Network Models with Unsplittable Node Flows with Application to Unit Train Scheduling *

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Abstract

We study network models where flows cannot be split or merged when passing through certain nodes, *i.e.,* for such nodes, each incoming arc flow must be matched to an outgoing arc flow of identical value. This requirement, which we call *no-split no-merge* (NSNM), appears in railroad applications where train compositions can only be modified at yards where necessary equipment is available. This combinatorial requirement is crucial when formulating problems occurring in the unit train business. We propose modeling approaches to represent the NSNM requirement. In particular, we give a linear formulation of the requirement on a single node that describes the convex hull in a lifted space. We present a cut-generating linear program to obtain valid inequalities in the original space of variables, and introduce a polynomial-time procedure to lift strong inequalities of lower-dimensional models into strong inequalities of the original model. In addition, we identify an exponential family of facet-defining inequalities that can be separated efficiently. To evaluate our results computationally, we study a stylized unit train problem. We compare a solution approach based on our results with one that relies on column generation. We then show that our results significantly reduce relaxation times and gaps when compared to leading commercial branch-and-cut software.

Keywords: Network optimization; Unit trains; Mixed integer programs; Cutting planes.

1 Introduction

We study a combinatorial requirement that occurs in the formulation of practical unit train problems. This requirement is that for some nodes of a network, the flow of incoming arcs must be sent through outgoing arcs without being split or merged.

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1.1 Motivation

Rail transportation of freight, which uses 140,000 miles of tracks in the United States, is a $60 billion industry. Since the rail infrastructure is enormous, even small improvements in operation can lead to large cost savings and substantial increases in customer satisfaction. For this reason, the use of optimization models has become pervasive in railroads, and problems at different levels from design to operation have been tackled; see [14]. Example applications include line planning, train timetabling and crew scheduling; see [10] for a classification. Such problems are now routinely modeled and solved using optimization techniques; see [12] for a survey, and [1] for a tutorial.

Several factors contribute to making railroad planning problems difficult. First, railroads have numerous customers who wish to send cars between different origin-destination pairs in the network. Since it is more economical to operate long trains than short ones, small shipments must be consolidated in ways that also allow for timely delivery. Second, cars can only be switched from trains to trains at particular yards of the rail network where skilled manpower and appropriate equipment is available to perform these changes. Third, railtracks inherently limit the flow of trains and the routes they can follow. Finally, limited resources in the form of cars, trains and crews force tradeoffs to be made between the various goals of the railroad.

To circumvent the difficulties associated with scheduling the movement of cars and trains in such a complex environment, US railroads typically optimize different components of the system independently. An important such component is the design of an efficient *blocking plan* that determines where cars will be switched from train to train on the way to their destination. Another crucial step is the design of a periodic train schedule between selected origin and destination pairs to allow for cars to be transported according to the blocking plan; see [23].

One important exception to the use of blocking in the US occurs in the transport of high volume of commodities and raw materials such as metal, coal, rock, and grain; see [22] for an example. Rail has long been a preferred choice for the transportation of such bulk commodities as it is cost-effective, energy-efficient and environment-friendly; see [14]. Unlike other trains used for freight transport, such *unit trains* are kept intact between their load origin and load destination, *i.e.*, no car is attached to or detached from the trains between the time they are loaded and the time they are unloaded. Train formation/busting can only be accomplished at certain pre-determined terminals (*build locations*) having the necessary equipment, machinery, and manpower. Blocking plans are therefore not relevant to unit trains, whose schedules and routes are more flexible, and for which planning is typically performed over longer periods of time.

At a high level, the problem of scheduling unit trains is a traditional network flow problem over a heterogeneous network that consists of a large number of terminals/nodes where train consists cannot be split or merged, together with a few others that have the necessary amenities to handle train busting and formation. A key to solving practical unit train problems is therefore to understand how to model such *no-split no-merge* (NSNM) requirements. Although there is abundant literature dealing with various features of train scheduling, dedicated studies of the NSNM requirement have been lacking. In this paper, we investigate how to represent and impose NSNM constraints in optimization models.

Even though our study is motivated by rail transportation, our results apply to other contexts. We describe some of these applications in the next section.

1.2 Literature Review

Problems specific to unit trains have been studied in [21], where the authors seek to determine schedules for unit rock trains that respect yard and track requirements, while taking into account customers preferences and hours of operations. In [18], we studied the scheduling of unit coal trains for a major class I railroad. In this problem, customers place requests at the beginning of the month for trains to be sent between specified load origins and destinations. The railroad then has to decide which reservations to satisfy during the month, and what cars to assign to them. In order to create a high-quality plan, the railroad has to anticipate when and where cars will be exchanged from one train to another. We developed a two-stage heuristic since our initial attempt at formulating the problem as a mixed-integer program (MIP) proved to be unsuccessful as the models were too large to be solved using commercial solvers.

In this paper, we perform a polyhedral study of NSNM and develop cutting plane techniques to improve the formulation of problems that contain this requirement. Such an approach has been successfully used in the past for the study of other MIPs defined on networks. For instance, cutting plane approaches have proven beneficial in the solution of network design problems; see [7, 9, 16, 2, 3, 24, 26]. Cutting plane approaches have also been studied for transportation problems related to railroads, including locomotive assignment; see [31]. Polyhedral results have also shown helpful in the design of heuristic procedures. [11] proposes a matheuristic mechanism for a class of service network design problems that combines a cutting plane procedure and a variable-fixing procedure to obtain high quality solutions efficiently.

In addition to train scheduling problems, the NSNM requirement also appears in the *unsplittable flow problem* (UFP) where demand must be satisfied via flows that cannot be split or merged throughout the network. Several variants of UFP have been studied in the computer science literature, most of which are NP-hard; see [4]. Approximation algorithms have been proposed for some of these problems; see [19] for a review. Many of these algorithms proceed by rounding a solution to a linear programming (LP) relaxation of the problem; see [13] and [20]. The NSNM requirement also appears in information infrastructure and telecommunication problems arising in bandwidth allocation and virtual-circuit routing; see [17]. Additional applications in packing, partitioning and load balancing are described in [19].

