

Iterative weighted thresholding method for sparse solution of underdetermined linear equations

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Abstract

Recently, iterative reweighted methods have attracted much interest in compressed sensing, since they perform better than unweighted ones in most cases. Currently, weights are chosen heuristically in existing iterative reweighted methods, and finding an optimal weight is an open problem since we do not know the exact support set beforehand. In this paper, we present a novel weighted l_1 -norm minimization problem for the sparsest solution of underdetermined linear equations, whose solution is also the sparsest under some given conditions. We propose an iterative weighted thresholding method for the weighted l_1 -norm minimization problem, where the weight w and variable x are optimized simultaneously, and prove that the iteration process will converge eventually. Moreover, we enhance the performance of our iterative weighted thresholding method using the homotopy technique. Extensive computational experiments show that our method performs better both in running time and recovery accuracy comparing with some state-of-the-art methods.

Keywords: Sparse optimization, iterative weighted thresholding method, homotopy method.
AMS subject classifications. 15A06, 15A29, 65K05, 90C25, 90C26, 90C59

1 Introduction

In this paper, we consider finding the sparsest solution to underdetermined linear equations. That is

$$\min_{x \in \mathbb{R}^n} \|x\|_0 \quad s.t. \quad Ax = b, \quad (1)$$

where x is an n -dimensional vector, A is an $m \times n$ matrix (usually $m < n$, say the measurement matrix), and b is an m -dimensional vector (say observations). $\|x\|_0$ denotes the number of nonzero components of x (for convenience, we call it l_0 -norm of x).

Problem (1) is important in Compressed Sensing (CS). It has attracted much interest after the pioneering work of Donoho et al. [5], who showed that the sparse signal may be reconstructed with even fewer samples than the sampling theorem requires. Up to now it has been applied to signal processing, image processing, machine learning, model selection, statistics, inverse problems, and other engineering fields.

1.1 Previous work

Problem (1) is an NP-hard combinatorial optimization problem [19], and even cannot be solved approximately within a fixed ratio unless P=NP [8]. However, a large number of methods have been proposed to approximately solve the problem. These methods can be categorized into four classes: (i) Greedy based methods, e.g., Matching Pursuit (MP) [18, 23], Subspace Pursuit [9], and the improved MP methods, such as regularized Orthogonal MP (OMP) [21], Stagewise

OMP [10], and Compressive Sampling Matching Pursuit [20]; (ii) l_0 -norm based methods, e.g., Penalty Decomposition (PD) method [17], Iterative Hard Thresholding methods [3]; (iii) l_p -norm based methods, e.g., $L_{1/2}$ Regularization method [27]; (iv) l_1 -norm based methods, e.g., Alternating Direction Method [29], Fast Iterative Shrinkage-Thresholding Method [1], Fixed-Point Continuation (FPC) [13], Proximal-Gradient Homotopy method (PGH) [26], and reweighted l_1 -minimization methods [30, 31, 2, 24].

The l_1 -norm based methods solve the following problem,

$$\min_{x \in \mathbb{R}^n} \|w \circ x\|_1 \quad s.t. \quad Ax = b, \quad (2)$$

where $w \geq 0$ and $w \circ x = (w_1 x_1, w_2 x_2, \dots, w_n x_n)^T$. For the special case $w_i = 1, i = 1, 2, \dots, n$, many methods have been proposed, such as alternative direction algorithms for l_1 -norm minimization (YALL1) ([29], [30]), Proximal-Gradient Homotopy method [26], and Fixed-Point Continuation [13].

To make problem (2) well approximate to the original problem (1), many strategies have been proposed. The reweighted methods assign different weight w_i^k to each component of x during the iteration process. By noting that

$$\sum_{i=1}^n \frac{|x_i|}{|x_i| + \delta} \rightarrow \|x\|_0 \quad \text{as } \delta \rightarrow 0^+, \quad (3)$$

the reweighted methods iteratively solve problem (2) by setting $w_i^{k+1} = 1/(\delta + |x_i^k|)$ for $i = 1, \dots, n$ at the k -th iteration. In this way, each subproblem is convex and is easy to solve.

Zhao et al. [30] showed that most reweighted l_1 -norm minimizations can be reformulated as

$$\min_{x \in \mathbb{R}^n} F_\delta(x) = \sum_{i=1}^n \phi_i(|x_i| + \delta) \quad s.t. \quad Ax = b, \quad (4)$$

where $F_\delta(x)$ is an approximate to the l_0 -norm. Then the reweighted methods obtain a solution of problem (4) by iteratively solving the subproblem

$$\min_{x \in \mathbb{R}^n} \|\nabla F_\delta(|x^k|) \circ x\|_1 \quad s.t. \quad Ax = b, \quad (5)$$

where $\nabla F_\delta(|x^k|)$ can be regarded as w^k . For example, the reweighted strategy $w_i^{k+1} = 1/(\delta + |x_i^k|)$ could be induced by setting $F_\delta(x) = \sum_{i=1}^n \log(|x_i| + \delta)$. Table 1 lists some more reweighted methods. More variants can refer to [30] and its references.

Alternatively, the Iterative Support Detection (ISD) method [24] assigns weight 0 to components in the support set and 1 to the other components during iteration. At each iteration, the support set is detected by some heuristic strategies, e.g., all components whose absolute values are larger than a thresholding. This reweighted strategy has also been adopted by Bi et al. [2] for their subproblems. In [31], Zhao et al. proposed a new method based on the weighted range space property to compute an optimal w from the dual of a linear program. More precisely, it alternatively solves the weighted primal problem with a fixed weight to obtain a new solution x , and then it solves a dual problem to obtain a new weight w .

Note that all reweighted methods presented above either adopt heuristic strategies for weight values, or alternatively solve a weighted linear program to obtain a solution x and solve a dual problem to obtain a new weight w in the iteration process. In this paper, we propose a new approach for the sparsest solution of underdetermined linear equations, which tries to optimize the solution x and the weight w simultaneously, and obtain a better recovery accuracy.

Table 1: variants of weighted method

relaxed objective	weighted method	corresponding $F_\delta(x)$
$\min_x \sum_{i=1}^n w_i x_i^2$ [6, 22]	$w_i = ((x_i^{k-1})^2 + \delta) \frac{p}{2}^{-1}$	-
$\min_x \sum_{i=1}^n w_i x_i $ [4]	$w_i = (x_i^{k-1} + \delta)^{-1}$	$\sum_{i=1}^n \log(x_i + \delta)$
$\min_x \sum_{i=1}^n w_i x_i $ [11]	$w_i = \frac{1}{(x_i^{k-1} + \delta)^{1-p}}$	$\frac{1}{p} \sum_{i=1}^n (x_i + \delta)^p$
$\min_x \sum_{i=1}^n w_i x_i $ [30]	$w_i = \frac{1}{(x_i^{k-1} + \delta)^{1-p} [x_i^{k-1} + \delta + (x_i^{k-1} + \delta)^p]}$	$\sum_{i=1}^n \log(x_i + \delta + (x_i + \delta)^p)$
$\min_x \sum_{i=1}^n w_i x_i $ [30]	$w_i = \frac{q + (x_i^{k-1} + \delta)^{1-q}}{(x_i^{k-1} + \delta)^{1-q} [x_i^{k-1} + \delta + (x_i^{k-1} + \delta)^q]^{1-p}}$	$\frac{1}{p} \sum_{i=1}^n (x_i + \delta + (x_i + \delta)^q)^p$
$\min_x \sum_{i=1}^n w_i x_i $ [30]	$w_i = \frac{1 + 2(x_i^{k-1} + \delta)}{(x_i^{k-1} + \delta + (x_i^{k-1} + \delta)^2)^{1-p}}$	$\frac{1}{p} \sum_{i=1}^n (x_i + \delta + (x_i + \delta)^2)^p$
$\min_x \sum_{i=1}^n w_i x_i $ [30]	$w_i = \frac{1 + (x_i^{k-1} + \delta)^p}{(x_i^{k-1} + \delta)^{p+1}}$	$\sum_{i=1}^n [\log(x_i + \delta) - \frac{1}{(x_i + \delta)^p}]$
$\min_x \sum_{i=1}^n w_i x_i $ [30]	$w_i = \frac{1 + (x_i + \delta)^{p+1}}{(x_i + \delta)^{p+1}}$	$\sum_{i=1}^n (x_i - \frac{1}{(x_i + \delta)^p})$
$\min_x \sum_{i=1}^n w_i x_i $ [4]	$w_i = \frac{1}{ x_i^{k-1} + \delta^2}$	$\sum_{i=1}^n \operatorname{atan}(\frac{ x_i }{\delta})$
$\min \ x_{K_1}\ _1 + w \ x_{K_2}\ _1$ [15]	$w \geq 1$	-
$\min_x \sum_{i=1}^n w_i x_i $ [7]	$w_i = \frac{1}{(x_i^{k-1} + \delta)^{1-p}}, i \in T \text{ and } w_i = 0, i \in T^c$	$\frac{1}{p} \ x_T\ _p^p$
$\min_x \sum_{i=1}^n w_i x_i $ [28]	$w_i = W_1, \text{ if } i \in K_1 \text{ and } w_i = W_2, \text{ if } i \in K_2$	-

1.2 Our contribution

First, we reformulate problem (1) as an equivalent weighted l_1 -norm minimization problem, where weights are binary variables. To solve the reformulated problem, we adopt homotopy technique and solve a sequence of penalty problems. Since the weight w is not easy to handle, we linearize the penalty term and turn to solve a subproblem which exists a closed-form solution though the weight w is boolean. Then we can optimize w and x simultaneously, which is different from all existing reweighted methods, in which w^k is determined after x^k . Extensive computational experiments show that our method performs better than some of state-of-the-art methods both in running time and recovery accuracy.

The purpose of compressed sensing is to reduce sampling as much as possible within a satisfactory recovery accuracy. In this paper, experimental results show that our method performs better on instances with higher difficulty levels than other methods. Here higher difficulty level means higher sparsity level K when m and n are fixed. By the definition of difficulty level of random instance in Section 4, it is easy to understand that our method will have a similar good performance on instances with smaller m when n and K are fixed. This is consistent with the original purpose of compressed sensing.

The rest of this paper is organized as follows. In Section 2, we reformulate the problem and depict the iterative weighted thresholding method. Moreover, convergence analysis is presented. The homotopy based iterative weighted thresholding method is described in Section 3. In Section 4, we consider some practical issues and present a practical method. In addition, a large number of experiments are conducted both in noiseless and noisy cases. Conclusions are drawn in Section 5.

1.3 Notations and assumption

Unless otherwise stated, $\|\cdot\|$ represents the Euclidean norm. The transposes of a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{m \times n}$ are denoted by x^T and A^T , respectively. Given an index set $I \subseteq \{1, \dots, n\}$, x_I denotes the subvector formed by the components of x indexed by I . The index set of nonzero components of a vector x is denoted by $S(x) = \{i : x_i \neq 0\}$ (called support set). Let $S^c(x)$ be the complement of $S(x)$, i.e., $S^c(x) = \{1, 2, \dots, n\} - S(x) = \{i : x_i = 0\}$. We also define $T(w) = \{i : w_i = 0\}$, where w_i is the weight corresponding to x_i . Let $w \circ x = (w_1 x_1, w_2 x_2, \dots, w_n x_n)^T$. For a set S , $|S|$ is the number of elements in S . $|x|^\downarrow$ means the vector of components of $|x| \in \mathbb{R}^n$ being arranged in nonincreasing order, i.e.,

$|x|_1^\downarrow \geq |x|_2^\downarrow \geq \dots \geq |x|_n^\downarrow$, and $|x|_r^\downarrow$ denotes the r -th component of $|x|^\downarrow$. $\|A\|_1^2 = \max_i \{\|A_i\|_2^2\}$, where A_i is the i -th column of matrix A . Without loss of generality, we suppose that the minimum of problem (1) is not less than 0.

2 Iterative weighted thresholding method

In this section, first we present an equivalent problem of problem (1). Then we establish its penalty problem and propose an iterative weighted thresholding method.

2.1 Equivalent problem

We reformulate problem (1) as the following equivalent minimization problem,

$$\min_{x,w} \sum_{i=1}^n w_i (|x_i| - \varepsilon) \quad s.t. \quad Ax = b, w \in \{0, 1\}^n, \quad (6)$$

where ε is some small positive number. Before proving the equivalence, the following property is needed.

