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Long-Step Path-Following Algorithm for Solving Symmetric Programming Problems with Nonlinear Objective Functions

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Abstract We describe a long-step path-following algorithm for a class of symmetric programming problems with nonlinear convex objective functions. The complexity estimates similar to the case of a linear-quadratic objective function are established. The results of numerical experiments for the class of optimization problems involving quantum entropy are presented.

Keywords Symmetric programming · Nonlinear objective functions · Interior-point methods

1 Introduction

In this paper we consider a long-step path-following algorithm for a symmetric programming problem with a nonlinear convex objective function f. We assume that f is compatible (in the sense of [17], p.66) with the standard self-concordant barrier $-\ln\det(x)$, $x \in \Omega$, where Ω is a symmetric (i.e., self-dual, homogeneous) cone. Unlike [17] we develop a direct (i.e., without any reductions) long-step path-following algorithm and prove complexity estimates similar to the case of linear-quadratic objective function considered in [4].

The direct approach we are using allows us to consider linear equality constraints in explicit form, work directly with a nonlinear objective function and avoid imposing conditions of the type (4.2.3) in [18] in an explicit form. In other words, our approach allows one to use all of the structural properties of a natural formulation of the problem.

The plan of the paper is as follows. In section 2 we briefly describe Jordan-algebraic concepts used in the paper. In section 3 we collected necessary properties

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of self-concordant functions and described Newton method applied to such functions in the presence of linear equality constraints. In section 4 we described properties of the central path and pertinent properties of self-concordant families. In section 5 we consider a long-setp path-following algorithm and establish complexity estimates.

We follow quite closely to the scheme of [4] where the case of a linear-quadratic objective function has been considered (but see also [11, Section 2.5] and [12] where the case of minimization of a convex objective function subject to finite number of convex inequality constraints has been analyzed in a similar fashion). Section 6 is devoted to numerical experiments for optimization problems involving minimization of quantum (von Neumann) entropy [2].

2 Jordan-algebraic Concepts

We adhere to the notation of an excellent book [3].

Let $\mathbf{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. A vector space V over $\mathbf{F}$ is called an algebra over $\mathbf{F}$ if a bilinear mapping $(x, y) \to xy$ from $V \times V$ into V is defined. For an element x in V let $L(x) : V \to V$ be the linear map such that

$$L(x)y = xy.$$

An algebra V over $\mathbf{F}$ is a Jordan algebra if

$$xy = yx, \quad x(x^2 y) = x^2(xy), \ \forall x, y \in V.$$

In other words, Jordan algebra is always commutative but typically not associative. In an algebra V one defines x^n recursively by $x^n = x \cdot x^{n-1}$. An algebra V is said to be power associative if $x^p \cdot x^q = x^{p+q}$ for any $x \in V$ and integers p, q.

Proposition 2.1 *A Jordan algebra is power associative. Besides,*

$$[L(x^p), L(x^q)] = 0, \ \forall x \in V,$$

and any positive integers p and q. (In other words, the corresponding linear operators commute).

This is Proposition II.1.2 in [3]. We will always assume that the Jordan algebra has an identity element e (i.e., $xe = x, \ \forall x \in V$).

Let V be a finite-dimensional power associative algebra over $\mathbf{F}$ with an identity element e, and let $F[X]$ denote the algebra over $\mathbf{F}$ of polynomials in one variable with coefficients in $\mathbf{F}$. For $x \in V$ we define

$$\mathbf{F}[x] = \{p(x) : p \in \mathbf{F}[X]\}.$$

A nonzero polynomial $p \in \mathbf{F}[X]$ of minimal possible degree such that $p(x) = 0$ is called the minimal polynomial of x. Given $x \in V$, let $m(x)$ be the degree of the minimal polynomial of x. We define the rank of V as

$$r = \max\{m(x) : x \in V\}.$$

An element x is called regular if $m(x) = r$.

Proposition 2.2 *The set of regular elements is open and dense in V. There exist polynomials $a_1, \ldots, a_r$ on V such that the minimal polynomial of every regular element x is given by*

$$f(\lambda; x) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \ldots + (-1)^r a_r(x).$$

The polynomials $a_1, \ldots, a_r$ are unique and a_j is homogeneous of degree j.

This is Proposition II.2.1 in [3]. The coefficient $a_1(x)$ is called the trace of x and is denoted $\mathrm{tr}(x)$ (in particular, trace is linear). The coefficient $a_r(x)$ is called the determinant of x and is denoted $\det(x)$. An element x is said to be invertible if there exists an element $y \in \mathbf{F}[x]$ such that $xy = e$. The set $\lambda \in \mathbf{F}$ such that $x - \lambda e$ is not invertible is called the spectrum of x and is denoted $\mathrm{spec}(x)$.

Given $x \in V$, we define

$$P(x) = 2L(x)^2 - L(x^2).$$

The map P is called the quadratic representation of V. We denote $DP(x)y$ by $2P(x, y)$. Here $DP(x)y$ is the Fréchet derivative of the map P at point $x \in V$ evaluated on $y \in V$. It is easy to see that

$$P(x, y) = L(x)L(y) + L(y)L(x) - L(xy), \; x, y \in V.$$

Proposition 2.3 *Let V be a finite-dimensional Jordan algebra over $\mathbf{F}$. An element $x \in V$ is invertible if and only if $P(x)$ is invertible. In this case*

$$P(x)x^{-1} = x, \quad P(x)^{-1} = P(x^{-1}).$$

This is Proposition II.3.1 in [3].

Proposition 2.4 *Let $\mathcal{J}$ be the (open) set of invertible elements in V. The map $x \to x^{-1} : \mathcal{J} \to \mathcal{J}$ is Fréchet differentiable and*

(i) $D(x^{-1})u = -P(x^{-1})u, x \in \mathcal{J}, u \in V$.
(ii) *If x and y are invertible, then $P(x)y$ is invertible and* $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.
(iii) $P(P(x)y) = P(x)P(y)P(x), \; \forall x, y \in V$.
(iv) $P(P(x)y, P(x)z) = P(x)P(y, z)P(x), \; \forall x, y, z \in V$.

This is Proposition II.3.3 in [3].

A bilinear form β on V is called associative if

$$\beta(xy, z) = \beta(x, yz), \; \forall x, y, z \in V.$$

Proposition 2.5 *The symmetric bilinear forms* $\mathrm{Tr}(L(xy))$ *and* $\mathrm{tr}(xy)$ *are associative.*

This is Proposition II.4.3 in [3]. Here $\mathrm{Tr}(L(xy))$ is the usual trace of the linear operator $L(xy) : V \to V$.

In case where $\mathbf{F} = \mathbb{R}$, we consider an important class of Euclidean Jordan algebras. A Jordan algebra V over $\mathbb{R}$ is called Euclidean if $\mathrm{tr}(x^2) > 0, \forall x \in V \setminus \{0\}$. An element $c \in V$ is called idempotent if $c^2 = c$. Two idempotents are orthogonal if $cd = 0$. A system of idempotents $c_1, \ldots, c_k$ is a complete system of orthogonal idempotents if $c_i^2 = c_i, \; c_i c_j = 0, \; i \neq j$, and $c_1 + \ldots + c_k = e$.

Theorem 2.1 *Let V be an Euclidean Jordan algebra. Given $x \in V$, there exist unique real numbers $\lambda_1, \dots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $c_1, \dots, c_k$ such that*

$$x = \lambda_1 c_1 + \cdots + \lambda_k c_k.$$

In this case $\operatorname{spec}(x) = \{\lambda_1, \dots, \lambda_k\},\ c_1, \dots, c_k \in \mathbb{R}[x]$.

This is Theorem III.1.1 in [3].

An idempotent is primitive if it is non-zero and cannot be written as a sum of two non-zero idempotents. We say that $c_1, \dots, c_m$ is a complete system of orthogonal primitive idempotents, or Jordan frame, if each c_j is primitive idempotent and if

$$c_j c_k = 0,\ j \neq k,\ c_1 + \ldots + c_m = e.$$

Note that in this case $m = r$ (rank of V).