1.3 Contribution

We study the NSNM requirement as it is one of the main sources of difficulty in the modeling of certain optimization problems in transportation, computer science, and information technology. In Section 2, we present formulations of this requirement in both the original and lifted spaces of variables. In particular, we introduce an extended formulation whose LP relaxation provides the convex hull of single node flow problems with NSNM requirement. We further show that the strength of extended formulations can be harnessed in the space of original variables through the solution of a cut-generation linear program. We then propose lifting procedures that use seed inequalities for easily constructed restrictions of the problem, and convert them into strong inequalities through a polynomial-time procedure. We derive a family of closed-form lifted facet-defining inequalities that can be used in the space of original variables, and can be separated in polynomial time. In Section 3, we present a stylized optimization model for unit train scheduling. We perform computational experiments on this model to evaluate the impact of the approaches we propose. Our results suggest that cutting plane techniques are better suited than column generation approaches for this problem, especially when the underlying network is large. In addition to yielding LP relaxations that are smaller in size and faster to solve, a rudimentary implementation of our results produces bounds better than those generated by CPLEX through its arsenal of cuts and preprocessing techniques.

Notation. We use the following notation throughout the paper. Consider a set $\mathcal{S} := \{(\boldsymbol{x};\boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{Z}^m \mid A\boldsymbol{x} + B\boldsymbol{y} = c\}$ where matrices A and B, and vector c are of conforming dimensions. We denote the convex hull of $\mathcal{S}$ by $\mathcal{PS}$. We represent the LP relaxation of $\mathcal{S}$ obtained by removing integrality requirement on integer variables by $\mathcal{LS}$. Further, we use $\text{proj}_{\boldsymbol{x}}(\mathcal{S})$ to represent the projection of $\mathcal{S}$ onto the space of variables $\boldsymbol{x}$. Finally, we refer to the dimension of $\mathcal{S}$ by $\dim(\mathcal{S})$.

2 Polyhedral Analysis

In this section, we study the single node flow problem with n incoming arcs and m outgoing arcs under the NSNM requirement. All proofs can be found in the Appendix. We denote the flow on incoming arc i by x_i for $i \in N := \{1,\ldots,n\}$ and the flow on outgoing arc j by y_j for $j \in M := \{1,\ldots,m\}$. We denote the capacity of incoming arc i by u_i for $i \in N$ and the capacity of outgoing arc j by v_j for $j \in M$. We define $w_{i,j} := \min\{u_i, v_j\}$. In addition to flow balance, we require that each nonzero incoming flow x_i be paired with a single outgoing flow y_j of identical value (*i.e.*, $x_i = y_j$) and that each nonzero outgoing flow y_j be paired with a single incoming flow x_i of identical value. We say that such a flow satisfies the NSNM requirement and use the notation $\boldsymbol{x} \rightleftharpoons \boldsymbol{y}$ to represent it.

Using the above notation, we formulate the single node set with NSNM requirement as

$$\mathcal{S}^{n,m} = \left\{ (x;y) \in \mathbb{R}^{n+m} \;\middle|\; \begin{array}{ll} \sum_{i\in N} x_i = \sum_{j \in M} y_j & \\ 0 \le x_i \le u_i, & \forall i \in N \\ 0 \le y_j \le v_j, & \forall j \in M \\ \boldsymbol{x} \rightleftharpoons \boldsymbol{y} & \end{array} \right\}.$$

In the remainder of this paper, we impose

Assumption 1. $0 < u_1 \le u_2 \le \ldots \le u_n$ *and* $0 < v_1 \le v_2 \le \ldots \le v_m$.

Assumption 2. $u_n = v_m$.

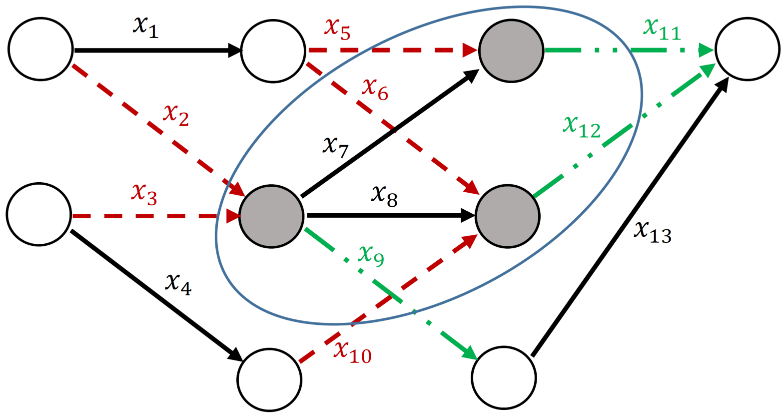

Figure 1: (Color online) An aggregation of multiple nodes with NSNM requirements.

There is no loss of generality in Assumption 1 since arcs can be reordered. When $u_n > v_m$, no outgoing arc can accommodate a flow of value u_n. The capacity of arc n can therefore be reduced to v_m without altering $\mathcal{S}^{n,m}$. Similarly, when $v_m > u_n$, v_m can be reduced to u_n, justifying Assumption 2.

It is clear that $\mathcal{PS}^{n,m}$ is a polytope since $\mathcal{S}^{n,m}$ is a finite union of polytopes. We next describe a useful property of the extreme points of $\mathcal{PS}^{n,m}$.

Proposition 1. *Let $(\bar{\boldsymbol{x}}; \bar{\boldsymbol{y}})$ be an extreme point of $\mathcal{PS}^{n,m}$. Then, for any $i \in N$, $\bar{x}_i = 0$ or $\bar{x}_i = w_{i,j}$ for some $j \in M$. Similarly, for any $j \in M$, $\bar{y}_j = 0$ or $\bar{y}_j = w_{i,j}$ for some $i \in N$.* □

Even though practical problems including those appearing in train scheduling contain several nodes with NSNM requirements, we claim that deriving strong formulations for the single node model $\mathcal{S}^{n,m}$ is crucially important. First, even when multiple nodes are subject to NSNM restrictions, these requirements are expressed for each node individually. Improved formulations for each node can therefore lead to better relaxation bounds for the full model. An example of the success of approaches that study single node relaxations is the fixed-charge network problem where solvers routinely apply cuts such as flow covers derived from single node structures; see [15, 25, 29]. Second, through suitable aggregation, single node models can be used to capture some of the complexity associated with multiple NSNM requirements. We elaborate on this comment next.