Proposition 2.1. *Let s be the minimum of problem (1), and let $\varepsilon_0 = \min\{|x|_s^\downarrow : Ax = b\}$. Then $\varepsilon_0 > 0$.*

Proof. We prove by contradiction. Suppose $\varepsilon_0 = 0$. By the definition of ε_0 , if $\varepsilon_0 = 0$, then there exists \hat{x} satisfying $A\hat{x} = b$ and $|\hat{x}|_s^\downarrow = 0$. Then we have $\|\hat{x}\|_0 \leq s - 1$, which contradicts the definition of s . Hence $\varepsilon_0 > 0$. \square

Next, we show that if ε is less than the ε_0 in Proposition 2.1, then problem (6) is equivalent to the l_0 -norm minimization problem (1).

Theorem 2.1. *Let $s = \min_x \{\|x\|_0 : Ax = b\}$, $\varepsilon_0 = \min\{|x|_s^\downarrow : Ax = b\}$, and $0 < \varepsilon < \varepsilon_0$. If x^o is a minimizer of problem (1), then (x^o, w^o) is a minimizer of problem (6), where*

$$w_i^o = 0, \text{ if } i \in S(x^o), \text{ otherwise } w_i^o = 1, i = 1, 2, \dots, n. \quad (7)$$

Conversely, if $(x_\varepsilon^, w_\varepsilon^*)$ is a minimizer of problem (6), then x_ε^* is a minimizer of problem (1).*

Proof. For any optimal solution $(x_\varepsilon^*, w_\varepsilon^*)$ of problem (6), it is obvious that $[w_\varepsilon^*]_i = 0$ if $|x_\varepsilon^*|_i > \varepsilon$ and $[w_\varepsilon^*]_i = 1$ if $|x_\varepsilon^*|_i \leq \varepsilon$. Define $T(w_\varepsilon^*) = \{i | [w_\varepsilon^*]_i = 0\}$. Then

$$f_v = \sum_{i=1}^n [w_\varepsilon^*]_i (|[x_\varepsilon^*]_i| - \varepsilon) = \sum_{i \in T^c(w_\varepsilon^*)} (|[x_\varepsilon^*]_i| - \varepsilon).$$

Since $0 < \varepsilon < \varepsilon_0$, and by the definition of ε_0 , it is easy to know that $|T(w_\varepsilon^*)| \geq s$ and $|T^c(w_\varepsilon^*)| \leq n - s$. Thus we can get a lower bound on f_v , i.e., $-(n - s)\varepsilon$.

Since the underdetermined equations has an s sparse solution, we can easily verify that the lower bound $-(n - s)\varepsilon$ can be achieved by any s sparse solution x^o and the weight defined in (7). Hence, any optimal solution of problem (1) with the weight defined in (7) provides an optimal solution of problem (6).

Furthermore, for any optimal solution $(x_\varepsilon^*, w_\varepsilon^*)$ of problem (6), if $|T(w_\varepsilon^*)| \geq s + 1$ or there exists a component $[x_\varepsilon^*]_i$ such that $0 < [x_\varepsilon^*]_i < \varepsilon$, then the lower bound of f_v cannot be achieved by $(x_\varepsilon^*, w_\varepsilon^*)$, which contradicts the definition of $(x_\varepsilon^*, w_\varepsilon^*)$. So $|T(w_\varepsilon^*)| = s$ and $[x_\varepsilon^*]_i = 0$ for all $i \in T^c(w_\varepsilon^*)$. Hence $\|x_\varepsilon^*\|_0 = s$, which indicates x_ε^* is an optimal solution of problem (1). \square

2.2 Iterative weighted thresholding method

By adding constraints to the objective function, problem (6) can be rewritten as

$$\min_{x,w} \Phi_{\mu,\varepsilon}(x,w) = \mu f(x) + \Psi_\varepsilon(x,w) \quad s.t. \quad w \in \{0,1\}^n, \quad (8)$$

where $f(x) = \frac{1}{2}\|Ax - b\|^2$ is a differentiable convex function whose gradient is Lipschitz continuous (denote its Lipschitz constant as L_f), $\Psi_\varepsilon(x,w) = \sum_i w_i(|x_i| - \varepsilon)$. Then by linearizing the penalty term $f(x)$ at a point y and adding the proximal term $\frac{L}{2}\|x - y\|^2$, we obtain

$$\min_{x,w} P_{L,\mu,\varepsilon}(y,x,w) := \mu h_L(y,x) + \Psi_\varepsilon(x,w) \quad s.t. \quad w \in \{0,1\}^n, \quad (9)$$

where $h_L(y,x) = f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}\|x - y\|^2$.

Furthermore, problem (9) can be reformulated as

$$\sum_{i=1}^n \min_{x_i, w_i} \frac{\mu L}{2} (x_i - \bar{y}_i)^2 + w_i (|x_i| - \varepsilon) \quad s.t. \quad w_i \in \{0,1\}, \quad (10)$$

where $\bar{y} = y - \frac{1}{L}\nabla f(y) = y - \frac{1}{L}A^T(Ay - b)$. Though problem (10) is nonconvex, we can deduce a closed-form solution of this problem, which is presented as the following lemma.

Lemma 2.1. *The solution of problem (10) is given by*

i) *if $\varepsilon \geq \frac{1}{2\mu L}$, then*

$$[(\bar{x}, \bar{w})]_i := [T_{(\mu,L,\varepsilon)}(y)]_i = \begin{cases} (\bar{y}_i, 0), & \text{if } |\bar{y}_i| \geq \varepsilon + \frac{1}{2\mu L}; \\ (\text{soft}(\bar{y}_i), 1), & \text{if } |\bar{y}_i| < \varepsilon + \frac{1}{2\mu L}, \end{cases} \quad (11)$$

where $\text{soft}(\cdot)$ is a soft thresholding operator defined as

$$\text{soft}(\bar{y}_i) := \text{sign}(\bar{y}_i) \max(|\bar{y}_i| - \frac{1}{\mu L}, 0);$$

ii) *if $\varepsilon < \frac{1}{2\mu L}$, then*

$$[(\bar{x}, \bar{w})]_i := [T_{(\mu,L,\varepsilon)}(y)]_i = \begin{cases} (\bar{y}_i, 0), & \text{if } |\bar{y}_i| \geq \sqrt{\frac{2\varepsilon}{\mu L}}, \\ (0, 1), & \text{otherwise.} \end{cases} \quad (12)$$

Proof. For convenience, we omit the index i in problem (10) and consider the problem

$$\min_{x,w} p(x,w) := \frac{\mu L}{2}(x - \bar{y})^2 + w(|x| - \varepsilon) \quad s.t. \quad w \in \{0,1\}, \quad (13)$$

where $x \in \mathbb{R}^1$. When $w = 0$, the optimal solution of problem (13) is $\hat{x} = \bar{y}$ and its objective value is $p(\hat{x}, 0) = 0$. When $w = 1$, the optimal solution of problem (13) is

$$\tilde{x} = \text{sign}(\bar{y}) \max(|\bar{y}| - \frac{1}{\mu L}, 0).$$

In the following, we compare the objective values of the two cases for obtaining the minimizer of problem (13).

i) Suppose $\varepsilon \geq \frac{1}{2\mu L}$. If $|\bar{y}| \geq \varepsilon + \frac{1}{2\mu L}$, then $|\bar{y}| \geq \frac{1}{\mu L}$, and the objective value $p(\tilde{x}, 1) = |\bar{y}| - \varepsilon - \frac{1}{2\mu L} \geq 0 = p(\hat{x}, 0)$. Hence, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\hat{x}, 0) = (\bar{y}, 0)$.

If $|\bar{y}| < \varepsilon + \frac{1}{2\mu L}$, then there are two cases: 1) if $\varepsilon + \frac{1}{2\mu L} > |\bar{y}| \geq \frac{1}{\mu L}$, then the objective value $p(\tilde{x}, 1) = |\bar{y}| - \varepsilon - \frac{1}{2\mu L} < 0 = p(\hat{x}, 0)$. In this case, the optimal solution of problem (13) is

$(\bar{x}, \bar{w}) = (\tilde{x}, 1)$; 2) if $|\bar{y}| < \frac{1}{\mu L}$, then the objective value $p(\tilde{x}, 1) = p(0, 1) = \frac{\mu L}{2} |\bar{y}|^2 - \varepsilon < \frac{1}{2\mu L} - \varepsilon < 0 = p(\hat{x}, 0)$. In this case, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (0, 1) = (\tilde{x}, 1)$. Hence, if $|\bar{y}| < \varepsilon + \frac{1}{2\mu L}$, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\tilde{x}, 1)$.

ii) Suppose $\varepsilon < \frac{1}{2\mu L}$. If $|\bar{y}| \geq \sqrt{\frac{2\varepsilon}{\mu L}}$, we consider two cases: 1) if $\frac{1}{\mu L} > |\bar{y}| \geq \sqrt{\frac{2\varepsilon}{\mu L}}$, then the objective value $p(\tilde{x}, 1) = p(0, 1) = \frac{\mu L}{2} |\bar{y}|^2 - \varepsilon \geq 0 = p(\hat{x}, 0)$. In this case, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\hat{x}, 0) = (\bar{y}, 0)$; 2) if $|\bar{y}| > \frac{1}{\mu L}$, then the objective value $p(\tilde{x}, 1) = |\bar{y}| - \varepsilon - \frac{1}{2\mu L} > \frac{1}{\mu L} - \varepsilon > 0 = p(\hat{x}, 0)$. In this case, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\hat{x}, 0) = (\bar{y}, 0)$. Hence, if $|\bar{y}| \geq \sqrt{\frac{2\varepsilon}{\mu L}}$, then the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\bar{y}, 0)$.

Furthermore, if $|\bar{y}| < \sqrt{\frac{2\varepsilon}{\mu L}}$, then $|\bar{y}| < \frac{1}{\mu L}$ since $\varepsilon < \frac{1}{2\mu L}$. And the objective value $p(\tilde{x}, 1) = p(0, 1) = \frac{\mu L}{2} |\bar{y}|^2 - \varepsilon < 0 = p(\hat{x}, 0)$. Hence, the optimal solution of problem (13) is $(\bar{x}, \bar{w}) = (\tilde{x}, 1) = (0, 1)$. \square

Remark. If $\varepsilon < \frac{1}{2\mu L}$, the closed-form solution given by Lemma 2.1 in (12) is similar to the hard thresholding operator [17]. The difference is that in (12) there is a parameter ε . When $\varepsilon = \frac{1}{2\mu L} = \sqrt{\frac{2\varepsilon}{\mu L}} = \frac{1}{\mu L}$, the closed-form solution is exactly the hard thresholding operator.

Basing on the thresholding operator given by Lemma 2.1, we propose the following Iterative Weighted Thresholding method for problem (8). We set the stopping criterion as the infinite norm of the subgradient of function (6) with respect to x being small enough, i.e.,

$$\min_{\xi} \|\mu \nabla f(x^{k+1}) + \xi \circ w^{k+1}\|_{\infty} < \epsilon$$

where $\xi_i = \text{sign}(x_i^{k+1})$ for $x_i^{k+1} \neq 0$ and $\xi_i \in [-1, 1]$ for $x_i^{k+1} = 0$.

Algorithm 1: Iterative Weighted Thresholding Method (IWT)

Input: $\mu > 0$, x^0 , $\varepsilon, \eta \geq 0$, $L \geq L_f + \eta$, $\epsilon > 0$;

Output: \hat{x} , \hat{w} ;

1: initialize $k \leftarrow 0$;

2: **repeat**

3: $(x^{k+1}, w^{k+1}) = T_{\mu, L, \varepsilon}(x^k)$;

4: $k \leftarrow k + 1$;

5: **until** $\min_{\xi} \|\mu \nabla f(x^{k+1}) + \xi \circ w^{k+1}\|_{\infty} < \epsilon$

6: $\hat{x} \leftarrow x^k$, $\hat{w} \leftarrow w^k$.

