there exists v such that $\|v\| \leq \epsilon$ and $x + v \in \mathcal{K}$. By Lemma 26 and (15), there exists a positive constant κ_1 such that

$$\|x\| - \|v\| \leq \|x + v\| \leq \kappa_1\langle x + v, z \rangle \leq 2\kappa_1\|z\|\epsilon. \tag{16}$$

From (16), we conclude that the proposition holds with $\kappa = \epsilon(1 + 2\kappa_1\|z\|)$. □

4.1 Error bounds for symmetric cones

In this subsection, we use Theorem 23 to prove error bounds for symmetric cones. First, we need to review a few aspects of Jordan algebras. More details can be found in the books by Koecher [26], Faraut and Korányi [15] and also in the survey article by Faybusovich [18]. A *Euclidean Jordan algebra* is a finite dimensional real vector space \mathcal{E} equipped with a bilinear product $\circ : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ (the Jordan product) and an inner product $\langle \cdot, \cdot \rangle$ satisfying the following axioms:

- (1) $x \circ y = y \circ x$,
- (2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$,
- (3) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$,

for all $x, y, z \in \mathcal{E}$. We will denote the identity element of \mathcal{E} by e and we recall that $e \circ x = x$, for all $x \in \mathcal{E}$. The cone of squares associated to a Jordan algebra is given by

$$\mathcal{K} = \{x^2 \mid x \in \mathcal{E}\}.$$

Under this setting, \mathcal{K} becomes a symmetric cone, i.e., a homogeneous² self-dual cone. Reciprocally, every symmetric cone arises as the cone of squares of some Euclidean Jordan algebra. Key examples of symmetric cones include the $n \times n$ positive semidefinite matrices \mathcal{S}_+^n , the nonnegative orthant \mathbb{R}_+^n and the second order cone.

We say that $c \in \mathcal{E}$ is an *idempotent* if $c \circ c = c$. Moreover, c is *primitive* if it is nonzero and there is no way of writing $c = a + b$, with nonzero idempotents a and b satisfying $a \circ b = 0$. We can now state the spectral theorem.

Theorem 28 (Spectral Theorem, see Theorem III.1.2 in [15]). *Let (\mathcal{E}, \circ) be a Euclidean Jordan algebra and let $x \in \mathcal{E}$. Then there are primitive idempotents c_1, \dots, c_r satisfying $c_1 + \dots + c_r = e$, $c_i \circ c_j = 0$ for $i \neq j$ and unique real numbers $\lambda_1, \dots, \lambda_r$ satisfying*

$$x = \sum_{i=1}^r \lambda_i c_i. \quad (17)$$

The λ_i appearing in Theorem 28 are called the *eigenvalues* of x . We will write $\lambda_{\min}(x)$ and $\lambda_{\max}(x)$ for the minimum and maximum eigenvalues of x , respectively. For an element $x \in \mathcal{E}$, we define the *rank* of x as the number of nonzero eigenvalues. The trace of x is defined as the sum of eigenvalues, i.e.,

$$\text{tr } x = \sum_{i=1}^r \lambda_i.$$

The rank of \mathcal{K} is defined by

$$\text{rank } \mathcal{K} = \max\{\text{rank } x \mid x \in \mathcal{K}\}.$$

With that, we have $\text{rank } \mathcal{K} = r = \text{tr}(e)$.

Throughout Section 4.1 and its subsections, we will assume that \mathcal{E} is a Euclidean Jordan algebra and that the inner product is given by

$$\langle x, y \rangle = \text{tr}(x \circ y). \quad (18)$$

With that, the corresponding norm is

$$\|x\| = \sqrt{\text{tr}(x^2)} = \left(\sum_{i=1}^r \lambda_i^2 \right)^{1/2} \quad (19)$$

Under this inner product, the primitive idempotents c_i appearing in Theorem 28 satisfy $\langle c_i, c_j \rangle = 0$ for $i \neq j$ and $\|c_i\| = 1$.

The next result follows from various propositions that appear in [15], such as Proposition III.2.2 and Exercise 3 in Chapter III. See also Equation (10) in [48].

²A cone is homogeneous if for every $x, y \in \text{ri } \mathcal{K}$ there is a linear bijection Q such that $Q(x) = y$ and $Q(\mathcal{K}) = \mathcal{K}$.

Proposition 29. *Let $x \in \mathcal{E}$.*

- (i) $x \in \mathcal{K}$ if and only if the eigenvalues of x are nonnegative.
- (ii) $x \in \text{ri}\mathcal{K}$ if and only if the eigenvalues of x are positive.
- (iii) Suppose $x, y \in \mathcal{K}$. Then, $x \circ y = 0$ if and only if $\langle x, y \rangle = 0$.

We will also need the following well-known fact on the function $\text{dist}(\cdot, \mathcal{K})$. Given $x \in \mathcal{E}$, we consider the spectral decomposition given by Theorem 28. Then, the element in \mathcal{K} closest to x is given by

$$y = \sum_{i=1}^r \max(\lambda_i, 0) c_i,$$

where the c_i are the primitive idempotents associated to λ_i (a proof can be found in Proposition 2.2 of [31]). Therefore,

$$\text{dist}(x, \mathcal{K})^2 = \sum_{i=1}^r \max(-\lambda_i(x), 0)^2. \quad (20)$$

Given $x \in \mathcal{E}$, the *Lyapunov operator* of x is the linear function $L_x : \mathcal{E} \rightarrow \mathcal{E}$ satisfying $L_x(y) = x \circ y$, for all $y \in \mathcal{E}$. The *quadratic representation* of x is the linear function $Q_x : \mathcal{E} \rightarrow \mathcal{E}$ such that $Q_x = 2L_x^2 - L_{x^2}$. We have

$$Q_x(e) = x^2, \quad \forall x \in \mathcal{E}. \quad (21)$$

Let c be an idempotent and $\alpha \in \mathbb{R}$. We define the following linear subspace of \mathcal{E} .

$$V(c, \alpha) = \{x \in \mathcal{E} \mid c \circ x = \alpha x\}.$$

Theorem 30 (Peirce Decomposition, see Proposition IV.1.1 and pg. 64 in [15]). *Let $c \in \mathcal{E}$ be an idempotent. Then \mathcal{E} is decomposed as the orthogonal direct sum*

$$\mathcal{E} = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus V(c, 0).$$

In addition, $V(c, 1)$ and $V(c, 0)$ are Euclidean Jordan algebras under the same Jordan product \circ . The orthogonal projections onto $V(c, 1)$ and $V(c, 0)$ are given by Q_c and Q_{e-c} , respectively. Furthermore, $V(c, 1/2) \circ V(c, 1/2) \subseteq V(c, 1) + V(c, 0)$.

We conclude this review with our assumptions for Section 4.1.

Assumption 2 (Overall assumptions for Section 4.1). *Throughout Section 4.1, \mathcal{E} is a Euclidean Jordan algebra, \mathcal{K} is its cone of squares, the inner product is given by (18), the norm is given by (19) and the distance function is the one induced by (19).*

4.1.1 Facial structure of symmetric cones

One important property of symmetric cones is that all faces can be seen as smaller symmetric cones. To explain that, we first take an arbitrary idempotent c . Then, the algebras $V(c, 1)$ and $V(c, 0)$ appearing in Theorem 30 also give rise to symmetric cones. In fact, if we define

$$\mathcal{F} = \{x^2 \mid x \in V(c, 1)\},$$

we have that \mathcal{F} is a face of \mathcal{K} and $\text{span}\mathcal{F} = V(c, 1)$. As \mathcal{F} is the cone of squares of $V(c, 1)$, it is also a symmetric cone on its own right. Therefore, it must be self-dual in some sense. However, if \mathcal{F} is a proper

face of \mathcal{K} then it cannot possibly satisfy $\mathcal{F}^* = \mathcal{F}$. The correct way of understanding the self-duality of \mathcal{F} is by restricting ourselves to $V(c, 1)$. It holds that

$$\mathcal{F}^* \cap V(c, 1) = \mathcal{F}.$$

Because of that, we say that \mathcal{F} is *self-dual on its span*. Since all faces of \mathcal{K} are self-dual on their span, \mathcal{K} is a *perfect cone*, following the definition by Barker [4].

Under these conditions, we have $c \in \text{ri } \mathcal{F}$ and c is the identity element in $V(c, 1)$. The conjugate face of \mathcal{F} is given as follows

$$\mathcal{F}^\Delta = \mathcal{K} \cap \{c\}^\perp = \{x^2 \mid x \in V(c, 0)\}.$$

That is, the faces generated by the algebras $V(c, 0)$ and $V(c, 1)$ are conjugate to each other. We remark that $e - c$ is the identity element in $V(c, 0)$ and $\text{span } \mathcal{F}^\Delta = V(c, 0)$.

Reciprocally, given a face \mathcal{F} of \mathcal{K} , there exists an idempotent c such that \mathcal{F} is the cone of squares of $V(c, 1)$. We summarize these facts in the next proposition, which is a consequence of Theorem 2 in [17], due to Faybusovich.