Consider the network of Figure 1 where nodes that are shaded have NSNM requirements while the others only need to satisfy balance equations. The NSNM requirement on individual nodes also implies an NSNM requirement between the red dashed arcs (associated with variables x_2, x_3, x_5, x_6, x_{10}) and the green semi-dotted arcs (associated with variables x_9, x_{11} and x_{12}). More generally, for a subset E of nodes of a network, we refer to the set of arcs whose heads belong to E and whose tails do not belong to E as $\delta^-(E)$. Similarly, we define $\delta^+(E)$ to be the set of arcs whose tails belong to E and whose heads do not belong to E. If E is a subset of nodes all satisfying the NSNM requirement, then the resulting single node set with incoming arcs $\delta^-(E)$ and outgoing arcs $\delta^+(E)$, which we denote by $\mathcal{S}_E$, satisfies NSNM. Since $\mathcal{S}_E$ is a relaxation of the initial problem,

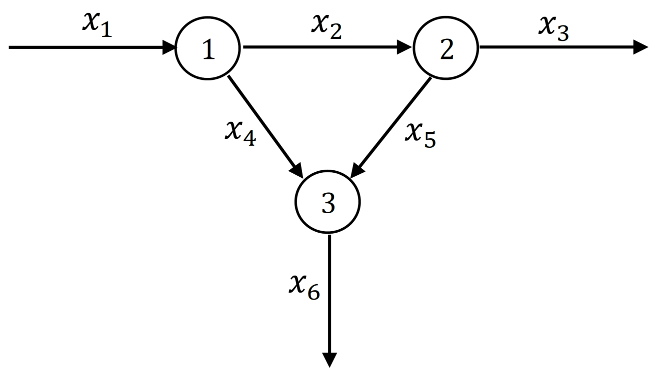

Figure 2: Network of Example 1.

valid inequalities obtained for this set can be used as cutting planes for the initial model.

We next argue that adding inequalities obtained for $\mathcal{PS}_E$ where E contains multiple nodes can tighten relaxations that only consider single node sets with NSNM requirements. Similarly, we show that inequalities obtained from single node sets with NSNM requirements are not always implied by the inequalities derived for $\mathcal{PS}_E$ where E contains multiple nodes.

Example 1. Consider the network of Figure 2, where all three nodes satisfy the NSNM requirement. The arcs associated with variables $x_1, \ldots, x_6$ have capacities 4, 4, 2, 2, 4, and 4, respectively. First, we show that the relaxation obtained by convexifying single node sets contains points that are cut by inequalities derived for a multi-node aggregation. Consider $\bar{\boldsymbol{\omega}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) = (3, 2, 2, 1, 0, 1)$. We show that $\bar{\boldsymbol{\omega}}$ belongs to the intersection of convex hulls of $\mathcal{S}_{\{1\}}$, $\mathcal{S}_{\{2\}}$ and $\mathcal{S}_{\{3\}}$. Note that there is a slight abuse of notation in the above statement as $\mathcal{S}_{\{i\}}$ has three variables while $\mathcal{S}$ has six. However, in such a case, we understand $\mathcal{S}_{\{i\}}$ to be the set in six variables obtained by reintroducing the missing variables and by considering them to be free. Proposition 1 implies that $\mathcal{PS}_{\{1\}}$ has extreme points $\boldsymbol{\nu}^1 = (x_1, x_2, x_4) = (0, 0, 0)$, $\boldsymbol{\nu}^2 = (4, 4, 0)$ and $\boldsymbol{\nu}^3 = (2, 0, 2)$. Therefore, $(\bar{x}_1, \bar{x}_2, \bar{x}_4) = \frac{1}{2}\boldsymbol{\nu}^2 + \frac{1}{2}\boldsymbol{\nu}^3$ belongs to $\mathcal{PS}_{\{1\}}$. Similar arguments show that $(\bar{x}_2, \bar{x}_3, \bar{x}_5) \in \mathcal{PS}_{\{2\}}$, and $(\bar{x}_4, \bar{x}_5, \bar{x}_6) \in \mathcal{PS}_{\{3\}}$. On the other hand, it is easy to verify that the subvector $(\bar{x}_1, \bar{x}_3, \bar{x}_6)$ of $\bar{\boldsymbol{\omega}}$ cannot be represented as a convex combination of the extreme points $\bar{\boldsymbol{\nu}}^1 = (x_1, x_3, x_6) = (0, 0, 0)$, $\bar{\boldsymbol{\nu}}^2 = (2, 2, 0)$ and $\bar{\boldsymbol{\nu}}^3 = (4, 0, 4)$ of of $\mathcal{PS}_{\{1,2,3\}}$. Proposition 1 implies that $(\bar{x}_1, \bar{x}_3, \bar{x}_6) \notin \mathcal{PS}_{\{1,2,3\}}$.

Next, we show that the relaxation obtained by aggregating nodes 1 and 3 in the network of Figure 2 is not contained in $\mathcal{PS}_{\{1\}}$. It is clear that the point $\hat{\boldsymbol{\omega}} = (\hat{x}_1, \hat{x}_2, \hat{x}_5, \hat{x}_6) = (3, 4, 4, 3)$ belongs to $\mathcal{S}_{\{1,3\}} \subseteq \mathcal{PS}_{\{1,3\}}$. However, $(\hat{x}_1, \hat{x}_2, x_4)$ does not satisfy the balance equation in the description of $\mathcal{S}_{\{1\}}$ for any $x_4 \geq 0$. We conclude that this point does not belong to $\mathcal{PS}_{\{1\}}$.

Example 1 suggests that cutting planes derived from different node aggregations can improve LP relaxation bounds. Our numerical experiments in Section 3 support this claim.