Theorem 2.2 *Suppose V has rank r. Then for $x \in V$ there exists a Jordan frame $c_1, \dots, c_r$ and real numbers $\lambda_1, \dots, \lambda_r$ such that*

$$x = \sum_{j=1}^{r} \lambda_j c_j.$$

The numbers λ_j (with multiplicities) are uniquely determined by x. Furthermore,

$$\det(x) = \prod_{j=1}^{r} \lambda_j, \quad \operatorname{tr}(x) = \sum_{j=1}^{r} \lambda_j.$$

This is Theorem III.1.2 in [3].

There exists a canonical scalar product on V defined as $\langle x, y \rangle = \operatorname{tr}(xy)$, $x, y \in V$. We will use the notation $\|x\|$ for $\langle x, x \rangle^{\frac{1}{2}}$, $x \in V$.

Given a function f which is defined at least on $\operatorname{spec}(x)$, we can define

$$f(x) = \sum_{i=1}^{r} f(\lambda_i) c_i,$$

if $x = \sum_{i=1}^{r} \lambda_i c_i$. In particular,

$$\exp(x) = \sum_{i=1}^{r} \exp(\lambda_i) c_i, \quad \ln x = \sum_{i=1}^{r} \ln \lambda_i c_i,\ \lambda_i > 0.$$

Convexity and differentiability of such functions on Euclidean Jordan algebras have been studied in [5]. We extensively use these properties in the paper.

Let

$$\overline{\Omega} = \{x^2 : x \in V\}.$$

Theorem 2.3 *Let V be an Euclidean Jordan algebra.The interior Ω of $\overline{\Omega}$ is a symmetric (i.e., self-dual, homogeneous) convex cone. Furthermore, Ω is the connected component of e in the set $\mathcal{J}$ of invertible elements, and also Ω is the set of elements x in V for which $L(x)$ is positive definite. In particular, the group of linear automorphisms $GL(\Omega)$ of Ω acts transitively on it. Moreover, $P(x) \in GL(\Omega)$ for any invertible x.*

This is Proposition III.2.2 in [3].

Let $c_1, \ldots, c_k$ be complete system of orthogonal idempotents. For each idempotent c, denote $V(c,0)$, $V(c,1)$, $V(c,1/2)$ the eigenspaces of $L(c)$ corresponding to eigenvalues 0, 1, 1/2 respectively. Then $L(c_1), \ldots, L(c_k)$ pairwise commute and

$$V = \bigoplus_{1 \le i \le j} V_{ij},$$

where $V_{ii} = V(c_i, 1)$, $V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2)$. Such a decomposition of V corresponding to a complete system of orthogonal idempotents is called the Peirce decomposition. It is studied in detail in Section 1 of Chapter IV in [3]. A typical example of a Jordan algebra over a field $\mathbf{F}$ is the vector space of symmetric matrices over $\mathbf{F}$ with multiplication operation

$$A \cdot B = \frac{AB + BA}{2},$$

where on the right we have a usual matrix multiplication. In case when $\mathbf{F} = \mathbb{R}$ we get an exmple of an Euclidean Jordan algebra.

Throughout this paper we will use small letters for the elements of Euclidean Jordan algebras. By capital letters we will usually denote elements of matrix algebras. Respectively, $\mathrm{Det}(X)$, $\mathrm{Tr}(X)$ will be used for matrix trace and matrix determinant.

3 Properties of self-concordant functions

Let $F : \Omega \to \mathbb{R}$ be three times continuously differentiable convex function on Ω. We say that F is self-concordant, if

$$|D^3F(x)(\xi,\xi,\xi)| \le 2[D^2F(x)(\xi,\xi)]^{3/2}, \; x \in \Omega, \; \xi \in V. \tag{1}$$

We will assume throughout this section that

$$F(x) \to +\infty, \; x \to \partial\Omega \tag{2}$$

Recall that gradient $\nabla F(x) \in V$ and Hessian $H_F(x) : V \to V$ are defined as follows:

$$DF(x)\xi = \langle \nabla F(x), \xi \rangle, \; x \in \Omega, \; \xi \in V, \tag{3}$$

$$D^2F(x)(\xi,\eta) = \langle H_F(x)\xi, \eta \rangle, \; x \in \Omega, \; \xi, \eta \in V. \tag{4}$$

Here $D^kF(x)$ is the k-th Fréchet derivative of F at x. We will assume throughout this section that $H_F(x)$ is positive definite symmetric linear operator on V for any $x \in \Omega$. Given $\xi \in V$, $x \in \Omega$, we will use the following notation:

$$[D^2F(x)(\xi,\xi)]^{1/2} = \langle H_F(x)\xi, \xi \rangle^{1/2} = \|\xi\|_x . \tag{5}$$

Proposition 3.1 *Given $y \in \Omega$ such that $\rho = \|x - y\|_x < 1$, we have:*

$$(1-\rho)^2 \langle H_F(x)\xi, \xi\rangle \le \langle H_F(y)\xi, \xi\rangle \le \frac{1}{(1-\rho)^2}\langle H_F(x)\xi, \xi\rangle, \ \xi \in V. \tag{6}$$

Moreover, (under the assumption (2)*)*

$$W_r(x) = \{y \in \Omega : \|y - x\|_x \le r\} \subset \Omega, \ \forall r < 1. \tag{7}$$

This is a particular case of Theorem 2.1.1 in [17].

Lemma 3.1 *Let G_1, $G_2 : V \to V$ be linear symmetric operators such that*

$$|\langle \xi, G_1\xi\rangle| \le \langle \xi, G_2\xi\rangle, \ \forall \xi \in V.$$

Then

$$|\langle \xi_1, G_1\xi_2\rangle| \le \sqrt{\langle \xi_1, G_2\xi_1\rangle}\sqrt{\langle \xi_2, G_2\xi_2\rangle}, \ \forall \xi_1, \xi_2 \in V.$$

See Lemma B.1 in [11].

Lemma 3.2 *Let $x \in \Omega$, $\xi_1 \in V$, $\|\xi_1\|_x < 1$ and $\xi_2 \in V$. Consider the function $\varphi : [0, 1] \to \mathbb{R}$,*

$$\varphi(t) = \langle \xi_2, \nabla F(x + t\xi_1)\rangle + (1-t)\langle H_F(x)\xi_1, \xi_2\rangle. \tag{8}$$

Then

$$|\varphi'(t)| \le \left[\frac{1}{(1 - t\,\|\xi_1\|_x)^2} - 1\right] \|\xi_1\|_x \, \|\xi_2\|_x \,. \tag{9}$$

In particular,

$$|\varphi(t) - \varphi(0)| \le \int_0^t |\varphi'(\tau)| \,\mathrm{d}\tau \le \|\xi_1\|_x^2 \, \|\xi_2\|_x \, \frac{t^2}{1 - t\,\|\xi_1\|_x}. \tag{10}$$

Proof. By (8),

$$\varphi'(t) = \langle [H_F(x + t\xi_1) - H_F(x)]\, \xi_1, \xi_2\rangle.$$

By Proposition 3.1,

$$\langle [H_F(x + t\xi_1) - H_F(x)]\, \xi, \xi\rangle \le \left[\frac{1}{(1 - t\,\|\xi_1\|_x)^2} - 1\right] \langle H_F(x)\xi, \xi\rangle, \tag{11}$$

and

$$\left[(1 - t\,\|\xi_1\|)^2 - 1\right] \langle H_F(x)\xi, \xi\rangle \le \langle [H_F(x + t\xi_1) - H_F(x)]\, \xi, \xi\rangle, \ \forall \xi \in V. \tag{12}$$

Hence,

$$|\langle [H_F(x + t\xi_1) - H_F(x)]\, \xi, \xi\rangle| \le \left[\frac{1}{(1 - t\,\|\xi_1\|_x)^2} - 1\right] \langle H_F(x)\xi, \xi\rangle \tag{13}$$

By Lemma 3.1, inequality (13) implies (9). Integrating inequality (9), we obtain (10). □

Let $a \in V$, X be a vector subspace in V. Given $x \in (a+X) \cap \Omega$, we define Newton direction $p(x) \in X$ of F as follows:

$$H_F(x)p(x) = -(\nabla F(x) + \mu(x)), \tag{14}$$

where $\mu(x) \in X^{\perp}$ (orthogonal complement of X with respect to scalar product $\langle , \rangle$). We have the following (obvious) characterization of $\mu(x)$.