2.3 Convergence of IWT when $\varepsilon \geq \frac{1}{2\mu L}$

Next, we show some results for Algorithm 1 for the case $\varepsilon \geq \frac{1}{2\mu L}$. Before proceeding, the following Lemma 2.2 describing some properties of Algorithm 1 is presented, which will be used later.

Lemma 2.2. Suppose $\varepsilon \geq \frac{1}{2\mu L}$. Let (x^{k+1}, w^{k+1}) be generated by Algorithm 1, then the following statements hold:

i) $|x_i^{k+1}| \geq \varepsilon + \frac{1}{2\mu L}$ for all $i \in \{i : w_i^{k+1} = 0\}$, and $|x_i^{k+1}| < \varepsilon - \frac{1}{2\mu L}$ for all $i \in \{i : w_i^{k+1} = 1\}$;

ii) if $w_i^k \neq w_i^{k+1}$, then $|x_i^k - x_i^{k+1}| \geq \frac{1}{\mu L}$;

iii) $x^{k+1} = x^k$ if and only if $\mathbf{0} \in \partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)$.

Proof. i) By case i) of Lemma 2.1, if $i \in \{i : w_i^{k+1} = 0\}$, then it is easy to see that $|x_i^{k+1}| \geq \varepsilon + \frac{1}{2\mu L}$. Moreover, if $i \in \{i : w_i^{k+1} = 1\}$, then

$$x_i^{k+1} = \text{sign}(\bar{x}_i^k) \max(|\bar{x}_i^k| - \frac{1}{\mu L}, 0), \text{ and } \bar{x}_i^k < \varepsilon + \frac{1}{2\mu L},$$

where $\bar{x}^k = x^k - \frac{1}{L}\nabla f(x^k)$, which implies that

$$|x_i^{k+1}| = \max(|\bar{x}_i^k| - \frac{1}{\mu L}, 0) < \max(\varepsilon - \frac{1}{2\mu L}, 0) = \varepsilon - \frac{1}{2\mu L}.$$

ii) This statement follows immediately from statement i) of Lemma 2.2.

iii) By statement ii), it is obvious that if $x^{k+1} = x^k$, then $w^{k+1} = w^k$. Furthermore, by statement i), $|x_i^{k+1}| \geq \varepsilon + \frac{1}{2\mu L}$, or $|x_i^{k+1}| < \varepsilon - \frac{1}{2\mu L}$. If $|x_i^k| = |x_i^{k+1}| \geq \varepsilon + \frac{1}{2\mu L}$, then by (11), $w_i^{k+1} = w_i^k = 0$, and $x_i^{k+1} = x_i^k - \frac{1}{L}[\nabla f(x^k)]_i$. Thus $[\nabla f(x^k)]_i = [\nabla f(x^{k+1})]_i = 0$. Hence $\mu[\nabla f(x^k)]_i + [\xi \circ w^k]_i = 0$, where $\xi_i = \text{sign}(x_i^k)$.

If $0 < |x_i^k| = |x_i^{k+1}| < \varepsilon - \frac{1}{2\mu L}$, then by (11), $w_i^{k+1} = w_i^k = 1$, and $x_i^{k+1} = \text{soft}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i)$. Since $|x_i^{k+1}| > 0$, by the definition of the soft thresholding operator, $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| > \frac{1}{\mu L}$. Hence,

$$\begin{aligned} x_i^{k+1} &= \text{sign}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i)(|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| - \frac{1}{\mu L}) \\ &= x_i^k - \frac{1}{L}[\nabla f(x^k)]_i - \frac{1}{\mu L}\text{sign}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i). \end{aligned}$$

The above first equality combined with $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| > \frac{1}{\mu L}$ indicates that $\text{sign}(x_i^{k+1}) = \text{sign}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i)$. Moreover, by $x_i^{k+1} = x_i^k$, we have

$$x_i^{k+1} = x_i^k - \frac{1}{L}[\nabla f(x^k)]_i - \frac{1}{\mu L}\text{sign}(x_i^k).$$

Thus $\mu[\nabla f(x^k)]_i + \text{sign}(x_i^k) = 0$, since $x_i^{k+1} = x_i^k$. Hence $\mu[\nabla f(x^k)]_i + [\xi \circ w^k]_i = 0$, where $\xi_i = \text{sign}(x_i^k)$.

If $x_i^{k+1} = x_i^k = 0$, then by (11), $w_i^k = w_i^{k+1} = 1$, and $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| - \frac{1}{\mu L} \leq 0$. Thus $\mu|[\nabla f(x^k)]_i| \leq 1$. Hence

$$0 \in \{\mu[\nabla f(x^k)]_i + [\xi \circ w^k]_i : \xi \in [-1, 1]\}.$$

Note that

$$\begin{aligned} \partial_x \Phi_{\mu, \varepsilon}(x^k, w^k) &= \{\mu\nabla f(x^k) + \xi \circ w^k : \xi_i = \\ &\text{sign}(x_i^k) \text{ for } x_i^k \neq 0, \text{ and } \xi_i \in [-1, 1] \text{ for } x_i^k = 0\}. \end{aligned}$$

The above three cases indicate that $\mathbf{0} \in \partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)$.

Conversely, by statement i), $|x_i^k| \geq \varepsilon + \frac{1}{2\mu L}$, or $|x_i^k| < \varepsilon - \frac{1}{2\mu L}$. If $|x_i^k| \geq \varepsilon + \frac{1}{2\mu L}$, then by statement i), we know $w_i^k = 0$. Hence

$$0 \in [\partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)]_i = \{\mu[\nabla f(x^k)]_i + \xi_i w_i^k\} = \{\mu[\nabla f(x^k)]_i\}.$$

So $[\nabla f(x^k)]_i = 0$, and $x_i^k - \frac{1}{L}[\nabla f(x^k)]_i = x_i^k$. Thus, by (11) and $|x_i^k| \geq \varepsilon + \frac{1}{2\mu L}$, we have $x_i^{k+1} = x_i^k - \frac{1}{L}[\nabla f(x^k)]_i = x_i^k$.

If $0 < |x_i^k| < \varepsilon - \frac{1}{2\mu L}$, then by statement i), $w_i^k = 1$. Hence

$$0 \in [\partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)]_i = \{\mu[\nabla f(x^k)]_i + \xi_i w_i^k\} = \{\mu[\nabla f(x^k)]_i + \text{sign}(x_i^k)\}.$$

So $-\frac{1}{L}[\nabla f(x^k)]_i = \frac{1}{\mu L}\text{sign}(x_i^k)$, and $x_i^k - \frac{1}{L}[\nabla f(x^k)]_i = x_i^k + \frac{1}{\mu L}\text{sign}(x_i^k)$. By $|x_i^k| < \varepsilon - \frac{1}{2\mu L}$, we have

$$\frac{1}{\mu L} < |x_i^k + \frac{1}{\mu L}\text{sign}(x_i^k)| < \varepsilon + \frac{1}{2\mu L}.$$

By (11),

$$\begin{aligned}
x_i^{k+1} &= \text{soft}\left(x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k)\right) \\
&= \text{sign}\left(x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k)\right) \max\left\{\left|x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k)\right| - \frac{1}{\mu L}, 0\right\} \\
&= x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k) - \frac{1}{\mu L} \text{sign}\left(x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k)\right) \\
&= x_i^k + \frac{1}{\mu L} \text{sign}(x_i^k) - \frac{1}{\mu L} \text{sign}(x_i^k) \\
&= x_i^k.
\end{aligned}$$

If $|x_i^k| = 0$, then by statement i), $w_i^k = 1$. Hence

$$0 \in [\partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)]_i = \{\mu[\nabla f(x^k)]_i + \xi_i w_i^k\} = \{\mu[\nabla f(x^k)]_i\} \cup [-1, 1].$$

Thus, there exists $\xi_i \in [-1, 1]$ such that $-\frac{1}{L}[\nabla f(x^k)]_i = \frac{1}{\mu L} \xi_i$, and $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| = |x_i^k + \frac{1}{\mu L} \xi_i| \leq \frac{1}{\mu L}$. Then by (11),

$$x_i^{k+1} = \text{sign}\left(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i\right) \max\left\{\left|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i\right| - \frac{1}{\mu L}, 0\right\} = 0 = x_i^k.$$

□

Theorem 2.2. *Let (x^k, w^k) be generated by Algorithm 1. Suppose that A is of full row rank. Let $\mu > 0$, $\eta \geq 0$, $L > L_f + \eta$ and $\varepsilon \geq \frac{1}{2\mu L}$. Then the following statements hold:*

i) *for all $k = 1, 2, \dots$, if $w^{k+1} \neq w^k$, we have*

$$\|x^{k+1} - x^k\| \geq \frac{1}{\mu L};$$

ii) *the sequence $\{\Phi_{\mu, \varepsilon}(x^k, w^k)\}$ is strictly decreasing and converges;*

iii) *there exists $K > 0$, for all $k > K$, w^k does not change, and the number of k at which $w^k \neq w^{k+1}$ is at most*

$$J^* = \left\lfloor \frac{2\mu L^2(\Phi_{\mu, \varepsilon}(x^1, w^1) - \Phi_{\mu, \varepsilon}^*)}{L - L_f} \right\rfloor,$$

where $\Phi_{\mu, \varepsilon}^* = \lim_{k \rightarrow \infty} \Phi_{\mu, \varepsilon}(x^k, w^k)$;

iv) *the sequence $\{P_{L, \mu, \varepsilon}(x^k, x^{k+1}, w^{k+1})\}$ is strictly decreasing.*

Proof. i) If $w^{k+1} \neq w^k$, then exists i such that $w_i^{k+1} \neq w_i^k$. By statement ii) of Lemma 2.2, we have $\|x^{k+1} - x^k\| \geq |x_i^{k+1} - x_i^k| \geq \frac{1}{\mu L}$.

ii) First, since $\nabla f(x)$ is Lipschitz continuous and $L > L_f + \eta$, we have

$$\begin{aligned}
\Phi_{\mu, \varepsilon}(x^{k+1}, w^{k+1}) &= \mu f(x^{k+1}) + \Psi_\varepsilon(x^{k+1}, w^{k+1}) \\
&\leq \underbrace{\mu(f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L_f}{2}\|x^{k+1} - x^k\|^2)}_a + \Psi_\varepsilon(x^{k+1}, w^{k+1}) \\
&\leq \underbrace{\mu(f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2)}_b + \Psi_\varepsilon(x^{k+1}, w^{k+1}) \\
&\leq \mu f(x^k) + \Psi_\varepsilon(x^k, w^k) \\
&= \Phi_{\mu, \varepsilon}(x^k, w^k),
\end{aligned}$$

where the last inequality follows from step 3 of Algorithm 1. From these inequalities, it is easy to obtain that

$$\begin{aligned}\Phi_{\mu,\varepsilon}(x^k, w^k) - \Phi_{\mu,\varepsilon}(x^{k+1}, w^{k+1}) &\geq b - a \\ &= \frac{\mu(L - L_f)}{2} \|x^{k+1} - x^k\|^2.\end{aligned}\quad (14)$$

If $x^{k+1} = x^k$, then by statement iii) of Lemma 2.2, Algorithm 1 will stop at the k -th iteration. If $x^{k+1} \neq x^k$, then (14) implies that

$$\Phi_{\mu,\varepsilon}(x^k, w^k) > \Phi_{\mu,\varepsilon}(x^{k+1}, w^{k+1}).$$

Hence $\{\Phi_{\mu,\varepsilon}(x^k, w^k)\}$ is strictly decreasing. Furthermore, it is obvious that $\Phi_{\lambda,\varepsilon}(x, w)$ is bounded below, e.g., $-\varepsilon$ is its lower bound. So the sequence of the objective function values $\{\Phi_{\mu,\varepsilon}(x^k, w^k)\}$ is strictly decreasing and converges.

iii) By (14) and convergence of the sequence $\{\Phi_{\mu,\varepsilon}(x^k, w^k)\}$, we have

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0, \quad (15)$$

which combined with i) implies that there exists $K > 0$, for all $k > K$, w^k does not change.