2.1 Extended Formulations

Before focusing on formulations of $\mathcal{S}^{n,m}$ in the space of original variables x_i and y_j, we first study a natural extended formulation, show that it is not *sharp*, and then introduce an alternative extended formulation that is *locally ideal*; see [30] for a discussion of sharp and locally ideal formulations. For $(i,j) \in N \times M$, we define binary variable $z_{i,j}$ to take value 1 if the flow on incoming arc i is routed onto outgoing arc j. The resulting MIP formulation is:

$$x_i - y_j \le u_i(1 - z_{i,j}), \qquad \forall (i,j) \in N \times M, \tag{1a}$$

$$y_j - x_i \le v_j(1 - z_{i,j}), \qquad \forall (i,j) \in N \times M, \tag{1b}$$

$$0 \le x_i \le u_i \sum_{j \in M} z_{i,j}, \qquad \forall i \in N, \tag{1c}$$

$$0 \le y_j \le v_j \sum_{i \in N} z_{i,j}, \qquad \forall j \in M, \tag{1d}$$

$$\sum_{j \in M} z_{i,j} \le 1, \qquad \forall i \in N, \tag{1e}$$

$$\sum_{i \in N} z_{i,j} \le 1, \qquad \forall j \in M, \tag{1f}$$

$$\sum_{i \in N} x_i = \sum_{j \in M} y_j \tag{1g}$$

$$z_{i,j} \ge 0, \qquad \forall (i,j) \in N \times M, \tag{1h}$$

$$z_{i,j} \in \mathbb{Z}, \qquad \forall (i,j) \in N \times M. \tag{1i}$$

Constraint (1a) in conjunction with (1b) ensures that when $z_{i,j}$ equals 1, then the incoming flow through arc i equals the outgoing flow through arc j. Constraints (1c) and (1d) ensure that the flow on an arc is positive only if it is paired with another arc. Constraints (1e) and (1f) impose that each incoming arc is matched with no more than one outgoing arc and vice-versa. Constraint (1g) is the balance equation. Even though it is naturally satisfied by mixed integer solutions, there are fractional solutions that do not satisfy it. We denote the set of feasible solutions to (1a)–(1i) by $\mathcal{T}^{n,m}$. As a result, $\mathcal{T}^{n,m}$ is an extended formulation of $\mathcal{S}^{n,m}$, *i.e.,*

Proposition 2. *It holds that* $\text{proj}_{(\boldsymbol{x},\boldsymbol{y})}\, \mathcal{T}^{n,m} = \mathcal{S}^{n,m}$. □

We next show through an example that $\mathcal{T}^{n,m}$ is not a sharp formulation of $\mathcal{S}^{n,m}$ *i.e.*, its LP relaxation $\mathcal{LT}^{n,m}$ contains extreme points whose projections do not belong to $\mathcal{PS}^{n,m}$.

Example 2. Consider an instance of $\mathcal{T}^{2,2}$ where $\boldsymbol{u} = (5,7)$ and $\boldsymbol{v} = (3,7)$. Consider a point $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ where $\hat{\boldsymbol{x}} = (5,5)$, $\hat{\boldsymbol{y}} = (3,7)$, $\hat{z}_{1,1} = \frac{3}{5}$, $\hat{z}_{1,2} = \frac{2}{5}$, $\hat{z}_{2,1} = \frac{2}{5}$ and $\hat{z}_{2,2} = \frac{3}{5}$. Vector $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ is an extreme point of $\mathcal{LT}^{2,2}$ yet $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}})$ does not satisfy the NSNM requirement in $\mathcal{S}^{2,2}$. Further, the projection of $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ onto the space of original variables $(\boldsymbol{x}; \boldsymbol{y})$ does not belong to $\mathcal{PS}^{2,2}$ since it cannot be obtained as a convex combination of its extreme points $(5,3;3,5)$, $(5,0;0,5)$, $(3,7;3,7)$, $(3,0;3,0)$ and $(0,0;0,0)$; see Proposition 1.

Example 2 implies that $\mathcal{T}^{n,m}$ does not yield the strongest relaxation of $\mathcal{PS}^{n,m}$, *i.e.*, $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\, \mathcal{L}\bar{\mathcal{T}}^{n,m} \supset \mathcal{PS}^{n,m}$. To obtain stronger LP relaxations, we introduce the following model where $z_{i,j}$ equals 1 if and only if $x_i = y_j = w_{i,j}$ for $(i,j) \in N \times M$:

$$x_i = \sum_{j \in M} w_{i,j} z_{i,j}, \qquad \forall i \in N, \tag{2a}$$

$$y_j = \sum_{i \in N} w_{i,j} z_{i,j}, \qquad \forall j \in M, \tag{2b}$$

$$\sum_{j \in M} z_{i,j} \leq 1, \qquad \forall i \in N, \tag{2c}$$

$$\sum_{i \in N} z_{i,j} \leq 1, \qquad \forall j \in M, \tag{2d}$$

$$z_{i,j} \geq 0, \qquad \forall (i,j) \in N \times M, \tag{2e}$$

$$z_{i,j} \in \mathbb{Z}, \qquad \forall (i,j) \in N \times M. \tag{2f}$$

We denote the set of feasible solutions to (2a)–(2f) by $\mathcal{D}^{n,m}$. We note that $\mathcal{D}^{n,m}$ is not an extended formulation for $\mathcal{S}^{n,m}$, as its projection onto the space of variables $(\boldsymbol{x};\boldsymbol{y})$ only contains extreme solutions of $\mathcal{S}^{n,m}$ and not all of its feasible solutions. Nevertheless, $\mathcal{D}^{n,m}$ is locally ideal, *i.e.*, it has an integral LP relaxation whose projection onto the space of variables $(\boldsymbol{x};\boldsymbol{y})$ gives $\mathcal{PS}^{n,m}$.

Proposition 3. *It holds that* $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})} \mathcal{LD}^{n,m} = \mathcal{PS}^{n,m}$. □

The proof of Proposition 3 establishes that the projection of convex hulls of $\mathcal{D}^{n,m}$ and $\mathcal{T}^{n,m}$ onto the space of variables $(\boldsymbol{x};\boldsymbol{y})$ are equal. The stronger claim that the convex hulls of these sets are identical does not hold in general. This stems from the fact that $\mathcal{T}^{n,m}$ may contain extreme points where $z_{i,j} = 1$ and $x_i = y_j = 0$ for some $(i,j) \in N \times M$, while such points do not belong to $\mathcal{D}^{n,m}$.