Proposition 3.2 *In* (14)*,* $\mu(x)$ *is a unique solution of the following optimization problem:*

$$\begin{aligned} &\psi(\lambda) = \langle H_F(x)^{-1}(\nabla F(x) + \lambda), (\nabla F(x) + \lambda)\rangle \rightarrow \min, \\ &\quad \lambda \in X^{\perp}. \end{aligned} \tag{15}$$

Moreover,

$$\psi(\mu(x)) = -\langle p(x), \nabla F(x)\rangle = \|p(x)\|_x^2 . \tag{16}$$

Proposition 3.3 *Let* $x \in \Omega$*,* $\|p(x)\|_x < 1$*. Then* $x^+ = x + p(x) \in \Omega$*, and*

$$\|p(x^+)\|_{x^+} \leq \frac{\|p(x)\|_x^2}{(1 - \|p(x)\|_x)^2}. \tag{17}$$

Proof. Consider the function $\varphi : [0, 1] \rightarrow \mathbb{R}$,

$$\varphi(t) = \langle p(x^+), \nabla F(x + tp(x))\rangle + (1 - t)\langle H_F(x)p(x), p(x^+)\rangle.$$

By Lemma 3.2

$$|\varphi(1) - \varphi(0)| \leq \frac{\|p(x)\|_x^2 \, \|p(x^+)\|_x}{1 - \|p(x)\|_x}. \tag{18}$$

Now,

$$\varphi(1) = \langle p(x^+), \nabla F(x^+)\rangle = - \|p(x^+)\|_{x^+}^2 , \tag{19}$$

where the last equality follows by (16). Furthermore,

$$\begin{aligned} \varphi(0) &= \langle p(x^+), \nabla F(x)\rangle + \langle H_F(x)p(x), p(x^+)\rangle \\ &= \langle p(x^+), \nabla F(x)\rangle - \langle \nabla F(x) + \mu(x), p(x^+)\rangle \\ &= \langle p(x^+), \nabla F(x)\rangle - \langle \nabla F(x), p(x^+)\rangle \\ &= 0, \end{aligned}$$

where we used (14) and $p(x^+) \in X$, $\mu(x) \in X^{\perp}$.

Hence, by (18), (19),

$$\|p(x^+)\|_{x^+}^2 \leq \frac{\|p(x)\|_x^2 \, \|p(x^+)\|_x}{1 - \|p(x)\|_x} \leq \frac{\|p(x)\|_x^2 \, \|p(x^+)\|_{x^+}}{(1 - \|p(x)\|_x)^2}, \tag{20}$$

where in the last inequality, we used (6) with $\xi = p(x^+)$. □

Proposition 3.4 *Given* $x \in \Omega$*, let* $\bar{t} = \frac{1}{1+\|p(x)\|_x}$*. Then,*

$$F(x) - F(x + \bar{t}p(x)) \leq \|p(x)\|_x - \ln(1 + \|p(x)\|_x).$$

To prove Proposition 3.4 we need the following.

Proposition 3.5 *Let* $x \in \Omega$, $\|y - x\|_x < 1$. *Then,*

$$F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle - \|y - x\|_x - \ln(1 - \|y - x\|_x). \tag{21}$$

For a proof see Theorem 4.7.8 in [18].

Proof of Proposition 3.4. According to Proposition 3.5

$$\begin{aligned} &F(x + tp(x)) \leq F(x) + t\langle \nabla F(x), p(x) \rangle - t\,\|p(x)\|_x - \ln(1 - t\,\|p(x)\|_x), \\ &\qquad t \in \left[0, \frac{1}{\|p(x)\|_x}\right). \end{aligned} \tag{22}$$

By (16),

$$\langle \nabla F(x), p(x) \rangle = -\,\|p(x)\|_x^2\,.$$

Introduce notation

$$\|p(x)\|_x = \delta(x). \tag{23}$$

Then (22) takes the form

$$F(x + tp(x)) \leq F(x) - t\delta(x)^2 - t\delta(x) - \ln(1 - t\delta(x)),\ t \in [0, \frac{1}{\delta(x)}). \tag{24}$$

The minimum of the right-hand side of (24) in t is attained at

$$\bar{t} = \frac{1}{1 + \delta(x)}$$

and is equal to $F(x) - \delta(x) + \ln(1 + \delta(x))$.

Hence,

$$F(x + \bar{t}p(x)) \leq F(x) - \delta(x) + \ln(1 + \delta(x)).$$

□

Corollary 3.1 *Let* $x \in \Omega$, $0 < \|p(x)\|_x = \delta(x) < \frac{1}{2}$. *Then* $x + p(x) \in \Omega$ *and* $F(x + p(x)) < F(x)$.

Proof. Note that $x + p(x) \in \Omega$ by Proposition 3.1. By (24)

$$F(x + p(x)) \leq F(x) - \delta(x)^2 - \delta(x) - \ln(1 - \delta(x)).$$

(Note that $\frac{1}{\delta(x)} > 2$ under our assumptions). The function $\delta \to \delta^2 + \delta + \ln(1 - \delta)$ is monotonically increasing for $0 \leq \delta \leq \frac{1}{2}$ and is equal to zero for $\delta = 0$.

Hence,

$$F(x + p(x)) < F(x), \quad \text{for} \quad 0 < \delta \leq \frac{1}{2}.$$

□

4 Properties of the central path and self-concordant families

Denote by $C^k(\Omega)$ the vector space of k times continuously differentiable real-valued functions on Ω. Let $f \in C^2(\Omega)$ be a convex function. We will further assume that f is continuous on $\bar{\Omega}$ and that $\mathcal{F} = (a+X)\cap\Omega$ is nonempty and bounded. This guarantees that for each $\beta \geq 0$, the function

$$F_\beta(x) = \beta f(x) - \ln\det x, \quad \beta \geq 0 \tag{25}$$

has a unique minimum $x(\beta) \in \mathcal{F}$.

Remark 4.1 The optimality condition for (25) takes the form

$$\beta\nabla f(x(\beta)) - x(\beta)^{-1} = -\eta(\beta) \in X^\perp. \tag{26}$$

Hence, for any $\beta > 0$, $\left(x(\beta), \nabla f(x(\beta)) + \frac{\eta(\beta)}{\beta}\right)$ can be used as a starting point for the primal-dual potential reduction algorithm in [4] (see [7], p.241).

Given $\beta > 0$, consider the function $\psi_\beta : \mathcal{F} \to X$,

$$\psi_\beta(x) = \pi(\nabla F_\beta(x)) = \pi(\beta\nabla f(x) - x^{-1}), \tag{27}$$

where $\pi : V \to X$ is the orthogonal projection.

Proposition 4.1 *The map* $(\beta, x) \to (\beta, \psi_\beta(x)) : (0,\infty)\times\mathcal{F} \to (0,\infty)\times X$ *is a* C^2 *diffeomorphism onto* $(0,\infty)\times X$.