Assume that w^k only changes at $k = n_1 + 1, \dots, n_J + 1$. That is,

$$w^{n_{j-1}+1} = \dots = w^{n_j} \neq w^{n_j+1} = \dots = w^{n_{j+1}}, j = 1, 2, \dots, J, \quad (16)$$

where $n_0 = 0$. Then by i), we have

$$\|x^{n_j+1} - x^{n_j}\| \geq \frac{1}{\mu L}, j = 1, 2, \dots, J,$$

which together with (14) implies that

$$\Phi_{\mu,\varepsilon}(x^{n_j}, w^{n_j}) - \Phi_{\mu,\varepsilon}(x^{n_j+1}, w^{n_j+1}) \geq \frac{L - L_f}{2\mu L^2}, j = 1, 2, \dots, J.$$

Summing up the above inequalities and using monotonicity of the sequence $\{\Phi_{\mu,\varepsilon}(x^k, w^k)\}$, we have

$$\begin{aligned}\Phi_{\mu,\varepsilon}(x^1, w^1) - \Phi_{\mu,\varepsilon}^* &\geq \Phi_{\mu,\varepsilon}(x^{n_1}, w^{n_1}) - \Phi_{\mu,\varepsilon}(x^{n_J+1}, w^{n_J+1}) \\ &\geq \frac{L - L_f}{2\mu L^2} J.\end{aligned}$$

Hence,

$$J \leq \left\lceil \frac{2\mu L^2 (\Phi_{\mu,\varepsilon}(x^1, w^1) - \Phi_{\mu,\varepsilon}^*)}{L - L_f} \right\rceil.$$

iv) By step 3 of Algorithm 1 and the assumption $L > L_f$, we have

$$P_{L,\mu,\varepsilon}(x^k, x^{k+1}, w^{k+1}) \leq P_{L,\mu,\varepsilon}(x^k, x^k, w^k) = \mu f(x^k) + \sum_i w_i^k (|x_i^k| - \varepsilon).$$

If $x^k = x^{k-1}$, then by statement iii) of Lemma 2.2, Algorithm 1 will stop at the $(k - 1)$ -th iteration. If $x^k \neq x^{k-1}$, then by the assumption $L > L_f$, $f(x^k) < h_L(x^{k-1}, x^k)$, the above inequalities imply that

$$P_{L,\mu,\varepsilon}(x^k, x^{k+1}, w^{k+1}) < \mu h_L(x^{k-1}, x^k) + \sum_i w_i^k (|x_i^k| - \varepsilon) = P_{L,\mu,\varepsilon}(x^{k-1}, x^k, w^k),$$

which completes the proof. \square

Suppose that all parameters satisfy the assumptions in Theorem 2.2. Then Algorithm 1 is reduced to solving the following problem when the number of iterations is large enough,

$$\min_x \left\{ \frac{\mu}{2} \|Ax - b\|^2 + \|x_{T_*^c}\|_1 \right\}, \quad (17)$$

where $T_* = \{i : w_i^* = 0\}$ and $T_*^c = \{i : w_i^* = 1\}$, $w^* = \lim_{k \rightarrow \infty} w^k$. Thus we can obtain the following theorem.

Theorem 2.3. *Let (x^k, w^k) be generated by Algorithm 1. Suppose that A is of full row rank. If $\mu > 0, \eta \geq 0, L > L_f + \eta$, and $\varepsilon \geq \frac{1}{2\mu L}$, then the following statements hold:*

i) there exists $K > 0$, such that for all $l \geq 1$, for any $\tilde{x} \in X^$,*

$$0 \leq \Phi_{\mu, \varepsilon}(x^{K+l}, w^*) - \Phi_{\mu, \varepsilon}(\tilde{x}, w^*) \leq \frac{\mu L \|\tilde{x} - x^K\|^2}{2l}, \quad (18)$$

where X^* is the set of optimal solutions of problem (17);

ii) $\{x^k\}$ converges;

iii) let $\lim_{k \rightarrow \infty} x^k = x^$. Then there exists $N > 0$, when $k > N$, $\text{sign}(x_i^k) = \text{sign}(x_i^*)$ for $i \in \{i : x_i^* \neq 0\}$;*

iv) let $\lim_{k \rightarrow \infty} x^k = x^$, and let $\lim_{k \rightarrow \infty} w^k = w^*$. Then $\mathbf{0} \in \{\mu \nabla f(x^*) + \xi^* \circ w^*\}$, where $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$, and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$. Moreover, (x^*, w^*) is a saddle point of problem (8);*

v) x^ defined in (iv) satisfies*

$$\|\nabla f(x^*)\| = \|A^T(Ax^* - b)\| \leq \frac{\|w^* \circ \xi^*\|}{\mu},$$

where $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$, and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$.

Remark: From statement v) of Theorem 2.3, we know that $\lim_{\mu \rightarrow \infty} \|\nabla f(x^*)\| = 0$. Furthermore, if matrix A is of full row rank, then it is easy to see that $\lim_{\mu \rightarrow \infty} \|Ax^* - b\| = 0$. Hence, we can obtain a feasible solution of problem (1) eventually if μ is large enough.

Proof. i) By statement iii) of Theorem 2.2, for the sequence $\{w^k\}$, there exists $K > 0$, when $k > K$, $w^k = w^*$, and minimizing $P_{L, \mu, \varepsilon}(x^k, x, w)$ is reduced to minimizing the following objective function

$$\bar{P}_{L, \mu, \varepsilon}(x^k, x) = \mu(f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2) + \|x_{T_*^c}\|_1.$$

Since x^{k+1} is a minimal solution of the function $\bar{P}_{L, \mu, \varepsilon}(x^k, x)$, we have $\mathbf{0} \in \partial \bar{P}_{L, \mu, \varepsilon}(x^k, x^{k+1})$. Denote $\mu(f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2)$ as $p(x)$. By noting that $p(x)$ is a strongly convex function with modulus μL , it holds that

$$p(x) \geq p(x^{k+1}) + \nabla p(x^{k+1})^T(x - x^{k+1}) + \frac{\mu L}{2}\|x - x^{k+1}\|^2. \quad (19)$$

Moreover, since $\|x_{T_*^c}\|_1 = \|w^* \circ x\|_1$ is a convex function, we obtain

$$\|w^* \circ x\|_1 \geq \|w^* \circ x^{k+1}\|_1 + (w^* \circ \xi)^T(x - x^{k+1}). \quad (20)$$

Adding (19) to (20), we get

$$\begin{aligned} p(x) + \|w^* \circ x\|_1 &\geq p(x^{k+1}) + \nabla p(x^{k+1})^T(x - x^{k+1}) + \frac{\mu L}{2}\|x - x^{k+1}\|^2 \\ &\quad + \|w^* \circ x^{k+1}\|_1 + (w^* \circ \xi)^T(x - x^{k+1}). \end{aligned} \quad (21)$$

Since $\|w^* \circ x\|_1 = \|x_{T_*^c}\|_1$, $\|w^* \circ x^{k+1}\|_1 = \|x_{T_*^c}^{k+1}\|_1$, $\partial \bar{P}_{L,\mu,\varepsilon}(x^k, x^{k+1}) = \{w^* \circ \xi + \nabla p(x^{k+1}) : \xi \in \partial \|x^{k+1}\|_1\}$, and $\mathbf{0} \in \partial \bar{P}_{L,\mu,\varepsilon}(x^k, x^{k+1})$, we can deduce from (21) that

$$\begin{aligned} & \mu(f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2) + \|x_{T_*^c}\|_1 \\ & \geq \mu(f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2) \\ & \quad + \|x_{T_*^c}^{k+1}\|_1 + \frac{\mu L}{2}\|x - x^{k+1}\|^2. \end{aligned} \quad (22)$$

Furthermore, by the convexity of function $f(x)$, Lipschitz continuity of ∇f and $L > L_f$, we obtain

$$\begin{aligned} f(x) & \geq f(x^k) + \nabla f(x^k)^T(x - x^k), \\ f(x^{k+1}) & \leq f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2, \end{aligned}$$

which combined with inequality (22) implies that

$$\Phi_{\mu,\varepsilon}(x, w^*) + \frac{\mu L}{2}\|x - x^k\|^2 \geq \Phi_{\mu,\varepsilon}(x^{k+1}, w^*) + \frac{\mu L}{2}\|x - x^{k+1}\|^2. \quad (23)$$

For any $\tilde{x} \in X^*$, substituting $x = \tilde{x} \in X^*$ into inequality (23) results in

$$\Phi_{\mu,\varepsilon}(x^{k+1}, w^*) - \Phi_{\mu,\varepsilon}(\tilde{x}, w^*) \leq \frac{\mu L}{2}(\|x^k - \tilde{x}\|^2 - \|x^{k+1} - \tilde{x}\|^2), \quad \forall k > K$$

Summing up the above inequalities from K to $K+l-1$ and by statement ii) of Theorem 2.2, we get

$$\begin{aligned} l(\Phi_{\mu,\varepsilon}(x^{K+l}, w^*) - \Phi_{\mu,\varepsilon}(\tilde{x}, w^*)) & \leq \sum_{i=K}^{K+l-1} [\Phi_{\mu,\varepsilon}(x^{i+1}, w^*) - \Phi_{\mu,\varepsilon}(\tilde{x}, w^*)] \\ & \leq \frac{\mu L}{2}(\|x^K - \tilde{x}\|^2 - \|x^{K+l} - \tilde{x}\|^2), \end{aligned} \quad (24)$$

which immediately yields i).

ii) One can conclude from (24) without any difficulty that

$$\|x^{K+l} - \tilde{x}\| \leq \|x^K - \tilde{x}\|,$$

for all $l \geq 1$. This inequality indicates that the sequence $\{x^k\}$ is bounded. On the other hand, the sequence $\{x^k\}$ satisfies (15). Hence, the sequence $\{x^k\}$ converges.

iii) We prove this statement by contradiction. If for any $N > 0$, there exist $k > N$ and $\hat{i} \in \{i : x_i^* \neq 0\}$ such that $\text{sign}(x_i^k) \neq \text{sign}(x_i^*)$, then

$$\|x^k - x^*\|_2 \geq |x_i^k - x_i^*| \geq |x_i^*| \geq \min\{|x_i^*| : |x_i^*| > 0\},$$

which contradicts the assumption $\lim_{k \rightarrow \infty} x^k = x^*$.

iv) By statement (iii) and $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$, there exists $N_1 \geq N$, such that when $k > N_1$, $\text{sign}(x_i^k) = \text{sign}(x_i^*)$ for $i \in \{i : x_i^* \neq 0\}$, and $\|x^{k+1} - x^k\|_\infty < \frac{1}{\mu L}$. Suppose $k > N_1$.

