Binary Extended Formulations of Polyhedral Mixed-integer Sets

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Abstract We analyze different ways of constructing binary extended formulations of polyhedral mixed-integer sets with bounded integer variables and compare their relative strength with respect to split cuts. We show that among all binary extended formulations where each bounded integer variable is represented by a distinct collection of binary variables, what we call "unimodular" extended formulations are the strongest. We also compare the strength of some binary extended formulations from the literature. Finally, we study the behavior of branch-and-bound on such extended formulations and show that branching on the new binary variables leads to significantly smaller enumeration trees in some cases.

1 Introduction

For a given formulation of an optimization problem, an extended formulation is one which uses additional variables to represent the same problem. In integer programming, it is common to use extended formulations that lead to stronger LP relaxations. (Ideally, the extended formulation may have an LP relaxation whose projection onto the original space is integral, see [17] for references to recent work on this topic.) For binary integer programs, the lift-and-project methods of Sherali and Adams [24], Lovász and Schrijver [19], and Balas, Ceria and Cornuéjols [4] yield such extended formulations. However, these extended formulations are, in general, too big to be practically useful as are those given by Bodur, Dash and Günlük [5] for general integer programs.

In this paper, we study extended formulations of bounded integer programs that are constructed by representing integer variables by a combination of new binary variables, possibly along with additional constraints on these binary variables. Such "binary extended formulations" have been studied by Glover [15], Sherali and Adams [25], and Roy [22]. Given a polyhedral mixed-integer set

$$P = \{(x, y) \in U \times \mathbb{R}^n : Ax + Cy \le b\}$$

where A, C, b are matrices of appropriate dimension and $U = \{0, \ldots, u_1\} \times \cdots \times \{0, \ldots, u_l\}$ with $u_1, \ldots, u_l \in \mathbb{Z}$, Sherali and Adams [25] studied the binary extended

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formulation:

$$Q = \{(x, y, z) \in \mathbb{R}^l \times \mathbb{R}^n \times \{0,1\}^q : Ax + Cy \le b, \\ x_i = \sum_{j=1}^{u_i} j z_{ij}, \sum_{j=1}^{u_i} z_{ij} \le 1 \text{ for } i = 1, \ldots, l\} \quad (1)$$

where $q = \sum_{i=1}^{l} u_i$. For $i \in \{1, \ldots, l\}$, the binary variables z_{ij} for $j \in \{0, \ldots, u_i\}$ are used to "binarize" variable x_i. Note that there is a one-to-one mapping between each $x_i \in \{0, \ldots, u_i\}$ and each $(z_{i1}, \ldots, z_{iu_i}) \in \{0,1\}^{u_i}$ satisfying $\sum_{j=1}^{u_i} z_{ij} \le 1$ and $x_i = \sum_{j=1}^{u_i} j z_{ij}$. More generally, Roy [22] defined a binary extended formulation of P to be a set S of the form

$$S = \{(x, y, z) \in \mathbb{R}^l \times \mathbb{R}^n \times \{0,1\}^q : Ax + Cy \le b, x = Tz, Dz \le f\},$$

for some $q > 0$, and some matrices D, T, f, where the linear mapping $x = Tz$ maps 0-1 points in $\{z \in \mathbb{R}^q : Dz \le f\}$ to U. In this paper we will study reformulations where each bounded integer variable is "binarized" separately, i.e., it is represented by a distinct collection of binary variables. Note that in both cases above, only the obvious domain of x_i is used to binarize x_i, and the constraints $Ax + Cy \le b$ do not play a role.

Owen and Mehrotra [21] proved some negative properties of two such binary extended formulations vis-a-vis the original integer program. In particular, they showed that 0-1 branching would perform worse on the extended formulation than on the original integer program in the sense that a much larger branch-and-bound tree would be generated in the former case, unless one branches in a specific manner. They thus argue that such binarization strategies are unlikely to be useful.

However, binary extended formulations have some attractive theoretical properties with respect to cutting planes. Cook, Kannan, and Schrijver [9] gave a mixed-integer program (MIP) with two bounded (between 0 and 2), integer variables and one bounded, continuous variable that cannot be solved in finite time by any cutting plane algorithm that only generates split cuts. But the binary extended formulation (1) can be solved in finite time with split cuts as all binary programs have this property, see Balas [3].

Bonami and Margot [7] showed that certain types of cutting planes were more effective (both theoretically and computationally) when generated on a binary extended formulation as opposed to the original formulation. More strikingly, Angulo and Van Vyve [1] showed that CPLEX [10] requires significantly more time to solve an MIP formulation of the fixed-charge transportation problem than a particular binary extended formulation (unlike Roy [22] and Sherali and Adams [25], they use the constraints $Ax+Cy \le b$ of the mixed-integer set to construct the extended formulation).

In practice, binarization changes the behavior of MIP solvers both in terms of branching and cut generation. In this paper we consider known binary extended formulations as well as more general ways to construct them and compare their relative strength with respect to adding certain families of cutting planes in the extended space. For some pairs of previously studied binary extended formulations, we show that the projection of the split closure of one extended formulation onto the original space of variables is strictly contained in the corresponding projection of the other. A natural question is whether it is possible to construct a strongest possible – in the above sense – binary extended formulation. Our main result is that among all binary extended formulations where each bounded integer variable is separately binarized, what we call "unimodular" extended formulations are strongest with respect to the projection of their split closures. Both the formulation in (1) and the extended formulation studied by Roy [22] and Bonami-Margot [7] belong to this class. Finally, we study the behavior of branch-and-bound on a certain binary extended formulation and show that the observation by Owen and Mehrotra [21] does not always hold.

The rest of the paper is organized as follows: in the next section, we formally define binary extended formulations and review split cuts. In Section 3, we study basic properties of binary extended formulations. In Section 4, we define unimodular binarization schemes and present our main result. In Section 5, we compare a number of binary extended formulations in terms of the strength of their split closures. Finally, in Section 6 we show that branching in the extended space can lead to smaller branch-and-bound trees when solving a mixed-integer program.

2 Preliminaries

We next formally define what we mean by binarization polytopes, binarization schemes, and binary extended formulations. We also review split cuts and define unimodular and integral affine transformations.

2.1 Notation

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron (all polyhedra in this paper are assumed to be rational). Let $I = \{1, \ldots, l\}$ be the index set of integer variables where $0 \leq l \leq n$. We call a set of the form

$$P^I = \{x \in P : x_i \in \mathbb{Z}, \text{ for } i \in I\}$$

a *polyhedral mixed-integer set*, and we call P the linear relaxation of P^I. For convenience, we assume that all integer variables defining P^I are bounded, i.e., P is defined by rational data as

$$P = \Big\{x \in \mathbb{R}^n \ : \ Ax \leq b, \ 0 \leq x_i \leq u_i \text{ for } i \in I\Big\}. \tag{2}$$

A set $X \subseteq \mathbb{R}^n \times \mathbb{R}^q$ is called an *extended formulation* of P if $P = \text{proj}_x(X)$ where $\text{proj}_x(X)$ stands for the orthogonal projection of points in X to the space of the variables x.

For positive integers q, u, let Γ_u^q be the set of all rational polytopes $B \subseteq \{(x, z) \in \mathbb{R} \times [0,1]^q \ : \ 0 \leq x \leq u\}$ such that

$$\text{proj}_x\{B \cap (\mathbb{R} \times \{0,1\}^q)\} = \{0, 1, \ldots, u\}. \tag{3}$$

Each polytope in Γ_u^q can be used to "binarize" a bounded integer variable $x \in \{0, \ldots, u\}$ using q new binary variables; setting the new variables to 0-1 values forces x to be an integer in $\{0, \ldots, u\}$. We refer to each polytope in Γ_u^q as a *binarization polytope*. We note that due to (3), if $B \in \Gamma_u^q$, then $B \cap (\mathbb{R} \times \{0,1\}^q)$ might contain points of the form (x,z) and (x,z') where $x \in \{0,\ldots,u\}$ and $z \neq z'$. However, $B \cap (\mathbb{R} \times \{0,1\}^q)$ does not contain two points of the form (x,z) and (x',z) where $x \neq x'$ as this would imply that the segment $\operatorname{conv}(\{x,x'\})$ belongs to the projection which contradicts the fact that (3) is a discrete set.

The following are some examples of binarization polytopes:

$$B^{\mathrm{F}}(u) = \{(x,z) \in \mathbb{R} \times [0,1]^u \;:\; x = \textstyle\sum_{j=1}^{u} j z_j,\;\; \sum_{j=1}^{u} z_j \leq 1\}, \tag{4}$$

$$B^{\mathrm{U}}(u) = \{(x,z) \in \mathbb{R} \times [0,1]^u \;:\; x = \textstyle\sum_{j=1}^{u} z_j,\; 1 \geq z_1 \geq z_2 \geq \ldots \geq z_u \geq 0\}, \tag{5}$$

$$B^{\mathrm{L}}(u) = \{(x,z) \in \mathbb{R} \times [0,1]^{\lceil \log_2(u+1) \rceil} \;:\; x = \textstyle\sum_{j=1}^{\lceil \log_2(u+1) \rceil} 2^{j-1} z_j,\; x \leq u\}. \tag{6}$$

Note that sets $B^{\mathrm{F}}(u)$ and $B^{\mathrm{U}}(u)$ are contained in Γ_u^u, whereas $B^{\mathrm{L}}(u)$ has only $\lceil \log_2(u+1) \rceil + 1$ variables. The set $B^{\mathrm{F}}(u)$, known as the *full*-binarization, was studied by Sherali and Adams [25] and by Angulo and Van Vyve [1]. The *unary*-binarization $B^{\mathrm{U}}(u)$ was studied by Roy [22] and by Bonami and Margot [7]. The *logarithmic*-binarization $B^{\mathrm{L}}(u)$ was studied by Owen and Mehrotra [21]. In addition, Gupte, Ahmed, Cheon, and Dey [16] study $B^{\mathrm{F}}(u)$ and $B^{\mathrm{L}}(u)$ in the context of bilinear programs.

Note that our definition does not require a bijection between integer points in $B \in \Gamma_u^q$ and $\{0,1,\ldots,u\}$. However, we will later show that this is a desirable property and is satisfied by $B^{\mathrm{F}}(u)$, $B^{\mathrm{U}}(u)$, and $B^{\mathrm{L}}(u)$. Also note that $B \cap (\mathbb{R} \times \{0,1\}^q)$ contains at least u distinct points and therefore $q \geq \lceil \log_2(u+1) \rceil$.

Let $\mathcal{B} = (B^1,\ldots,B^l)$ be an ordered set of l polytopes where each $B^i \in \Gamma_{u_i}^{q_i}$. We will call ordered sets of the form $\mathcal{B}$ *binarization schemes* and in particular if all B^i defining $\mathcal{B}$ are unary (or full or logarithmic) binarization polytopes, we will call the scheme a unary (respectively, full or logarithmic) binarization scheme. Let $q = \sum_{i \in I} q_i$. We define $P_{\mathcal{B}}$ to be the polyhedron

$$P_{\mathcal{B}} = \Big\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^q \;:\; x \in P,\; (x_i,z_i) \in B^i \text{ for } i \in I\Big\}. \tag{7}$$

Here we abuse notation, and let z be a vector in $\mathbb{R}^q$, and $z_1 \in \mathbb{R}^{q_1},\ldots,z_l \in \mathbb{R}^{q_l}$ be subvectors of z; i.e., $z^T = (z_1^T,\ldots,z_l^T)$, and z_{ij} is the jth component of the ith subvector of z. $P_{\mathcal{B}}$ is an extended formulation of P, i.e., $\operatorname{proj}_x(P_{\mathcal{B}}) = P$ since for every $x \in P$ and $i \in I$, $0 \leq x_i \leq u_i$, and hence there exists $z_i \in \mathbb{R}^{q_i}$ such that $(x_i,z_i) \in B^i$. In the rest of the paper we will call $P_{\mathcal{B}}$ a *binary extended formulation* of P. Let

$$I_{\mathcal{B}} = \{1,\ldots,l,n+1,\ldots,n+q\}. \tag{8}$$

As $P^I = \operatorname{proj}_x(P_{\mathcal{B}}^{I_{\mathcal{B}}})$, we call $P_{\mathcal{B}}^{I_{\mathcal{B}}}$ a binary extended formulation of P^I. Furthermore, as the integrality requirements on the new z variables force the x variables to be integral, one can drop the integrality requirements on the x variables in $P_{\mathcal{B}}^{I_{\mathcal{B}}}$ and still get a valid extended formulation of P^I. However, we will later argue that one may be able to obtain stronger split cuts by retaining the integrality of the x variables.