Unlike formulation $\mathcal{T}^{n,m}$, $\mathcal{D}^{n,m}$ cannot be used in MIP models since it only encodes the extreme points of $\mathcal{S}^{n,m}$. Its advantage over $\mathcal{T}^{n,m}$ however is that it is locally ideal. This property has two main practical implications that we use in our computational experiments in Section 3. First, a cut-generating linear program (CGLP) can be devised to obtain valid inequalities for $\mathcal{PS}^{n,m}$ in the space of original variables; see Section 2.2. Second, since $\mathcal{LD}^{n,m}$ gives the convex hull of individual NSNM in a higher dimension, its addition to the initial LP relaxation of the problem provides the best bound that can be obtained by convexifying NSNM for individual nodes.

For practical problems related to unit train scheduling, in which monthly planning is formulated over a time-space network, $\mathcal{LD}^{n,m}$ is too large to be embedded directly inside of the LP relaxation of the initial problem. An alternative approach is to use path-based models together with column generation. Our computational experiments in Section 3 show that the approach we investigate here has two main advantages over the one based on column generation. First, the column generation approach provides weaker primal and dual bounds for larger size networks, which are of more practical interest. Second, the column generation approach does not seem to be as effective as the cutting plane approach even when the only combinatorial constraints in the underlying problems are NSNM requirements. We therefore believe that the benefits of obtaining a strong formulation of the NSNM requirement in the space of original variables will become even more evident on large size practical problems where a variety of other combinatorial requirements on the flow variables are imposed, as cutting planes generated for various substructures can be handled with little effort inside of current commercial codes.

2.2 Formulations in the Space of Original Variables

We now study the convex hull of $\mathcal{S}^{n,m}$ in the space of original variables $\boldsymbol{x}$ and $\boldsymbol{y}$. We present a few observations to streamline our derivation. First, if $\sum_{i\in N}\alpha_i x_i + \sum_{j\in M}\beta_j y_j \le \gamma$ defines a face of $\mathcal{PS}^{n,m}$, then this face can also be represented by the inequality $\sum_{i\in N\setminus\{1\}}(\alpha_i-\alpha_1)x_i + \sum_{j\in M}(\beta_j + \alpha_1)y_j \le \gamma$ since $x_1 = \sum_{j\in M} y_j - \sum_{i\in N\setminus\{1\}} x_i$. In the remainder of this section, whenever considering a valid inequality for $\mathcal{PS}^{n,m}$ that contains variables other than x_1, we use the balance constraint to eliminate x_1. Second, because the definition of $\mathcal{S}^{n,m}$ is symmetric in variables $\boldsymbol{x}$ and $\boldsymbol{y}$, families of valid inequalities for $\mathcal{PS}^{n,m}$ come in mirroring pairs as described in the next remark.

Remark 1. Consider an instance of $\mathcal{PS}^{n,m}$ with variables x and y where incoming arcs have capacities $\boldsymbol{u}\in\mathbb{R}^n$ and outgoing arcs have capacities $\boldsymbol{v}\in\mathbb{R}^m$. Let $\alpha_i(\boldsymbol{u},\boldsymbol{v})$ for $i\in N$, $\beta_j(\boldsymbol{u},\boldsymbol{v})$ for $j\in M$ and $\gamma(\boldsymbol{u},\boldsymbol{v})$ be functions of arc capacities $\boldsymbol{u}$ and $\boldsymbol{v}$. Further, let $C(\boldsymbol{u},\boldsymbol{v})$ represent a set of conditions defined on these arc capacities. Assume that the inequality $\sum_{i\in N}\alpha_i(\boldsymbol{u},\boldsymbol{v})x_i + \sum_{j\in M}\beta_j(\boldsymbol{u},\boldsymbol{v})y_j \le \gamma(\boldsymbol{u},\boldsymbol{v})$ is valid for $\mathcal{PS}^{n,m}$ under conditions $C(\boldsymbol{u},\boldsymbol{v})$. Then, under conditions $C(\boldsymbol{u},\boldsymbol{v})$, the inequality $\sum_{j\in M}\beta_j(\boldsymbol{u},\boldsymbol{v})x'_j + \sum_{i\in N}\alpha_i(\boldsymbol{u},\boldsymbol{v})y'_i \le \gamma(\boldsymbol{u},\boldsymbol{v})$ is valid for the single node set with variables x' and y' where incoming arcs have capacities $\boldsymbol{v}$ and outgoing arcs have capacities $\boldsymbol{u}$.

2.2.1 A Cut-Generating Linear Program.

It follows from Proposition 3 that it is possible to obtain a description of $\mathcal{PS}^{n,m}$ in the space of original variables by projecting $\mathcal{LD}^{n,m}$ onto the space of variables $(\boldsymbol{x};\boldsymbol{y})$ through, say, Fourier-Motzkin elimination; see [32] for a description. Further, it follows from [6] that facet-defining inequalities of $\mathcal{PS}^{n,m}$ correspond to extreme rays of the projection cone, even though not all extreme rays of the projection cone correspond to facet-defining inequalities of $\mathcal{PS}^{n,m}$. Expressing free dual variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as differences of nonnegative variables, we obtain the following result.

Proposition 4. *Consider the pointed cone*

$$\mathcal{C} = \left\{ \boldsymbol{\xi}\in\mathbb{R}_+^{4(n+m)+nm} \;\middle|\; w_{i,j}(\alpha_i^- - \alpha_i^+ + \beta_j^- - \beta_j^+) + \gamma_i + \theta_j - \pi_{i,j} = 0, \qquad \forall (i,j)\in N\times M \right\},$$

where $\boldsymbol{\xi} = (\boldsymbol{\alpha}^+;\boldsymbol{\alpha}^-;\boldsymbol{\beta}^+;\boldsymbol{\beta}^-;\boldsymbol{\gamma};\boldsymbol{\theta};\boldsymbol{\pi})$. Then, for any $\boldsymbol{\xi}\in\mathcal{C}$,

$$\sum_{i\in N}(\alpha_i^+ - \alpha_i^-)x_i + \sum_{j\in M}(\beta_j^+ - \beta_j^-)y_j \le \sum_{i\in N}\gamma_i + \sum_{j\in M}\theta_j, \tag{3}$$

is valid for $\mathcal{PS}^{n,m}$. Further, every facet-defining inequality of $\mathcal{PS}^{n,m}$ is of the form (3) for some extreme ray of $\mathcal{C}$. □