If $|x_i^*| > 0$, then $|x_i^k| > 0$ and $|x_i^{k+1}| > 0$. If $|x_i^k| \geq \varepsilon + \frac{1}{2\mu L}$, then by $\|x^{k+1} - x^k\|_\infty < \frac{1}{\mu L}$ and statement i) of Lemma 2.2, we have $|x_i^{k+1}| \geq \varepsilon + \frac{1}{2\mu L}$, and $w_i^{k+1} = w_i^k = 0$. Then by (11), we have $x_i^{k+1} = x_i^k - \frac{1}{L}[\nabla f(x^k)]_i$, and $|\frac{1}{L}[\nabla f(x^k)]_i| = |x_i^{k+1} - x_i^k|$. Thus, for $\zeta_i \in [\partial_x \Phi_{\mu,\varepsilon}(x^k, w^k)]_i$,

$$|\zeta_i| = |\mu[\nabla f(x^k)]_i + w_i^k \text{sgn}(x_i^k)| = \mu L |x_i^{k+1} - x_i^k|. \quad (25)$$

If $|x_i^*| > 0$ and $0 < |x_i^k| < \varepsilon - \frac{1}{\mu L}$, then by $\|x^{k+1} - x^k\|_\infty < \frac{1}{\mu L}$ and statement i) of Lemma 2.2, we have $0 < |x_i^{k+1}| < \varepsilon - \frac{1}{\mu L}$ and $w_i^{k+1} = w_i^k = 1$. Then by (11), we have $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| > \frac{1}{\mu L}$, and

$$x_i^{k+1} = x_i^k - \frac{1}{L}[\nabla f(x^k)]_i - \frac{1}{\mu L} \text{sign}(x_i^{k+1}).$$

So $|x_i^{k+1} - x_i^k| = |\frac{1}{L}[\nabla f(x^k)]_i + \frac{1}{\mu L} \text{sign}(x_i^{k+1})|$. Thus, for $\zeta_i \in [\partial_x \Phi_{\mu, \varepsilon}(x^k, w^k)]_i$,

$$|\zeta_i| = |\mu[\nabla f(x^k)]_i + w_i^k \text{sgn}(x_i^k)| = \mu L |x_i^{k+1} - x_i^k|. \quad (26)$$

If $|x_i^*| = 0$, then for any $\varepsilon \in (0, \frac{1}{2\mu L})$, there exists $N_2 \geq N_1$, such that when $k > N_2$, it holds that $|x_i^k| < \varepsilon$ and $|x_i^{k+1}| < \varepsilon$. If $|x_i^{k+1}| > 0$, then by (11) we obtain $\varepsilon + \frac{1}{2\mu L} > |x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| > \frac{1}{\mu L}$, $w_i^{k+1} = w_i^k = 1$, and

$$\begin{aligned} x_i^{k+1} &= \text{sign}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i) \max(|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| - \frac{1}{\mu L}, 0) \\ &= (x_i^k - \frac{1}{L}[\nabla f(x^k)]_i) - \frac{1}{\mu L} \text{sign}(x_i^k - \frac{1}{L}[\nabla f(x^k)]_i) \\ &= x_i^k - \frac{1}{L}[\nabla f(x^k)]_i - \frac{1}{\mu L} \text{sign}(x_i^{k+1}). \end{aligned}$$

Thus,

$$|\mu[\nabla f(x^k)]_i + \text{sign}(x_i^{k+1})| = \mu L |x_i^{k+1} - x_i^k| \leq \mu L |x_i^{k+1}| + \mu L |x_i^k| < 2\mu L \varepsilon. \quad (27)$$

If $|x_i^{k+1}| = 0$, then by (11) we obtain $|x_i^k - \frac{1}{L}[\nabla f(x^k)]_i| \leq \frac{1}{\mu L}$, and $w_i^{k+1} = w_i^k = 1$. Thus,

$$|\frac{1}{L}[\nabla f(x^k)]_i| \leq \frac{1}{\mu L} + |x_i^k| \leq \frac{1}{\mu L} + \varepsilon,$$

and then

$$|\mu[\nabla f(x^k)]_i| \leq 1 + \mu L \varepsilon. \quad (28)$$

So there exists $\xi_i \in [-1, 1]$ such that

$$|\mu[\nabla f(x^k)]_i + \xi_i| \leq \mu L \varepsilon. \quad (29)$$

Summing up the above four cases (25),(26),(27) and(29), we have

$$\lim_{k \rightarrow \infty} \min_{\xi} \|\mu \nabla f(x^k) + w^k \circ \xi\|_\infty = 0,$$

where $\xi_i = \text{sign}(x_i^k)$ for $x_i^k \neq 0$, and $\xi_i \in [-1, 1]$ for $x_i^k = 0$. Then, since $\lim_{k \rightarrow \infty} x^k = x^*$, it holds by statement iii) that

$$|\mu \nabla f(x_i^*) + w_i^* \xi_i^*| = |\mu \nabla f(\lim_{k \rightarrow \infty} x_i^k) + \lim_{k \rightarrow \infty} w_i^k \xi_i^k| = \lim_{k \rightarrow \infty} |\mu \nabla f(x_i^k) + w_i^k \xi_i^k| = 0$$

for $i \in \{i : x_i^* \neq 0\}$, and

$$|\mu[\nabla f(x^*)]_i| = |\mu[\nabla f(\lim_{k \rightarrow \infty} x^k)]_i| = \lim_{k \rightarrow \infty} |\mu[\nabla f(x^k)]_i| \leq 1$$

for $i \in \{i : x_i^* = 0\}$, where the last inequality comes from (28).

Hence, we obtain

$$\min_{\xi^*} \|\mu \nabla f(x^*) + w^* \circ \xi^*\|_\infty = 0, \quad (30)$$

where $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$, and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$.

To prove that (x^*, w^*) is a saddle point of problem (8) is to prove that

$$\Phi_{\mu,\varepsilon}(x^*, w^*) \leq \Phi_{\mu,\varepsilon}(x, w^*), \text{ for all } x;$$

and

$$\Phi_{\mu,\varepsilon}(x^*, w^*) \leq \Phi_{\mu,\varepsilon}(x^*, w), \text{ for all } w \in \{0, 1\}^n.$$

By statement (ii), the sequence $\{w^k\}$ converges ahead of $\{x^k\}$ to w^* . And after that, problem (6) is a convex optimization problem, and by statement (iv), the sequence $\{x^k\}$ converges to a minimum solution x^* . Hence the first inequality holds.

On the other hand, since the objective function of

$$\min_{w \in \{0, 1\}^n} \Phi_{\mu,\varepsilon}(x^*, w)$$

is linear in w , and w is a boolean variable, the optimal solution w of the above problem is

$$w_i := \begin{cases} 0, & \text{if } |x_i^*| > \varepsilon, \\ 1, & \text{if } |x_i^*| \leq \varepsilon. \end{cases} \quad (31)$$

Furthermore, by Lemma 2.2, if (x^k, w^k) converges to (x^*, w^*) , then

$$|x_i^*| \geq \varepsilon + \frac{1}{2\mu L} \text{ for all } i \in \{i : w_i^* = 0\},$$

$$|x_i^*| < \varepsilon - \frac{1}{2\mu L} \text{ for all } i \in \{i : w_i^* = 1\}.$$

Thus w^* generated by our method satisfies

$$\Phi_{\mu,\varepsilon}(x^*, w^*) \leq \Phi_{\mu,\varepsilon}(x^*, w) \text{ for all } w.$$

Hence, we conclude that (x^*, w^*) is a saddle point of problem (8).

v) By (30), there exists ξ^* , $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$ and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$, such that $\mu \nabla f(x^*) + w^* \circ \xi^* = 0$. Then

$$\|\mu \nabla f(x^*)\| - \|w^* \circ \xi^*\| \leq \|\mu \nabla f(x^*) + w^* \circ \xi^*\| = 0.$$

Hence,

$$\|\nabla f(x^*)\| = \|A^T(Ax^* - b)\| \leq \frac{\|w^* \circ \xi^*\|}{\mu}.$$

□

2.4 Convergence of IWT when $\varepsilon < \frac{1}{2\mu L}$

Proving the convergence of IWT when $\varepsilon < \frac{1}{2\mu L}$ is similar to proving the convergence of IWT when $\varepsilon \geq \frac{1}{2\mu L}$ in Subsection 2.3. Here we only list the results and omit the proofs.

Lemma 2.3. *Suppose $\varepsilon < \frac{1}{2\mu L}$. Let (x^{k+1}, w^{k+1}) be generated by Algorithm 1, then the following statements hold:*

- i) $|x_i^{k+1}| \geq \sqrt{\frac{2\varepsilon}{\mu L}}$ for all $i \in \{i : w_i^{k+1} = 0\}$, and $|x_i^{k+1}| = 0$ for all $i \in \{i : w_i^{k+1} = 1\}$;
- ii) if $w_i^k \neq w_i^{k+1}$, then $|x_i^k - x_i^{k+1}| \geq \sqrt{\frac{2\varepsilon}{\mu L}}$;
- iii) if $x^{k+1} = x^k$ then $\mathbf{0} \in \partial_x \Phi_{\mu,\varepsilon}(x^k, w^k)$.

Theorem 2.4. Let (x^k, w^k) be generated by Algorithm 1. Suppose that A is of full row rank. Let $\mu > 0$, $\eta \geq 0$, $L > L_f + \eta$ and $\varepsilon < \frac{1}{2\mu L}$. Then the following statements hold:

i) for all $k = 1, 2, \dots$, if $w^{k+1} \neq w^k$, we have

$$\|x^{k+1} - x^k\| \geq \sqrt{\frac{2\varepsilon}{\mu L}};$$

ii) the sequence $\{\Phi_{\mu,\varepsilon}(x^k, w^k)\}$ is strictly decreasing and converges;

iii) there exists $K > 0$, for all $k > K$, w^k does not change, and the number of k at which $w^k \neq w^{k+1}$ is at most

$$J^* = \left\lfloor \frac{L(\Phi_{\mu,\varepsilon}(x^1, w^1) - \Phi_{\mu,\varepsilon}^*)}{\varepsilon(L - L_f)} \right\rfloor,$$

where $\Phi_{\mu,\varepsilon}^* = \lim_{k \rightarrow \infty} \Phi_{\mu,\varepsilon}(x^k, w^k)$;

iv) the sequence $\{P_{L,\mu,\varepsilon}(x^k, x^{k+1}, w^{k+1})\}$ is strictly decreasing.

Theorem 2.5. Let (x^k, w^k) be generated by Algorithm 1. Suppose that A is of full row rank. If $\mu > 0$, $\eta \geq 0$, $L > L_f + \eta$, and $\varepsilon < \frac{1}{2\mu L}$, then the following statements hold:

i) there exists $K > 0$, such that for all $l \geq 1$, for any $\tilde{x} \in X^*$,

$$0 \leq \Phi_{\mu,\varepsilon}(x^{K+l}, w^*) - \Phi_{\mu,\varepsilon}(\tilde{x}, w^*) \leq \frac{\mu L \|\tilde{x} - x^K\|^2}{2l}, \quad (32)$$

where X^* is the set of optimal solutions of problem (17);

ii) $\{x^k\}$ converges;

iii) let $\lim_{k \rightarrow \infty} x^k = x^*$. Then there exists $N > 0$, when $k > N$, $\text{sign}(x_i^k) = \text{sign}(x_i^*)$ for $i \in \{i : x_i^* \neq 0\}$;

iv) let $\lim_{k \rightarrow \infty} x^k = x^*$, and let $\lim_{k \rightarrow \infty} w^k = w^*$. Then $\mathbf{0} \in \{\mu \nabla f(x^*) + \xi^* \circ w^*\}$, where $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$, and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$. Moreover, (x^*, w^*) is a saddle point of problem (8);

v) x^* defined in (iv) satisfies

$$\|\nabla f(x^*)\| = \|A^T(Ax^* - b)\| \leq \frac{\|w^* \circ \xi^*\|}{\mu},$$

where $\xi_i^* = \text{sign}(x_i^*)$ for $x_i^* \neq 0$, and $\xi_i^* \in [-1, 1]$ for $x_i^* = 0$.

Remark: From statement v) of Theorem 2.5, we know that $\lim_{\mu \rightarrow \infty} \|\nabla f(x^*)\| = 0$. Furthermore, if matrix A is of full row rank, then it is easy to see that $\lim_{\mu \rightarrow \infty} \|Ax^* - b\| = 0$. Hence, we can obtain a feasible solution of problem (1) eventually if μ is large enough.

3 Homotopy method based on IWT

Homotopy approach has attracted increasing interest of researchers [14, 26]. Given a sequence of regularization parameter values, the approach solves a sequence of regularized subproblems, where the solution of the current regularized subproblem is used as an initial solution for solving the next regularized subproblem. In [26], the authors used the homotopy approach for the l_1 -regularized Least Squares problem, and gave theoretical analysis. Very recently, the authors in [14] applied the homotopy approach to the PDAS algorithm, and provided rigorous convergence analysis. The experimental results show significant improvements in recovery accuracy and running time over some other algorithms. These works motivate us using the homotopy approach for problem (8) based on our iterative weighted thresholding method.

Now we present the main outline of our homotopy method for problem (8). Given a sequence of regularization parameter values, the homotopy method solves a sequence of regularized problems (8) using the Iterative Weighted Thresholding Method (see Algorithm 1). We call the homotopy method as the HIWT method, in which ε takes a fixed value during the iteration process. In Algorithm 2, steps 4-7 are corresponding to Algorithm 1.