Proposition 31. *Let \mathcal{K} be a symmetric cone and \mathcal{F} be a face of \mathcal{K} .*

- (i) *There is an idempotent $c \in \text{ri } \mathcal{F}$ such that $V(c, 1)$ is a Euclidean Jordan algebra, \mathcal{F} is the cone of squares of $V(c, 1)$ and $\text{span } \mathcal{F} = V(c, 1)$.*
- (ii) *Let c be as in the previous item. The conjugate face of \mathcal{F} is $\mathcal{F}^\Delta = \mathcal{K} \cap \{c\}^\perp$ and is the cone of squares of $V(c, 0)$. Furthermore $\text{span } \mathcal{F}^\Delta = V(c, 0)$ and $V(c, 0) = V(e - c, 1)$.*
- (iii) *\mathcal{F} is self-dual on its span, i.e., $\mathcal{F} = \mathcal{F}^* \cap \text{span } \mathcal{F}$.*

Let $x \in \mathcal{E}$. If there exists x^{-1} such that $x \circ x^{-1} = e$ and $L_x L_{x^{-1}} = L_{x^{-1}} L_x$, we say that x^{-1} is the inverse of x in \mathcal{E} , see Chapter III of [26]. A sufficient condition for the existence of x^{-1} is “ $x \in \text{ri } \mathcal{K}$ ”. As in the case of symmetric matrices, the eigenvalues of x^{-1} are the reciprocals of the eigenvalues of x .

Now, let c be an idempotent and consider the algebra $V(c, 1)$ together with its cone of squares \mathcal{F} . If $x \in V(c, 1)$, x might have an inverse in $V(c, 1)$ even if it does not have an inverse in \mathcal{E} . In this case, x^{-1} would satisfy $x \circ x^{-1} = c$. Similarly, “ $x \in \text{ri } \mathcal{F}$ ” is a sufficient condition for the existence of an inverse in $V(c, 1)$. With that, we have the following proposition.

Proposition 32. *Let \mathcal{K} be a symmetric cone, $x \in \mathcal{E}$ and c be an idempotent. Following Theorem 30, write*

$$x = x_1 + x_2 + x_3,$$

with $x_1 \in V(c, 1)$, $x_2 \in V(c, 1/2)$, $x_3 \in V(c, 0)$. Let \mathcal{F} be the cone of squares of $V(c, 1)$.

- (i) *If $x \in \mathcal{K}$, then $x_1 \in \mathcal{F}$ and $x_3 \in \mathcal{F}^\Delta$.*
- (ii) *If $x \in \text{ri } \mathcal{K}$, then $x_1 \in \text{ri } \mathcal{F}$ and $x_3 \in \text{ri } \mathcal{F}^\Delta$.*
- (iii) *(Schur complement) Suppose $x_3 \in \text{ri } \mathcal{F}^\Delta$. Then $x \in \text{ri } \mathcal{K}$ if and only if*

$$x_1 - Q_{x_2}(x_3^{-1}) \in \text{ri } \mathcal{F},$$

where x_3^{-1} denotes the inverse of x_3 in $V(c, 0)$.

Proof. (i) Let $y \in \mathcal{F}$. Since $x \in \mathcal{K}$, we have $\langle x, y \rangle = \langle x_1, y \rangle \geq 0$. This shows that $x_1 \in \mathcal{F}^* \cap V(c, 1)$. Since \mathcal{F} is self-dual over its span, we conclude that $x_1 \in \mathcal{F}$. A similar argument holds for x_3 .

(ii) Let $y \in \mathcal{F} \setminus \{0\}$. Since $x \in \text{ri } \mathcal{K}$, we have $\langle x, y \rangle = \langle x_1, y \rangle > 0$. This shows that $x_1 \in \text{ri } (\mathcal{F}^* \cap V(c, 1))$. Since \mathcal{F} is self-dual over its span, we conclude that $x_1 \in \text{ri } \mathcal{F}$. A similar argument holds for x_3 .

(iii) See Corollary 5 in the article by Gowda and Sznajder [20].

□

4.1.2 Amenability and facial residual functions for symmetric cones

We will first show that symmetric cones are amenable.

Proposition 33 (Symmetric cones are amenable and orthogonal projectionally exposed). *Let $\mathcal{F} \triangleleft \mathcal{K}$, where \mathcal{K} is a symmetric cone. There exists an orthogonal projection Q such $Q(\mathcal{K}) = \mathcal{F}$. In particular, \mathcal{K} is amenable and we have*

$$\text{dist}(x, \mathcal{F}) = \text{dist}(x, \mathcal{K}), \quad \forall x \in \text{span } \mathcal{F}.$$

Proof. Let \mathcal{K} be a symmetric cone and \mathcal{F} be a face of \mathcal{K} . First, we observe that since $\mathcal{F} \subseteq \mathcal{K}$, we have $\text{dist}(x, \mathcal{F}) \geq \text{dist}(x, \mathcal{K})$, for every $x \in \mathcal{E}$.

Next, following Proposition 31, let c be an idempotent such that $V(c, 1)$ is the Euclidean Jordan algebra whose cone of squares is \mathcal{F} and such that $\text{span } \mathcal{F} = V(c, 1)$. By Theorem 30, the orthogonal projection onto $V(c, 1)$ is given by Q_c . Together with item (i) of Proposition 32, we obtain that

$$Q_c(x) \in \mathcal{F}, \quad \forall x \in \mathcal{K}.$$

This shows that \mathcal{K} is orthogonal projectionally exposed. Since $\|Q_c\| \leq 1$, we have that item (i) of Proposition 9 implies $\text{dist}(x, \mathcal{F}) = \text{dist}(x, \mathcal{K})$ for all $x \in \text{span } \mathcal{F}$. □

Next, we will show that symmetric cones admit FRFs of the form $\kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}$, where κ is some positive constant. We first need a few auxiliary results.

Lemma 34. *Let \mathcal{E} be a Euclidean Jordan algebra, let c be an idempotent and $w \in V(c, 1/2)$. Then, there are $w_0 \in V(c, 0)$, $w_1 \in V(c, 1)$ such that*

$$\begin{aligned} w^2 &= w_0 + w_1 \\ \text{tr}(w_0) &= \text{tr}(w_1) = \frac{\text{tr}(w^2)}{2}. \end{aligned}$$

Proof. From Theorem 30, we can write $w^2 = w_0 + w_1$, with $w_0 \in V(c, 0)$, $w_1 \in V(c, 1)$. On one hand, taking the inner product with c , we obtain

$$\langle w^2, c \rangle = \langle w, w \circ c \rangle = \frac{\langle w, w \rangle}{2} = \frac{\text{tr}(w^2)}{2},$$

where the first equality follows from axiom (3) in Section 4.1 and the second equality follows from the assumption that $w \in V(c, 1/2)$. On the other hand, we have

$$\langle w_0 + w_1, c \rangle = \langle w_1, c \rangle,$$

since $w_0 \in V(c, 0)$. To conclude, we recall that $e - c$ belongs to $V(c, 0)$, so that

$$\text{tr}(w_1) = \langle w_1, c + (e - c) \rangle = \langle w_1, e \rangle.$$

□

At last, we recall the following variational characterization of λ_{\min} , which can be found, for instance, in Equation (9) in [48].

$$\lambda_{\min}(x) = \min\{\langle x, y \rangle \mid y \in \mathcal{K}, \langle y, e \rangle = 1\}.$$

Then, since $\text{tr}(y) = \langle y, e \rangle$, we obtain

$$\langle x, y \rangle \geq \lambda_{\min}(x)\text{tr}(y), \quad \forall x \in \mathcal{E}, \forall y \in \mathcal{K}. \quad (22)$$

Theorem 35 (Facial residual functions for symmetric cones). *Let \mathcal{K} be a symmetric cone and let $\mathcal{F} \trianglelefteq \mathcal{K}$ be an arbitrary face. Let $z \in \mathcal{F}^*$ and $\hat{\mathcal{F}} = \mathcal{F} \cap \{z\}^\perp$. Then, there is a positive constant κ (depending on $\mathcal{K}, \mathcal{F}, z$) such that whenever x satisfies the inequalities*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$$

we have

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}.$$

That is, we can take $\psi_{\mathcal{F}, z}(\epsilon, \|x\|) = \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}$ as a facial residual function for \mathcal{F} and z .