2.2 Integral, affine transformations and split cuts

For a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\text{conv}(X)$. Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let $1 \leq l \leq n$ and $I = \{1, \ldots, l\}$. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the *split set* associated with (π, π_0) is defined to be

$$S(\pi, \pi_0) = \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}.$$

We call a valid inequality for $\text{conv}(P \setminus S(\pi, \pi_0))$ a *split cut* for P *derived from* $S(\pi, \pi_0)$. If $\pi \in \mathbb{Z}^l \times \{0\}^{n-l}$ and $\pi_0 \in \mathbb{Z}$, then $\mathbb{Z}^l \times \mathbb{R}^{n-l} \subseteq \mathbb{R}^n \setminus S(\pi, \pi_0)$, and split cuts derived from the associated split set are valid for P^I. Let $\mathcal{S}_n(I) = \{S(\pi, \pi_0) : \pi \in \mathbb{Z}^l \times \{0\}^{n-l}, \pi_0 \in \mathbb{Z}\}$. We define the *split closure of* P *with respect to* I as

$$\text{SC}(P, I) = \bigcap_{S \in \mathcal{S}_n(I)} \text{conv}\,(P \setminus S)\,.$$

When $l = n$, we simply use $\text{SC}(P)$ instead of $\text{SC}(P, I)$. It is easy to see that for all $P, Q \subseteq \mathbb{R}^n$,

$$P \subseteq Q \implies \text{SC}(P, I) \subseteq \text{SC}(Q, I). \tag{9}$$

For $k = 2, 3, \ldots$, we define $\text{SC}^k(P, I) = \text{SC}(\text{SC}^{k-1}(P, I), I)$ where $\text{SC}^1(P, I) = \text{SC}(P, I)$. Split closures were first studied in [9] and play an important role in the theory and practice of integer programming.

For a given polyhedral mixed-integer set P^I and two binarization schemes $\mathcal{B}$ and $\mathcal{C}$, we will later compare these binarization schemes with respect to the "strength" of the split closures of the associated extended formulations. As $P_{\mathcal{B}}$ and $P_{\mathcal{C}}$ may not belong to the same Euclidean space, we will compare the projections of their split closures onto the original space. For this comparison we need a generalization of unimodular transformations that we describe below.

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *unimodular transformation* if $f(x) = Ux + v$ where U is a $n \times n$ unimodular matrix (i.e., an integral matrix with determinant ± 1) and $v \in \mathbb{Z}^n$. The split closure operation is invariant under unimodular transformations, see [11, Proposition 3] and also [12]. We generalize this result in Theorem 1 by giving a result on integral, affine transformations, i.e., functions $f : \mathbb{R}^m \to \mathbb{R}^n$ of the form $f(x) = Vx + v$ where V is an integral $n \times m$ matrix, and $v \in \mathbb{Z}^n$. For such an f, and $S \subseteq \mathbb{R}^n$, we define $f^{-1}(S) = \{x \in \mathbb{R}^m : f(x) \in S\}$, and for a collection $\mathcal{S}$ of subsets of $\mathbb{R}^n$, we define $f^{-1}(\mathcal{S}) = \{f^{-1}(S) : S \in \mathcal{S}\}$.

3 Basic properties of binarizations

In this section, we study how to get a stronger relaxation than the split closure of a polyhedron by applying split cuts to a binary extended formulation.

Let P be defined as in (2) and let $P_{\mathcal{B}}$ be defined as in (7). We observed in Section 2.1 that $\text{proj}_x(P_{\mathcal{B}}) = P$. For any extended formulation Q of P, it is shown in [6] that even when the new variables in Q are treated as continuous variables, the projection of the split closure of Q is contained in the split closure of P:

Proposition 1 *[6, Lemma 2] Let P be a polyhedron and let Q be an extended formulation of P, then* $\text{proj}_x(\text{SC}(Q, I)) \subseteq \text{SC}(P, I)$.

In addition, for some P, Q the containment above can be strict. This happens only when $\mathrm{SC}(Q, I)$ is defined by split cuts derived from multiple split sets [6, Lemma 1].

For $P_{\mathcal{B}}$ the above result implies that $\mathrm{proj}_x(\mathrm{SC}(P_{\mathcal{B}}, I)) \subseteq \mathrm{SC}(P, I)$. We will next show that this containment is not strict. For $i \in I$, let $w_0^i, w_{u_i}^i \in \{0,1\}^{q_i}$ be such that $(0, w_0^i), (u_i, w_{u_i}^i) \in B^i$. Clearly, $T^i = \mathrm{conv}\{(0, w_0^i), (u_i, w_{u_i}^i)\}$ is contained in B^i and is 1 dimensional. Furthermore,

$$P_{\mathcal{T}} = \Big\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^q \;:\; x \in P,\; (x_i, z_i) \in T^i \text{ for } i \in I\Big\}$$

is an extended formulation of P contained in $P_{\mathcal{B}}$. Since $P_{\mathcal{T}}$ has the same dimension as P, by [5, Corollary 4.5] we have $\mathrm{proj}_x(\mathrm{SC}(P_{\mathcal{B}}, I)) \supseteq \mathrm{proj}_x(\mathrm{SC}(P_{\mathcal{T}}, I)) = \mathrm{SC}(P, I)$. Therefore, to get stronger split cuts from $P_{\mathcal{B}}$, the new variables should be explicitly declared as binary variables.

Another natural question is whether the original variables need to be declared integral in the extended formulation as the integrality of the new z variables implies the integrality of the original x variables. We will next argue that in some cases, this is necessary as one gets weaker split cuts otherwise.

3.1 Linear binarizations

Let $B \in \Gamma_u^q$ be a binarization polytope. We say that B is *affine* if all $(x, z) \in B$ satisfy $x = \alpha^T z + \alpha_0$ for some $\alpha \in \mathbb{R}^n$ and $\alpha_0 \in \mathbb{R}$; in this case for all $(j, w^j) \in B$ with $w^j \in \{0,1\}^q$, we have $j = \alpha^T w^j + \alpha_0$ for $j \in \{0, \ldots, u\}$. We call B *linear* if it is affine and $\alpha_0 = 0$. The binarization polytopes $B^{\mathrm{U}}(u), B^{\mathrm{F}}(u)$ and $B^{\mathrm{L}}(u)$ are all linear. However not all binarizations are affine, as we show in the next example.

Example 1 The binarization polytope associated with $x \in \{0, \ldots, 3\}$ given by

$$\begin{aligned} B &= \mathrm{conv}\{(0,\ (0,0)), (1,\ (1,0)), (2,\ (1,1)), (3,\ (0,1))\} \\ &= \big\{(x, z) \in [0,3] \times [0,1]^2 : x - z_1 - z_2 \geq 0, \quad x + z_1 - 3z_2 \geq 0, \\ &\qquad\qquad -x + z_1 + 3z_2 \geq 0, \quad -x - z_1 + z_2 \geq -2\big\} \end{aligned} \tag{10}$$

is not affine. If it were, then for some $\alpha \in \mathbb{R}^2$ and $\alpha_0 \in \mathbb{R}$, we would have $j = \alpha^T w^j + \alpha_0$ for $j = 0, 1, 2, 3$ where $w^0 = (0,0), w^1 = (1,0), w^2 = (1,1)$, and $w^3 = (0,1)$. This would imply that $\alpha_0 = 0$ (from $j = 0$) and α satisfies $\alpha_1 = 1$, $\alpha_1 + \alpha_2 = 2$, and $\alpha_2 = 3$ simultaneously, which is not possible.

Let $I = \{1, \ldots, l\}$ and consider a binarization scheme $\mathcal{B} = (B^1, \ldots, B^l)$ defined by affine binarization polytopes. For each B^i let $x_i = a_i^T z_i + b_i$ hold where z_i denotes the vector of binary variables associated with x_i. Furthermore, if all $a_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}$, then

$$\mathrm{SC}(P_{\mathcal{B}}, I_{\mathcal{B}}) = \mathrm{SC}(P_{\mathcal{B}}, I'), \tag{11}$$

where $I_{\mathcal{B}}$, defined in (8), contains the indices of the original integer variables as well as the indices of the binarization variables, whereas $I' = I_{\mathcal{B}} \setminus I$ contains the indices of the binarization variables only. To see this, simply substitute each x_i in the inequalities defining a split set $S \in \mathcal{S}(I_{\mathcal{B}})$ by $a_i^T z_i + b_i$ to obtain an equivalent split set in $\mathcal{S}(I')$. We next observe that (11) does not necessarily hold for non-affine binarization schemes.

Proposition 2 *There exists a polyhedral mixed-integer set P^I and a binarization scheme $\mathcal{B} = (B^1, \ldots, B^{|I|})$ composed of non-affine binarization polytopes such that*

$$\text{conv}(P^I) = \text{SC}(P, I) = \text{proj}_x(SC(P_{\mathcal{B}}, I_{\mathcal{B}})) \subsetneq \text{proj}_x(\text{SC}(P_{\mathcal{B}}, I')), \text{ where } I' = I_{\mathcal{B}} \setminus I.$$

Proof Let $P = \{x \in [0,3]^2 : 0 \le x_2 - x_1 \le 0.5\}$. Let $\mathcal{B} = (B, B)$ where B is defined by (10). Let $I = \{1,2\}, I' = \{3,4,5,6\}$. By definition $\text{SC}(P) = \text{SC}(P, I)$ and,

$$P_{\mathcal{B}} = \{(x, z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \ : \ x \in P, \ (x_i, z_i) \in B, \ \text{for } i = 1, 2\}.$$

Clearly, we have $\text{conv}(P \cap \mathbb{Z}^2) = \{x \in [0,3]^2 : x_2 - x_1 = 0\} = \text{SC}(P)$. Therefore, $\text{proj}_x(\text{SC}(P_{\mathcal{B}}, I')) \supseteq \text{SC}(P)$. Let $(\bar{x}, \bar{z}) \in P_{\mathcal{B}}$ be defined by

$$\bar{x} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}, \bar{z} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Clearly $\bar{x} \notin \text{SC}(P)$. We will show that $(\bar{x}, \bar{z}) \in \text{SC}(P_{\mathcal{B}}, I')$ and thus $\text{proj}_x(\text{SC}(P_{\mathcal{B}}, I')) \neq \text{SC}(P)$.

Suppose $(\bar{x}, \bar{z}) \notin \text{SC}(P_{\mathcal{B}}, I')$. Then there exists a split set $S \in \mathcal{S}(I')$ such that

$$(\bar{x}, \bar{z}) \notin \text{conv}(P_{\mathcal{B}} \setminus S). \tag{12}$$

Let $S = \{(x, z) : \delta < az_{11} + bz_{12} + cz_{21} + dz_{22} < \delta + 1\}$ with $a, b, c, d, \delta \in \mathbb{Z}$. Then $(\bar{x}, \bar{z}) \in S$ which implies that $a\bar{z}_{11} + b\bar{z}_{12} + c\bar{z}_{21} + d\bar{z}_{22} = \delta + .5$ (as $\bar{z}$ is half-integral).

The points $(\bar{x}, z')$ and $(\bar{x}, z'')$ defined by $z' = \bar{z} + d_z$ and $z'' = \bar{z} - d_z$ where

$$d_z = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}$$

are both contained in $P_{\mathcal{B}}$ and $(\bar{x}, \bar{z}) = .5(\bar{x}, z') + .5(\bar{x}, z'')$. If $|c| \ge 1$, then S contains neither $(\bar{x}, z')$ nor $(\bar{x}, z'')$, contradicting (12); therefore $c = 0$.

The points (x', z') and (x'', z'') defined by $(x', z') = (\bar{x}, \bar{z}) + (d_x, d_z)$ and $(x'', z'') = (\bar{x}, \bar{z}) - (d_x, d_z)$, where

$$d_x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, d_z = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{pmatrix},$$

are both contained in $P_{\mathcal{B}}$ and $(\bar{x}, \bar{z}) = .5(x', z') + .5(x'', z'')$. For z', the expression $az'_{11} + bz'_{12} + dz'_{22}$ is integral as a, b, d are integral, $c = 0$, and z'_{11}, z'_{12} and z'_{22} are integral. Therefore $(x', z') \notin S$. Similarly, $(x'', z'') \notin S$, and $x \in \text{conv}(P_{\mathcal{B}} \setminus S)$, a contradiction. ∎

We next show that an affine binarization polytope can be transformed into a linear binarization polytope using a unimodular transformation. As split closures are invariant under unimodular transformations, this observation implies that there is no additional benefit in using affine binarizations in terms of cutting.