Proposition 4 shows that there is a polynomial time algorithm to identify a valid inequality for $\mathcal{PS}^{n,m}$ that is violated at $(\boldsymbol{x}^*;\boldsymbol{y}^*)$ for each $(\boldsymbol{x}^*;\boldsymbol{y}^*)\notin\mathcal{PS}^{n,m}$. The coefficients of this inequality can be obtained by solving the following CGLP

$$\phi^* = \max \quad \sum_{i\in N} \left(\alpha_i^+ - \alpha_i^-\right) x_i^* + \sum_{j\in M} \left(\beta_j^+ - \beta_j^-\right) y_j^* - \sum_{i\in N} \gamma_i - \sum_{j\in M} \theta_j \tag{4a}$$

$$s.t. \quad (\boldsymbol{\alpha}^+; \boldsymbol{\alpha}^-; \boldsymbol{\beta}^+; \boldsymbol{\beta}^-; \boldsymbol{\gamma}; \boldsymbol{\theta}; \boldsymbol{\pi}) \in \mathcal{C}, \tag{4b}$$

$$\sum_{i\in N} \left(\alpha_i^- + \gamma_i\right) + \sum_{j\in M} \left(\beta_j^- + \theta_j\right) \leq 1, \tag{4c}$$

where the normalization equality (4c) is added to ensure that the problem is bounded when a violated inequality exists. In particular, a positive value for ϕ^* indicates the existence of a violated cutting plane, while a nonpositive value establishes that $(\boldsymbol{x}^*; \boldsymbol{y}^*) \in \mathcal{PS}^{n,m}$. We evaluate the effectiveness of CGLP computationally in Section 3. We mention however that CGLP can be relatively slow to solve for large problems and that the cuts it produces are not guaranteed to be facet-defining inequalities for $\mathcal{PS}^{n,m}$. For these reasons, we next identify structured families of facet-defining inequalities for $\mathcal{PS}^{n,m}$.

2.2.2 Convex Hull Descriptions for Special Cases.

We first study special cases of $\mathcal{S}^{n,m}$ for which we can derive a complete convex hull description. One such situation occurs when all arcs have equal capacities. In the ensuing statement, $\mathcal{RS}^{n,m}$ is the relaxation of $\mathcal{S}^{n,m}$ obtained by removing the NSNM requirement $\boldsymbol{x} \rightleftharpoons \boldsymbol{y}$.

Proposition 5. *Assume that $u_i = v_j = u$ for all $i \in N$ and $j \in M$. Then $\mathcal{PS}^{n,m} = \mathcal{RS}^{n,m}$.* □

Next, we consider the case where there is a single outgoing arc. In this case, $u_n = v_1$ according to Assumption 2.

Proposition 6. *A linear description of $\mathcal{PS}^{n,1}$ is given by*

$$\mathcal{PS}^{n,1} = \left\{ (\boldsymbol{x}; y_1) \in \mathbb{R}_+^{n+1} \;\middle|\; \begin{array}{l} \sum_{i\in N} x_i = y_1 \\ \sum_{i\in N} \frac{x_i}{u_i} \leq 1 \end{array} \right\}.$$

□

The proof of Proposition 6 relies on the fact that $\mathcal{PS}^{n,1}$ has $n+1$ extreme points and therefore their convex combinations define a simplex. Through Remark 1, Proposition 6 also provides a linear description for $\mathcal{PS}^{1,m}$. Next, we consider the case with two incoming and two outgoing arcs. We give results for the situation where $u_1 \leq v_1 \leq v_2 = u_2$, as the result for the other case where $v_1 \leq u_1 \leq u_2 = v_2$ can be obtained using Remark 1.

Proposition 7. *A linear description of $\mathcal{PS}^{2,2}$ where $u_1 \le v_1 \le v_2 = u_2$ is given by*

$$\mathcal{PS}^{2,2} = \left\{ (\boldsymbol{x}; \boldsymbol{y}) \in \mathbb{R}^4_+ \;\middle|\; \begin{array}{l} x_1 + x_2 = y_1 + y_2 \\ y_1 + y_2 - x_2 \le u_1 \\ (u_1 - u_2)x_2 + u_2 y_2 \le u_1 u_2 \\ (u_1 - u_2 + v_1)x_2 + (u_2 - v_1)y_2 \le u_1 u_2 \\ (u_2 - u_1)x_2 + (v_1 - u_2 + u_1)y_2 \le u_2 v_1 \\ u_2 x_2 + (v_1 - u_2)y_2 \le u_2 v_1 \\ (v_1 - u_1)x_2 + (u_2 - v_1)y_1 \le v_1(u_2 - u_1) \\ (u_1 - v_1)x_2 + v_1 y_1 \le v_1 u_1 \end{array} \right\}.$$

□

Proposition 7 can be proven by projecting over $(\boldsymbol{x}; \boldsymbol{y})$ a higher-dimensional representation of the convex hull obtained using disjunctive programming.

2.2.3 Lifting Results.

We next focus on the derivation of facet-defining inequalities for $\mathcal{PS}^{n,m}$ where $n \ge 2$ and $m \ge 2$, since the other cases are studied in Section 2.2.2. We present in Example 3 an instance of a single node model with three incoming and three outgoing arcs that we use in the remainder of the paper to illustrate our results.

Example 3. Consider

$$\widehat{\mathcal{S}}^{3,3} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = y_1 + y_2 + y_3 \\ x_1 \le 3, x_2 \le 7, x_3 \le 13 \\ y_1 \le 5, y_2 \le 11, y_3 \le 13 \\ \boldsymbol{x} \rightleftharpoons \boldsymbol{y} \end{array} \right\}.$$

The facet-defining inequalities of $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ are diverse in structure. In particular, the 45 equalities/inequalities necessary in the description of $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ are listed in Table 7.

Next, we introduce a procedure to lift strong inequalities of lower-dimensional restrictions of $\mathcal{S}^{n,m}$ into strong inequalities for $\mathcal{PS}^{n,m}$. Motivated by Proposition 1, we consider restrictions of $\mathcal{S}^{n,m}$ where some variables x_i are fixed either at 0 or w_{ij} for some $j \in M$, and then projected out of the description. These restrictions of $\mathcal{S}^{n,m}$ have fewer variables, which simplifies the derivation of facet-defining *seed* inequalities for lifting; see [27] for a survey of lifting results in MIP.