Proof. Let \mathcal{F} be a face of \mathcal{K} , $z \in \mathcal{F}^*$ and let $\hat{\mathcal{F}} = \mathcal{F} \cap \{z\}^\perp$. By item (i) of Proposition 31, there is an idempotent $c \in \text{ri } \mathcal{F}$ such that $V(c, 1)$ is a Jordan algebra satisfying

$$\mathcal{F} = \{u^2 \mid u \in V(c, 1)\}.$$

Furthermore, we have $V(c, 1) = \text{span } \mathcal{F}$. Now, suppose that we have $x \in \mathcal{E}$ such that

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon.$$

By Theorem 30, we can decompose x and z as

$$\begin{aligned} x &= x_1 + x_2 + x_3 \\ z &= z_1 + z_2 + z_3, \end{aligned}$$

where $x_1, z_1 \in V(c, 1)$, $x_2, z_2 \in V(c, 1/2)$, $x_3, z_3 \in V(c, 0)$. We recall that $V(c, 1)$, $V(c, 1/2)$ and $V(c, 0)$ are orthogonal subspaces. In particular, this implies that x_1 is the orthogonal projection of x onto $V(c, 1) = \text{span } \mathcal{F}$. Therefore, $\text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon$ implies $\|x - x_1\| \leq \epsilon$. As x_2 and x_3 are orthogonal, we obtain

$$\|x_2\| \leq \epsilon \tag{23}$$

$$\|x_3\| \leq \epsilon. \tag{24}$$

Now we turn our attentions to $\hat{\mathcal{F}}$. First, since $z \in \mathcal{F}^*$, we have

$$z_1 \in \mathcal{F}, \quad \hat{\mathcal{F}} = \mathcal{F} \cap \{z_1\}^\perp. \tag{25}$$

As $V(c, 1)$ is a *bona fide* Jordan algebra and $\hat{\mathcal{F}}$ is a face of \mathcal{F} , again by Proposition 31 there is some idempotent \hat{c} such that $\hat{V}(\hat{c}, 1)$ is the Jordan algebra contained in $V(c, 1)$ that generates $\hat{\mathcal{F}}$, i.e.,

$$\hat{\mathcal{F}} = \{u^2 \mid u \in \hat{V}(\hat{c}, 1)\},$$

where

$$\hat{V}(\hat{c}, \alpha) = \{u \in V(c, 1) \mid \hat{c} \circ u = \alpha u\} = V(\hat{c}, \alpha) \cap V(c, 1).$$

We remark that $\hat{V}(\hat{c}, \alpha)$ might be smaller than $V(\hat{c}, \alpha)$ and we use the symbol \hat{V} to emphasize that $\hat{V}(\hat{c}, \alpha)$ is a subalgebra of $V(c, 1)$.

Given the idempotent \hat{c} , we apply Theorem 30 substituting \mathcal{E} by $V(c, 1)$ and c by \hat{c} . It follows that

$$V(c, 1) = \hat{V}(\hat{c}, 1) \oplus \hat{V}(\hat{c}, 1/2) \oplus \hat{V}(\hat{c}, 0).$$

Then, we further decompose x_1 as

$$x_1 = x_{11} + x_{12} + x_{13},$$

with $x_{11} \in \hat{V}(\hat{c}, 1)$, $x_{12} \in \hat{V}(\hat{c}, 1/2)$, $x_{13} \in \hat{V}(\hat{c}, 0)$. Our goal is to bound to x_{12} and x_{13} .

³To see that, first recall that \mathcal{F} is self-dual on its span, i.e., $\mathcal{F} = \{v \in V(c, 1) \mid \langle u, v \rangle \geq 0, \forall u \in \mathcal{F}\}$. Now, let $v \in \mathcal{F}$ be arbitrary. Since $z \in \mathcal{F}^*$, we have $\langle v, z \rangle = \langle v, z_1 \rangle \geq 0$, due to the orthogonality among $V(c, 0)$, $V(c, 1/2)$ and $V(c, 1)$. This shows that $z_1 \in \mathcal{F}$. Similarly, we can show that $\mathcal{F} \cap \{z\}^\perp = \mathcal{F} \cap \{z_1\}^\perp$.

We first bound x_{13} by invoking Lemma 26 appropriately. To do so, first recall (25), so that $z_1 \in \mathcal{F}$ and $\hat{\mathcal{F}} = \mathcal{F} \cap \{z_1\}^\perp$. We restrict ourselves to $V(c, 1)$ and let $\hat{\mathcal{F}}^\Delta$ denote the conjugate face of $\hat{\mathcal{F}}$ with respect to \mathcal{F} . That is,

$$\hat{\mathcal{F}}^\Delta = \mathcal{F} \cap \hat{\mathcal{F}}^\perp = V(c, 1) \cap \mathcal{F}^* \cap \hat{\mathcal{F}}^\perp.$$

Recalling that $\langle x, z \rangle \leq \epsilon$, we have

$$\begin{aligned} \langle z_1, x_1 \rangle &\leq \epsilon - \langle z_2, x_2 \rangle - \langle z_3, x_3 \rangle \\ &\leq \epsilon + \epsilon \|z_2\| + \epsilon \|z_3\| && \text{(From (23) and (24))} \\ &\leq \epsilon(1 + \|z_2\| + \|z_3\|). \end{aligned} \tag{26}$$

Recall that symmetric cones are nice because they are amenable (Proposition 13), see also Proposition 4 and Section 4.1 in the work by Chua and Tunçel [11] or Theorem 4.1 in the work by Pólik and Terlaky [42]. Since \mathcal{F} is a symmetric cone (see Section 4.1.1), \mathcal{F} must be nice as well. Therefore, we can apply Proposition 1 to \mathcal{F} and z_1 . This shows that $z_1 \in \text{ri } \hat{\mathcal{F}}^\Delta$ and, in particular, $z_1 \in \hat{V}(\hat{c}, 0)$.⁴ As $\hat{\mathcal{F}}^\Delta$ is a symmetric cone whose Jordan algebra is $\hat{V}(\hat{c}, 0)$, we have $\hat{\mathcal{F}}^\Delta = \hat{\mathcal{F}}^{\Delta*} \cap \hat{V}(\hat{c}, 0)$, by items (ii) and (iii) of Proposition 31. This shows that $z_1 \in \text{ri } \hat{\mathcal{F}}^{\Delta*}$.

Now, since $\text{dist}(x, \mathcal{K}) \leq \epsilon$, we have⁵

$$\text{dist}(x_1, \mathcal{F}) \leq \epsilon, \quad \text{dist}(x_{13}, \hat{\mathcal{F}}^\Delta) \leq \epsilon. \tag{27}$$

Therefore, there is $u \in \hat{V}(\hat{c}, 0)$ such that $x_{13} + u \in \hat{\mathcal{F}}^\Delta$ and $\|u\| \leq \epsilon$. Since $z_1 \in \hat{V}(\hat{c}, 0)$, we have the following inequalities

$$\begin{aligned} \langle z_1, x_{13} + u \rangle &= \langle z_1, x_{11} + x_{12} + x_{13} + u \rangle && (z_1 \text{ is orthogonal to } x_{11}, x_{12}) \\ &= \langle z_1, x_1 + u \rangle \\ &\leq \epsilon(1 + \|z_2\| + \|z_3\| + \|z_1\|) && \text{(From (26)).} \end{aligned}$$

We apply Lemma 26 to $\hat{\mathcal{F}}^\Delta$ and z_1 , which tells us that there is $\kappa_1 > 0$ such that $\|w\| \leq \kappa_1 \langle w, z_1 \rangle$ whenever $w \in \hat{\mathcal{F}}^\Delta$. It follows that

$$\|x_{13} + u\| \leq \epsilon \kappa_1 (1 + \|z_2\| + \|z_3\| + \|z_1\|).$$

As $\|u\| \leq \epsilon$, we conclude that

$$\|x_{13}\| \leq \hat{\kappa}_1 \epsilon, \tag{28}$$

where $\hat{\kappa}_1 = (\kappa_1((1 + \|z_2\| + \|z_3\| + \|z_1\|) + 1))$.

The next task is to bound x_{12} . First, we apply Lemma 34 to x_{12} , with $V(c, 1)$ in place of \mathcal{E} , thus obtaining $w_0 \in \hat{V}(\hat{c}, 0)$ and $w_1 \in \hat{V}(\hat{c}, 1)$ such that

$$x_{12}^2 = w_0 + w_1 \tag{29}$$

$$\text{tr}(w_0) = \frac{\text{tr}(x_{12}^2)}{2}. \tag{30}$$

From (27), and since c is the identity element in $V(c, 1)$, we have

$$x_1 + \epsilon c \in \mathcal{F}.^6$$

⁴Rigorously, the argument so far only shows that $z_1 \in \text{ri}(\mathcal{F}^* \cap \hat{\mathcal{F}}^\perp)$. However, since $z_1 \in V(c, 1)$, we can put ‘‘ri’’ outside and conclude that $V(c, 1) \cap \text{ri}(\mathcal{F}^* \cap \hat{\mathcal{F}}^\perp) = \text{ri}(\mathcal{F}^* \cap \hat{\mathcal{F}}^\perp \cap V(c, 1))$. Therefore, as remarked, $z_1 \in \text{ri } \hat{\mathcal{F}}^\Delta$. Furthermore, since $z_1 \in \mathcal{K}$ and $\hat{c} \in \hat{\mathcal{F}}$, we have $\langle \hat{c}, z \rangle = 0$. By item (iii) of Proposition 29, we have $\hat{c} \circ z = 0$ and $z_1 \in \hat{V}(\hat{c}, 0)$ as claimed.

⁵Let $u \in \mathcal{K}$ be such that $\text{dist}(x, \mathcal{K}) = \|x - u\|$. Decompose u following the same decomposition of x . We have $u = u_{11} + u_{12} + u_{13} + u_2 + u_3$. By item (i) of Proposition 32, we have that $u_{13} \in \hat{\mathcal{F}}^\Delta$. Therefore $\text{dist}(x_{13}, \hat{\mathcal{F}}^\Delta) \leq \|x_{13} - u_{13}\| \leq \|x - u\| \leq \epsilon$. Similarly, we have $\text{dist}(x_1, \mathcal{F}) \leq \|x_1 - u_1\| \leq \epsilon$.