Algorithm 2: The homotopy method based on IWT (HIWT)

Input: $\mu_0 = 1, x^0, \varepsilon, \eta \geq 0, \rho > 1, L \geq L_f + \eta, \bar{\mu} > 0, \epsilon > 0$;
Output: \hat{x}, \hat{w} ;
1: initialization $k \leftarrow 0$;
2: **repeat**
3: $i \leftarrow 0, x^{k,0} = x^k$;
4: **repeat**
5: $(x^{k,i+1}, w^{k,i+1}) = T_{\mu_k, L, \varepsilon}(x^{k,i})$;
6: $i \leftarrow i + 1$;
7: **until** $\min_{\xi \in \partial \Psi_\varepsilon(x^{k,i+1}, w^{k,i+1})} \|\mu_k \nabla f(x^{k,i+1}) + w^{k,i+1} \circ \xi\|_\infty < \epsilon$
8: $x^{k+1} \leftarrow x^{k,i}$;
9: $\mu_{k+1} \leftarrow \rho \mu_k$;
10: $k \leftarrow k + 1$;
11: **until** μ_k reaches $\bar{\mu}$
12: $\hat{x} \leftarrow x^k, \hat{w} \leftarrow w^k$.

Next, we analyze some properties of Algorithm 2.

Theorem 3.1. *Let the sequence $\{x^{k,i}, w^{k,i}\}$ be generated by Algorithm 2. If*

$$1 > \frac{1}{\rho} = \frac{\mu_k}{\mu_{k+1}} > \frac{h_{L_f}(x^{k,n_k-1}, x^{k+1,0})}{h_L(x^{k,n_k-1}, x^{k+1,0})},$$

then the sequence $\{\Phi_{\mu_k, \varepsilon}(x^{k,0}, w^{k,0})\}$ is strictly decreasing and converges, where n_k is the number of iterations between steps 4-7.

Proof. First,

$$\begin{aligned} \Phi_{\mu_k, \varepsilon}(x^{k,0}, w^{k,0}) &= \mu_k f(x^{k,0}) + \Psi_\varepsilon(x^{k,0}, w^{k,0}) \\ &\geq \mu_k h_L(x^{k,0}, x^{k,1}) + \Psi_\varepsilon(x^{k,1}, w^{k,1}) \\ &> \dots \\ &> \mu_k h_L(x^{k,n_k-1}, x^{k,n_k}) + \Psi_\varepsilon(x^{k,n_k}, w^{k,n_k}) \\ &= \mu_k h_L(x^{k,n_k-1}, x^{k+1,0}) + \Psi_\varepsilon(x^{k+1,0}, w^{k+1,0}), \end{aligned} \quad (33)$$

where the first inequality holds by the choice of $(x^{k,1}, w^{k,1})$, and the other inequalities hold by statement iv) of Theorem 2.2.

Then, since ∇f is Lipschitz continuous, it holds that

$$\begin{aligned} \Phi_{\mu_{k+1}, \varepsilon}(x^{k+1,0}, w^{k+1,0}) &= \mu_{k+1} f(x^{k+1,0}) + \Psi_\varepsilon(x^{k+1,0}, w^{k+1,0}) \\ &\leq \mu_{k+1} h_{L_f}(x^{k,n_k-1}, x^{k+1,0}) + \Psi_\varepsilon(x^{k+1,0}, w^{k+1,0}), \end{aligned}$$

which together with (33) indicates that

$$\begin{aligned} &\Phi_{\mu_k, \varepsilon}(x^{k,0}, w^{k,0}) - \Phi_{\mu_{k+1}, \varepsilon}(x^{k+1,0}, w^{k+1,0}) \\ &> \mu_k h_L(x^{k,n_k-1}, x^{k+1,0}) - \mu_{k+1} h_{L_f}(x^{k,n_k-1}, x^{k+1,0}). \end{aligned}$$

By the assumption of this theorem and the fact that $\Phi_{\mu, \varepsilon}(x, w)$ is bounded below, we can conclude that the sequence $\{\Phi_{\mu_k, \varepsilon}(x^{k,0}, w^{k,0})\}$ is strictly decreasing and converges. \square

It must be remarked that, $h_{L_f}(x^{k,n_k-1}, x^{k+1,0}) < h_L(x^{k,n_k-1}, x^{k+1,0})$, if $L > L_f$. This result holds trivially from the definition of h_L in (9). Hence there always exists ρ such that

$$1 > \frac{1}{\rho} = \frac{\mu_k}{\mu_{k+1}} > \frac{h_{L_f}(x^{k,n_k-1}, x^{k+1,0})}{h_L(x^{k,n_k-1}, x^{k+1,0})}.$$

From Theorem 3.1, the sequence $\{\Phi_{\mu_k, \varepsilon}(x^{k,0}, w^{k,0})\}$ is strictly decreasing and converges. Furthermore, from the definition of $\Psi_\varepsilon(x, w)$ in (8), we have $-n\varepsilon \leq \Psi_\varepsilon(x, w) \leq 0$. Hence the sequence $\{\mu_k f(x^{k,0})\}$ is also bounded. Then there exists a subsequence of $\{\mu_k f(x^{k,0})\}$ which converges.

Let (x^*, w^*) be an accumulation point of the sequence $\{(x^{k,0}, w^{k,0})\}$ corresponding to the subsequence of $\{\mu_k f(x^{k,0})\}$. For convenience, we suppose $\lim_{k \rightarrow \infty} x^{k,0} = x^*$, and $\lim_{k \rightarrow \infty} w^{k,0} = w^*$. Let $T_* = \{i : w_i^* = 0\}$, $T_*^c = \{i : w_i^* = 1\}$, and $(x^{k,0}, w^{k,0}) = (x^k, w^k)$.

Corollary 3.1. *Let $C = \lim_{k \rightarrow \infty} \mu_k f(x^k)$, then*

$$\Psi_\varepsilon(x^*, w^*) \leq \Psi_\varepsilon(x^0, w^0) + \mu_0 f(x^0) - C.$$

Moreover, if $x^0 = \mathbf{0}$ and $w^0 = \mathbf{1}$, then $|T_*| \leq \frac{\mu_0 \|b\|^2 - C}{\varepsilon}$.

Proof. By Theorem 3.1, the sequence $\{\Phi_{\mu_k, \varepsilon}(x^k, w^k)\}$ is strictly decreasing and converges. Hence,

$$\Psi_\varepsilon(x^0, w^0) + \mu_0 f(x^0) > \Psi_\varepsilon(x^k, w^k) + \mu_k f(x^k).$$

Taking $k \rightarrow \infty$ in the above inequality, we get

$$\Psi_\varepsilon(x^0, w^0) + \mu_0 f(x^0) \geq \Psi_\varepsilon(x^*, w^*) + C.$$

Furthermore, if $x^0 = \mathbf{0}$ and $w^0 = \mathbf{1}$, then the above inequality reduces to

$$\|x_{T_*^c}^*\|_1 - |T_*^c| \varepsilon \leq -n\varepsilon + \mu_0 f(x^0) - C.$$

Since $|T_*^c| + |T_*| = n$, we furthermore have

$$|T_*| \leq \frac{\mu_0 \|b\|^2 - C - \|x_{T_*^c}^*\|_1}{\varepsilon} \leq \frac{\mu_0 \|b\|^2 - C}{\varepsilon}.$$

□

Remark: (1) Corollary 3.1 indicates that, the number of i satisfying $w_i^* = 0$ depends on the parameter ε . In other words, parameter ε controls the size of components of x^* with weight 0. When ε is large, the size is small. On the contrary, when ε is small, the size may be large.

(2) The process of weighting 0 to components is almost similar to the process of adding components to the support set. The objectives of the two processes are the same, i.e., controlling the size of the larger components whose absolute values are larger than some thresholding value. Comparing the iteration operators (11) and (12) of our algorithm with that of IHT [3], the first case of (11) or (12) represents weighting 0 to some components, while the first case of the operator in IHT adds some components to the support set respectively. But both of our operators and IHT retains relatively larger components to the respective thresholding values.

The authors in [14, 26] have shown that, during iterations controlling the number of components being added to the support set properly will efficiently enhance the effect of their homotopy algorithms. This approach might be equally effective for Algorithm 2.

Corollary 3.1 tells us that we can properly control the number of components being weighted 0 by adjusting ε . Thus dynamically adjusting ε may be a feasible method to enhance the effect of Algorithm 2. We will give a concrete method of dynamically adjusting ε in our experiments part of the next section.

Theorem 3.2. For Algorithm 2, the number of outer loops is not more than $\lceil \log_\rho \frac{\bar{\mu}}{\mu_0} \rceil$.

Proof. Let p be the number of the outer loops of Algorithm 2. Since ρ is the increasing rate of μ_k , Algorithm 2 is terminated when $\mu_k > \bar{\mu}$, and one can easily deduce that

$$\mu_0 \rho^{p-1} < \bar{\mu},$$

and

$$\mu_0 \rho^p \geq \bar{\mu},$$

which implies that

$$p \leq \lceil \log_\rho \frac{\bar{\mu}}{\mu_0} \rceil.$$

□

4 Experiments

In this section, we conduct computational experiments for testing the performance of our HIWT method on the compressed sensing problem. All experiments are performed on a personal computer with 2.5GHz Intel(R) Core(TM) i7-4710MQ CPU and 12GB memory, using MATLAB R2012b.

First, we describe a practical algorithm of Algorithm 2 in Subsection 4.1. Then we perform a large number of experiments in Subsections 4.2, 4.3, and 4.4 to verify efficiency and effectiveness of the practical algorithm.

4.1 Practical algorithm

During testing the effectiveness of Algorithm 2, we observed that an appropriate ε will make the algorithm recover more random signals. Theorem 2.1 shows that an ideal value of ε should be no more than ε_0 . Our computational experience showed that, when ε is set too small, the computational results are not good enough. Hence, the value of ε should be smaller than ε_0 but must not be too small. However, ε_0 defined in Theorem 2.1 is unknown in practice. Hence we propose a self-adaptive strategy to periodically estimate ε , which is defined as

$$\varepsilon_{k+1} := \max\{\alpha^k \|x^k\|_{\infty, u}, |x^k|_t^\downarrow\}, \quad (34)$$

where $0 < \alpha < 1$, $t = 1.1|T(w^k)|$, and $\|x^k\|_{\infty, u}$ denotes the average value of u largest absolute values of x^k . Parameter u is set as $\lceil \frac{n}{2048} \rceil$ in this paper.

The first term of (34) can be referred to the first method of threshold determination in ISDM [24], where u is set as 1. It is heuristic, but does work. In order to make our algorithm more applicable to large scale instances, we change the value of u from 1 to $\lceil \frac{n}{2048} \rceil$, which can make the algorithm achieve desired results. The second term of (34) is to estimate ε_0 as $|x^k|_t^\downarrow$.

At the beginning of iterations, if we just set ε_k as $|x^k|_t^\downarrow$ defined above, then we cannot achieve desired results. This is due to that, the current solution is a very poor approximation to the optimal solution, which makes $|x^k|_t^\downarrow$ a big difference from ε_0 . Note that $\alpha^k \|x^k\|_{\infty, u} \rightarrow 0$ as $k \rightarrow \infty$ for $0 < \alpha < 1$. If we only set ε_k as $\alpha^k \|x^k\|_{\infty, u}$, then ε_k may be too close to zero after some iterations, and at this time the second term of (34) may be a more appropriate value for ε_k . So we combine the two strategies in (34) for obtaining better performance of our algorithm.

Considering the self-adaptive choice of ε_k , we modify Algorithm 2 as the following Algorithm 3, by updating ε_k when every inner iteration begins.

Algorithm 3: Practical homotopy method based on IWT (PHIWT)

Input: $\mu_0 = 1, x^0, \varepsilon_0, \rho > 1, \eta \geq 0, L \geq L_f + \eta, \bar{\mu} > 0, \epsilon > 0;$

Output: $\hat{x}, \hat{w};$

- 1: initialization $k \leftarrow 0;$
 - 2: **repeat**
 - 3: $i \leftarrow 0, x^{k,0} = x^k;$
 - 4: **repeat**
 - 5: $(x^{k,i+1}, w^{k,i+1}) \in T_{\mu_k, L, \varepsilon_k}(x^{k,i});$
 - 6: $i \leftarrow i + 1;$
 - 7: **until** $\min_{\xi \in \partial \Psi_{\varepsilon_k}(x^{k,i+1}, w^{k,i+1})} \|\mu_k \nabla f(x^{k,i+1}) + w^{k,i+1} \circ \xi\|_{\infty} < \epsilon$
 - 8: $x^{k+1} \leftarrow x^{k,i};$
 - 9: $\mu_{k+1} \leftarrow \rho \mu_k;$
 - 10: $\varepsilon_{k+1} = \max(\alpha^k \|x^k\|_{\infty, u}, |x^k|_t^{\downarrow});$
 - 11: $k \leftarrow k + 1;$
 - 12: **until** μ_{k+1} reaches $\bar{\mu}$
 - 13: $\hat{w} \leftarrow w^k;$
 - 14: $\hat{x}(I) = \arg \min_{x_I} \|A_I x_I - b\|^2$, where $I = \{i : x_i^k \neq 0\}$ if $\|x^k\|_0 < \frac{m}{2}$, otherwise $I = \{i : |x_i^k| \geq |x^k|_{\frac{m}{2}}^{\downarrow}\}.$
-

In Algorithm 3, we use step 14 to further enhance the solution, which can reduce the relative error and improve feasibility of the solution. In the experiments, the parameter settings of Algorithm 3 are summarized in Table 2.