Proof. Let $c \in X$. The problem

$$-\langle c, x\rangle + \beta f(x) - \ln\det(x) \to min, \ x \in \mathcal{F} \tag{28}$$

has a unique solution $x(c,\beta) \in \mathcal{F}$, since $\mathcal{F}$ is bounded, F_β is strictly convex on Ω and

$$F_\beta(x) \to \infty, \ x \in \mathcal{F}, \ x \to \partial\Omega.$$

The optimality condition for (28) takes the form:

$$-c + \beta\nabla f(x) - x^{-1} \in X^\perp, \tag{29}$$

which is equivalent to

$$\psi_\beta(x(c,\beta)) = c.$$

Hence ψ_β is surjective for each $\beta > 0$. It is also injective, since

$$c = \psi_\beta(x_1) = \psi_\beta(x_2), \ x_1, \ x_2 \in \mathcal{F}$$

implies that both x_1 and x_2 are optimal solutions to (28) which is unique due to strict convexity of $-\ln\det(x)$. Furthermore, for $\xi \in X$, $x \in \mathcal{F}$, we have

$$D\psi_\beta(x)\xi = \pi(\beta H_f(x)\xi + P(x^{-1})\xi).$$

Thus, $D\psi_\beta(x)\xi = 0$ is equivalent to $\beta H_f(x)\xi + P(x^{-1})\xi \in X^\perp$, i.e.,

$$\beta\langle\xi, H_f(x)\xi\rangle + \langle\xi, P(x)^{-1}\xi\rangle = 0, \ \forall\xi \in X,$$

i.e., $\xi = 0$. Hence,

$$D\psi_\beta(x) : X \to X$$

is a linear isomorphism for any $\beta > 0$, $x \in \mathcal{F}$. Hence, by implicit function theorem the map $(\beta, x) \to (\beta, \psi_\beta(x)) : (0, \infty) \times \mathcal{F} \to (0, \infty) \times X$ is a C^2 diffeomorphism. □

Remark 4.2 Note that

$$\psi_\beta(x(\beta)) = 0, \ \beta > 0 \text{ (see (29))}. \tag{30}$$

Hence, $x(\beta) = \psi_\beta^{-1}(0)$ is a twice continuously differentiable function on $(0, +\infty)$. Moreover, differentiating (30) with respect to β, we obtain

$$D\psi_\beta(x(\beta))\frac{dx}{d\beta}(\beta) + \pi(\nabla f(x(\beta)) = 0,$$

which is the same as

$$\pi(\beta H_f(x(\beta))\frac{dx}{d\beta}(\beta) + P(x(\beta))^{-1}\frac{dx}{d\beta}(\beta) + \nabla f(x(\beta))) = 0.$$

Since $\frac{dx}{d\beta}(\beta) \in X$, there exists a unique $\nu(\beta) \in X^\perp$ such that

$$H_{F_\beta}(x(\beta))\frac{dx}{d\beta}(\beta) = -(\nabla f(x(\beta)) + \nu(\beta)),$$

i.e.,

$$\frac{dx}{d\beta}(\beta) = -H_{F_\beta}(x(\beta))^{-1}(\nabla f(x(\beta)) + \nu(\beta)). \tag{31}$$

Note that by (31)

$$\begin{aligned}\frac{df}{d\beta}(x(\beta)) &= \langle \nabla f(x(\beta)), \frac{dx}{d\beta}(\beta)\rangle \\ &= -\langle \nabla f(x(\beta)) + \nu(\beta), H_{F_\beta}(x(\beta))^{-1}(\nabla f(x(\beta)) + \nu(\beta))\rangle \\ &\leq 0,\end{aligned} \tag{32}$$

where we used the fact that

$$\nu(\beta) \in X^\perp, \quad \frac{dx}{d\beta}(\beta) \ \in X. \tag{33}$$

Hence,

Corollary 4.1 *The function f is monotonically decreasing along the path $\beta \to x(\beta)$, $\beta > 0$.*

Notice that (31), (33) implies that $\nu(\beta)$ is the minimizer of the optimization problem

$$\varphi_\beta(\lambda) = \langle \nabla f(x(\beta)) + \lambda, H_{F_\beta}(x(\beta))^{-1}(\nabla f(x(\beta)) + \lambda)\rangle \to \min \quad \lambda \in X^\perp \tag{34}$$

On the other hand, by (26)

$$\beta \nabla f(x(\beta)) - x(\beta)^{-1} = -\eta(\beta),$$

for some $\eta(\beta) \in X^\perp$. By (34),

$$\varphi_\beta(\frac{\eta(\beta)}{\beta}) \geq \varphi_\beta(\nu(\beta)), \ \beta > 0.$$

But

$$\nabla f(x(\beta)) + \frac{\eta(\beta)}{\beta} = \frac{x(\beta)^{-1}}{\beta}$$

Consequently,

$$\begin{aligned} \varphi_\beta(\nu(\beta)) &\leq \varphi_\beta(\frac{\eta(\beta)}{\beta}) \\ &= \frac{\langle x(\beta)^{-1}, H_{F_\beta}(x(\beta))^{-1}x(\beta)^{-1}\rangle}{\beta^2} \\ &\leq \frac{\langle x(\beta)^{-1}, P(x(\beta))x(\beta)^{-1}\rangle}{\beta^2}. \end{aligned} \tag{35}$$

The last inequality follows from

$$H_{F_\beta}(x) = \beta H_f(x) + P(x)^{-1} \geq P(x)^{-1}$$

and consequently

$$H_{F_\beta}(x)^{-1} \leq P(x).$$

But

$$\langle x(\beta)^{-1}, \frac{P(x(\beta))x(\beta)^{-1}}{\beta^2}\rangle = \frac{\langle e, e\rangle}{\beta^2} = \frac{r}{\beta^2}. \tag{36}$$

Combining (32), (35) and (36), we obtain

$$\frac{df}{d\beta}(x(\beta)) \geq -\frac{r}{\beta^2}, \ \beta > 0. \tag{37}$$

Integrating (37) from β_1 to β_2, we obtain

$$f(x(\beta_2)) - f(x(\beta_1)) \geq r(\frac{1}{\beta_2} - \frac{1}{\beta_1}), \ \text{for } \beta_2 > \beta_1 > 0. \tag{38}$$

Note also that for any $x \in (a + X) \cap \overline{\Omega}$, using convexity of f, we obtain

$$\begin{aligned} f(x) - f(x(\beta)) &\geq \langle \nabla f(x(\beta)), x - x(\beta)\rangle \\ &= \langle \frac{x^{-1}(\beta)}{\beta} - \frac{\eta(\beta)}{\beta}, x - x(\beta)\rangle \end{aligned}$$

for some $\eta(\beta) \in X^{\perp}$, where we used (26). Since $x - x(\beta) \in X$, we obtain

$$\begin{aligned} f(x) - f(x(\beta)) &\geq \langle \frac{x^{-1}(\beta)}{\beta}, x - x(\beta) \rangle \\ &= \langle x^{-1}(\beta), x \rangle - \frac{r}{\beta} \\ &\geq -\frac{r}{\beta}, \end{aligned}$$

where we used $x^{-1}(\beta) \in \Omega$, $x \in \overline{\Omega}$ and self-duality of the cone Ω. In particular if x^* is an optimal solution to the problem

$$f(x) \to \min,\ x \in (x + X) \cap \overline{\Omega}, \tag{39}$$

then

$$f(x^*) \geq f(x(\beta)) - \frac{r}{\beta}.$$

Since $f(x^*) \leq f(x(\beta))$ $(x(\beta) \in (a + X) \cap \overline{\Omega})$, we obtain

$$f(x(\beta)) \geq f(x^*) \geq f(x(\beta)) - \frac{r}{\beta}. \tag{40}$$

In particular,

Corollary 4.2 *We have*

$$f(x(\beta)) \to f(x^*),\ \beta \to +\infty.$$

From now on we assume that $f \in C^3(\Omega)$ and f is continuous on $\overline{\Omega}$. Let

$$B(x) = -\ln\det(x),\ x \in \Omega. \tag{41}$$

We say that f is compatible with B in the sense of [17], p.66, if

$$|D^3 f(x)(\xi, \xi, \xi)| \leq 2D^2 f(x)(\xi, \xi)[D^2 B(x)(\xi, \xi)]^{1/2},\ \xi \in V. \tag{42}$$

Proposition 4.2 *Let f be compatible with B in the sense of* (42). *Then*

$$F_\beta(x) = \beta f(x) + B(x),\ x \in \Omega,\ \beta \geq 0, \tag{43}$$

is self-concordant in the sense of (1), (2).