Proposition 3 *Let $B \in \Gamma_u^q$ be an affine binarization polytope. Then there exists a linear binarization polytope $B' \in \Gamma_u^q$ that is a unimodular transformation of B.*

Proof Let $(0, \bar{w}) \in B \cap (\mathbb{R} \times \{0,1\}^q)$. Since B is an affine binarization, there exist $\alpha \in \mathbb{R}^q$ and $\alpha_0 \in \mathbb{R}$ such that $B \subseteq \{(x,z) : x = \alpha^T z + \alpha_0\}$. By definition, we must have $\alpha_0 = -\alpha^T \bar{w}$.

Now define $f : \mathbb{R}^q \to \mathbb{R}^q$ as

$$f(z) = \bar{w} + Dz,$$

where D is the diagonal matrix with $D_{jj} = (1 - 2\bar{w}_j)$. Since $\bar{w} \in \{0,1\}^q$, $D_{jj} \in \{-1, 1\}$, and therefore D is unimodular with $D^{-1} = D$ and $D\bar{w} = -\bar{w}$. Consequently, f is invertible, with

$$f^{-1}(z) = D^{-1}(z - \bar{w}) = Dz + \bar{w}.$$

Furthermore, $f(\bar{w}) = \bar{w} + D\bar{w} = \bar{w} - \bar{w} = 0$. Define $B' = \{(x, f(z)) : (x,z) \in B\}$. As $f(\{0,1\}^q) = \{0,1\}^q$, B' is a binarization polytope. For any $(x, f(z)) \in B'$, we have

$$x = \alpha^T z + \alpha_0 = \alpha^T f^{-1}(f(z)) - \alpha^T \bar{w} = \alpha^T (D^{-1} f(z) + \bar{w}) - \alpha^T \bar{w} = \alpha^T D^{-1} f(z)$$

Thus, $B' \subseteq \{(x, z') : x = \alpha^T D^{-1} z'\}$, and hence B' is a linear binarization polytope. ∎

3.2 Perfect binarizations

The set Γ_u^q, as defined in (3), contains infinitely many polytopes, and therefore one can define infinitely many binary extended formulations of P of the form (7). We next look at a natural finite subset of Γ_u^q.

A binarization polytope $B \in \Gamma_u^q$ is *exact* if for each $x \in \{0, 1, \ldots, u\}$ there is a unique $z \in \{0,1\}^q$ such that $(x,z) \in B$. We say that a binarization polytope $B \in \Gamma_u^q$ is *perfect* if it is exact and $B = \text{conv}(B \cap (\mathbb{R} \times \{0,1\}^q))$. Thus, if B is perfect, then it is the convex hull of $u+1$ points of the form (k, w^k) for $k = 0, \ldots, u$. As each $w^k \in \{0,1\}^q$ there are at most $2^{q(u+1)}$ distinct perfect binarization polytopes in Γ_u^q. Also note that if $B \in \Gamma_u^q$, then it has dimension at most u.

Proposition 4 *Consider the extended formulation $P_{\mathcal{B}}$ of P where $\mathcal{B} = (B^1, \ldots, B^l)$ and B^i is not perfect for some i. Then $P_{\hat{\mathcal{B}}} \subseteq P_{\mathcal{B}}$ for some $\hat{\mathcal{B}}$ obtained from $\mathcal{B}$ by replacing B^i with a perfect binarization polytope $\hat{B}^i$.*

Proof Without loss of generality, suppose that $B^1 \in \Gamma_{u_1}^{q_1}$ is not perfect. For each $k \in \{0, \ldots, u_1\}$, choose a corresponding $w_k^1 \in \{0,1\}^{q_1}$ such that $(k, w_k^1) \in B^1$. If B^1 is not exact, this choice may not be unique for some values of k. Let $\hat{B}^1 = \text{conv}(\{(k, w_k^1) : k = 0, \ldots, u_1\})$. Then $\hat{B}^1$ is in $\Gamma_{u_1}^{q_1}$, perfect and contained in B^1. It follows that $P_{\hat{\mathcal{B}}} \subseteq P_{\mathcal{B}}$. ∎

The following is an example of a binarization polytope that is not exact.

Example 2 Let $P \subset \mathbb{R}^n$ be a polyhedron with $0 \le x_i \le 7$ for $i \in I$. Consider the binarization polytope $B = \{(x,z) \in \mathbb{R} \times [0,1]^4 : x = 5z_4 + \sum_{j=1}^3 2^{j-1} z_j\}$ and the associated extended formulation

$$P_{\mathcal{B}} = \{(x,z) \in \mathbb{R}^n \times [0,1]^q : x \in P, x_i = 5z_{i4} + \sum_{j=1}^{3} 2^{j-1} z_{ij} \text{ for all } i \in I\},$$

where $q = 4|I|$. Notice that $x_i \in \{5, 6, 7\}$ has two possible representations, one with $z_{i4} = 0$ and a second with $z_{i4} = 1$. Therefore we can define another valid binary extended formulation by setting z_{i4} to zero: $P_{\mathcal{B}'} = P_{\mathcal{B}} \cap \{(x, z) : z_{i4} = 0,\ i \in I\}$. Since $P_{\mathcal{B}'} \subseteq P_{\mathcal{B}}$, by Equation (9), we have $SC(P_{\mathcal{B}'}, I_{\mathcal{B}}) \subseteq SC(P_{\mathcal{B}}, I_{\mathcal{B}})$.

Note that $B^{\mathrm{F}}(u)$ and $B^{\mathrm{U}}(u)$ are perfect binarization polytopes whereas $B^{\mathrm{L}}(u)$ is exact but not perfect unless $u+1$ is a power of 2. By Proposition 4 perfect binarization polytopes are more desirable as they lead to stronger extended formulations. We define the perfect version of $B^{\mathrm{L}}(u)$ as (here $q = \lceil \log_2(u + 1)\rceil$)

$$B^{\mathrm{L}^+}(u) = \mathrm{conv}(B^{\mathrm{L}}(u) \cap (\mathbb{R} \times \{0, 1\}^q)).$$

We next show that the binarization polytope $B^{\mathrm{L}}(u)$ can be made perfect by adding at most u inequalities to $B^{\mathrm{L}}(u)$. For this, we adapt a result from [18, Corollary 2.6] about knapsack polytopes with *superincreasing* coefficients. We give a proof here to explicitly construct the required inequalities in this context.

Proposition 5 *$B^{L^+}(u)$ can be described by one equation and at most $q-1$ inequalities for $q = \lceil \log_2(u + 1)\rceil$ and the simple bound constraints. These inequalities can be computed in polynomial time.*

Proof Let $P = \mathrm{conv}(\{z \in \{0, 1\}^q : \sum_{j=1}^q 2^{j-1} z_j \le u\})$ and note that

$$\begin{aligned} B^{\mathrm{L}^+}(u) &= \mathrm{conv}\Big(\Big\{(x, z) \in \mathbb{R} \times \{0, 1\}^q : x = \sum_{j=1}^{q} 2^{j-1} z_j \le u\Big\}\Big) \\ &= \Big\{(x, z) \in \mathbb{R} \times P : x = \sum_{j=1}^{q} 2^{j-1} z_j\Big\}. \end{aligned}$$

We will next give an explicit inequality description of P. Let $u = \sum_{j=1}^q 2^{j-1} c_j$ for $c \in \{0, 1\}^q$. Let J be the set of indices such that $c_j = 0$ and note that $|J| \le q - 1$ as $u > 0$. For each $j \in J$, define the vector $a_j \in \{0, 1\}^q$ as

$$a_{jk} = \begin{cases} c_k & \text{if } k > j, \\ 1 & \text{if } k = j, \\ 0 & \text{if } k < j, \end{cases}$$

where a_{jk} stands for the kth coefficient of the vector a_j. Note that $\sum_{i=1}^q 2^{i-1} a_{ji} > u$ for all $j \in J$. Define $b_j = \sum_{k=1}^q a_{jk} - 1$. The inequalities $a_j^T x \le b_j$ are known as cover inequalities for the knapsack polytope. Let A be the matrix whose rows are a_j^T for $j \in J$ and define $Q = \{x \in [0, 1]^q : Ax \le b\}$. Since A^T has the so-called *consecutive 1's property*, it follows that A is totally unimodular. In addition, as b is also integral, Q is an integral polytope.

We now show that $P = Q$. Since P is also an integral polytope, it suffices to show that $P \cap \{0, 1\}^q = Q \cap \{0, 1\}^q$. First consider $\bar{z} \in \{0, 1\}^q$ such that $\bar{z} \notin Q$. Then, for some $j \in J$, $a_j^T \bar{z} > b_j$ and consequently $\bar{z} \ge a_j$. Then $\sum_{i=1}^q 2^{i-1} \bar{z}_i \ge \sum_{i=1}^q 2^{i-1} a_{ji} > u$. Thus, $\bar{z} \notin P$.

Conversely, consider $\bar{z} \in \{0, 1\}^q$ such that $\bar{z} \notin P$. Clearly, $\sum_{i=1}^q 2^{i-1} \bar{z}_i > u = \sum_{i=1}^q 2^{i-1} c_i$. Therefore there exists an index j such that $\bar{z}_j > c_j$ and $\bar{z}_k = c_k$ for all $k > j$. As $\bar{z}, c \in \{0,1\}^q$, we have $\bar{z}_j = 1$ and $c_j = 0$. Therefore $j \in J$ and the first $j-1$ components of a_j are zero and the jth component is one. Therefore, $a_j^T \bar{z} = a_j^T c + 1 > a_j^T c = b_j$. Thus, $\bar{z} \notin Q$.

Hence $P \cap \{0,1\}^q = Q \cap \{0,1\}^q$ and $P = Q$. Finally as $|J| \leq q-1$, the proof is complete. ∎

3.3 Strength of single disjunctions in extended space

The following result shows that if a split disjunction in the extended space only involves the binarization variables associated with a single original variable, then it is not more useful than a split disjunction involving the original variable itself.

Proposition 6 *Let P^I be a given polyhedral mixed-integer set, with $I = \{1,\ldots,l\}$ and let $\mathcal{B} = (B^1,\ldots,B^l)$ a binarization scheme. Let $B^1 \in \Gamma_u^{q_1}$ and let z_1 be the auxiliary binary variables associated with x_1. Then, for any split set $S = \{(x,z) \in \mathbb{R}^{n+q} : \pi_0 < \pi^T z_1 < \pi_0 + 1\}$ in the extended space, there exists a split set $S' = \{x \in \mathbb{R}^n : \pi_0' < x_1 < \pi_0' + 1\}$ in the original space such that*

$$\mathrm{proj}_x(P_{\mathcal{B}} \setminus S) \supseteq P \setminus S'.$$

Proof Let

$$P_{\mathcal{B}} \setminus S = P_{\mathcal{B}} \cap \{(x,z) \in \mathbb{R}^{n+q} : (x_1, z_1) \in A_0 \cup A_1\},$$

where $A_0 = \{(x_1,z_1) \in B^1 : \pi^T z_1 \leq \pi_0\}$ and $A_1 = \{(x_1,z_1) \in B^1 : \pi^T z_1 \geq \pi_0 + 1\}$. As B^1 is a binarization polytope, for each $t \in \{0,\ldots,u\}$ there exists a point $p^t = (t, w^t) \in B^1$ where $w^t \in \{0,1\}^{q_1}$. Without loss of generality, assume that $p^0 \in A_0$ and let $s \in \{0,\ldots,u\}$ be the largest index such that $p^s \in A_0$. We will show that choosing $\pi_0' = s$ is sufficient.

If $P \subseteq \mathrm{proj}_x(P_{\mathcal{B}} \setminus S)$, then the result holds trivially. Otherwise, there exists a point $\hat{x} \in P \setminus \mathrm{proj}_x(P_{\mathcal{B}} \setminus S)$. For any distinct $r, t \in \{0,\ldots,u\}$ with $t \geq \hat{x}_1 \geq r$ we can construct a point $p^{rt} = (\hat{x}_1, w^{rt}) = \lambda p^t + (1-\lambda)p^r$ where $\lambda = (\hat{x}_1 - r)/(t-r)$. As $\lambda \in [0,1]$, we have $p^{rt} \in B^1$. Consequently, there exists a point $(\hat{x},\hat{z}) \in P_{\mathcal{B}}$ where $\hat{z}_1 = w^{rt}$. If $p^r, p^t \in A_0$, then, by convexity, we also have $p^{rt} \in A_0$, implying that $(\hat{x},\hat{z}) \in \{(x,z) \in P_{\mathcal{B}} : (x_1,z_1) \in A_0\}$. Similarly if $p^r, p^t \in A_1$ then, $(\hat{x},\hat{z}) \in \{(x,z) \in P_{\mathcal{B}} : (x_1,z_1) \in A_1\}$. Therefore, if there exists r,t such that $t \geq \hat{x}_1 \geq r$ and $p^r, p^t \in A_0$ or $p^r, p^t \in A_1$, then $\hat{x} \in \mathrm{proj}_x(P_{\mathcal{B}} \setminus S)$, a contradiction.