Table 2: parameter settings of Algorithm 3

parameter	default value	description
μ_0	1	initial value for penalty parameter
x^0	$\text{zeros}(n, 1)$	initial solution
ε_0	$\ A^T b\ _{\infty} / \ A\ _1^2$	initial ε_k
ρ	2.2	increasing rate of penalty parameter
L	$10 + \lambda_{\max}(A^T A)$	L value for $n \leq 512$
	$\frac{2}{3} \ A\ _1^2$	L value for $n > 512$
$\bar{\mu}$	$\ A\ _1^2$	final penalty parameter for noiseless case
	$0.1 \ A\ _1^2$	final penalty parameter for noisy case
ϵ	0.01	parameter in inner stopping criterion
α	0.71	decreasing rate of ε
mxitr	3000	maximum number of inner iterations

In the experiments, we define the degree of recovery difficulty for a given instance. From chapter 9 of [12], we know that if $m > Ck \ln(\frac{n}{k})$, and A is a Gaussian random matrix or a partial Fourier transform, then the l_1 -norm minimization problem returns the original signal with high probability. Basing on this fact, we define the degree of recovery difficulty for a given instance as

$$DRD = \frac{k \ln(\frac{n}{k})}{m}. \quad (35)$$

Given two random instances $R1$ and $R2$, we say $R1$ is harder than $R2$ if $DRD(R1) > DRD(R2)$, i.e., the probability of exact recovery for $R1$ is less than that of $R2$.

4.2 Comparison with some state-of-the-art algorithms in the noiseless case

In this subsection, we show the performances of our algorithm (PHIWT) and some state-of-the-art algorithms, including MPECA [2], ISDM [24], FPC_AS [25], PGH [26], YALL1 [29], QPDM [17], FPC [13], and PDASC [14]. QPDM [17] and PDASC [14] are algorithms based on l_0 -norm. The others are algorithms based on l_1 -norm, in which MPECA [2], FPC_AS [25] and PHIWT are algorithms with reweighted or weighted strategies.

We mainly test the algorithms on two classes of random instances without noise. Signals of the two classes follow the Gaussian or uniform distributions, respectively. For the first class of instances, we generate each instance as follows: given the dimension n of a signal, the number of observations m and the number of nonzeros K , we generate a random Gaussian matrix $A \in \mathbb{R}^{m \times n}$, in which the components follow the normal distribution $N(0, 1)$. And we generate a random Gaussian signal $x^o \in \mathbb{R}^n$ with $\|x^o\|_0 = K$, in which the components follow the normal distribution $N(0, 1)$, and set $b = Ax^o$.

From the definition of DRD in (35), m , n and k are related to each other, and varying k with fixed m and n will change DRD . Hence, in the experiments, we only consider varying k with fixed m and n for simplicity. It is worth noting that varying n will change the scale of the problem.

In the first experiment, we compare different algorithms on a range of sparsity levels with fixed m and n , i.e., set $m = 512$, $n = 2048$, $K \in (0, \lfloor \frac{m}{2} \rfloor)$. For each triple (m, n, K) , we generate 200 random instances. The average results on 200 random instances are depicted in Fig. 1(a).

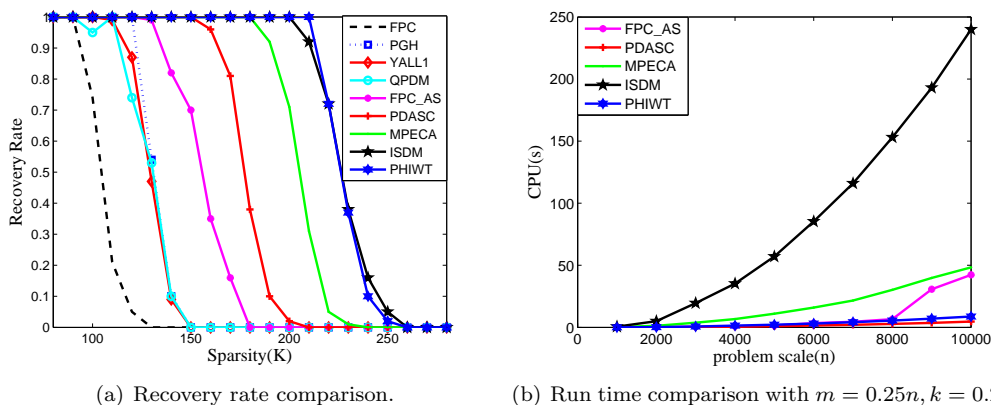


Figure 1: Computational comparisons of algorithms in the noiseless case with Gaussian matrix and Gaussian signal.

From Fig. 1(a), the recovery rate of every algorithm decreases as sparsity K increases when m and n are fixed, and ISDM and PHIWT have the highest recovery rate. Moreover, all algorithms without reweighted or weighted techniques like FPC, PGH, YALL1, QPDM are uniformly worse than ones with reweighted or weighted techniques like FPC_AS, PDASC, MPECA, ISDM, and PHIWT. Hence we only need to compare the later five algorithms as in Fig. 1(b).

Fig. 1(b) shows the average running time of the five algorithms on instances with different problem scales n , where $m = 0.25n$, and $k = 0.2m$. From this figure, we find that among all tested algorithms, PDASC is the fastest, and PHIWT is the second fastest. The running time of PHIWT is roughly 1/10 of that of ISDM, 1/3 of that of MPECA, and is comparable to that of PDASC. Though in Fig. 1(a), ISDM and PHIWT is similar in recovery rate, ISDM is very time-consuming.

For the second set of test, we generate every random instance with the same A as in the first set of test, while generate a random signal $x^o \in \mathbb{R}^n$ with $\|x^o\|_0 = K$, in which the components are uniformly distributed on $[-1, 1]$, and set $b = Ax^o$. Results are presented in Fig. 2.

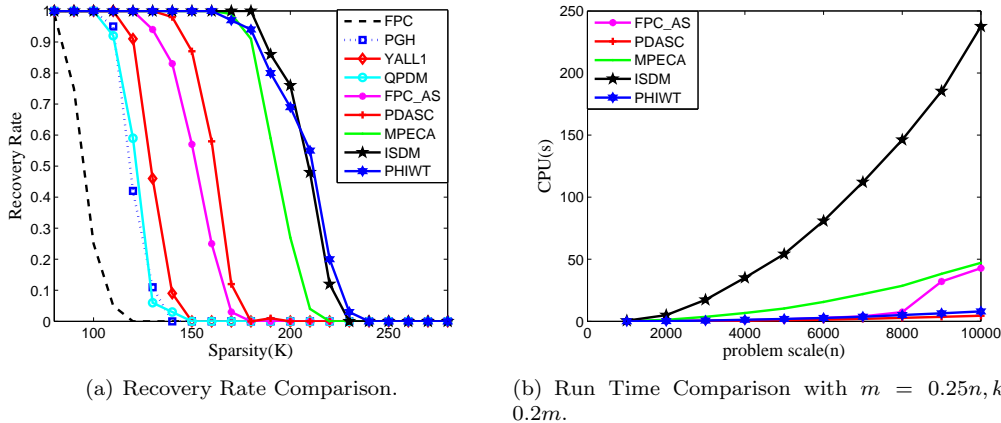


Figure 2: Computational comparisons of the methods in the noiseless case with Gaussian matrix and uniformly distributed signal.

From Fig. 2(a), the recovery rate of every algorithm decreases as sparsity K increases when m and n are fixed, and ISDM and PHIWT have the highest recovery rate. Moreover, all algorithms without reweighted or weighted techniques like FPC, PGH, YALL1, QPDM are uniformly worse than ones with reweighted or weighted techniques like FPC_AS, PDASC, MPECA, ISDM, and PHIWT. Hence we only need to compare the latter five algorithms as in Fig. 2(b).

Fig. 2(b) shows the average running time of the five algorithms on instances with different problem scales n , where $m = 0.25n$, and $k = 0.2m$. From this figure, we find that among all tested algorithms, PDASC is the fastest, and PHIWT is the second fastest. The running time of PHIWT is roughly 1/10 of that of ISDM, 1/3 of that of MPECA, and is comparable to that of PDASC. Though in Fig. 2(a), ISDM and PHIWT is similar in recovery rate, ISDM is very time-consuming.

4.3 Comparison with some state-of-the-art algorithms in the noisy case

A good algorithm should have a strong anti-noise capability for compressed sensing. So in this subsection, we test the performances of all compared algorithms in the noisy case. And for the noisy case, we set $\bar{\mu} = 0.1\|A\|_1^2$ in our algorithm PHIWT, and keep the other parameter settings unchanged as in Table 2.

We mainly test the algorithms on two classes of random instances, signals of which follow the Gaussian or uniform distributions, and compare the performances. For the first class of instances, we generate each instance as follows: given the dimension n of a signal, the number of observations m and the number of nonzeros K , we generate a random Gaussian matrix $A \in \mathbb{R}^{m \times n}$, in which all elements follow the normal distribution $N(0, 1)$. We generate a random Gaussian signal $x^o \in \mathbb{R}^n$, $\|x^o\|_0 = K$, and a random Gaussian noise $z \in \mathbb{R}^m$, in which all elements follow the normal distribution $N(0, 1)$, and set $b = Ax^o + \sigma z$, where $\sigma = 0.01$. We only consider the case with fixed m and n while varying K by the same reason as in Subsection 4.2.

In the first experiment, we compare different algorithms on a range of sparsity levels with

fixed m and n , i.e., $m = 512$, $n = 2048$, $K \in (0, \lfloor \frac{m}{2} \rfloor]$. For each triple (m, n, K) , we generate 200 random instances. The average results on 200 random instances are shown as Fig. 3(a).

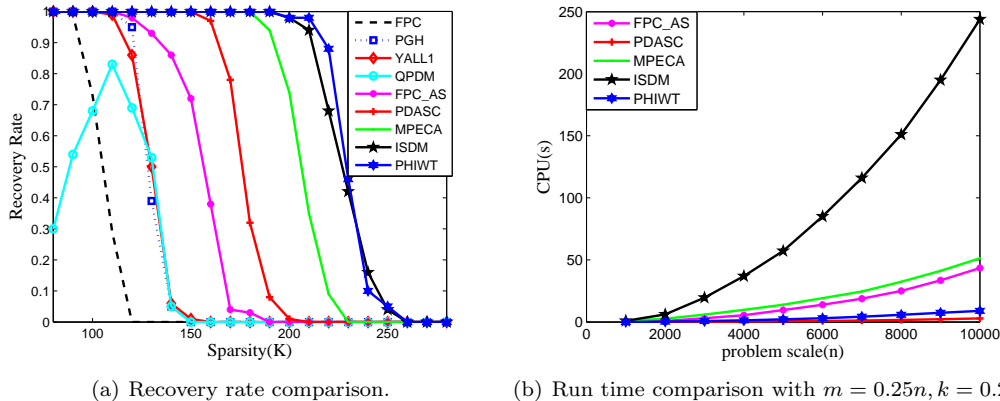


Figure 3: Computational comparisons of algorithms in the noisy case with Gaussian matrix, Gaussian signal and Gaussian noise.

From Fig. 3(a), the recovery rate of every algorithm decreases as sparsity K increases when m and n are fixed, and ISDM and PHIWT have the highest recovery rate. Moreover, all algorithms without reweighted or weighted techniques like FPC, PGH, YALL1 and QPDM are uniformly worse than ones with reweighted or weighted techniques like FPC_AS, PDASC, MPECA, ISDM, and PHIWT. Hence we only need to compare the latter five algorithms as in Fig. 3(b).