For a proof, see e.g. Theorem 3.5 in [8]. In particular, Propositions 1-5 and Lemma 3.2 are applicable to the function F_β, $\beta > 0$. We will use notation $p_\beta(x)$, $x \in (a + X) \cap \Omega$ for the Newton direction of F_β at x, i.e.,

$$\begin{gathered} H_{F_\beta}(x) p_\beta(x) = -(\nabla F_\beta(x) + \mu_\beta(x)), \\ \mu_\beta(x) \in X^{\perp},\ p_\beta(x) \in X. \end{gathered} \tag{44}$$

We will denote by

$$\delta_\beta(x) = \langle p_\beta(x), H_{F_\beta}(x) p_\beta(x) \rangle^{1/2},\ x \in (a + X) \cap \Omega. \tag{45}$$

Note that

$$\delta_\beta(x)^2 = -\langle \nabla F_\beta(x), p_\beta(x) \rangle. \tag{46}$$

Proposition 4.3 *Given $x \in (a+X)\cap\Omega$, let $\delta_\beta(x) \le 1/3$. Then*

$$F_\beta(x) - F_\beta(x(\beta)) \le \frac{\delta_\beta(x)^2}{1-[\frac{9}{4}\delta_\beta(x)]^2}. \tag{47}$$

Proof. Set $x^{(0)} = x$. Since $\delta_\beta(x) \le 1/3$, we have by Proposition 3.1, $x^{(1)} = x^{(0)} + p_\beta(x^{(0)}) \in (a+X)\cap\Omega$ and by Proposition 3.3

$$\delta_\beta(x^{(1)}) \le \frac{9}{4}\delta_\beta(x^{(0)})^2 < \delta_\beta(x^{(0)}),$$

since $\delta_\beta(x^{(0)}) \le \frac{1}{3}$ (unless $\delta_\beta(x^{(0)}) = 0$). Continuing

$$x^{(i+1)} = x^{(i)} + p_\beta(x^{(i)}), \tag{48}$$

we obtain the sequence $x^{(i)} \in (a+X)\cap\Omega,\ i = 0, 1, \ldots$, such that

$$\delta_\beta(x^{(i+1)}) < \delta_\beta(x^{(i)}) \le \frac{1}{3}, \tag{49}$$

$$\delta_\beta(x^{(i)}) \le (\frac{9}{4})^{2^i-1}\delta_\beta(x^{(0)})^{2^i},\ i = 1, 2, \ldots \tag{50}$$

In particular,

$$\delta_\beta(x^{(i)}) \le \frac{1}{3}(\frac{9}{4}\cdot\frac{1}{3})^{2^i-1} = \frac{1}{3}(\frac{3}{4})^{2^i-1},$$

i.e.,

$$\delta_\beta(x^{(i)}) \to 0,\ i \to \infty.$$

Note that due to (44), (45) for $x \in (a+X)\cap\Omega$

$$\delta_\beta(x) = 0 \Leftrightarrow \nabla F_\beta(x) \in X^\perp,$$

i.e., $x = x(\beta)$. By Corollary 3.1

$$F_\beta(x^{(i+1)}) < F_\beta(x^{(i)}),\ i = 0, 1, \ldots, \tag{51}$$

Consider

$$L = \{x \in \Omega : F_\beta(x) \le F_\beta(x^{(0)})\}.$$

Since $F_\beta(x) \to +\infty,\ x \to \partial\Omega$, L is a closed subset in V. Since $x^{(i)} \in L,\ i = 0, 1, \ldots$, due to (51), any limit point of the sequence $x^{(i)}$ belongs to L and consequently to Ω. Since $\delta_\beta(x^{(i)}) \to 0,\ i \to \infty$, we conclude that $\delta_\beta(\overline{x}) = 0$ for any limit point $\overline{x}$ of the sequence $x^{(i)}$, i.e., $\overline{x} = x(\beta)$. Now $\mathcal{F} = (a+X)\cap\overline{\Omega}$ is compact by our assumptions. Since $x^{(i)} \in \mathcal{F}$ for all i, we conclude that $x^{(i)} \to x(\beta),\ i \to \infty$.

Using convexity of F_β for any $x \in (a+X)\cap\Omega$

$$F_\beta(x + p_\beta(x)) - F_\beta(x) \ge \langle\nabla F_\beta(x), p_\beta(x)\rangle = -\delta_\beta(x)^2,$$

(See (46)), i.e.,

$$F_\beta(x) - F_\beta(x + p_\beta(x)) \le \delta_\beta(x)^2.$$

Taking into account that $x^{(i)} \to x(\beta),\ i \to \infty$, we obtain

$$\begin{aligned} F_\beta(x) - F_\beta(x(\beta)) &= \sum_{i=0}^{\infty} (F_\beta(x^{(i)}) - F_\beta(x^{(i+1)})) \\ &\leq \sum_{i=0}^{\infty} \delta_\beta(x^{(i)})^2 \\ &\leq \sum_{i=0}^{\infty} (\frac{9}{4})^{2^{i+1}-2} [\delta_\beta(x)]^{2^{i+1}}, \end{aligned}$$

where in the last inequality we used (50). Hence,

$$F_\beta(x) - F_\beta(x(\beta)) \leq \frac{\delta_\beta(x)^2}{1 - [\frac{9}{4}\delta_\beta(x)]^2}.$$

□

Proposition 4.4 *Given $x \in (a + X) \cap \Omega$, $\delta_\beta(x) \leq 1/3$, we have*

$$|f(x) - f(x(\beta))| \leq \left[\frac{\delta_\beta(x)}{1 - \frac{9}{4}\delta_\beta(x)} \cdot \frac{1 + \delta_\beta(x)^2}{1 - \delta_\beta(x)} \right] \frac{\sqrt{r}}{\beta}. \tag{52}$$

Proof. Denote in this proof $p_\beta(x) = p,\ \delta_\beta(x) = \delta$. We obviously have

$$\nabla F_\beta(x) = \beta \nabla f(x) - x^{-1}.$$

Hence,

$$\nabla f(x) = \frac{\nabla F_\beta(x) + x^{-1}}{\beta}. \tag{53}$$

Using convexity of f:

$$\langle \nabla f(x+p), p \rangle \geq f(x+p) - f(x) \geq \langle \nabla f(x), p \rangle \tag{54}$$

By (53)

$$\begin{aligned} \beta \langle \nabla f(x), p \rangle &= \langle \nabla F_\beta(x), p \rangle + \langle x^{-1}, p \rangle \\ &= -\delta_\beta(x)^2 + \langle x^{-1}, p \rangle, \end{aligned}$$

where in the second equality we used (46). Now

$$\langle x^{-1}, p \rangle = \langle P(x^{-\frac{1}{2}})e, p \rangle = \langle e, P(x^{-\frac{1}{2}})p \rangle$$

Consequently,

$$|\langle x^{-1}, p \rangle| \leq |\langle e, P(x)^{-\frac{1}{2}} p \rangle| \leq \|e\| \left\| P(x)^{-\frac{1}{2}} p \right\| \leq \sqrt{r}\delta_\beta(x),$$

where in the last inequality we used

$$\langle P(x)^{-1} p, p \rangle \leq \langle H_{F_\beta}(x) p, p \rangle = \delta_\beta(x)^2.$$

Consequently,

$$|\langle \nabla f(x), p\rangle| \leq \frac{\delta_\beta(x)^2 + \sqrt{r}\delta_\beta(x)}{\beta} \tag{55}$$