In the remainder of this section, we use $[I, J, K]$ to represent $I \subseteq N$, $J \subseteq M$, and $K \subseteq (N \setminus I) \times (M \setminus J)$ such that for each distinct pair (i_1, j_1) and (i_2, j_2) in K, $i_1 \ne i_2$ and $j_1 \ne j_2$. We define

$$\mathcal{S}^{n,m}_{[I,J,K]} = \left\{ (\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m} \;\middle|\; \begin{array}{ll} x_i = 0, & \forall i \in I \\ y_j = 0, & \forall j \in J \\ x_i = y_j = w_{i,j}, & \forall (i,j) \in K \end{array} \right\},$$

for any $[I, J, K]$. In words, $\mathcal{S}^{n,m}_{[I,J,K]}$ is a restriction of $\mathcal{S}^{n,m}$ where I (*resp.* J) denotes the indices of variables $\boldsymbol{x}$ (*resp.* $\boldsymbol{y}$) that are fixed at zero, while K denotes the indices of pair of matching variables that are fixed at their maximum joint capacity values. Further, for any $[I, J, K]$, we define $K_x = \{i \,|\, (i, j) \in K, \text{ for some } j \in M \setminus J\}$ as the set of indices of variables $\boldsymbol{x}$ that are in a matching pair of K. Similarly, we define $K_y = \{j \,|\, (i, j) \in K, \text{ for some } i \in N \setminus I\}$. Finally, we introduce $\mathcal{S}^{n,m}_{X,Y}$ with $X = N \setminus (I \cup K_x)$ and $Y = M \setminus (J \cup K_y)$ to be the projection of $\mathcal{S}^{n,m}_{[I,J,K]}$, for $[I, J, K]$, onto the space of variables x_i for $i \in X$ and y_j for $j \in Y$. In words, $\mathcal{S}^{n,m}_{X,Y}$ corresponds to the single node model with NSNM requirement obtained by retaining arcs in X and Y and by removing all others.

We next describe two polynomial lifting procedures. The first lifts variable x_i (*resp.* y_j) for $i \in I$ (*resp.* $j \in J$) from zero. The second simultaneously lifts variables x_i and y_j for $(i, j) \in K$.

For any $[I, J, K]$, we denote a valid inequality of $\mathcal{S}^{n,m}_{X,Y}$ by $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma$, where $\boldsymbol{x}$ (*resp.* $\boldsymbol{y}$) indicates a vector in the space of variables x_i (*resp.* y_j) for $i \in X$ (*resp.* $j \in Y$.) We refer to $\boldsymbol{y}_{-j}$ as the vector with all elements of $\boldsymbol{y}$ except y_j. The next proposition describes a procedure for lifting a single variable x_i fixed at zero. Lifting a variable y_j fixed at zero is similar.

Proposition 8. *Assume that* $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma$ *defines a d-dimensional face of* $\mathcal{PS}^{n,m}_{X,Y}$ *for some* $[I, J, K]$. *For* $k \in I$, *define*

$$\mu^k = \min_{j \in Y} Z^k_j, \tag{5}$$

where

$$Z^k_j = \min \left\{ \frac{\gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y}}{w_{k,j}} \,\middle|\, (\boldsymbol{x}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X, Y\setminus\{j\}},\, y_j = w_{k,j} \right\}, \tag{6}$$

for $j \in Y$. *Then,*

$$\mu^k x_k + \boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma, \tag{7}$$

defines a face of $\mathcal{PS}^{n,m}_{X\cup\{k\},Y}$ *of dimension at least* $d + 1$. □

Proposition 9 describes a procedure to simultaneously lift a pair of variables x_k and y_l with indices (k, l) in K.

Proposition 9. *Assume that* $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma$ *defines a d-dimensional face of* $\mathcal{PS}^{n,m}_{X,Y}$ *for some* $[I, J, K]$. *Consider* $(k, l) \in K$. *For any* $i \in X$ *and* $j \in Y$, *define*

$$Z^{k,l}_{i,j} = \min \left\{ \gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y} \,\middle|\, (\boldsymbol{x}_{-i}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X\setminus\{i\}, Y\setminus\{j\}},\, x_i = w_{i,l},\, y_j = w_{k,j} \right\}. \tag{8a}$$

For $i \in X$, *define*

$$Z^{k,l}_{i,0} = \min \left\{ \gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y} \,\middle|\, (\boldsymbol{x}_{-i}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X\setminus\{i\}, Y},\, x_i = w_{i,l} \right\}. \tag{8b}$$

Further, for $j \in Y$, *define*

$$Z^{k,l}_{0,j} = \min \left\{ \gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y} \,\middle|\, (\boldsymbol{x}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X, Y\setminus\{j\}},\, y_j = w_{k,j} \right\}. \tag{8c}$$

Finally, define $\mathcal{X} = \big\{(\lambda, \mu) \in \mathbb{R}^2 \,\big|\, (9a) - (9d)\big\}$ *where*

$$w_{k,l}\lambda + w_{k,l}\mu \leq 0, \tag{9a}$$

$$w_{k,l}\lambda + (w_{k,l} - w_{i,l})\mu \leq Z^{k,l}_{i,0}, \qquad \forall i \in X, \tag{9b}$$

$$(w_{k,l} - w_{k,j})\lambda + w_{k,l}\mu \leq Z^{k,l}_{0,j}, \qquad \forall j \in Y, \tag{9c}$$

$$(w_{k,l} - w_{k,j})\lambda + (w_{k,l} - w_{i,l})\mu \leq Z^{k,l}_{i,j}, \qquad \forall (i,j) \in X \times Y. \tag{9d}$$

Then, for any extreme point $(\bar{\lambda}, \bar{\mu})$ *of* $\mathcal{X}$*, the inequality*

$$\bar{\lambda}(w_{k,l} - x_k) + \bar{\mu}(w_{k,l} - y_l) + \boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma, \tag{10}$$

defines a face of $\mathcal{PS}^{n,m}_{X\cup\{k\},Y\cup\{l\}}$ *of dimension at least* $d+2$. □

The lifting problems described in Propositions 8 and 9 require the solution of a polynomial number of optimization problems with a linear objective function over a single node problem with NSNM requirement. We next show that optimizing a linear function over $\mathcal{PS}^{n,m}$ is equivalent to solving an assignment problem, and hence it can be solved in strongly polynomial time; see [28].