As $p^0 \in A_0$, for all pairs $r, t \in \{0,\ldots,u\}$ with $r \leq \hat{x}_1 \leq t$, we have $p^r \in A_0$ and $p^t \in A_1$. Therefore $p^0,\ldots,p^s \in A_0$, $p^{s+1},\ldots,p^u \in A_1$ and $s < \hat{x}_1 < s+1$. Since $\hat{x}$ was chosen arbitrarily in $P \setminus \mathrm{proj}_x(P_{\mathcal{B}} \setminus S)$, this shows that $P \setminus \mathrm{proj}_x(P_{\mathcal{B}} \setminus S) \subseteq \{x \in P : s < x_1 < s+1\}$. ∎

Note that this result also implies that

$$\mathrm{proj}_x(\mathrm{conv}(P_{\mathcal{B}} \setminus S)) \supseteq \mathrm{conv}(P \setminus S')$$

and we observe that split disjunctions in the extended space must involve binarization variables associated with multiple original variables in order to generate cuts that cannot be obtained using original variables. We next give an example where cuts from a single split set in the extended space can give the convex hull of a mixed-integer set while there is no split set, or more generally, no lattice-free convex set in the original space that can do the same.

Example 3 Let $P = \{x \in [0,2]^2 : x_2 = \frac{1}{2}x_1 + \frac{1}{2}\}$ and $I = \{1,2\}$. Then P^I consists of a single point $p = (1,1)$. As p is contained in the relative interior of P, there is no lattice-free convex set (e.g., a split set) $S \subseteq \mathbb{R}^2$ that satisfies $P^I = \text{conv}(P \setminus S)$. Let $\mathcal{B} = (B_1, B_2)$, where $B_i = B^{\text{F}}(2)$ is the full binarization polytope (4) with $u = 2$ for $i = 1,2$, i.e.,

$$B_i = \{(x_i, z_i) \in \mathbb{R} \times [0,1]^2 \ : \ x_i = z_{i1} + 2z_{i2}, \ z_{i1} + z_{i2} \leq 1\}.$$

Let $P_{\mathcal{B}}$ be the binary extended formulation of P defined by $\mathcal{B}$. For $(x,z) \in P_{\mathcal{B}}$,

$$x_2 - z_{12} = \frac{1}{2}x_1 + \frac{1}{2} - z_{12} = \frac{1}{2}(z_{11} + 2z_{12}) + \frac{1}{2} - z_{12} = \frac{1}{2}(z_{11} + 1) > 0. \quad (13)$$

Let $S = \{(x,z) \in \mathbb{R}^2 \times \mathbb{R}^{(2+2)} : 0 < x_2 - z_{12} < 1\}$ be a split set in the space of $P_{\mathcal{B}}$. Then $P_{\mathcal{B}} \setminus S$ consists of points in $P_{\mathcal{B}}$ that satisfy $x_2 - z_{12} \leq 0$ or $x_2 - z_{12} \geq 1$. Because of (13), there are are no points in $P_{\mathcal{B}}$ that satisfy the first inequality, and all points in $P_{\mathcal{B}}$ satisfying $x_2 - z_{12} \geq 1$ also satisfy $(z_{11} + 1) \geq 2$ and thus the equations $z_{11} = 1, z_{12} = 0$, $x_1 = 1$, and $x_2 = 1$. Therefore

$$\text{proj}_x(\text{conv}(P_{\mathcal{B}} \setminus S)) = \{(1,1)\} = P^I.$$

Bonami and Margot [7] have already observed that the rank-2 simple split closure of an extended formulation obtained using the unary binarization scheme (5) always leads to the integer hull in the original space when the original set has only two integer variables. The example above shows that in some cases this might happen even with a single split cut in the extended space when the split disjunction combines binarization variables associated with different original variables.

4 Unimodular binarization schemes

We next characterize a class of binarization schemes that contain the full and unary-binarization schemes and show that they all have equally strong projected split closures. Moreover, by this measure, we show that these schemes lead to the strongest extended formulations for polyhedral mixed-integer sets.

Definition 1 Let $B \in \Gamma_u^u$, for some integer $u > 0$, be a perfect binarization polytope such that B is the convex hull of points (j, w^j) for $j = 0, \ldots, u$ where $w^j \in \{0,1\}^u$. We say that B is *unimodular* if the $u \times u$ matrix with columns $w^j - w^0$ for $j = 1, \ldots, u$ is unimodular.

Recall that the full binarization polytope $B^{\text{F}}(u)$ and the unary binarization polytope $B^{\text{U}}(u)$ are perfect. Moreover, $B^{\text{F}}(u)$ is equal to the convex hull of points (j, e^j), where e^j is the jth standard unit vector for $j = 1, \ldots, u$, and e^0 is the all-zeros vector. Similarly, $B^{\text{U}}(u)$ is equal to the convex hull of points (j, d^j), where $d^j = \sum_{i=0}^{j} e^i$ for $j = 0, \ldots, u$. Consequently, both these polytopes are unimodular.

We next present some technical results that we need for the main result. We start off by generalizing a result in [11, Proposition 3] on unimodular transformations to integral, affine transformations.

Theorem 1 *Let $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ be polyhedral sets and let $I = \{1, \dots, l\}$ and $I' = \{1, \dots, l'\}$ where $l \leq m$ and $l' \leq n$ and $n - l' = m - l$. Let $g\colon \mathbb{R}^l \to \mathbb{R}^{l'}$ be an integral, affine transformation and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be defined as $f(x) = (g(x^1), x^2)$ where $x = (x^1, x^2) \in \mathbb{R}^l \times \mathbb{R}^{m-l}$. If $f(P) \subseteq Q$, then for any integer $k \geq 1$,*

$$f(\mathrm{SC}^k(P, I)) \subseteq SC^k(Q, I').$$

Proof Let $\mathcal{S}$ be a collection of sets in $\mathbb{R}^n$. For any $S \in \mathcal{S}$, if $x \in S$ then $f(x) \in f(S)$, and therefore

$$f\left(\bigcap_{S\in\mathcal{S}} S\right) \subseteq \bigcap_{S\in\mathcal{S}} f(S). \tag{14}$$

Furthermore, note that $f(\sum_{i=1}^t \lambda_i x_i) = \sum_{i=1}^t \lambda_i f(x_i)$ for any $x_1, \dots, x_t \in \mathbb{R}^m$ and any $\lambda_1, \dots, \lambda_t \in \mathbb{R}$ satisfying $\sum_{i=1}^t \lambda_i = 1$. Therefore, for any $T \subseteq \mathbb{R}^m$

$$f(\mathrm{conv}(T)) = \mathrm{conv}(f(T)). \tag{15}$$

In addition,

$$f(P) \setminus S = \{f(x) : x \in P, f(x) \notin S\} = \{f(x) : x \in P, x \notin f^{-1}(S)\} = f(P \setminus f^{-1}(S))$$

Taking $T = P \setminus f^{-1}(S)$ in (15), we see that

$$f(\mathrm{conv}(P \setminus f^{-1}(S))) = \mathrm{conv}(f(P) \setminus S). \tag{16}$$

Let $g(x_1) = Vx_1 + v$ where $V \in \mathbb{Z}^{l' \times l}$ and $v \in \mathbb{Z}^{l'}$. Consider the split set $S \in \mathcal{S}_n(I')$ given by $S = \{(y_1, y_2) \in \mathbb{R}^{l'} \times \mathbb{R}^{n-l'} : \pi_0 < \pi_1^T y_1 + \pi_2^T y_2 < \pi_0 + 1\}$, where π_1, π_2 and π_0 are integral and $\pi_2 = 0$. Then

$$\begin{aligned} f^{-1}(S) &= \{(x_1, x_2) \in \mathbb{R}^l \times \mathbb{R}^{m-l} : \pi_0 < \pi_1^T(Vx_1 + v) < \pi_0 + 1\} \\ &= \{(x_1, x_2) \in \mathbb{R}^l \times \mathbb{R}^{m-l} : \pi_0 - \pi_1^T v < \pi_1^T V x_1 < \pi_0 + 1 - \pi_1^T v\}. \end{aligned}$$

As $\pi_1^T V$ and $\pi_1^T v$ are integral, we see that $f^{-1}(S)$ is a split set in $\mathcal{S}_m(I)$ unless $\pi_1^T V = 0$, in which case $f^{-1}(S) = \emptyset$. Therefore, $\{f^{-1}(S) : S \in \mathcal{S}_n(I')\} \subseteq \mathcal{S}_m(I) \cup \{\emptyset\}$, and

$$\mathrm{SC}(P, I) = \bigcap_{S\in\mathcal{S}_m(I)} \mathrm{conv}(P \setminus S) \subseteq \bigcap_{S\in\mathcal{S}_n(I')} \mathrm{conv}(P \setminus f^{-1}(S)). \tag{17}$$

Then

$$\begin{aligned} f(\mathrm{SC}(P, I)) &\subseteq f(\bigcap_{S\in\mathcal{S}_n(I')} \mathrm{conv}(P \setminus f^{-1}(S))) \\ &\subseteq \bigcap_{S\in\mathcal{S}_n(I')} f(\mathrm{conv}(P \setminus f^{-1}(S))) \\ &= \bigcap_{S\in\mathcal{S}_n(I')} \mathrm{conv}(f(P) \setminus S) \ \subseteq \ SC(Q, I'), \end{aligned}$$

where the first inclusion follows from (17) and the second one follows from (14). The next equality follows from (16) and the final inclusion follows from (9) and the fact that $P \subseteq Q$.

Therefore the claim holds for $k = 1$ and the result follows by induction on k. ∎

Lemma 1 *Let q, u be positive integers and let $B \in \Gamma_u^u$ be unimodular and $C \in \Gamma_u^q$. Then there exists an integral, affine transformation $g : \mathbb{R}^{u+1} \to \mathbb{R}^{q+1}$ such that $g(B) \subseteq C$ where $g(x, z) = (x, h(z))$ for some $h : \mathbb{R}^u \to \mathbb{R}^q$.*

Proof As B is perfect, $B = \text{conv}(\{(j, v^j) : j = 0, \dots, u\})$ for some $v^j \in \{0,1\}^u$ and C contains points (j, w^j) for some $w^j \in \{0,1\}^q$ for all $j = 0, \dots, u$. Let V be the $u \times u$ unimodular matrix with columns $v^j - v^0$ and let W be the integral matrix with columns $w^j - w^0$.

Define the integral affine transformation $h : \mathbb{R}^u \to \mathbb{R}^q$ as $h(z) = WV^{-1}z - WV^{-1}v^0 + w^0$ and note that WV^{-1} is an integral matrix and v^0, w^0 are integral vectors. Furthermore,

$$h(v^j) = WV^{-1}v^j - WV^{-1}v^0 + w^0 \;=\; WV^{-1}(v^j - v^0) + w^0.$$

As $WV^{-1}V = W$, we have $h(v^j) = w^j$.

Clearly, $g(x, z) = (x, h(z))$ is an integral affine transformation and it commutes with the convex hull operator $\text{conv}(\cdot)$:

$$\begin{aligned} g(B) = g(\text{conv}(\{(j, v^j) : j = 0, \dots, u\})) &= \text{conv}(\{g(j, v^j) : j = 0, \dots, u\}) \\ &= \text{conv}(\{(j, w^j) : j = 0, \dots, u\}) \subseteq C. \end{aligned}$$ ∎

We now prove the main result of this section.