⁶If $x_1 \in \mathcal{F}$, then we have $x_1 + \epsilon c \in \mathcal{F}$. If not, then $\lambda_{\min}(x_1) < 0$. Here, we are considering the minimum eigenvalue of x_1 with respect to the algebra $V(c, 1)$. In this case, from Proposition 20, we have that $\epsilon \geq \text{dist}(x_1, \mathcal{F}) \geq -\lambda_{\min}(x_1)$. Then, since c is the identity in $V(c, 1)$, adding ϵc to x_1 has the effect of adding ϵ to $\lambda_{\min}(x_1)$.

In addition, since $c \in \text{ri } \mathcal{F}$, the following holds for every $\alpha > 0$,

$$x_1 + (\alpha + \epsilon)c \in \text{ri } \mathcal{F}.$$

We write $c = \hat{c} + (c - \hat{c})$ and recall that $\hat{c} \in V(\hat{c}, 1)$ and $(c - \hat{c}) \in \hat{V}(\hat{c}, 0)$. Then, we obtain from item (ii) of Proposition 32 that

$$\begin{aligned} x_{11} + (\epsilon + \alpha)\hat{c} &\in \text{ri } \hat{\mathcal{F}} \\ x_{13} + (\epsilon + \alpha)(c - \hat{c}) &\in \text{ri } \hat{\mathcal{F}}^\Delta. \end{aligned} \quad (31)$$

Now, we apply item (iii) of Proposition 32, which tells us that the following Schur complement must be a relative interior point of $\hat{\mathcal{F}}$.

$$(x_{11} + (\epsilon + \alpha)\hat{c}) - Q_{x_{12}}((x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}) \in \text{ri } \hat{\mathcal{F}}, \quad (32)$$

where $(x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}$ is the inverse of $x_{13} + (\epsilon + \alpha)(c - \hat{c})$ in $\hat{V}(\hat{c}, 0)$.

The next subgoal is to bound from below the minimum eigenvalue⁷ of $(x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}$ in the algebra $\hat{V}(\hat{c}, 0)$. Since $x_{13} + (\epsilon + \alpha)(c - \hat{c}) \in \hat{\mathcal{F}}^\Delta$ and $c - \hat{c}$ is the unit element in $\hat{V}(\hat{c}, 0)$, we have

$$\lambda_{\max}(x_{13} + (\epsilon + \alpha)(c - \hat{c})) = \lambda_{\max}(x_{13}) + \epsilon + \alpha.$$

In addition, from (28) and (19), we have that $\lambda_{\max}(x_{13}) \leq \hat{\kappa}_1 \epsilon$. Thus, we obtain

$$\lambda_{\min}(x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1} = \frac{1}{\lambda_{\max}(x_{13} + (\epsilon + \alpha)(c - \hat{c}))} \geq \frac{1}{(\hat{\kappa}_1 + 1)\epsilon + \alpha}. \quad (33)$$

We now return to (32). As the Schur complement is a relative interior point of $\hat{\mathcal{F}}^\Delta$, its inner product with c must be nonnegative. Recalling that $Q_{x_{12}}$ is self-adjoint, it follows that

$$\begin{aligned} \langle x_{11} + (\epsilon + \alpha)\hat{c}, c \rangle &\geq \langle Q_{x_{12}}((x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}), c \rangle \\ &= \langle (x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}, x_{12}^2 \rangle && \text{(From (21))} \\ &= \langle (x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}, w_0 \rangle && \text{(From (29))} \\ &\geq \lambda_{\min}((x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}) \text{tr}(w_0) && \text{(From (22))}^8 \\ &= \lambda_{\min}((x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1}) \frac{\text{tr}(x_{12}^2)}{2} && \text{(From (30))} \\ &\geq \frac{1}{(\hat{\kappa}_1 + 1)\epsilon + \alpha} \frac{\|x_{12}\|^2}{2}. && \text{(From (33) and (19))} \end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\|x_{12}\|^2 \leq 2((\hat{\kappa}_1 + 1)\epsilon + \alpha) \|c\| \|x_{11} + (\epsilon + \alpha)\hat{c}\|.$$

Since α is an arbitrary positive number, we get

$$\|x_{12}\|^2 \leq 2((\hat{\kappa}_1 + 1)\epsilon) \|c\| \|x_{11} + \epsilon\hat{c}\|.$$

Therefore,

$$\|x_{12}\| \leq \hat{\kappa}_2 \sqrt{\|\epsilon x_{11} + \epsilon^2 \hat{c}\|},$$

⁷The subtlety here is that $x_{13} + (\epsilon + \alpha)(c - \hat{c})$ and its inverse, seen as elements of $\hat{V}(\hat{c}, 0)$, have no zero eigenvalues, since they belong to $\text{ri } \hat{\mathcal{F}}^\Delta$. If we see them as elements of \mathcal{E} , zero eigenvalues might appear, but the corresponding idempotents certainly do not belong to $\hat{V}(\hat{c}, 0)$.

⁸(22) is invoked with $\hat{V}(\hat{c}, 0)$ in place of \mathcal{E} , so that $\lambda_{\min}((x_{13} + (\epsilon + \alpha)(c - \hat{c}))^{-1})$ refers to the minimum eigenvalue in the algebra $\hat{V}(\hat{c}, 0)$ and that is also why we can use (33) at the end.

where $\hat{\kappa}_2 = \sqrt{2(\hat{\kappa}_1 + 1)\|c\|}$. Finally, using the triangle inequality, we get

$$\|x_{12}\| \leq \epsilon \hat{\kappa}_2 \sqrt{\|\hat{c}\|} + \hat{\kappa}_2 \sqrt{\epsilon \|x_{11}\|}. \quad (34)$$

We are now ready to bound $\text{dist}(x, \hat{\mathcal{F}})$. From (31), we have that $x_{11} - \epsilon \hat{c} \in \hat{\mathcal{F}}$. It follows that

$$\begin{aligned} \text{dist}(x, \hat{\mathcal{F}}) &\leq \|x - x_{11} - \epsilon \hat{c}\| \\ &\leq \|x_{12} + x_{13} + x_2 + x_3\| + \epsilon \|\hat{c}\| \\ &\leq \epsilon \hat{\kappa}_2 \sqrt{\|\hat{c}\|} + \hat{\kappa}_2 \sqrt{\epsilon \|x_{11}\|} + \hat{\kappa}_1 \epsilon + 2\epsilon + \epsilon \|\hat{c}\| \quad (\text{From (23), (24), (28), (34)}) \\ &\leq \kappa \epsilon + \kappa \sqrt{\epsilon \|x\|}, \end{aligned}$$

where $\kappa = \max(\hat{\kappa}_2 \sqrt{\|\hat{c}\|} + \hat{\kappa}_1 + 2 + \|\hat{c}\|, \hat{\kappa}_2)$. \square

4.1.3 Hölderian error bounds for symmetric cones

Following Theorem 35, our first step is to bound by above the composition of facial residual functions of \mathcal{K} .

Lemma 36. *Suppose that $\psi_i(\epsilon, \|x\|) = \kappa_i \epsilon + \kappa_i \sqrt{\epsilon \|x\|}$ for $i = 1, \dots, \ell - 1$, where the κ_i are positive constants and $\ell \geq 2$. Then, there is a positive constant κ such that*

$$\psi_{\ell-1} \diamond \dots \diamond \psi_1(\epsilon, \|x\|) \leq \kappa \sum_{j=0}^{\ell-1} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}},$$

for every $\epsilon \geq 0$ and every x .

Proof. We proceed by induction on ℓ . For $\ell = 2$, it is enough to take $\kappa = \kappa_1$. Now, suppose that the proposition is true for some $\ell > 2$. We will show that it is also true for $\ell + 1$. Let $\varphi = \psi_\ell \diamond \dots \diamond \psi_1$. Since ψ_ℓ is monotone nondecreasing in each argument, we have by the induction hypothesis that there exists some $\tilde{\kappa}$ such that

$$\begin{aligned} \varphi(\epsilon, \|x\|) &= \psi_\ell(\epsilon + \psi_{\ell-1} \diamond \dots \diamond \psi_1(\epsilon, \|x\|), \|x\|) \\ &\leq \psi_\ell \left(\epsilon + \tilde{\kappa} \sum_{j=0}^{\ell-1} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}}, \|x\| \right) \\ &\leq \kappa_\ell \epsilon + \sum_{j=0}^{\ell-1} \kappa_\ell \tilde{\kappa} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}} + \kappa_\ell \sqrt{\epsilon \|x\|} + \sum_{j=0}^{\ell-1} \kappa_\ell \sqrt{\tilde{\kappa}} \epsilon^{(2^{-j-1})} \|x\|^{1-(2^{-j-1})}, \end{aligned}$$

where we used the fact that the square root satisfies $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$, when u and v are nonnegative. Looking at the terms that appear in both summations, we see that it is possible to group the coefficients and so we obtain

$$\varphi(\epsilon, \|x\|) \leq \kappa \sum_{j=0}^{\ell} \epsilon^{(2^{-j})} \|x\|^{1-(2^{-j})},$$

for $\kappa = \kappa_\ell + \kappa_\ell \tilde{\kappa} + \kappa_\ell \sqrt{\tilde{\kappa}}$. \square

With that we have the following theorem.