Fig. 3(b) shows the average running time of the five algorithms on instances with different problem scales n , where $m = 0.25n$, and $k = 0.2m$. From this figure, we find that among all tested algorithms, PDASC is the fastest, and PHIWT is the second fastest. The running time of PHITW is roughly 1/10 of that of ISDM, 1/3 of that of MPECA, and is comparable to that of PDASC. Though in Fig. 3(a), ISDM and PHIWT is similar in recovery rate, ISDM is very time-consuming.

For the second set of test, we generate every random instance with the same A and z as in the first set of test, while generate a random signal $x^o \in \mathbb{R}^n$ with $\|x^o\|_0 = K$, in which the components are uniformly distributed on $[-1, 1]$, and set $b = Ax^o + \sigma z$, where $\sigma = 0.01$. Results are presented as in Fig. 4.

From Fig. 4(a), the recovery rate of every algorithm decreases as sparsity K increases when m and n are fixed, and ISDM and PHIWT have the highest recovery rate. Moreover, all algorithms without reweighted or weighted techniques like FPC, PGH, YALL1, QPDM are uniformly worse than ones with reweighted or weighted techniques like FPC_AS, PDASC, MPECA, ISDM, and PHIWT. Hence we only need to compare the latter five algorithms as in Fig. 4(b).

Fig. 4(b) shows the average running time of the five algorithms on instances with different problem scales n , where $m = 0.25n$, and $k = 0.2m$. From this figure, we find that among all tested algorithms, PDASC is the fastest, and PHIWT is the second fastest. The running time of PHITW is roughly 1/10 of that of ISDM, 1/3 of that of MPECA, and is comparable to that of PDASC. Though in Fig. 4(a), ISDM and PHIWT is similar in recovery rate, ISDM is very time-consuming.

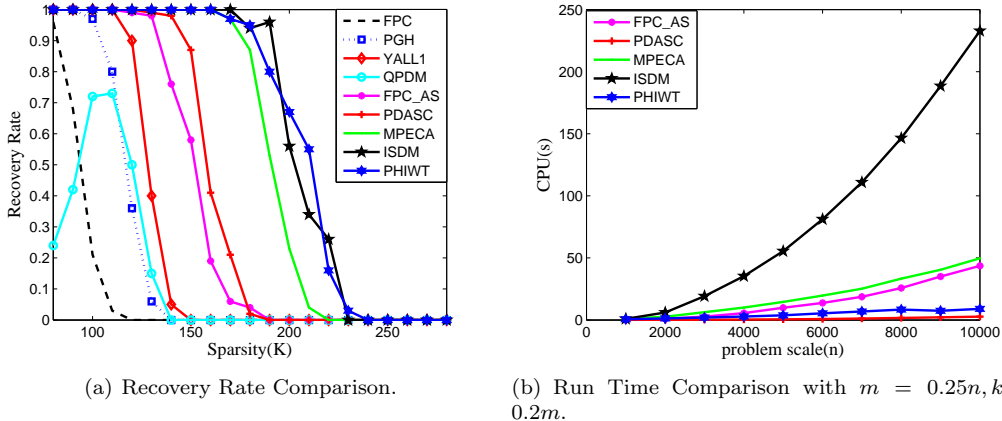


Figure 4: Computational comparisons of the methods in the noisy case with Gaussian matrix, uniformly distributed signal and Gaussian noise.

4.4 Comparison with some state-of-the-art algorithms in large scale case

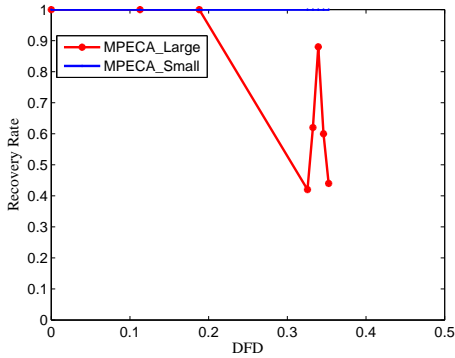
By the observations obtained in the previous experiments, we conclude that PHIWT, MPECA and ISDM are the algorithms whose recovery rates are higher than the other algorithms and are robust against Gaussian noise. However, a good algorithm should have good performance in large-scale case as well as that in small-scale case. Hence, in this subsection we compare the performances of MPECA, ISDM and PHIWT in small-scale case, i.e., $n = 2048$, and those of MPECA, ISDM and PHIWT in large-scale case, i.e., $n = 32768$. We still set $m = n/4$. We generate 50 instances for each test in the same way as in the previous subsections, and calculate the average computational results.

We compare the recovery rates of the three algorithms in small and large scale cases when varying DRD . We show the performances of MPECA and ISDM in Figs. 5 and 6, respectively. From Fig. 5(a), we see MPECA performs remarkably worse in large-scale case than in small-scale case for Gaussian signal in the noiseless case. From Fig. 5(b), we see MPECA performs remarkably worse in large-scale case than in small-scale case for uniformly distributed signal in the noiseless case. From Fig. 6(a), we see the performance of ISDM in large-scale case is slightly worse than in small-scale case for Gaussian signal in the noiseless case. From Fig. 6(b), we see the performance of ISDM in large-scale case is similar to that in small-scale case for uniformly distributed signal in the noiseless case. From the experiment in the previous section, we have seen that ISDM is too time consuming, and the running time of MPECA is three times more than ours. So we skip systematic experiment on recovery quality of ISDM and MPECA. Now we show the performance of our algorithm PHIWT in Figs. 7 and 8.

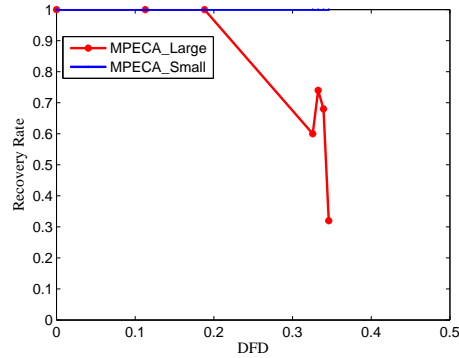
From Figs. 7 and 8, we see the curves of PHIWT in large-scale case are all similar to the curves of PHIWT in small-scale of the four different cases. This verifies that our algorithm PHIWT is also effective in the large-scale case.

5 Conclusions

Iterative reweighted methods are state-of-the-art methods for compressed sensing, where weights are determined heuristically before solving a weighted l_1 -norm minimization problem. In this paper, we have proposed a novel weighted method for solving the compressed sensing problem, in which the weight w and the variable x are optimized simultaneously. More specifically, we have proposed a new weighted l_1 -norm minimization problem, and proved that

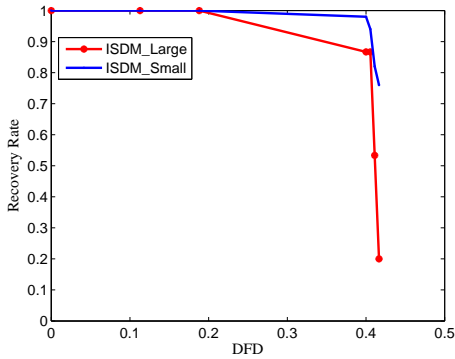


(a) Recovery rate comparisons in small and large scales with Gaussian signal and without noise.

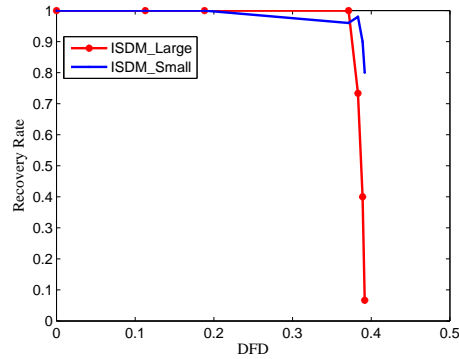


(b) Recovery rate comparisons in small and large scales with uniformly distributed signal and without noise.

Figure 5: Recovery rate comparisons of MPECA in small and large scales without noise.



(a) Recovery rate comparisons in small and large scales with Gaussian signal and without noise.



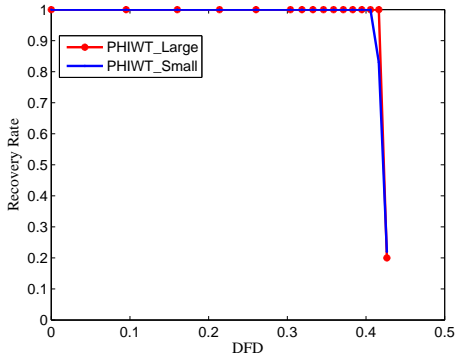
(b) Recovery rate comparisons in small and large scales with uniformly distributed signal and without noise.

Figure 6: Recovery rate comparisons of ISDM in small and large scales without noise.

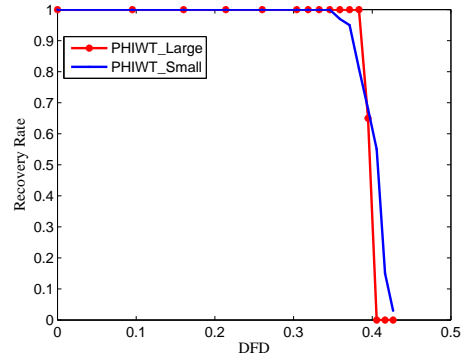
it is equivalent to the compressed sensing problem under some given conditions. By linearizing the penalty term $f(x)$ and adding a proximal term, we have obtained a closed-form solution. And thus we have obtained an iterative weighted thresholding method for the compressed sensing problem, which is essentially a hybrid of the soft thresholding method and the hard thresholding method. We have proved convergence of the iterative weighted thresholding method. To improve the performance of the iterative weighted thresholding method, we have proposed a homotopy algorithm based on the iterative weighted thresholding method for the compressed sensing problem.

Considering noise, problem scale, structure of signal, we have designed a series of computational experiments to verify the effectiveness of our homotopy algorithm. Experimental results show that our algorithm is the best comparing with the state-of-the-art algorithms no matter in recovery rate or in recovery quality, and comparable to the fastest algorithm in running time. Furthermore, we find that our homotopy algorithm based on the iterative weighted thresholding method can solve instances in the large scale case very well.

Note that, there is a parameter ε in our iterative weighted thresholding method, which plays an important role in our algorithm. Hence one can try to find a better strategy for taking the value of ε , such that it is adaptive to the structure of the signal. Furthermore, it

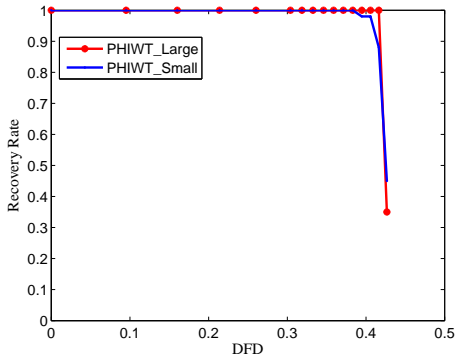


(a) Recovery rate comparisons in small and large scales with Gaussian signal and without noise.

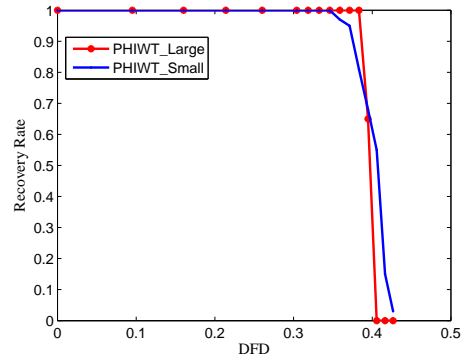


(b) Recovery rate comparisons in small and large scales with uniformly distributed signal and without noise.

Figure 7: Recovery rate comparisons of PHIWT in small and large scales without noise.



(a) Recovery rate comparisons in small and large scales with Gaussian signal and with noise.



(b) Recovery rate comparisons in small and large scales with uniformly distributed signal and with noise.

Figure 8: Recovery rate comparisons of PHIWT in small and large scales with noise.

will be interesting to extend the method in this paper to solve the rank minimization problem, and others.

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