Theorem 2 *Let P be defined as in (2), and let $I = \{1, \dots, l\}$. Consider a binarization scheme $\mathcal{B} = (B^1, \dots, B^l)$ where each B^i is unimodular and let $\mathcal{C} = (C^1, \dots, C^l)$ be an arbitrary binarization scheme. Then for all integers $k \geq 1$,*

$$\text{proj}_x(\text{SC}^k(P_{\mathcal{B}}, I_{\mathcal{B}})) \subseteq \text{proj}_x(\text{SC}^k(P_{\mathcal{C}}, I_{\mathcal{C}})).$$

Proof Lemma 1 implies that for each $i = 1, \dots, l$, there exists an integral affine transformation h_i such that the transformation $g_i : (x_i, z_i) \to (x_i, h_i(z_i))$ is integral, affine, and $g_i(B^i) \subseteq C^i$. Therefore, if $(x, z_1, \dots, z_l) \in P_{\mathcal{B}}$, then $(x_i, z_i) \in B^i$ and $(x_i, h_i(z_i)) \in C^i$. Let f be the integral affine function from the space of $P_{\mathcal{B}}$ to the space of $P_{\mathcal{C}}$ defined as

$$f(x, z_1, \dots, z_l) = (x, h_1(z_1), \dots, h_l(z_l)).$$

Then $f(P_{\mathcal{B}}) \subseteq P_{\mathcal{C}}$ and Theorem 1 implies that $f(\text{SC}^k(P_{\mathcal{B}}, I_{\mathcal{B}})) \subseteq \text{SC}^k(P_{\mathcal{C}}, I_{\mathcal{C}})$ for all $k \geq 1$.

Let $\bar{x} \in \text{proj}_x(\text{SC}^k(P_{\mathcal{B}}, I_{\mathcal{B}}))$ for some $k \geq 1$. By definition, there exists vectors $\bar{z}_1, \dots, \bar{z}_l$ such that $(\bar{x}, \bar{z}_1, \dots, \bar{z}_l) \in \text{SC}^k(P_{\mathcal{B}}, I_{\mathcal{B}})$. Therefore

$$f(\bar{x}, \bar{z}_1, \dots, \bar{z}_l) = (\bar{x}, h_1(\bar{z}_1), \dots, h_l(\bar{z}_l)) \in \text{SC}^k(P_{\mathcal{C}}, I_{\mathcal{C}}).$$

This implies that $\bar{x} \in \text{proj}_x(\text{SC}^k(P_{\mathcal{C}}, I_{\mathcal{C}}))$, and the proof is complete. ∎

The following is a consequence of Theorem 2.

Corollary 1 *Let P be defined as in (2), and let $I = \{1, \dots, l\}$. If $\mathcal{B}$ and $\mathcal{C}$ are two binarization schemes defined by unimodular binarization polytopes, then*

$$\text{proj}_x(\text{SC}(P_{\mathcal{B}}, I_{\mathcal{B}})) = \text{proj}_x(\text{SC}(P_{\mathcal{C}}, I_{\mathcal{C}})).$$

In particular we conclude that full and unary binarization schemes are stronger than all other binarization schemes in the sense that the projection of their split closures are equal to each other and are contained in all other projected split closures. Moreover, the proof of Theorem 2 implies that the unimodular transformation that maps one unimodular binarization scheme to another also maps its split closure (in the extended space) to the split closure of the other.

As an other application of Theorem 2, let $\mathcal{B}$ be a binarization scheme defined by a unimodular binarization polytope and let $\mathcal{C}$ be the logarithmic binarization scheme. Therefore we have $\mathrm{proj}_x(\mathrm{SC}^k(P_{\mathcal{B}}, I_{\mathcal{B}})) \subseteq \mathrm{proj}_x(\mathrm{SC}^k(P_{\mathcal{C}}, I_{\mathcal{C}}))$ for all $k \geq 1$. Furthermore, as $P_{\mathcal{C}}^{I_{\mathcal{C}}}$ is defined by $q = \sum_{i=1}^{l} \lceil \log_2(u_i + 1) \rceil$ binary variables, all vertices of $\mathrm{SC}^q(P_{\mathcal{C}}, I_{\mathcal{C}})$ have integral z values by a result of Balas [3] on disjunctive cuts. Therefore they also have integral coordinates for the variables $x_1, \ldots, x_l$. Consequently, $\mathrm{proj}_x(\mathrm{SC}^q(P_{\mathcal{B}}, I_{\mathcal{B}})) = \mathrm{conv}(P^I)$ and we have the following observation.

Corollary 2 *Let P, I and $\mathcal{B}$ be defined as in Theorem 2. Then* $\mathrm{proj}_x(\mathrm{SC}^q(P_{\mathcal{B}}, I_{\mathcal{B}})) = \mathrm{conv}(P^I)$ *where* $q = \sum_{i=1}^{l} \lceil \log_2(u_i + 1) \rceil$.

5 Relative strength of binarization schemes

By Proposition 1, the projection of the split closure of any extended formulation is contained in the split closure of the original formulation. In this section, we first give an example for which the containment is strict for the logarithmic extended formulation. We then give a hierarchy of other schemes considered earlier in the paper.

5.1 The logarithmic binarization is better than the original formulation

Consider

$$P = \Big\{ x \in [0,2]^2 \ : \ 2x_1 + x_2 \leq 5, \ \ -2x_1 + 3x_2 \leq 3 \Big\},$$

and the associated integer set $P^I \subset \mathbb{Z}^2$ where $I = \{1, 2\}$. Now consider the extended formulation of P obtained by using the logarithmic binarization scheme:

$$P_L = \Big\{ (x,z) \in [0,2]^2 \times [0,1]^4 \ : \ x \in P, \ x_i = z_{i1} + 2z_{i2}, \quad \text{for } i = 1,2 \Big\}.$$

Note that the logarithmic binarization polytope $B^{\mathrm{L}}(u)$ is not perfect for $u = 2$.

Theorem 3 *For P and P_L defined above, we have* $\mathrm{proj}_x(\mathrm{SC}(P_L)) \subsetneq \mathrm{SC}(P)$.

Proof We will show that the point $\bar{x} = (1.25, 1.5) \in P$ is contained in the split closure of P but not in the projection of the split closure of the associated logarithmic extended formulation P_L. See Figure 1.

Suppose $\bar{x} \notin \mathrm{SC}(P)$. Then $\bar{x} \notin \mathrm{conv}(P \setminus S)$ for some split set $S = \{x \in \mathbb{R}^2 : \pi_0 < \pi^T x < \pi_0 + 1\}$ where π, π_0 are integral. By definition, $P \setminus S = P \cap (S_1 \cup S_2)$ where $S_1 = \{x \in \mathbb{R}^2 : \pi^T x \leq \pi_0\}$ and $S_2 = \{x \in \mathbb{R}^2 : \pi^T x \geq \pi_0 + 1\}$ and $\mathbb{Z}^2 \subseteq S_1 \cup S_2$. Note that $\bar{x}$ lies in the convex hull of $(2,1) \in P \cap \mathbb{Z}^2$ and $p_1 = (1, 5/3) \in P$ and therefore, $p_1 \in S$. In a similar manner, we can conclude that $p_2 = (1.5, 2) \in P$ and

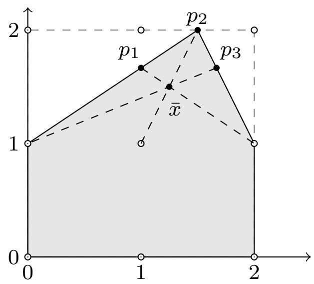

Fig. 1 Polytope P

$p_3 = (5/3, 5/3) \in P$ are both contained in S as $\bar{x}$ lies in the convex hull of p_2 and $(1,1) \in P$, and also in the convex hull of p_3 and $(0,1) \in P$.

Consequently $(1,1)$ and $(1,2)$ are not contained in the same S_i, otherwise p_1 (a convex combination of the previous two points) would be contained in the same S_i, contradicting $p_1 \in S$. Similarly, we can conclude that $(1,2)$ and $(2,2)$ are not in the same S_i (otherwise p_2 would be contained in the same S_i), and $(1,1)$ and $(2,2)$ are not in the same S_i (otherwise p_3 would be contained in the same S_i). Given that there are only two choices for S_i, we get a contradiction.

We next show that the inequality $x_2 \le 1.4$ is valid for the split closure of P_L. We will first argue that the following inequalities are split cuts for P_L:

$$z_{21} + \quad z_{22} \le 1 \tag{18}$$

$$z_{22} - \quad z_{12} \le 0 \tag{19}$$

$$-2z_{11} + 3x_2 \le 3 \tag{20}$$

$$2x_1 + 3x_2 \le 7. \tag{21}$$

The inequality (18) can be obtained as a Gomory-Chvátal cut (and therefore as a split cut [9]) from the inequality $2z_{21} + 2z_{22} \le 3$ which is implied by $z_{21} + 2z_{22} = x_2 \le 2$ and $z_{21} \le 1$. To obtain inequality (19), replace x_i in $-2x_1 + 3x_2 \le 3$ by $x_i = z_{i1} + 2z_{i2}$ to get $-2z_{11} - 4z_{12} + 3z_{21} + 6z_{22} \le 3$. Adding the valid inequalities $2z_{11} \le 2$, $-3z_{21} \le 0$ and $-2z_{12} \le 0$ for P_L to the previous inequality, we get $6z_{22} - 6z_{12} \le 5$ which yields (19) as a Gomory-Chvátal cut for P_L.

To see that inequality (20) is a split cut, consider the disjunction $z_{12} \le 0$ or $z_{12} \ge 1$. If $z_{12} \ge 1$, then $z_{11} = 0$ and $x_1 = 2$, and therefore $x_2 \le 1$ as $2x_1 + x_2 \le 5$ for all $x \in P$ and (20) is satisfied. On the other hand, if $z_{12} \le 0$, then $x_1 = z_{11}$ and (20) is implied by the second inequality defining P, namely, $-2x_1 + 3x_2 \le 3$. Finally, inequality (21) can be obtained as a split cut from the disjunction $x_1 \le 1$ or $x_1 \ge 2$. If $x_1 \le 1$, then the inequality $-2x_1 + 3x_2 \le 3$ defining P implies $3x_2 \le 5$ and (21) is satisfied. On the other hand, if $x_1 = 2$, then the inequality $2x_1 + x_2 \le 5$ defining P implies that $x_2 \le 1$ and therefore (21) holds.

Combining inequalities (18)-(21) with the multipliers 4,4,1, and 1, respectively, gives the following inequality:

$$2x_1 + 6x_2 - 2z_{11} - 4z_{12} + 4z_{21} + 8z_{22} \;\le\; 14.$$

As $-2z_{11} - 4z_{12} = -2x_1$ and $4z_{21} + 8z_{22} = 4x_2$, this simplifies to $10x_2 \le 14$ and therefore, $x_2 \le 1.4$ is indeed valid for the split closure of P_L. ∎

5.2 Perfect logarithmic binarization is better than logarithmic binarization

We next give an example for which the projection of the split closure of the perfect logarithmic extended formulation is strictly contained in the projection of the split closure of the logarithmic formulation. Consider

$$P = \{x \in [0,2]^2 \;:\; x_1 + 10x_2 \le 20,\; 10x_1 + x_2 \le 20\},$$

and the associated integer set $P^I \subset \mathbb{Z}^2$ where $I = \{1,2\}$. Now consider the extended formulation of P obtained by using the logarithmic binarization scheme:

$$P_L = \{(x,z) \in \mathbb{R}^2 \times [0,1]^4 \;:\; x \in P,\; x_i = z_{i1} + 2z_{i2}, \quad \text{for } i = 1,2\},$$

and the extended formulation of P obtained by using the perfect logarithmic binarization scheme:

$$P_{L+} = \{(x,z) \in \mathbb{R}^2 \times [0,1]^4 \;:\; x \in P,\; x_i = z_{i1} + 2z_{i2},\; z_{i1} + z_{i2} \le 1, \quad \text{for } i = 1,2\}.$$

Theorem 4 *For P defined above, we have* $\text{proj}_x(\text{SC}(P_{L+})) \subsetneq \text{proj}_x(\text{SC}(P_L))$.

Proof We will show that the point $\bar{x} = (6/5, 6/5)$ belongs to $\text{proj}_x(\text{SC}(P_L))$ but not to $\text{proj}_x(\text{SC}(P_{L+}))$. We will first argue that the following inequalities are split cuts for P_{L+}:

$$z_{11} + z_{12} \qquad\quad + z_{22} \le 1 \tag{22}$$

$$z_{12} + z_{21} + z_{22} \le 1 \tag{23}$$

To see (22) is a split cut, consider the disjunction $z_{22} \le 0$ or $z_{22} \ge 1$. When $z_{22} \le 0$, (22) holds as $z_{11} + z_{12} \le 1$ is valid for P_{L+}. On the other hand, if $z_{22} \ge 1$, then $x_2 = 2$ and $x_1 = 0$. Consequently, $z_{11} + z_{12} = 0$ and the inequality (22) holds. The argument for (23) is similar using the disjunction $z_{12} \le 0$ or $z_{12} \ge 1$.

Adding inequalities (22) and (23), we get $z_{11} + 2z_{12} + z_{21} + 2z_{22} \le 2$, which is the same as $x_1 + x_2 \le 2$ and therefore $\bar{x} \notin \text{proj}_x(\text{SC}(P_{L+}))$.