Theorem 37 (Error bounds for symmetric cones - 1st form). *Let \mathcal{K} be a symmetric cone, \mathcal{L} a subspace and $a \in \mathcal{E}$ such that $(\mathcal{K}, \mathcal{L}, a)$ is feasible. Then, there is a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a$) such that whenever x and ϵ satisfy the inequalities*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa \|x\| + \kappa) \left(\sum_{j=0}^{d_{\text{PPS}}(\mathcal{L}, a)} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}} \right).$$

Proof. \mathcal{K} is an amenable cone, because of Proposition 33. Therefore, we may apply Theorem 23 and Proposition 24, which tell us that there exists a positive constant $\tilde{\kappa}$ such that

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\tilde{\kappa} \|x\| + \tilde{\kappa})(\epsilon + \varphi(\epsilon, \|x\|)) \quad (35)$$

where

$$\varphi(\epsilon, \|x\|) = \begin{cases} \psi_{d_{\text{PPS}}(\mathcal{L}, a)} \diamond \cdots \diamond \psi_1(\epsilon, \|x\|) & \text{if } d_{\text{PPS}}(\mathcal{L}, a) > 0 \\ \epsilon & \text{if } d_{\text{PPS}}(\mathcal{L}, a) = 0, \end{cases}$$

and the ψ_i are facial residual functions as in Theorem 35. Then, we apply Lemma 36, to obtain a constant κ' such that

$$\varphi(\epsilon, \|x\|) \leq \kappa' \sum_{j=0}^{d_{\text{PPS}}(\mathcal{L}, a)} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}}.$$

We then let $\hat{\kappa} = \kappa' + 1$ so that

$$\epsilon + \varphi(\epsilon, \|x\|) \leq \hat{\kappa} \sum_{j=0}^{d_{\text{PPS}}(\mathcal{L}, a)} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}}. \quad (36)$$

Using (36) in (35) and letting $\kappa = \tilde{\kappa} \hat{\kappa}$ gives the desired error bound. \square

We observe that if \mathcal{K} is a symmetric cone and d is taken to be the identity element e , then the generalized eigenvalue function $\lambda_{\mathcal{K}}^e(\cdot)$ discussed in Section 2.5 coincides with the minimum eigenvalue function $\lambda_{\min}(\cdot)$. Following Remark 25, we may also substitute the condition “ $\text{dist}(x, \mathcal{K}) \leq \epsilon$ ” in Theorem 37 by “ $\lambda_{\min}(x) \geq -\epsilon$ ”. Furthermore, if x lies in some compact set and $\epsilon \leq 1$ then we can give a better looking error bound, where the sum is replaced by the term with smallest exponent. This leads to the second form of our error bounds results, which is closer to the way Sturm stated his error bound result.

Proposition 38 (Error bounds for symmetric cones - 2nd form). *Let \mathcal{K} be a symmetric cone, \mathcal{L} a subspace and $a \in \mathcal{E}$ such that $(\mathcal{K}, \mathcal{L}, a)$ is feasible. Let ρ be a positive real number. Then, there exists a positive constant κ (depending on $\mathcal{K}, \mathcal{L}, a, \rho$) such that for every x and $\epsilon \leq 1$ satisfying*

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \|x\| \leq \rho,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa \epsilon^{(2^{-d_{\text{PPS}}(\mathcal{L}, a)})}.$$

Furthermore, the proposition is still valid if we replace the inequality “ $\text{dist}(x, \mathcal{K}) \leq \epsilon$ ” by “ $\lambda_{\min}(x) \geq -\epsilon$ ”.

Proof. We apply Theorem 37 to $(\mathcal{K}, \mathcal{L}, a)$. Let $\hat{\kappa}$ be the obtained constant. Since $\epsilon \leq 1$, we have

$$\epsilon^{(2^{-d_{\text{PPS}}(\mathcal{L}, a)})} \geq \epsilon^{(2^{-j})},$$

for all $j = 0, \dots, \ell - 1$. Recalling that $\|x\| \leq \rho$, we have

$$\begin{aligned} \text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) &\leq (\hat{\kappa} \|x\| + \hat{\kappa}) \left(\sum_{j=0}^{d_{\text{PPS}}(\mathcal{L}, a)} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}} \right) \\ &\leq (\hat{\kappa} \rho + \hat{\kappa}) \left(\sum_{j=0}^{d_{\text{PPS}}(\mathcal{L}, a)} \epsilon^{(2^{-d_{\text{PPS}}(\mathcal{L}, a)})} \rho^{1-2^{-j}} \right) \\ &\leq \kappa \epsilon^{(2^{-d_{\text{PPS}}(\mathcal{L}, a)})}, \end{aligned}$$

where κ is the square of the maximum among all the constants so far, that is,

$$\sqrt{\kappa} = \max\{\hat{\kappa}\rho + \hat{\kappa}, \max\{\rho^{1-\frac{1}{2^j}} \mid j = 0, \dots, d_{\text{PPS}}(\mathcal{L}, a)\}\}.$$

The fact that error bound is still valid if we replace the inequality “ $\text{dist}(x, \mathcal{K}) \leq \epsilon$ ” by “ $\lambda_{\min}(x) \geq -\epsilon$ ” follows from Remark 25. \square

Remark 39 (Bounds on the distance to the PPS condition). *We can use Proposition 24 to bound the quantity $d_{\text{PPS}}(\mathcal{L}, a)$ in both Theorem 37 and Proposition 38. For that, we need the following facts on a symmetric cone \mathcal{K} .*

(i) *The length $\ell_{\mathcal{K}}$ of the longest chain of faces of \mathcal{K} satisfies $\ell_{\mathcal{K}} = \text{rank } \mathcal{K} + 1$.*

(ii) *The distance to polyhedrality of \mathcal{K} satisfies $\ell_{\text{poly}}(\mathcal{K}) \leq \text{rank } \mathcal{K} - 1$.*

Item (i) is a result due to Ito and Lourenço, see Theorem 14 in [23]. Then, Theorem 11 in [30] tell us that $1 + \ell_{\text{poly}}(\mathcal{K}) \leq \ell_{\mathcal{K}} - 1$. It follows that $\ell_{\text{poly}}(\mathcal{K}) \leq \text{rank } \mathcal{K} - 1$, which is item (ii). Therefore, if $\mathcal{K} = \mathcal{K}^1 \times \dots \times \mathcal{K}^s$ is the direct product of s symmetric cones, we have the the following bound.

$$d_{\text{PPS}}(\mathcal{L}, a) \leq \min \left\{ \dim(\mathcal{L}^\perp \cap \{a\}^\perp), \sum_{i=1}^s (\text{rank } \mathcal{K}^i - 1), d_S(\mathcal{L}, a) \right\}.$$

4.2 Intersection of cones

Suppose $\mathcal{K}^1 \subseteq \mathcal{E}$ and $\mathcal{K}^2 \subseteq \mathcal{E}$ are amenable cones. It is not clear whether $\mathcal{K}^1 \cap \mathcal{K}^2$ is also amenable. Even if it turns out that $\mathcal{K}^1 \cap \mathcal{K}^2$ is indeed amenable, it is also not clear how to construct FRFs for $\mathcal{K}^1 \cap \mathcal{K}^2$ from the FRFs of \mathcal{K}^1 and \mathcal{K}^2 . Therefore, at first glance, the results in Theorem 23 are not directly applicable. Nevertheless, we will show in this subsection that it is still possible to give error bounds while sidestepping these issues.

Suppose $(\mathcal{K}^1 \cap \mathcal{K}^2, \mathcal{L}, a)$ is feasible. Let $\hat{\mathcal{L}}, \hat{a}$ be such that

$$\hat{\mathcal{L}} + \hat{a} = \{(x, x) \mid x \in \mathcal{L} + a\}.$$

Due to Propositions 11 and 17, $\mathcal{K}^1 \times \mathcal{K}^2$ is an amenable cone. Furthermore, we can use as FRFs the sum of facial residual functions for \mathcal{K}^1 and \mathcal{K}^2 . We will show in this subsection that it is possible to obtain error bounds for $(\mathcal{K}^1 \cap \mathcal{K}^2, \mathcal{L}, a)$ through $(\mathcal{K}^1 \times \mathcal{K}^2, \hat{\mathcal{L}}, \hat{a})$. Recall that, by convention (see Section 2.1), the inner product in $\mathcal{E} \times \mathcal{E}$ is such that if $(x, y), (\hat{x}, \hat{y}) \in \mathcal{E} \times \mathcal{E}$, we have $\langle (x, y), (\hat{x}, \hat{y}) \rangle = \langle x, \hat{x} \rangle + \langle y, \hat{y} \rangle$. Then, for $x \in \mathcal{E}$, it can be verified that

$$\text{dist}(x, \mathcal{K}^1 \cap \mathcal{K}^2) \leq \epsilon \Rightarrow \text{dist}((x, x), \mathcal{K}^1 \times \mathcal{K}^2) \leq \sqrt{2}\epsilon \quad (37)$$

$$\text{dist}(x, \mathcal{L} + a) \leq \epsilon \Rightarrow \text{dist}((x, x), \hat{\mathcal{L}} + \hat{a}) \leq \sqrt{2}\epsilon \quad (38)$$

$$\text{dist}(x, \mathcal{K}^1 \cap \mathcal{K}^2 \cap (\mathcal{L} + a)) \leq \frac{1}{\sqrt{2}} \text{dist}((x, x), (\mathcal{K}^1 \times \mathcal{K}^2) \cap (\hat{\mathcal{L}} + \hat{a})). \quad (39)$$

Then, the next proposition follows immediately from Theorem 23.