Applying Lemma 3.2 to $F = F_\beta$, $\xi_1 = \xi_2 = p$, and noticing in notations of Lemma 3.2 that $\varphi(1) = \langle \nabla F_\beta(x+p), p\rangle$, $\varphi(0) = 0$, we obtain

$$|\langle p, \nabla F_\beta(x+p)\rangle| \leq \frac{\delta_\beta(x)^3}{1-\delta_\beta(x)},\ \delta_\beta(x) < 1.$$

Furthermore,

$$\begin{aligned}\langle p, (x+p)^{-1}\rangle &= \langle p, (P(x^{\frac{1}{2}})e + p)^{-1}\rangle \\ &= \langle p, [P(x^{\frac{1}{2}})(e + P(x^{-\frac{1}{2}})p)]^{-1}\rangle \\ &= \langle p, P(x^{-\frac{1}{2}})(e + P(x^{-\frac{1}{2}})p)^{-1}\rangle \\ &= \langle P(x)^{-\frac{1}{2}}p, (e + P(x)^{-\frac{1}{2}}p)^{-1}\rangle\end{aligned} \tag{56}$$

Let

$$P(x)^{-\frac{1}{2}}p = \sum_{i=1}^{r} \lambda_i e_i$$

be the spectral decomposition. By (56)

$$\langle p, (x+p)^{-1}\rangle = \sum_{i=1}^{r} \frac{\lambda_i}{1+\lambda_i}.$$

Hence,

$$\begin{aligned}|\langle p, (x+p)^{-1}\rangle| &\leq \sum_{i=1}^{r} \frac{|\lambda_i|}{|1+\lambda_i|} \\ &\leq \sum_{i=1}^{r} \frac{|\lambda_i|}{1-|\lambda_i|} \\ &\leq (\sum_{i=1}^{r}|\lambda_i|^2)^{\frac{1}{2}}(\sum_{i=1}^{r}\frac{1}{(1-|\lambda_i|)^2})^{\frac{1}{2}} \\ &\leq \left\|P(x^{-\frac{1}{2}})p\right\| \left[\sum_{i=1}^{r} \frac{1}{\left(1-\left\|P(x^{-\frac{1}{2}})p\right\|\right)^2}\right]^{\frac{1}{2}} \\ &= \frac{\left\|P(x^{-\frac{1}{2}})p\right\|\sqrt{r}}{1-\left\|P(x^{-\frac{1}{2}})p\right\|} \\ &\leq \frac{\delta_\beta(x)\sqrt{r}}{1-\delta_\beta(x)},\end{aligned} \tag{57}$$

where in the last inequality we used $\langle P(x)^{-1}p, p\rangle \leq \delta_\beta(x)^2$ and monotonicity of the function $t \to \frac{t}{1-t}$ on $(0,1)$. Combining (55), (57) and taking into account (53), we obtain

$$|\langle \nabla f(x+p), p\rangle| \leq \frac{\delta_\beta(x)^3 + \delta_\beta(x)\sqrt{r}}{(1-\delta_\beta(x))\beta}. \tag{58}$$

By (54), (55), (58), we obtain

$$\frac{\delta_\beta(x)^3 + \delta_\beta(x)\sqrt{r}}{(1-\delta_\beta(x))\beta} \geq f(x+p) - f(x) \geq -\frac{\delta_\beta(x)^2 + \sqrt{r}\delta_\beta(x)}{\beta} \tag{59}$$

From (59),

$$|f(x+p) - f(x)| \leq \frac{\delta_\beta(x)\sqrt{r}}{\beta} \cdot \frac{1+\delta_\beta(x)^2}{1-\delta_\beta(x)} \tag{60}$$

Let $x^{(0)} = x$ and $x^{(1)}$, $x^{(2)}$, . . ., be the sequence obtained by repeating Newton steps. Then

$$\begin{aligned}
|f(x(\beta)) - f(x)| &= |\sum_{i=0}^{\infty} f(x^{(i+1)}) - f(x^{(i)})| \\
&\leq \sum_{i=0}^{\infty} |f(x^{(i+1)}) - f(x^{(i)})| \\
&\leq \sum_{i=0}^{\infty} \frac{\delta_\beta(x^{(i)})\sqrt{r}}{\beta} \cdot \frac{1+\delta_\beta(x^{(i)})^2}{1-\delta_\beta(x^{(i)})} \\
&\leq \frac{\sqrt{r}}{\beta} \cdot \frac{1+\delta_\beta(x)^2}{1-\delta_\beta(x)} \sum_{i=0}^{\infty} \delta_\beta(x^{(i)}) \\
&\leq \frac{\sqrt{r}}{\beta} \cdot \frac{1+\delta_\beta(x)^2}{1-\delta_\beta(x)} \sum_{i=0}^{\infty} (\frac{9}{4})^{2^i-1} \delta_\beta(x)^{2^i} \\
&\leq \frac{\sqrt{r}}{\beta} \cdot \frac{1+\delta_\beta(x)^2}{1-\delta_\beta(x)} \cdot \frac{\delta_\beta(x)}{1-\frac{9}{4}\delta_\beta(x)},
\end{aligned}$$

where the second inequality follows from (60) and the fourth inequality from (50). □

5 Complexity estimates

We are now in position to describe a version of the long-step path-following algorithm (compare with [4]). Suppose we are given $\beta_0 > 0$ and $x^{(0)} \in (a+X) \cap \Omega$ such that $\delta_\beta(x^{(0)}) \leq \frac{1}{3}$. Given $\theta > 0$, we perform so-called outer iterations: $\beta_1 = (1+\theta)\beta_0, \ldots, \beta_i = (1+\theta)^i\beta_0$. Suppose we were able to construct a sequence of points $x^{(1)}$, $x^{(2)}$, . . ., such that $x^{(i)} \in (a+X) \cap \Omega$ and $\delta_{\beta_i}(x^{(i)}) \leq \frac{1}{3}$.

Theorem 5.1 *Given* $\epsilon > 0$ *and*

$$i \geq \frac{\ln(\frac{4r}{\epsilon\beta_0})}{\ln(1+\theta)} \tag{61}$$

Then

$$f(x^{(i)}) - f(x^*) \leq \epsilon,$$

where x^* *is an optimal solution to the problem* (39).

Proof. Condition (61) is equivalent to

$$\frac{r}{\beta_i} \leq \frac{\epsilon}{4}.$$

We have

$$f(x^{(i)}) - f(x^*) = (f(x^{(i)}) - f(x(\beta_i))) + (f(x(\beta_i)) - f(x^*)).$$

By Proposition 4.4,

$$\begin{aligned} f(x^{(i)}) - f(x(\beta_i)) &\leq \frac{\frac{1}{3}}{1 - \frac{9}{4}\cdot\frac{1}{3}} \cdot \frac{1 + (\frac{1}{3})^2}{1 - \frac{1}{3}} \cdot \frac{\sqrt{r}}{\beta_i} \\ &\leq \frac{20}{9} \cdot \frac{\sqrt{r}}{\beta_i} \\ &\leq \frac{3\sqrt{r}}{\beta_i}, \end{aligned}$$

where we used $\delta_{\beta_i}(x^{(i)}) \leq \frac{1}{3}$ and monotonicity of the function

$$\delta \to \frac{\delta}{1 - \frac{9}{4}\delta} \cdot \frac{1 + \delta^2}{1 - \delta}$$

on the interval $[0, \frac{4}{9})$. Furthermore,

$$f(x(\beta_i)) - f(x^*) \leq \frac{r}{\beta_i}$$

by (40). Hence,

$$f(x^{(i)}) - f(x^*) \leq \frac{3\sqrt{r}}{\beta_i} + \frac{r}{\beta_i} \leq \frac{4r}{\beta_i} \leq \epsilon.$$

□

We must describe the procedure for updating $x^{(i)}$. Suppose that we have $x^{(i)} \in (a + X) \cap \Omega$ such that $\delta_{\beta_i}(x^{(i)}) \leq \frac{1}{3}$. We have to find $x^{(i+1)} \in (a + X) \cap \Omega$ such that $\delta_{\beta_{i+1}}(x^{(i+1)}) \leq \frac{1}{3}$ by performing several Newton steps for the function $F_{\beta_{i+1}}$, using $x^{(i)}$ as the starting point. Each such Newton step is called an inner iteration.