To that $\bar{x} \in \text{proj}_x(\text{SC}(P_L))$, we first show that $\bar{x} \in \text{SC}(P)$. Let

$$p_1 = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix},$$

and note we can write $\bar{x}$ as a convex combination of any one of these points and an integral point in P, see Figure 2. More precisely: $\bar{x} = 4/5p_1 + 1/5(2,0) = 2/5p_2 + 3/5(1,1) = 4/5p_3 + 1/5(0,2)$. Therefore, if $\bar{x} \notin \text{SC}(P)$, then for some split set S we have $\bar{x} \notin \text{conv}(P \setminus S)$ and $p_1, p_2, p_3, \bar{x} \in S$. Let $\mathbb{R}^2 \setminus S = A \cup B \supset \mathbb{Z}^2$ where A and B are half spaces denoting the two sides of the split disjunction. Without loss of generality, assume $(1,1) \in A$. As $p_1 \notin A$, we have $(1,2) \in B$ and as $p_3 \notin A$, we have $(2,1) \in B$. But then, $p_2 \in B$ as $p_2 = 1/2(2,1) + 1/2(1,2)$, a contradiction. Therefore, $\bar{x} \in \text{SC}(P)$.

To prove that $\bar{x}$ belongs to $\text{proj}_x(\text{SC}(P_L))$, we will show that $\bar{p} = (\bar{x}, \bar{z}) \in \text{SC}(P_L)$ where

$$\bar{x} = \begin{pmatrix} 6/5 \\ 6/5 \end{pmatrix}, \bar{z} = \begin{pmatrix} 2/5 & 2/5 \\ 2/5 & 2/5 \end{pmatrix}.$$

As $\bar{x} \in P$ and $\bar{p}$ satisfies $x_i = z_{i1} + 2z_{i2}$ for $i = 1,2$, we have $\bar{p} \in P_L$. Suppose $\bar{p} \notin \text{SC}(P_L)$. Then $\bar{p} \notin \text{conv}(P_L \setminus S)$ for some split set $S = \{(x,z) \in \mathbb{R}^{2+4} : d <$

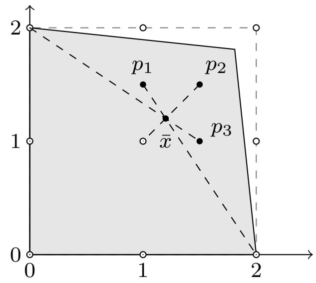

Fig. 2 Polytope P

$a^T x + c \cdot z < d + 1\}$ where c is an integral matrix, a is an integral vector and d is an integer, and $c \cdot z = \sum_{ij} c_{ij} z_{ij}$. Subtracting appropriate multiples of the equations $x_i = z_{i1} + 2z_{i2}$ (valid for P_L) from $a^T x + c \cdot z$, we can assume $a = 0$ and

$$S = \{(x, z) : d < c \cdot z < d + 1\}$$

for some nonzero c. As c is integral and $\bar{z}$ is $(1/5)$-integral, it follows that $c \cdot \bar{z} = d + \delta$ where $\delta \in \{1/5, 2/5, 3/5, 4/5\}$.

We will next construct several pairs of points $p', p'' \in P_L$ with the property that $\bar{p}$ is a convex combination of p', p''; if both p' and p'' are not contained in S, then $\bar{p} \in \text{conv}(P_L \setminus S)$, a contradiction. Therefore the split set must contain at least one of p' or p'' for each pair, and we will use this fact to impose conditions on c till we get a contradiction.

First note that if $3/5 \leq x_1 \leq 9/5$ and $3/5 \leq x_2 \leq 9/5$, then $(x_1, x_2) \in P$. Therefore, for any such (x_1, x_2), choosing z_{ij} values for $i = 1, 2$ and $j = 1, 2$ such that $x_1 = z_{11} + 2z_{12}$ and $x_1 = z_{21} + 2z_{22}$, we get a point in P_L.

(i) Let $d^1 = (d^1_x, d^1_z)$ and $d^2 = (d^2_x, d^2_z)$ where

$$d^1_x = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix}, \; d^1_z = \begin{pmatrix} 2/5 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } d^2_x = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}, \; d^2_z = \begin{pmatrix} 0 & 0 \\ 2/5 & 0 \end{pmatrix}. \tag{24}$$

Let $d = d^1 + d^2$ and consider the pair of points $p' = \bar{p} + d$ and $p'' = \bar{p} - d$ in P_L. Clearly, $\bar{p} = p'/2 + p''/2$, and for p', we have

$$c \cdot z' = c \cdot \bar{z} + c \cdot (d^1_z + d^2_z) = d + \delta + 2/5(c_{11} + c_{21}),$$

and for p'', we have

$$c \cdot z'' = c \cdot \bar{z} - c \cdot (d^1_z + d^2_z) = d + \delta - 2/5(c_{11} + c_{21}).$$

Therefore, unless $|c_{11} + c_{21}| \leq 1$ both p', p'' lie outside the split set S (recall that $1/5 \leq \delta \leq 4/5$). Therefore, we conclude that $|c_{11} + c_{21}| \leq 1$. Similarly, letting $d = d^1 - d^2$, we conclude that $|c_{11} - c_{21}| \leq 1$, which implies that $|c_{11}| + |c_{21}| \leq 1$. Furthermore, as both P and p are symmetric with respect to the coordinates x_1 and x_2, we can assume that $|c_{11}| \geq |c_{21}|$ and therefore, $c_{21} = 0$ and $|c_{11}| \leq 1$. As $\bar{x} \in \text{SC}(P)$, Proposition 6 implies that $c_{22} \neq 0$.

(ii) Now consider $d^3 = (d^3_x, d^3_z)$ and $d^4 = (d^4_x, d^4_z)$ where

$$d^3_x = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix}, \; d^3_z = \begin{pmatrix} -2/5 & 2/5 \\ 0 & 0 \end{pmatrix} \text{ and } d^4_x = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}, \; d^4_z = \begin{pmatrix} 0 & 0 \\ -2/5 & 2/5 \end{pmatrix}. \tag{25}$$

Letting $d = d^1 + d^4$ and considering the pair of points $p' = \bar{p} + d$ and $p'' = \bar{p} - d$ in P_L, we can now argue that

$$|c_{11} + c_{22} - c_{21}| \leq 1.$$

Similarly, using $d = d^1 - d^4$ we conclude that

$$|c_{11} - c_{22} + c_{21}| \leq 1.$$

As $c_{21} = 0$, inequalities (5.2) and (5.2) together imply that $|c_{11}| + |c_{22}| \leq 1$. As $|c_{22}| \geq 1$ we conclude that $c_{11} = 0$ and $|c_{22}| = 1$. Furthermore, as $c_{11} = 0$ we observe that $c_{12} \neq 0$ by Proposition 6.

(iii) Letting $d = d^3 + d^4$ and using points $p' = \bar{p} + d$ and $p'' = \bar{p} - d$ in P_L, we can argue that

$$|c_{12} - c_{11} + (c_{22} - c_{21})| \leq 1 \quad (\text{from } d = d^3 + d^4), \tag{26}$$

and letting $d = d^3 - d^4$, similarly, we can argue that

$$|c_{12} - c_{11} - (c_{22} - c_{21})| \leq 1 \quad (\text{from } d = d^3 - d^4). \tag{27}$$

As $c_{11} = c_{21} = 0$, these inequalities simplify to $|c_{12} + c_{22}| \leq 1$ and $|c_{12} - c_{22}| \leq 1$. Consequently $|c_{12}| + |c_{22}| \leq 1$ which gives the desired contradiction as $|c_{22}| = 1$ and $c_{12} \neq 0$. ■

5.3 Unary binarization is better than perfect logarithmic binarization

We next give an example for which the projection of the split closure of the unary extended formulation (which is equal to the projection of the split closure of the full extended formulation) is strictly contained in the projection of the split closure of the perfect logarithmic extended formulation. Consider

$$P = \Big\{(x, y) \in [0, 3]^3 \times [0, 1]^3 \ : \ \textstyle\sum_{i=1}^3 x_i = 4, \ x_i \leq 4y_i, \ \text{ for } i = 1, 2, 3\Big\},$$

and the associated integer set $P^I \subset \mathbb{Z}^6$ where $I = \{1, 2, 3, 4, 5, 6\}$. Now consider the unary extended formulation of P:

$$P_U = \Big\{(x, y, z) \in \mathbb{R}^{3+3+3\times 3} \ : \ (x, y) \in P, \ x_i = z_{i1} + 2z_{i2} + 3z_{i3}, \\ z_{i1} + z_{i2} + z_{i3} \leq 1 \ \text{ for } i = 1, 2, 3\Big\},$$

and the perfect logarithmic extended formulation of P:

$$P_{L+} = \Big\{(x, y, z) \in \mathbb{R}^{3+3+3\times 2} \ : \ (x, y) \in P, \ x_i = z_{i1} + 2z_{i2}, \ \text{ for } i = 1, 2, 3\Big\},$$

where y variables are not binarized as they are binary variables. Note that the logarithmic binarization polytope $B^{\mathrm{L}}(u)$ is perfect for $u = 3$.

Theorem 5 *For P defined above,* $\text{proj}_{x,y}(\text{SC}(P_U)) \subsetneq \text{proj}_{x,y}(\text{SC}(P_{L+}))$.

Proof Let $(\bar{x}, \bar{y}) = [(1.5, 1, 1.5), (.5, .5, .5)]$. We will argue that the point $(\bar{x}, \bar{y})$ belongs to $\text{proj}_{x,y}(\text{SC}(P_{L+}))$ but not to $\text{proj}_{x,y}(\text{SC}(P_U))$. First we will show that the following inequalities are Gomory-Chvátal cuts for P_U:

$$y_1 - z_{11} - z_{12} - z_{13} \geq 0, \tag{28}$$
$$y_2 - z_{21} - z_{22} - z_{23} \geq 0, \tag{29}$$
$$y_3 + z_{11} + z_{12} + z_{13} + z_{21} + z_{22} + z_{23} \geq 2. \tag{30}$$

To derive inequality (28), we take the combination of constraints

$$(4y_1 - x_1 \geq 0) + (x_1 - z_{11} - 2z_{12} - 3z_{13} = 0) - 3(z_{11} + z_{12} + z_{13} \leq 1) +$$
$$(z_{12} \geq 0) + 2(z_{13} \geq 0)$$

to obtain $4y_1 - 4z_{11} - 4z_{12} - 4z_{13} \geq -3$ as a valid inequality for P_U. Dividing this inequality by 4 and rounding up the resulting right-hand-side, we obtain (28) as a Gomory-Chvátal cut for P_U. We can obtain (29) in a similar manner by taking constraints involving $y_2, x_2, z_{21}, z_{22}, z_{23}$. Taking the combination of constraints

$$-\frac{1}{12}(x_3 \leq 3) + \frac{1}{3}(x_1 + x_2 + x_3 = 4) + \frac{1}{4}(4y_3 - x_3 \geq 0) +$$
$$\frac{1}{3}(-x_1 + z_{11} + 2z_{12} + 3z_{13} = 0) + \frac{1}{3}(-x_2 + z_{21} + 2z_{22} + 3z_{23} = 0)$$

and rounding up the nonzero coefficients of the variables and rounding up the right-hand-side, we obtain (30) as a Gomory-Chvátal cut for P_U.

Adding Inequalities (28)-(30), we obtain $y_1 + y_2 + y_3 \geq 2$ is a valid inequality for $\text{SC}(P_U)$ which is violated by $(\bar{x}, \bar{y})$.

To prove that $(\bar{x}, \bar{y}) \in \text{proj}_{x,y}(\text{SC}(P_{L+}))$, we will show that $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) \in \text{SC}(P_{L+})$ where

$$\bar{x} = \begin{pmatrix} 1.5 \\ 1 \\ 1.5 \end{pmatrix}, \bar{y} = \begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}, \bar{z} = \begin{pmatrix} .5 & .5 \\ 0 & .5 \\ .5 & .5 \end{pmatrix}.$$

It is easy to verify that $\bar{p} \in P_{L+}$. Suppose $\bar{p} \notin \text{SC}(P_{L+})$. Then $\bar{p} \notin \text{conv}(P_{L+} \setminus S)$ for some split set $S = \{(x, y, z) \in \mathbb{R}^{3+3+6} : d < a^T x + b^T y + c \cdot z < d + 1\}$ where c is an integral matrix, a, b are integral vectors and d is an integer, and $c \cdot z = \sum_{ij} c_{ij} z_{ij}$. As in the proof of Theorem 4, we can argue that $a = 0$ and

$$\bar{p} \in S = \left\{(x, y, z) : d < b^T y + c \cdot z < d + 1\right\}.$$

As b and c are integral and $\bar{y}$ and $\bar{z}$ are half-integral, it follows that $b^T \bar{y} + c \cdot \bar{z}$ is half-integral and

$$b^T \bar{y} + c \cdot \bar{z} = d + 0.5.$$

Moreover, all points in P_{L+} satisfy $x_1 + x_2 + x_3 = 4$ and therefore $\sum_{i=1}^3 z_{i1} + 2\sum_{i=1}^3 z_{i2} = 4$. We can add multiples of this equation to $b^T y + c \cdot z$ to eliminate the coefficient of z_{11}. Therefore without loss of generality, we can assume that $c_{11} = 0$.