Proposition 40 (Error bound for the intersection of amenable cones). *Suppose $\mathcal{K}^1 \subseteq \mathcal{E}$ and $\mathcal{K}^2 \subseteq \mathcal{E}$ are amenable cones. Suppose also that $(\mathcal{K}^1 \cap \mathcal{K}^2, \mathcal{L}, a)$ is feasible. Let $\hat{\mathcal{L}}, \hat{a}$ be such that*

$$\hat{\mathcal{L}} + \hat{a} = \{(x, x) \mid x \in \mathcal{L} + a\}.$$

The following hold.

(i) Let

$$\mathcal{F}_\ell \subsetneq \dots \subsetneq \mathcal{F}_1 = \mathcal{K}^1 \times \mathcal{K}^2$$

be a chain of faces of $\mathcal{K}^1 \times \mathcal{K}^2$ together with $z_i \in \mathcal{F}_i^* \cap \hat{\mathcal{L}}^\perp \cap \{\hat{a}\}^\perp$ such that that $(\mathcal{F}_\ell, \hat{\mathcal{L}}, \hat{a})$ satisfies the PPS condition and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i .

For $i = 1, \dots, \ell - 1$, let ψ_i be a facial residual function of $\mathcal{K}^1 \times \mathcal{K}^2$ with respect to \mathcal{F}_i, z_i . Then, after positive rescaling the ψ_i , there is a positive constant κ (depending on $\mathcal{K}^1, \mathcal{K}^2, \mathcal{L}, a$) such that whenever $x \in \text{span}(\mathcal{K}^1 \cap \mathcal{K}^2)$ satisfies the inequalities

$$\text{dist}(x, \mathcal{K}^1 \cap \mathcal{K}^2) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}^1 \cap \mathcal{K}^2) \leq (\kappa\|x\| + \kappa)(\epsilon + \varphi(\epsilon, \|x\|)),$$

where $\varphi = \psi_{\ell-1} \diamond \dots \diamond \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

(ii) There exists at least one chain satisfying the requirements in item (i) of length $d_{PPS}(\hat{\mathcal{L}}, \hat{a}) + 1$, where $d_{PPS}(\hat{\mathcal{L}}, \hat{a})$ is the minimum number of facial reduction steps necessary to find a face satisfying the PPS condition for the problem $(\mathcal{K}^1 \times \mathcal{K}^2, \hat{\mathcal{L}}, \hat{a})$. The following inequality holds.

$$d_{PPS}(\hat{\mathcal{L}}, \hat{a}) \leq \min\{\ell_{\text{poly}}(\mathcal{K}^1) + \ell_{\text{poly}}(\mathcal{K}^2), \dim(\hat{\mathcal{L}}^\perp \cap \{a\}^\perp), d_S(\hat{\mathcal{L}}, \hat{a})\}.$$

Proof. Item (i) is a consequence of applying Theorem 23 to $(\mathcal{K}^1 \times \mathcal{K}^2, \hat{\mathcal{L}}, \hat{a})$ together with (37), (38) and (39), rescaling the functions ψ_i if necessary. Item (ii) is a direct consequence of Proposition 24. \square

We conclude this subsection with an application of Proposition 40. Let \mathcal{N}^n denote the cone of $n \times n$ symmetric matrices with nonnegative entries. Then, the doubly nonnegative cone \mathcal{D}^n is defined as the intersection $\mathcal{S}_+^n \cap \mathcal{N}^n$. It corresponds to the matrices that are simultaneously positive semidefinite and nonnegative. The cone \mathcal{D}^n has found many applications recently, see [54, 25, 1, 2].

Proposition 41 (Error bound for the doubly nonnegative cone). *Suppose $(\mathcal{D}^n, \mathcal{L}, a)$ is feasible. Then, there is a positive constant κ (depending on n, \mathcal{L}, a) such that whenever x satisfies the inequalities*

$$\text{dist}(x, \mathcal{D}^n) \leq \epsilon, \quad \text{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{D}^n) \leq (\kappa\|x\| + \kappa) \left(\sum_{j=0}^{d_{PPS}(\hat{\mathcal{L}}, \hat{a})} \epsilon^{(2^{-j})} \|x\|^{1-2^{-j}} \right),$$

where $\hat{\mathcal{L}}$ and \hat{a} are as in Proposition 40. Furthermore,

$$d_{PPS}(\hat{\mathcal{L}}, \hat{a}) \leq \min\{n - 1, \dim(\hat{\mathcal{L}}^\perp \cap \{\hat{a}\}^\perp), d_S(\hat{\mathcal{L}}, \hat{a})\}.$$

Proof. We apply Proposition 40 to $\mathcal{D}^n = \mathcal{S}_+^n \cap \mathcal{N}^n$. From item (i) of Proposition 17, we know that facial residual functions for $\mathcal{S}_+^n \times \mathcal{N}^n$ can be taken to be positive rescalings of the sum of facial residual functions for \mathcal{S}_+^n and \mathcal{N}^n . From Theorem 35 and Proposition 18, we conclude that facial residual functions for $\mathcal{S}_+^n \times \mathcal{N}^n$ can be taken to be

$$\psi_i(\epsilon, \|x\|) = \kappa_i \epsilon + \kappa_i \sqrt{\epsilon \|x\|}.$$

Then, we apply Lemma 36 and proceed as in the proof of Theorem 37.

The bound on $d_{PPS}(\hat{\mathcal{L}}, \hat{a})$ follows from item (ii) of Proposition 40 and the fact the $\ell_{\text{poly}}(\mathcal{S}_+^n) \leq n - 1$ (see Remark 39) and $\ell_{\text{poly}}(\mathcal{N}^n) = 0$, since \mathcal{N}^n is a polyhedral cone. \square

We can also prove a result similar to Proposition 38. For example, if we impose $\epsilon \leq 1$ and $\|x\| \leq \rho$ and use the fact that $d_{\text{PPS}}(\hat{\mathcal{L}}, \hat{a}) \leq n - 1$, we may adjust the constant κ so that the bound becomes

$$\text{dist}(x, (\mathcal{L} + a) \cap \mathcal{D}^n) \leq \kappa \epsilon^{(2^{1-n})}, \quad (40)$$

where, analogous to Proposition 38, κ depends on $\mathcal{D}^n, \mathcal{L}, a, \rho$.

Remark 42. In Example 2 of [47], Sturm constructed a subspace $\mathcal{L} \subseteq \mathcal{S}^n$ and a sequence of matrices $\{x_\epsilon \mid \epsilon > 0\}$ contained in \mathcal{S}_+^n , such that $\text{dist}(x_\epsilon, \mathcal{L}) < \epsilon$ but $\text{dist}(x_\epsilon, \mathcal{S}_+^n \cap \mathcal{L}) \geq \epsilon^{1/2^{n-1}}$, for every ϵ . This happens because all matrices in $\mathcal{S}_+^n \cap \mathcal{L}$ are such that their $(1, n)$ entry is 0, while the $(1, n)$ entry of x_ϵ is $\epsilon^{1/2^{n-1}}$.

Therefore, apart from the constant κ , Sturm's error bound can be tight. However, a closer inspection shows that the x_ϵ are, in fact, doubly nonnegative matrices and the same reasoning shows that $\text{dist}(x_\epsilon, \mathcal{D}^n \cap \mathcal{L}) \geq \epsilon^{1/2^{n-1}}$. It follows that (40) is also tight in the same sense.

5 Conclusion and open problems

In this paper, we introduced the concepts of *amenable cones* and *facial residual functions* (FRFs), which makes it possible to derive error bound results for problems that do not satisfy regularity conditions. As applications, we gave Hölderian error bounds for symmetric cones (Theorem 37 and Proposition 38) and for doubly nonnegative cones (Proposition 41). In summary, given some pointed closed convex cone \mathcal{K} , we need the following ingredients for obtaining error bounds under our approach:

1. First, it is necessary to prove that \mathcal{K} is *amenable* (Definition 8).
2. Then, we must work out the facial residual functions as in Theorem 35.
3. Finally, we apply Theorem 23 and try to obtain an upper bound for the composition of facial residual functions as in Lemma 36. We can further restrict ϵ (as in Propositions 38) and/or change the distance functions (Remark 25).