Theorem 5.2 *Each outer iteration requires at most*

$$\frac{22}{3} + 22\theta\left(\frac{5}{2}\sqrt{r} + \frac{\theta r}{\theta + 1}\right)$$

inner iterations.

Proof. We start with a point $x^{(i)} \in (a+X)\cap\Omega$ such that $\delta_{\beta_i}(x^{(i)}) \leq \frac{1}{3}$. By Proposition 3.4,

$$\overline{x} = x^{(i)} + \frac{p_{\beta_{i+1}}(x^{(i)})}{1 + \delta_{\beta_{i+1}}(x^{(i)})} \in (a + X) \cap \Omega \tag{62}$$

and

$$\begin{aligned} F_{\beta_{i+1}}(x^{(i)}) - F_{\beta_{i+1}}(\overline{x}) &\geq \delta_{\beta_{i+1}}(x^{(i)}) - \ln(1 + \delta_{\beta_{i+1}}(x^{(i)})) \\ &\geq \frac{1}{3} - \ln(1 + \frac{1}{3}) \\ &> \frac{1}{22} \end{aligned} \tag{63}$$

provided $\delta_{\beta_{i+1}}(x^{(i)}) \geq \frac{1}{3}$. Here we used the fact that the function $\delta \to \delta - \ln(1 + \delta)$ is monotonically increasing on $(0, \infty)$. If $\delta_{\beta_{i+1}}(x^{(i)}) \leq \frac{1}{3}$, set $x^{(i+1)} = x^{(i)}$. We continue to perform iterations (62) until $\delta_{\beta_{i+1}}(\overline{x}) < \frac{1}{3}$. If N is the number of such iterations, then by (63)

$$F_{\beta_{i+1}}(x^{(i)}) - F_{\beta_{i+1}}(x(\beta_{i+1})) \geq \frac{N}{22}. \tag{64}$$

Consider the function

$$\gamma(x, \beta) = F_\beta(x) - F_\beta(x(\beta)),\ x \in (a + X) \cap \Omega.$$

We wish to estimate $\gamma(x^{(i)}, \beta_{i+1})$. By the mean value theorem,

$$\gamma(x, \beta_{i+1}) = \gamma(x, \beta_i) + \frac{\partial \gamma}{\partial \beta}(x, \hat{\beta})\theta\beta_i, \tag{65}$$

for some $\hat{\beta}$ (which depends on x) in (β_i, β_{i+1}).

Recalling

$$F_\beta(x) = \beta f(x) - \ln\det(x),$$

we have

$$\frac{\partial \gamma}{\partial \beta}(x, \beta) = f(x) - f(x(\beta)) - \langle \nabla F_\beta(x(\beta)), \frac{dx(\beta)}{d\beta} \rangle.$$

But

$$\nabla F_\beta(x(\beta)) \in X^\perp,\ \frac{dx(\beta)}{d\beta} \in X.$$

Hence,

$$\frac{\partial \gamma}{\partial \beta}(x, \beta) = f(x) - f(x(\beta)). \tag{66}$$

By (65), (66),

$$\begin{aligned}\gamma(x^{(i)}, \beta_{i+1}) - \gamma(x^{(i)}, \beta_i) &= \theta\beta_i(f(x^{(i)}) - f(x(\hat{\beta})))\\ &\leq \theta\beta(f(x^{(i)}) - f(x(\beta_{i+1}))\\ &\leq \theta\beta_i(|f(x^{(i)}) - f(x(\beta_i))|) + f(x(\beta_i)) - f(x(\beta_{i+1}))\end{aligned} \tag{67}$$

where in the first inequality we used the fact that the function $\beta \to f(x(\beta))$ is monotonically decreasing (see Corollary 4.1). By Proposition 4.3,

$$\begin{aligned}\gamma(x^{(i)}, \beta_i) &= F_{\beta_i}(x^{(i)}) - F_{\beta_i}(x(\beta_i))\\ &\leq \frac{(\frac{1}{3})^2}{1 - (\frac{9}{4}\cdot\frac{1}{3})^2}\\ &= \frac{1}{3},\end{aligned} \tag{68}$$

since $\delta_{\beta_i}(x^{(i)}) \leq \frac{1}{3}$. By Propostion 4.4,

$$|f(x^{(i)}) - f(x(\beta_i))| \leq \frac{5}{2}\frac{\sqrt{r}}{\beta_i}. \tag{69}$$

By (38),

$$f(x(\beta_i)) - f(x(\beta_{i+1})) \leq r\left(\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right) = \frac{\theta r}{\beta_{i+1}}. \tag{70}$$

Combining (67) - (70), we obtain

$$\begin{aligned}\gamma(x^{(i)}, \beta_{i+1}) &= F_{\beta_{i+1}}(x^{(i)}) - F_{\beta_{i+1}}(x(\beta_{i+1}))\\ &\leq \frac{1}{3} + \theta\beta_i\left(\frac{5}{2}\frac{\sqrt{r}}{\beta_i} + \frac{\theta r}{\beta_{i+1}}\right)\\ &= \frac{1}{3} + \theta\left(\frac{5}{2}\frac{\sqrt{r}}{\beta_i} + \frac{\theta r}{\beta_{i+1}}\right).\end{aligned}$$

Combining this with (64), we obtain

$$N \leq \frac{22}{3} + 22\theta\left(\frac{5}{2}\sqrt{r} + \frac{\theta r}{\theta + 1}\right).$$

□

Theorem 5.3 *An upper bound for the total number of Newton iterations is given by*

$$\frac{\ln(\frac{4r}{\epsilon\beta_0})}{\ln(1+\theta)}\left(\frac{22}{3} + 22\theta\left(\frac{5}{2}\sqrt{r} + \frac{\theta r}{1+\theta}\right)\right).$$

6 Numerical experiments

Let $g : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$h(\lambda) = g'(\lambda)$$

is matrix monotone (in the sense of [1], p.112). Given a simple Euclidean Jordan algebra V, consider the function

$$f : \overline{\Omega} \to \mathbb{R},\ f(x) = \mathrm{tr}(g(x)),\ x \in V.$$

Proposition 6.1 *Given $x \in \Omega$, $\xi \in V$, we have*

$$|D^3 f(x)(\xi, \xi, \xi)| \le 2D^2 f(x)(\xi, \xi)[D^2 B(x)(\xi, \xi)]^{1/2}.$$

For the proof, see [8] Theorem 3.2.

Consequently, according to Proposition 4.2 the function

$$F_\beta(x) = \beta f(x) - \ln \det(x),\ x \in \Omega,\ \beta \ge 0,$$

satisfies (43). For the implementation of our algorithm, we need to have complete expressions for $\nabla F_\beta(x)$ and $H_{F_\beta}(x)$, $x \in \Omega$. We obviously have

$$\nabla F_\beta(x) = \beta h(x) - x^{-1}.$$

For the calculation of $H_{F_\beta}(x)$ consider a spectral decomposition:

$$x = \sum_{i=1}^{r} \lambda_i e_i$$

and the corresponding Peirce decomposition of V:

$$V = \bigoplus_{i \le j} V_{ij}. \tag{71}$$

Given $\xi \in V$, let

$$\xi = \sum_{i=1}^{r} \xi_i e_i + \sum_{j<k} \xi_{jk} \tag{72}$$

be the decomposition of ξ according to (71). Introduce a notation

$$[\lambda_j, \lambda_k]_h = \begin{cases} \dfrac{h(\lambda_j) - h(\lambda_k)}{\lambda_j - \lambda_k}, & \lambda_j \ne \lambda_k \\ h'(\lambda_j), & \lambda_j = \lambda_k. \end{cases}$$