We next construct several pairs of points $p', p'' \in P_{L+}$ such that $\bar{p} = 0.5p' + 0.5p''$; then if both p' and p'' lie outside S, then $\bar{p} \in \text{conv}(P_{L+} \setminus S)$, a contradiction. Therefore S must contain at least one of p' or p'' for each pair, and we will use this fact to impose conditions on b and c, till we show that there cannot exist such a split set.

(i) Consider the pair of points $p' = (x', \bar{y}, z')$ and $p'' = (x'', \bar{y}, z'')$ in P_{L+} defined by

$$x' = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, z' = \begin{pmatrix} 1 & .5 \\ 0 & .5 \\ 0 & .5 \end{pmatrix} \text{ and } x'' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, z'' = \begin{pmatrix} 0 & .5 \\ 0 & .5 \\ 1 & .5 \end{pmatrix}$$

and note that $\bar{p} = 0.5p' + 0.5p''$. For p', we have $b^T\bar{y} + c \cdot z' = b^T\bar{y} + c \cdot \bar{z} - .5c_{31} = d + 1/2 - .5c_{31}$ and for p'', we have $b^T\bar{y} + c \cdot z'' = b^T\bar{y} + c \cdot \bar{z} + .5c_{31} = d + 1/2 + .5c_{31}$ and clearly unless $c_{31} = 0$ both p', p'' lie outside the split set S. Therefore, we conclude that $c_{31} = 0$.

(ii) Next consider $p' = (x', \bar{y}, z')$ and $p'' = (x'', \bar{y}, z'')$ in P_{L+} defined by

$$x' = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, z' = \begin{pmatrix} 0 & 1 \\ 0 & .5 \\ 1 & 0 \end{pmatrix} \text{ and }, x'' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, z'' = \begin{pmatrix} 1 & 0 \\ 0 & .5 \\ 0 & 1 \end{pmatrix}$$

and note that $\bar{p} = 0.5p' + 0.5p''$. For p', we have $b^T\bar{y} + c \cdot z' = b^T\bar{y} + c \cdot \bar{z} + .5(c_{12} - c_{32}) = d + 1/2 + .5(c_{12} - c_{32})$ and for p'' we have $b^T\bar{y} + c \cdot z'' = b^T\bar{y} + c \cdot \bar{z} - .5(c_{12} - c_{32}) = d + 1/2 - .5(c_{12} - c_{32})$. Therefore, unless $c_{12} - c_{32} = 0$, both p', p'' lie outside the split set S and we conclude that $c_{12} = c_{32}$.

(iii) Next consider $p' = (\bar{x}, \bar{y}, z')$ and $p'' = (\bar{x}, \bar{y}, z'')$ in P_{L+} defined by

$$z' = \begin{pmatrix} 0 & .75 \\ 0 & .5 \\ 0 & .75 \end{pmatrix} \text{ and } z'' = \begin{pmatrix} 1 & .25 \\ 0 & .5 \\ 1 & .25 \end{pmatrix}.$$

For p', we have $b^T\bar{y} + c \cdot z' = b^T\bar{y} + c \cdot \bar{z} + .25(c_{12} + c_{32}) = d + 1/2 + .5c_{12}$ and for p'' we have $b^Ty'' + c \cdot z'' = d + 1/2 - .5c_{12}$. Unless $c_{12} = 0$, both $p', p'' \notin S$ and we conclude that $c_{12} = c_{32} = 0$.

(iv) Next consider $p' = (x', \bar{y}, z')$ and $p'' = (x'', \bar{y}, z'')$ in P_{L+} defined by

$$x' = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, z' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } x'' = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, z'' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For p', we have $b^T\bar{y} + c \cdot z' = d + 1/2 + .5c_{22}$ and for p'', we have $b^Ty'' + c \cdot z'' = d + 1/2 - .5c_{22}$. Both p' and p'' lie outside S unless $c_{22} = 0$. Therefore, $c_{22} = 0$.

(v) Next consider $p' = (x', y', z')$ and $p'' = (x'', y', z'')$ in P_{L+} defined by

$$x' = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, y' = \begin{pmatrix} .5 \\ 0 \\ .5 \end{pmatrix}, z' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } x'' = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, y'' = \begin{pmatrix} .5 \\ 1 \\ .5 \end{pmatrix}, z'' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For p', we have $b^Ty' + c \cdot z' = d + 1/2 - .5b_2$ and for p'', we have $b^Ty'' + c \cdot z'' = d + 1/2 + .5b_2$. Both p' and p'' lie outside S unless $b_2 = 0$. Therefore, $b_2 = 0$.

(vi) Next consider $p' = (x', y', z')$ and $p'' = (x'', y', z'')$ in P_{L+} defined by

$$x' = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, y' = \begin{pmatrix} 1 \\ 0 \\ .5 \end{pmatrix}, z' = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & .5 \end{pmatrix} \text{ and } x'' = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, y'' = \begin{pmatrix} 0 \\ 1 \\ .5 \end{pmatrix}, z'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & .5 \end{pmatrix}$$

For p', we have $b^T y' + c \cdot z' = d + 1/2 + .5b_1$ and for p'', we have $b^T y'' + c \cdot z'' = d + 1/2 - .5b_1$. Both p' and p'' lie outside S unless $b_1 = 0$. Therefore, $b_1 = 0$. Similarly we can argue that $b_3 = 0$.

Combining these observations, we conclude that all components of b and c have to be zero except c_{21}. Therefore, $b^T \bar{y} + c \cdot \bar{z} = c_{21}\bar{z}_{21} = 0$, and we obtain a contradiction to the fact that $b^T \bar{y} + c \cdot \bar{z}$ is half integral, and the proof is complete. ∎

6 Branching

We next consider binarization in the context of branch and bound (B&B) trees for integer programs. We will construct a polyhedral set such that the description of its integer hull can be obtained with a much smaller tree when a binary extended formulation is used instead of the original formulation.

To simplify notation, we will consider a pure-integer set $P^I = P \cap \mathbb{Z}^n$ where $P \subseteq [0, u]^n$ is a polytope and $I = \{1, \ldots, n\}$. A B&B tree for P^I is a rooted binary tree where each node has either zero or 2 successor nodes. Nodes in the tree without successor nodes are called leaf nodes and the only node without a predecessor is called the root node. We label the root node with $D = [0, u]^n$. The labels of the non-root nodes are obtained from their parent node via "branching". More precisely, if a node is labeled with D', its successor nodes are labeled with $D' \cap L$ and $D' \cap R$ where $L = \{x \in \mathbb{R}^n : x_i \le t\}$ and $R = \{x \in \mathbb{R}^n : x_i \ge t + 1\}$ for some $i \in I$ and $t \in \mathbb{Z}$. We refer to $D' \cap L$ as the left successor of D' and $D' \cap R$ as the right successor.

We define a B&B tree for the binary extended formulation $P_{\mathcal{B}}^{I_{\mathcal{B}}} \subseteq [0, u]^n \times [0, 1]^q$ similarly. In this case we label the root node with $[0, u]^n \times [0, 1]^q$ and create the remaining nodes by branching on the auxiliary variables as well as the original ones.

Let $\mathcal{T}$ be a B&B tree for P^I, and let $\text{leaf}(\mathcal{T})$ denote the labels of the leaf nodes of $\mathcal{T}$. From now on we will refer to a node by its label. Note that

$$P^I \subseteq \bigcup_{N \in \text{leaf}(\mathcal{T})} N \cap P \subseteq P, \quad \text{and} \quad \text{conv}(P^I) \subseteq \text{conv}\Big(\bigcup_{N \in \text{leaf}(\mathcal{T})} N \cap P \Big) \subseteq P.$$

We will call $\mathcal{T}$ a *complete* B&B tree if optimizing any linear function over P^I is the same as optimizing it over P intersected with the leaf nodes of $\mathcal{T}$. In other words, $\mathcal{T}$ is called complete if

$$\text{conv}(P^I) = \text{conv}\Big(\bigcup_{N \in \text{leaf}(\mathcal{T})} N \cap P \Big).$$

Let $\mathcal{T}_{\mathcal{B}}$ be a B&B tree for $P_{\mathcal{B}}^{I_{\mathcal{B}}}$. We call $\mathcal{T}_{\mathcal{B}}$ a *complete* B&B tree with respect to P^I if

$$\text{conv}(P^I) = \text{proj}_x \left(\text{conv}\Big(\bigcup_{N \in \text{leaf}(\mathcal{T}_{\mathcal{B}})} N \cap P_{\mathcal{B}} \Big) \right).$$

In an earlier paper, Owen and Mehrotra [20] studied the binary extended formulation $P_{\mathcal{B}}$ using the full binarization scheme as defined in equations (4). They argue that given a B&B tree $\mathcal{T}_{\mathcal{B}}$ for $P_{\mathcal{B}}^{I_{\mathcal{B}}}$, one can construct a B&B tree $\mathcal{T}$ for P^I with the same number of leaves such that

$$\bigcup_{N \in \text{leaf}(\mathcal{T})} (N \cap P) \subseteq \bigcup_{N \in \text{leaf}(\mathcal{T}_{\mathcal{B}})} \text{proj}_x(N \cap P_{\mathcal{B}}). \tag{31}$$

They also prove a similar result for the logarithmic binarization scheme (6). Thus, it seems that there is no benefit in branching on the auxiliary binary variables and they conclude that *"remodeling of mixed-integer programs by binary variables should be avoided in practice unless special techniques are used to handle these variables."*

We also point out that equation (31) holds for the unary binarization scheme as well. To see this, first note that for the unary binarization scheme

$$(z_{it} \le 0) \implies (x_i \le t-1) \text{ and } (z_{it} \ge 1) \implies (x_i \ge t)$$

and therefore any B&B tree $\mathcal{T}_{\mathcal{B}}$ can be constructed by branching only on the auxiliary variables. Consequently, any leaf node N of the B&B tree has the form

$$N = \{z_{it} = 0,\ \forall (i,t) \in S_0,\ \text{and},\ z_{it} = 1,\ \forall (i,t) \in S_1\}$$

for some index sets S_0 and S_1. Now consider a B&B tree $\mathcal{T}$ for P^I constructed from $\mathcal{T}_{\mathcal{B}}$ as follows: if two node in $\mathcal{T}_{\mathcal{B}}$ are created from their common predecessor by adding the conditions $(z_{it} = 0)$ and $(z_{it} = 1)$, then we create two nodes in $\mathcal{T}$ by adding the conditions $(x_i \le t-1)$ and $(x_i \ge t)$, respectively. Note that for every leaf node N of $\mathcal{T}_{\mathcal{B}}$, there is a corresponding leaf node N' of $\mathcal{T}$:

$$N' = \{x_i \le t-1,\ \forall (i,t) \in S_0,\ \text{and},\ x_i \ge t,\ \forall (i,t) \in S_1\} = \{a_i \le x_i \le b_i,\ \forall i \in I\}$$

for some integer vectors a and b.

Given a point $\bar{x} \in P \cap N'$, we construct a point $(\bar{x}, \bar{z}) \in P_{\mathcal{B}}$ where for all $i \in I$

$$\bar{z}_{it} = \begin{cases} 1 & 1 \le t \le a_i \\ (\bar{x}_i - a_i)/(b_i - a_i) & a_i < t \le b_i \\ 0 & b_i < t \le u_i. \end{cases}$$

It is easy to see that the point $(\bar{x}, \bar{z}) \in N$ and therefore $\text{proj}_x(N \cap P_{\mathcal{B}}) \supset (N \cap P)$. Consequently, branching on the auxiliary binary variables associated with the unary binarization scheme does not seem useful.