If it is hard to prove that \mathcal{K} is amenable, we might be able to express \mathcal{K} as an intersection of amenable cones and apply Proposition 40.

We will now point out some open questions and directions for future work.

1. *Which cones are amenable? Do they admit simple facial residual functions?* We proved that symmetric cones are amenable, but how about homogeneous cones? An answer to this question might follow from the fact that homogeneous cones are “slices of positive semidefinite cones”, see the work by Chua [10] and Proposition 1 together with section 4 of the work by Faybusovich [16].

Another interesting family of cones to investigate are the p -cones. They are defined as

$$\mathcal{L}_p^n = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid t \geq \|x\|_p\},$$

where $\|\cdot\|_p$ denotes the p -norm. For $p = 1$ or $p = \infty$ or $n < 3$, \mathcal{L}_p^n is polyhedral and hence it is amenable by Proposition 9. If $p = 2$, then \mathcal{L}_p^n becomes the second order cone which is a symmetric cone, so Theorem 35 applies. It remains to analyze the case $1 < p < \infty$, $p \neq 2$ and $n \geq 3$. For those p , since \mathcal{L}_p^n is strictly convex, we know from Proposition 9 that \mathcal{L}_p^n is amenable. However, we do not know how obtain FRFs that are simpler than the canonical one. We remark that it was recently shown that \mathcal{L}_p^n is not homogeneous for those p , see the work by Ito and Lourenço [24]. Therefore, computing FRFs for homogeneous cones will not be helpful here. We conjecture that \mathcal{L}_p^n admit FRFs of the form $\kappa\epsilon + \kappa(\|x\|\epsilon)^{1/p}$.

2. It might be possible to relax Definition 8 and obtain error bound results for cones that are not amenable. For example, we could require that for every face \mathcal{F} of \mathcal{K} , there should be some $\kappa > 0$ and $\gamma \in (0, 1]$ such that

$$\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K})^\gamma,$$

for every $x \in \text{span } \mathcal{F}$ satisfying $\text{dist}(x, \mathcal{K}) \leq 1$. For cones satisfying this property, it seems that a result similar to Proposition 19 might hold. In this case, it could be possible to obtain a result analogous to Theorem 23.

A Miscellaneous proofs

Proof of Proposition 11

- (i) Let \mathcal{F} be a face of $\mathcal{K}^1 \times \mathcal{K}^2$. We have $\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^2$, where \mathcal{F}^1 and \mathcal{F}^2 are faces of \mathcal{K}^1 and \mathcal{K}^2 respectively. From our assumptions in Section 2, Equation (2) and the amenability of \mathcal{K}^1 and \mathcal{K}^2 , it follows that there are positive constants κ_1, κ_2 such that

$$\text{dist}((x_1, x_2), \mathcal{F}) \leq \sqrt{\kappa_1^2 \text{dist}(x_1, \mathcal{K}^1)^2 + \kappa_2^2 \text{dist}(x_2, \mathcal{K}^2)^2},$$

whenever $x_1 \in \text{span } \mathcal{F}^1$ and $x_2 \in \text{span } \mathcal{F}^2$. Therefore,

$$\begin{aligned} \text{dist}((x_1, x_2), \mathcal{F}) &\leq \max\{\kappa_1, \kappa_2\} \sqrt{\text{dist}(x_1, \mathcal{K}^1)^2 + \text{dist}(x_2, \mathcal{K}^2)^2} \\ &= \max\{\kappa_1, \kappa_2\} \text{dist}((x_1, x_2), \mathcal{K}^1 \times \mathcal{K}^2), \end{aligned}$$

whenever $(x_1, x_2) \in \text{span}(\mathcal{F}^1 \times \mathcal{F}^2) = (\text{span } \mathcal{F}^1) \times (\text{span } \mathcal{F}^2)$.

- (ii) If \mathcal{A} is the zero map, we are done, since $\{0\}$ is amenable. So, suppose that \mathcal{A} is a nonzero injective linear map. Then, the faces of $\mathcal{A}(\mathcal{K})$ are images of faces of \mathcal{K} by \mathcal{A} . Accordingly, let $\mathcal{F} \trianglelefteq \mathcal{A}(\mathcal{K})$. Because \mathcal{K} is amenable, there is κ such that

$$\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K}), \quad \forall x \in \text{span } \mathcal{F}. \quad (41)$$

As \mathcal{A} is a linear map, we have $\text{span } \mathcal{A}(\mathcal{F}) = \mathcal{A}(\text{span } \mathcal{F})$. Let $\sigma_{\min}, \sigma_{\max}$ denote, respectively, the minimum and maximum singular values of \mathcal{A} . We have

$$\sigma_{\min} = \min\{\|Ax\| \mid \|x\| = 1\}, \quad \sigma_{\max} = \max\{\|Ax\| \mid \|x\| = 1\}.$$

They are both positive since \mathcal{A} is injective and nonzero. Now, let $x \in \text{span } \mathcal{F}$, then

$$\begin{aligned} \text{dist}(\mathcal{A}(x), \mathcal{A}(\mathcal{F})) &\leq \sigma_{\max} \text{dist}(x, \mathcal{F}) \\ &\leq \kappa \sigma_{\max} \text{dist}(x, \mathcal{K}) && \text{(From (41))} \\ &\leq \frac{\kappa \sigma_{\max}}{\sigma_{\min}} \text{dist}(\mathcal{A}(x), \mathcal{A}(\mathcal{K})). \end{aligned}$$

Proof of Proposition 12

Proof. $(i) \Rightarrow (ii)$ Let $x, u \in \mathcal{E}$ be such that $x + u \in \text{span } \mathcal{F}$ and $\|u\| = \text{dist}(x, \text{span } \mathcal{F})$. Since $\text{dist}(\cdot, \mathcal{K})$ and $\text{dist}(\cdot, \mathcal{F})$ are sublinear functions, we have that (4) implies that

$$\begin{aligned} \text{dist}(x, \mathcal{F}) &\leq \text{dist}(-u, \mathcal{F}) + \text{dist}(x + u, \mathcal{F}) \\ &\leq \text{dist}(x, \text{span } \mathcal{F}) + \kappa(\text{dist}(x + u, \mathcal{K})) \\ &\leq \text{dist}(x, \text{span } \mathcal{F}) + \kappa(\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F})) \\ &\leq (1 + \kappa)(\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F})), \quad \forall x \in \mathcal{E}. \end{aligned} \quad (42)$$

Here we used the fact that $\text{dist}(-u, \mathcal{F}) \leq \| -u \|$, since $0 \in \mathcal{F}$. This shows that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ Since \mathcal{K} and $\text{span } \mathcal{F}$ intersect at 0 subtransversally, there is $\delta > 0$ such that

$$\text{dist}(z, \mathcal{F}) \leq \kappa(\text{dist}(z, \mathcal{K}) + \text{dist}(z, \text{span } \mathcal{F})), \quad \forall z \text{ with } \|z\| \leq \delta.$$

Therefore, if $x \in \mathcal{E}$ is nonzero, we have

$$\text{dist}\left(\delta \frac{x}{\|x\|}, \mathcal{F}\right) \leq \kappa\left(\text{dist}\left(\delta \frac{x}{\|x\|}, \mathcal{K}\right) + \text{dist}\left(\delta \frac{x}{\|x\|}, \text{span } \mathcal{F}\right)\right).$$

Now, we recall that if C is a convex cone, then $\text{dist}(\alpha x, C) = \alpha \text{dist}(x, C)$ for every positive α . We conclude that

$$\text{dist}(x, \mathcal{F}) \leq \kappa(\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F})), \quad \forall x \in \mathcal{E}.$$

Therefore, if $x \in \text{span } \mathcal{F}$, then $\text{dist}(x, \mathcal{F}) \leq \kappa \text{dist}(x, \mathcal{K})$.

$(i) \Rightarrow (iii)$ The inequality in (42) shows that

$$\text{dist}(x, \mathcal{F}) \leq (2 + 2\kappa) \max(\text{dist}(x, \mathcal{K}), \text{dist}(x, \text{span } \mathcal{F})), \quad \forall x \in \mathcal{E}.$$

Therefore, \mathcal{K} and $\text{span } \mathcal{F}$ are boundedly linearly regular.