Then according to (4) in [13], we have

$$H_f(x)(\xi) = \sum_{i=1}^{r} h'(\lambda_i)\xi_i e_i + \sum_{j<k} [\lambda_j, \lambda_k]_h \xi_{jk}.$$

Taking into account $D(x^{-1}) = -P(x^{-1})$, we obtain

$$H_{F_\beta}(x)(\xi) = \sum_{i=1}^{r}(\beta h'(\lambda_i) + \frac{1}{\lambda_i^2})\xi_i e_i + \sum_{j<k}(\beta[\lambda_j, \lambda_k]_h + \frac{1}{\lambda_j \lambda_k})\xi_{jk}.$$

Consequently,

$$H_{F_\beta}(x)^{-1}(\xi) = \sum_{i=1}^{r}(\beta h'(\lambda_i) + \frac{1}{\lambda_i^2})^{-1}\xi_i e_i + \sum_{j<k}(\beta[\lambda_j, \lambda_k]_h + \frac{1}{\lambda_j \lambda_k})^{-1}\xi_{jk}.$$

For a concrete example, we consider the minimization of quantum entropy in the Euclidean Jordan algebra of real symmetric matrices. More precisely, we consider

$$\begin{aligned} f(X) = \operatorname{Tr}(CX) + \operatorname{Tr}(X \ln X) &\to \min \\ \operatorname{Tr}(A_i X) = b_i,\ i = 1, 2, \ldots, m&, \\ X \geq 0&, \end{aligned} \tag{73}$$

where C, and A_i, $i = 1, 2, \ldots, m$, are real symmetric matrices and $X \geq 0$ means that X is positive semidefinite. We assume that the feasible set is bounded and has a nonempty interior.

To solve (73), we consider

$$\begin{aligned} F_\beta(X) = \beta(\operatorname{Tr}(CX) + \operatorname{Tr}(X \ln X)) - \ln \operatorname{Det} X &\to \min \\ \operatorname{Tr}(A_i X) = b_i,\ i = 1, 2, \ldots, m&, \\ X \geq 0&. \end{aligned} \tag{74}$$

We first find the analytic center S by solving

$$\begin{aligned} F(X) = -\ln \operatorname{Det} X &\to \min \\ \operatorname{Tr}(A_i X) = b_i,\ i = 1, 2, \ldots, m&, \\ X \geq 0&. \end{aligned} \tag{75}$$

which is well-known and can be efficiently solved by SDP solvers like SDPT3 [19]; we used YALMIP [14] and SDPT3 in our test.

Without loss of generality, we always assume $\operatorname{Tr}(X) = 1$; indeed, within the formalism of Euclidean Jordan algebras one can always assume that $\operatorname{tr}(x) = k$ and then use the rescaling $y = \frac{x}{k}$ ([6]). We set $\beta_0 = 0.0001$, $\theta = 10$ and $\epsilon = 0.0001$ for all of our tests. To our knowledge, there are very few efficient methods which can solve this problem. Here we compare to the results obtained by using the *quantum_entr* function from CVXQUAD [9], which are implemented in CVX [10] and solved using MOSEK [16]. In running our algorithm, we use a line search on each Newton step. Table 1 shows the results for randomly generated data $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, m$ and $b \in \mathbb{R}^m$ (Here we always assume $A_m = I$ and $b_m = 1$). All of the data can be accessed from the website: `https://www.researchgate.net/publication/321746649_Data_qentropy`.

Table 1 Numerical results for randomly generated instances of the quantum entropy minimization problem (73), which are obtained on a 13-inch Macbook pro with 2.4GHz Dual-Core Intel i5 and 8GB RAM. Here n is dimension of X, m is the number of linear equality constraints, T_{ac} is the time for finding the analytic center (75) and N is the total number of Newton steps. Codes are implemented in MATLAB 9.0 [15]. We tried SDPT3 as a solver for CVXQUAD, but it gets much slower as the dimension grows, e.g., in the $n = 50$, $m = 50$ case, the time is about 960 seconds (compare to 68.5 using MOSEK). We see that our method is much faster than the one proposed in [9] starting from $n = 15$ and gives a comparable accuracy. It also takes into account the structure of the problem (compare the results for $m = 1$ and the running value of m).

		Interior Point Method				CVXQUAD using MOSEK	
n	m	$T_{ac}(s)$	Time(s)	$fmin$	N	Time(s)	$fmin$
5	1	0.38	0.006	-4.1538	10	0.244	-4.1538
5	5	0.41	0.008	-1.7651	11	0.278	-1.7651
10	1	0.38	0.016	-5.0940	15	0.338	-5.0940
10	10	0.43	0.03	-4.5971	17	0.37	-4.5971
15	1	0.40	0.04	-7.9795	21	0.62	-7.9795
15	10	0.45	0.06	-5.4021	19	0.62	-5.4021
20	1	0.44	0.07	-8.7401	22	1.37	-8.7401
20	10	0.49	0.11	-7.4114	21	1.57	-7.4114
30	1	0.53	0.21	-10.3297	24	7.42	-10.3297
30	10	0.71	0.39	-9.1865	26	6.59	-9.1865
40	1	0.56	0.47	-12.0387	25	27.70	-12.0387
40	20	1.60	1.48	-10.1613	25	22.83	-10.1613
50	1	0.68	0.97	-14.4099	26	74.25	-14.4100
50	50	4.78	6.76	-9.9558	26	68.50	-9.9559
60	1	0.90	1.70	-14.9266	24	152.54	-14.9267
60	50	9.83	11.74	-11.8059	24	180.59	-11.8060
70	1	1.37	3.62	-16.5574	29	416.79	-16.5574
70	50	17.19	24.70	-13.0724	30	339.15	-13.0724
80	1	1.83	5.57	-17.0010	29	582.31	-17.0010
80	50	33.32	40.08	-15.6651	30	604.22	-15.6651
90	1	2.63	9.24	-19.7464	29	1308.47	-19.7464
90	50	54.03	62.80	-16.3319	30	1190.54	-16.3319
100	1	4.17	13.58	-19.6382	29	2050.19	-19.6382
100	100	133.83	164.19	-15.9864	32	2407.63	-15.9864
150	1	20.57	104.74	-24.3021	31	N/A	failed
150	100	765.67	763.44	-20.4707	32	N/A	failed

Table 2 Statistics of the time usage in our Interior Point Method, where T_{tot} is the total time, T_{ass} the total assembling time of the linear system, and T_{eig} the total time of eigendecomposition included in finding Newton direction, performing line search, etc. Note that here we are using the profiling tool from Matlab, which usually shows more elapsed time compared to *tic toc* or *timeit* used in Table 1.

n	m	T_{tot} (s)	T_{ass} (s)	T_{eig} (s)
50	50	7.372	7.045 (96%)	0.084 (1%)
100	100	172.573	169.466 (98%)	0.374 (0.22%)

7 Concluding remarks

We developed a path-following algorithm for a class of symmetric programming problems with a convex nonlinear objective function. Complexity estimates (similar to the case of linear-quadratic objective function) has been obtained. For the class of optimization problems involving quantum entropy we compared our numerical results with the approach of [9]. While approach of [9] has an advantage of possibility of using powerful semidefinite programming packages (like MOSEK), our approach allows one to fully utilize a concrete structure of the problem. Having said that, we do not try to compare our approach with one of [9], since both approaches are applicable to two different classes of problems with (relatively) small intersection. We also did not try to develop a software package but rather showed the viability of our approach.

In comparison to semidefinite programming problems, the problems we analyzed numerically requires solving numerous eigendecomposition problems for symmetric positive definite matrices of a given size n. However, the computational time (at least for $n \leq 150$) required for performing eigendecomposition computations are negligible in comparison with the time for assembling linear system of equations for finding Newton directions (see Table 2). The latter procedure is quite similar to the one in semidefinite programming case. Its speed (and accuracy with which the corresponding linear system can be solved) can be substantially improved following the pattern of semidefinite programming.

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