Now consider an alternative binarization defined by the binarization polytope

$$B^{\text{o}}(u) = \{(x,z) \in \mathbb{R} \times [0,1]^u \ :\ x = u z_u + \textstyle\sum_{j=1}^{u-1} z_j, \\ 0 \le z_{u-1} \le \ldots \le z_1 \le 1, z_1 + z_u \le 1\},$$

and the polyhedron

$$P^n = \Big\{x \in [0,4]^n : \sum_{i \in S} x_i + \sum_{i \notin S} (4 - x_i) \ge \frac{1}{2},\ \ \forall S \subseteq \{1, \ldots, n\}\Big\}$$

obtained by cutting all of the corners of the hypercube $[0,4]^n$. Clearly

$$P^n \cap \mathbb{Z}^n = \{0,1,2,3,4\}^n \setminus \{0,4\}^n.$$

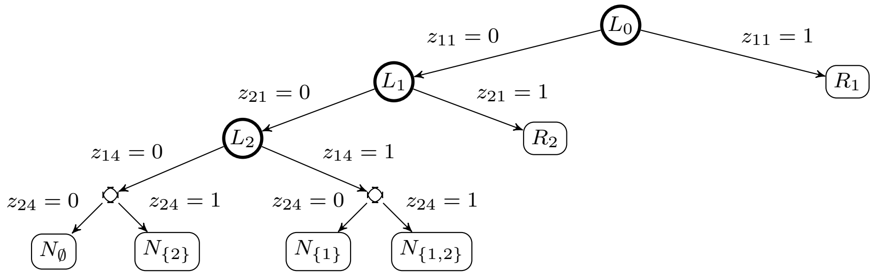

Fig. 3 The tree $\mathcal{T}_\mathcal{B}$ for $n = 2$.

Proposition 7 *Given the binarization scheme $\mathcal{B} = (B^o(4), \ldots, B^o(4))$, there exists a B&B tree $T_\mathcal{B}$ for $(P^n_\mathcal{B})^{I_\mathcal{B}}$ that is complete with respect to $(P^n)^I$ and has $2^n + n$ leaf nodes.*

Proof We will describe the construction of the B&B tree $\mathcal{T}_\mathcal{B}$. See Figures 3 and 4 for the B&B tree and the projection of its leaves when $n = 2$. We label the root node with

$$L_0 = \{(x, z) \in [0, 4]^n \times [0, 1]^{n\times 4}\}$$

where z_i denotes the vector of auxiliary variables associated with x_i. For $i = 1, \ldots, n$, node L_{i-1} has two successor nodes L_i and R_i obtained by branching on variable z_{i1} as follows:

$$L_i = \{(x, z) \in L_{i-1} : z_{i1} = 0\}, \quad R_i = \{(x, z) \in L_{i-1} : z_{i1} = 1\}$$

Nodes $R_1, \ldots, R_n$ are leaf nodes of the tree. Note that as $z_{i1} = 1$ for $(x, z) \in R_i$,

$$R_i = \{(x, z) \in L_0 \ : \ z_{k1} = 0 \text{ for } k < i, \ z_{i1} = 1\}.$$

Consequently $\text{proj}_x(P^n_\mathcal{B} \cap R_i) = \{x \in P^n \ : \ x_i \in [1, 3]\} \subseteq \text{conv}(P^n \cap \mathbb{Z}^n)$.

The rest of the tree consists of a complete binary tree of depth n rooted at node L_n obtained by branching on z_{i4} for all $i = 1, \ldots, n$. This leads to 2^n additional leaf nodes

$$\begin{aligned} N_S &= \{(x, z) \in L_n : z_{i4} = 1 \ \forall i \in S, z_{i4} = 0 \ \forall i \notin S\}, \\ &\subseteq \{(x, z) \in L_n : x_i = 4 \ \forall i \in S, x_i = 0 \ \forall i \notin S\} \end{aligned}$$

one for each subset S of $\{1, \ldots, n\}$, and notice that $P^n_\mathcal{B} \cap N_S = \emptyset$. The tree $\mathcal{T}_\mathcal{B}$ has a total of $2^n + n$ leaf nodes. Furthermore,

$$\text{conv}\Big(\bigcup_{N \in \text{leaf}(\mathcal{T}_\mathcal{B})} \text{proj}_x(P^n_\mathcal{B} \cap N) \Big) = \text{conv}\Big(\bigcup_{i=1}^{n} \text{proj}_x(P^n_\mathcal{B} \cap R_i) \Big) \subseteq \text{conv}(P^n \cap \mathbb{Z}^n),$$

implying $\mathcal{T}_\mathcal{B}$ is complete. ∎

We next show that any complete B&B tree in the original space is approximately at least twice as big as the one described in Proposition 7. We will use the following fact from convex analysis in the proof of the next claim: If $a^T x \leq b$ is a valid inequality for $X \subseteq \mathbb{R}^n$, then

$$\text{conv}(X \cap \{x : a^T x = b\}) = \text{conv}(X) \cap \{a^T x = b\}. \tag{32}$$

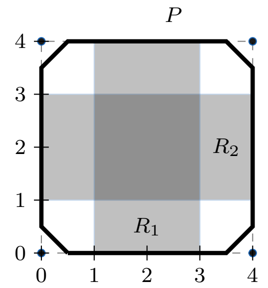

Fig. 4 The projection of $R_i \cap P^n_{\mathcal{B}}$ for $i = 1, 2$, to $\mathbb{R}^2$.

Proposition 8 *Any complete B&B tree for* $(P^n)^I$ *has at least* $2 \cdot 2^n - 1$ *leaf nodes.*

Proof Let $g(n)$ be the minimum size of a complete B&B tree for $(P^n)^I$. We will prove that $g(n) \geq 2 \cdot 2^n - 1$ by induction on n. For $n = 1$, P^1 is the line segment $[0.5, 3.5]$. Clearly, a single branch (with 2 leaves) does not lead to a complete tree and therefore $g(1) \geq 3 = 2 \cdot 2^1 - 1$. We now assume $n \geq 2$, and assume the result holds for P^k with $k < n$.

Let $\mathcal{T}$ be a complete B&B tree for $(P^n)^I$ with root node label $D = [0, 4]^n$. As P^n is symmetric we can rename the variables and assume that the first variable branched on is x_n and the successor nodes are $L = \{x \in D : x_n \leq t\}$ and $R = \{x \in D : x_n \geq t + 1\}$ for some $t \in \{0, 1, 2, 3\}$. Any other choice of t would lead to one of L or R being equal to D. In addition, as the hyperplane defined by $x_n = 2$ is a plane of symmetry for P^n, we can also assume that $t \in \{0, 1\}$. Let $\mathcal{T}_L$ be the subtree of $\mathcal{T}$ rooted at L and let $\mathcal{T}_R$ be the subtree of $\mathcal{T}$ rooted at R.

Consider first $\mathcal{T}_L$. Let $H = \{x \in \mathbb{R}^n : x_n = 0\}$ and let $\mathcal{T}'$ be the tree with the same choice of branches as $\mathcal{T}_L$ but with root node $L \cap H$. Thus, for each node $N \in \mathcal{T}_L$, there is a corresponding node in $\mathcal{T}'$ with the label $N \cap H$. If the successor nodes of node $A \in \mathcal{T}'$ are obtained by branching on x_n to $x_n = 0$ and $x_n = 1$, then clearly the left successor node has the same label as A and the right one has the label $\emptyset$. Therefore, it is possible to replace the branching conditions on x_n in $\mathcal{T}'$ with $x_1 \leq 4$ and $x_1 \geq 5$ to obtain the same labels. Consequently, one can obtain a new tree $\tilde{\mathcal{T}}$, with identical labels at every node as $\mathcal{T}'$, that branches only on variables $x_1, \ldots, x_{n-1}$. We will next show that $\tilde{\mathcal{T}}$ (and therefore $\mathcal{T}_L$) has at least $g(n-1)$ leaf nodes.

As $\mathcal{T}$ is a complete for $(P^n)^I$, it follows that

$$\mathrm{conv}(P^n \cap \mathbb{Z}^n) = \mathrm{conv}(\bigcup_{N \in \mathrm{leaf}(\mathcal{T})} N \cap P^n).$$

Intersecting both the left-hand and right-hand terms of the above equation with H, and then using equation (32) to take H inside the convex hull expressions, we obtain

$$\mathrm{conv}(P^n \cap \mathbb{Z}^n \cap H) = \mathrm{conv}(\bigcup_{N \in \mathrm{leaf}(\mathcal{T}_L)} N \cap (P^n \cap H)). \tag{33}$$

The equality above follows from the fact that the intersection of the label of any leaf node of $\mathcal{T}_R$ with H is the empty set. For each leaf node N of $\mathcal{T}_L$, the corresponding leaf node of $\mathcal{T}'$ is $N \cap H$, and therefore (33) implies

$$\text{conv}((P^n \cap H) \cap \mathbb{Z}^n) = \text{conv}(\bigcup_{N' \in \text{leaf}(\mathcal{T}')} N' \cap (P^n \cap H)).$$

Therefore $\mathcal{T}'$ is a complete B&B tree for $(P^n \cap H)^I$ and so is $\tilde{\mathcal{T}}$ as both $\tilde{\mathcal{T}}$ and $\mathcal{T}'$ have the same leaf node labels. Note that

$$(P^n \cap H)^I = P^n \cap H \cap \mathbb{Z}^n = (P^{n-1} \cap \mathbb{Z}^{n-1}) \times \{0\}$$

and $\tilde{\mathcal{T}}$ only branches on variables $x_1, \ldots, x_{n-1}$ and consequently, $\tilde{\mathcal{T}}$ yields a complete B&B tree for $P^{n-1} \cap \mathbb{Z}^{n-1}$ after dropping x_n. Therefore $\tilde{\mathcal{T}}$ has at least $g(n-1)$ leaf nodes implying that $\mathcal{T}_L$ also has at least $g(n-1)$ leaf nodes.

We now consider $\mathcal{T}_R$, the second part of the tree $\mathcal{T}$, which is rooted at R. We will next show that $\mathcal{T}_R$ has at least $g(n-1)+1$ leaf nodes. In this part of the proof, we let $H = \{x \in \mathbb{R}^n : x_n = 4\}$ and let $\mathcal{T}'$ be obtained from $\mathcal{T}_R$ by changing the label of its root node to $R \cap H$. Repeating the same arguments used for $\mathcal{T}_L$ earlier, it is easy to see that $\mathcal{T}'$ has at least $g(n-1)$ leaf nodes. Moreover, note that $p^1, p^2 \in P^n \cap R$ where $p^1 = (0, 0, \ldots, 3)$, $p^2 = (0, 0, \ldots, 3.5)$. As $\mathcal{T}$ is complete and p^1 is integral, $\mathcal{T}_R$ has a leaf node containing p^1. Furthermore, this leaf node cannot contain p^2 as it does not belong to $\text{conv}(P^n \cap \mathbb{Z}^n)$. Notice that the points p^1 and p^2 only differ in the last coordinate and therefore cannot be separated by a branching decision that involves the first $n-1$ variables. Consequently, one of the branching conditions in $\mathcal{T}_R$ (leading to this leaf node) must be on the variable x_n. Therefore, the tree $\mathcal{T}'$ must contain a node $\bar{N}$ whose successors are labeled $\emptyset$ and $\bar{N}$. Clearly, contracting the edge between these two nodes with the label $\bar{N}$ and deleting the node labeled $\emptyset$ still yields a complete tree for $(P^n \cap H)$ with at least one less leaf node than $\mathcal{T}'$. Therefore, $\mathcal{T}'$ has at least $g(n-1)+1$ leaf nodes as desired.

Combining the bounds on the leaf nodes of $\mathcal{T}_L$ and $\mathcal{T}_R$, we conclude that $g(n) \geq 2g(n-1) + 1 \geq 2 \cdot (2^n - 1) + 1 = 2 \cdot 2^n - 1$. ∎

Combining Propositions 7 and 8, we obtain the following result.

Corollary 3 *There exists a mixed-integer set P^I for which any complete B&B tree has strictly more leaves than a particular B&B tree for a binary extended formulation that is complete with respect to P^I.*

7 Concluding Remarks

In this paper, we showed that binary extended formulations augmented with split cuts can yield stronger bounds for mixed-integer programs than the original formulation augmented with all possible split cuts. This result is of potential interest because many of the cutting planes generated by state-of-the-art commercial MIP solvers are special types of split cuts. However, these solvers do not generate all possible split cuts as they use heuristics for separation. Therefore, it would be interesting to investigate the performance of different binarization schemes in practice and see how they fare with cut separation algorithms implemented in commercial solvers. It would also be interesting to systematically analyze the relative strength of binarizations that add fewer than u auxiliary variables per each bounded integer variable $0 \leq x \leq u$, unlike unimodular binarizations.

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