$(iii) \Rightarrow (ii)$ Let $U = \{x \in \mathcal{E} \mid \|x\| \leq 1\}$. Then, there exists κ_U such that

$$\begin{aligned} \text{dist}(x, \mathcal{F}) &\leq \kappa_U \max(\text{dist}(x, \mathcal{K}), \text{dist}(x, \text{span } \mathcal{F})) \\ &\leq \kappa_U (\text{dist}(x, \mathcal{K}) + \text{dist}(x, \text{span } \mathcal{F})), \quad \forall x \in U. \end{aligned}$$

Therefore, \mathcal{K} and $\text{span } \mathcal{F}$ intersect subtransversally at 0. □

Proof of Proposition 17

1. Suppose that $x \in \text{span } \mathcal{K}$ satisfies the inequalities

$$\text{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon. \quad (43)$$

Note that

$$\mathcal{F} \cap \{z\}^\perp = (\mathcal{F}^1 \cap \{z_1\}^\perp) \times (\mathcal{F}^2 \cap \{z_2\}^\perp).$$

Also, due to our assumptions (Section 2.1), we have

$$\|x - y\|^2 = \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2$$

for every $x, y \in \mathcal{E}^1 \times \mathcal{E}^2$. Thus we have the following implications:

$$\text{dist}(x, \mathcal{K}) \leq \epsilon \quad \Rightarrow \quad \text{dist}(x_1, \mathcal{K}^1) \leq \epsilon, \quad \text{dist}(x_2, \mathcal{K}^2) \leq \epsilon \quad (44)$$

$$\text{dist}(x, \text{span } \mathcal{F}) \leq \epsilon \quad \Rightarrow \quad \text{dist}(x_1, \text{span } \mathcal{F}^1) \leq \epsilon, \quad \text{dist}(x_2, \text{span } \mathcal{F}^2) \leq \epsilon \quad (45)$$

The first step is showing that there are positive constants κ_1 and κ_2 such that for all $x \in \mathcal{E}^1 \times \mathcal{E}^2$, we also have

$$x \text{ satisfies (43)} \quad \Rightarrow \quad \langle x_1, z_1 \rangle \leq \kappa_1 \epsilon \quad \text{and} \quad \langle x_2, z_2 \rangle \leq \kappa_2 \epsilon. \quad (46)$$

Suppose x satisfies (43). By (45), we have $\text{dist}(x_1, \text{span } \mathcal{F}^1) \leq \epsilon$. Therefore, there exists $y_1 \in \mathcal{E}^1$ such that $x_1 + y_1 \in \text{span } \mathcal{F}^1$ and $\|y_1\| \leq \epsilon$. Due to (44) and the amenability of \mathcal{K}^1 , there exists $\hat{\kappa}_1$ (not depending on x_1) such that

$$\text{dist}(x_1 + y_1, \mathcal{F}^1) \leq \hat{\kappa}_1 \text{dist}(x_1 + y_1, \mathcal{K}^1) \leq 2\epsilon \hat{\kappa}_1.$$

Therefore, there exists $v_1 \in \mathcal{E}^1$ such that $\|v_1\| \leq 2\epsilon\hat{\kappa}_1$ and

$$x_1 + y_1 + v_1 \in \mathcal{F}^1.$$

In a completely analogous manner, there is a constant $\hat{\kappa}_2 > 0$ and there are $y_2, v_2 \in \mathcal{E}^2$ such

$$x_2 + y_2 + v_2 \in \mathcal{F}^2,$$

with $\|y_2\| \leq \epsilon$ and $\|v_2\| \leq 2\epsilon\hat{\kappa}_2$. It follows that

$$\langle (x_1 + y_1 + v_1, x_2 + y_2 + v_2), (z_1, z_2) \rangle \leq M\epsilon,$$

for $M = 1 + \|z_1\| + 2\hat{\kappa}_1 + \|z_2\| + 2\hat{\kappa}_2$. Since $\langle x_1 + y_1 + v_1, z_1 \rangle \geq 0$ and $\langle x_2 + y_2 + v_2, z_2 \rangle \geq 0$, we get

$$\langle x_i + y_i + v_i, z_i \rangle \leq M\epsilon,$$

for $i = 1, 2$. We then conclude that

$$\langle x, z \rangle \leq \epsilon \quad \Rightarrow \quad \langle x_i, z_i \rangle \leq \kappa_i \epsilon, \quad (47)$$

whenever x satisfies (44) and (45), where $\kappa_i = M + \|z_i\| + \|z_i\|2\hat{\kappa}_i$.

Now, let $\psi_{\mathcal{F}_1, z_1}$ and $\psi_{\mathcal{F}_2, z_2}$ be arbitrary facial residual functions for \mathcal{F}_1, z_1 and \mathcal{F}_2, z_2 , respectively. We positive rescale $\psi_{\mathcal{F}_1, z_1}$ and $\psi_{\mathcal{F}_2, z_2}$ so that

$$\text{dist}(x_i, \mathcal{K}) \leq \epsilon, \quad \langle x_i, z_i \rangle \leq \kappa_i \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}_i) \leq \epsilon$$

implies $\text{dist}(x_i, \hat{\mathcal{F}}_i) \leq \psi_{\mathcal{F}_i, z_i}(\epsilon, \|x_i\|)$, for $i = 1, 2$.

Finally, from (44), (45), (47) and using the fact that $\psi_{\mathcal{F}_1, z_1}$ and $\psi_{\mathcal{F}_2, z_2}$ are monotone nondecreasing on the second argument we conclude that whenever x satisfies (43) we have

$$\begin{aligned} \text{dist}(x, \hat{\mathcal{F}}) &= \sqrt{\text{dist}(x_1, \hat{\mathcal{F}}^1)^2 + \text{dist}(x_2, \hat{\mathcal{F}}^2)^2} \\ &\leq \text{dist}(x_1, \hat{\mathcal{F}}^1) + \text{dist}(x_2, \hat{\mathcal{F}}^2) \\ &\leq \psi_{\mathcal{F}_1, z_1}(\epsilon, \|x\|) + \psi_{\mathcal{F}_2, z_2}(\epsilon, \|x\|). \end{aligned}$$

Therefore, $\psi_{\mathcal{F}_1, z_1} + \psi_{\mathcal{F}_2, z_2}$ is a facial residual function for \mathcal{F}, z .

2. The proposition is true if \mathcal{A} is the zero map, so suppose that \mathcal{A} is a nonzero injective linear map. First, we observe that

$$(\mathcal{A}(\mathcal{F})) \cap \{z\}^\perp = \mathcal{A}(\mathcal{F} \cap \{\mathcal{A}^\top z\}^\perp).$$

Let $\hat{\mathcal{F}} = \mathcal{F} \cap \{\mathcal{A}^\top z\}^\perp$. Let $\psi_{\mathcal{F}, \mathcal{A}^\top z}$ be a facial residual function for \mathcal{F} and $\mathcal{A}^\top z$. Let σ_{\min} denote the minimum singular value of \mathcal{A} . We note that σ_{\min} is positive because \mathcal{A} is injective. We positive rescale $\psi_{\mathcal{F}, \mathcal{A}^\top z}$ so that whenever x satisfies

$$\text{dist}(x, \mathcal{K}) \leq \frac{1}{\sigma_{\min}} \epsilon, \quad \langle x, \mathcal{A}^\top z \rangle \leq \epsilon, \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \frac{1}{\sigma_{\min}} \epsilon$$

we have:

$$\text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, \mathcal{A}^\top z}(\epsilon, \|x\|).$$

Then, we have the following implications:

$$\begin{aligned} \text{dist}(\mathcal{A}(x), \mathcal{A}(\mathcal{K})) \leq \epsilon &\quad \Rightarrow \quad \text{dist}(x, \mathcal{K}) \leq \frac{1}{\sigma_{\min}} \epsilon \\ \langle \mathcal{A}(x), z \rangle \leq \epsilon &\quad \Leftrightarrow \quad \langle x, \mathcal{A}^\top z \rangle \leq \epsilon \\ \text{dist}(\mathcal{A}(x), \text{span } \mathcal{A}(\mathcal{F})) \leq \epsilon &\quad \Rightarrow \quad \text{dist}(x, \text{span } \mathcal{F}) \leq \frac{1}{\sigma_{\min}} \epsilon \\ \text{dist}(\mathcal{A}(x), \mathcal{A}(\hat{\mathcal{F}})) \leq \sigma_{\max} \psi_{\mathcal{F}, \mathcal{A}^\top z}(\epsilon, \|\mathcal{A}x\|/\sigma_{\min}) &\quad \Leftrightarrow \quad \text{dist}(x, \hat{\mathcal{F}}) \leq \psi_{\mathcal{F}, \mathcal{A}^\top z}(\epsilon, \|x\|), \end{aligned}$$

where σ_{\max} is the maximum singular value of \mathcal{A} . This shows that we can use

$$\tilde{\psi}_{\mathcal{A}(\mathcal{F}),z}(\epsilon, \|\mathcal{A}x\|) = \sigma_{\max}\psi(\epsilon, \|\mathcal{A}x\|/\sigma_{\min})$$

as a facial residual function for $\mathcal{A}(\mathcal{F})$ and z .

Acknowledgements

We thank the editors and four referees for their insightful comments, which helped to improve the paper substantially. In particular, the discussion on tangentially exposed cones and subtransversality was suggested by Referee 1. Also, comments from Referees 1 and 2 motivated Remark 10. We would like to thank Prof. Gábor Pataki for helpful advice and for suggesting that we take a look at projectionally exposed cones. Incidentally, this was also suggested by Referee 4. Referee 4 also suggested the remark on the tightness of the error bound for doubly nonnegative matrices. Feedback and encouragement from Prof. Tomonari Kitahara, Prof. Masakazu Muramatsu and Prof. Takashi Tsuchiya were highly helpful and they provided the official translation of “amenable cone” to Japanese: 恭順錐 (*kyoujunsui*). This work was partially supported by the Grant-in-Aid for Scientific Research (B) (18H03206) and the Grant-in-Aid for Young Scientists (19K20217) from Japan Society for the Promotion of Science.

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