Proposition 10. *The optimization problems*

$$(\mathtt{Q}): \quad \psi^* = \min\left\{ \sum_{i\in N} \alpha_i x_i + \sum_{j \in M} \beta_j y_j \,\middle|\, (\boldsymbol{x};\boldsymbol{y}) \in \mathcal{PS}^{n,m} \right\},$$

$$(\bar{\mathtt{Q}}): \quad \bar{\psi}^* = \min\left\{ \sum_{i\in N}\sum_{j \in M} (\alpha_i + \beta_j) w_{i,j} z_{i,j} \,\middle|\, \begin{array}{ll} \sum_{j\in M} z_{i,j} \leq 1, & \forall i \in N \\ \sum_{i \in N} z_{i,j} \leq 1, & \forall j \in M \\ z_{i,j} \in \{0,1\}, & \forall (i,j) \in N \times M \end{array} \right\},$$

are such that $\psi^* = \bar{\psi}^*$. *Further, if* $\boldsymbol{z}^*$ *is an optimal solution of* $(\bar{\mathtt{Q}})$, *then* $(\boldsymbol{x}^*;\boldsymbol{y}^*)$ *is an optimal solution of* $(\mathtt{Q})$ *where* $x^*_i = \sum_{j\in M} w_{i,j} z^*_{i,j}$ *for* $i \in N$, *and* $y^*_j = \sum_{i \in N} w_{i,j} z^*_{i,j}$ *for* $j \in M$. *Conversely, if* $(\boldsymbol{x}^*;\boldsymbol{y}^*)$ *is an optimal extreme point of* $(\mathtt{Q})$ *with matching* $\mathcal{M}^*$, *then* $\boldsymbol{z}^*$ *is an optimal solution of* $(\bar{\mathtt{Q}})$ *where* $z^*_{i,j} = 1$ *for* $(i,j) \in \mathcal{M}^*$, *and* $z^*_{i,j} = 0$ *for* $(i,j) \notin \mathcal{M}^*$. □

Propositions 8, 9 and 10 lead to polynomial algorithms to derive facet-defining inequalities for $\mathcal{PS}^{n,m}$. We first form restrictions of $\mathcal{PS}^{n,m}$ for which we can readily identify facet-defining inequalities (*e.g.* $\mathcal{PS}^{n,1}$, $\mathcal{PS}^{1,m}$ and $\mathcal{PS}^{2,2}$ studied in Section 2.2.2). We then lift these inequalities to obtain facet-defining inequalities for $\mathcal{PS}^{n,m}$ as recorded next.

Proposition 11. *Assume that* $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma$ *defines a facet of* $\mathcal{PS}^{n,m}_{X,Y}$ *for some* $[I,J,K]$. *Then, when applied to seed inequality* $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \leq \gamma$ *and independent of the ordering of elements of* I, J, *and* K, *the lifting procedures of Proposition 8 for* I *and* J *and of Proposition 9 for* K *yield a facet-defining inequality for* $\mathcal{PS}^{n,m}$. *Further, each such lifted inequality can be computed in* $\mathcal{O}(\zeta^3(|I| + |J| + |K|nm))$ *where* $\zeta = \max\{n,m\}$.

In Example 5, we illustrate the use of the lifting procedures introduced in this section on the set $\widehat{\mathcal{S}}^{3,3}$ of Example 3. In particular, we show that all nontrivial facet-defining inequalities of $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ can be obtained through lifting. This example indicates that the above lifting procedure is a powerful algorithmic tool to obtain a rich pool of facet-defining inequalities for $\mathcal{PS}^{n,m}$.

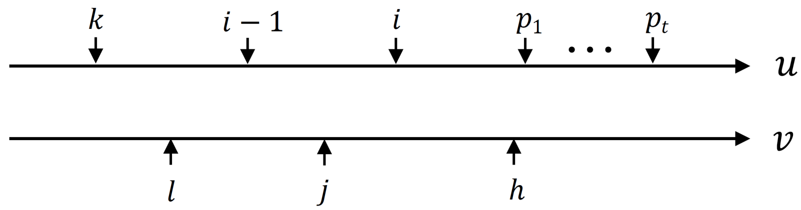

Figure 3: Graphical representation of the parameters of NSNM inequality (11).

2.2.4 Families of Closed-Form Facet-defining Inequalities.

From a practical perspective, even though the lifting problems of Section 2.2.3 can be solved in polynomial time for a given seed inequality and a given lifting order, the choices of variables fixing and lifting orders may lead to an exponential number of lifted inequalities. Separating these inequalities through enumeration would result in intense computational effort for large size problems. For this reason, we identify in this section a class of lifted inequalities that can be separated efficiently.

Before describing this class of facet-defining inequalities of $\mathcal{PS}^{n,m}$ in closed-form, we first give results about *trivial* inequalities, *i.e.*, constraints in the description of $\mathcal{LS}^{n,m}$. Proposition 12 shows that the balance constraint is the only equality holding for all solutions of $\mathcal{PS}^{n,m}$, and establishes that most trivial inequalities are facet-defining for $\mathcal{PS}^{n,m}$.

Proposition 12. *(i) It holds that* $\dim(\mathcal{PS}^{n,m}) = n + m - 1$.

(ii) Inequality $x_i \geq 0$ (resp. $y_j \geq 0$) for $i \in N$ (resp. $j \in M$) is facet-defining for $\mathcal{PS}^{n,m}$.

(iii) For any $i \in N$ (resp. $j \in M$), $x_i \leq u_i$ (resp. $y_j \leq v_j$) is facet-defining for $\mathcal{PS}^{n,m}$ if and only if $u_i \leq v_j$ (resp. $v_j \leq u_i$) for some $j \in M \setminus \{m\}$ (resp. $i \in N \setminus \{n\}$). □

When applied to Example 3, Proposition 12 shows that inequalities imposing lower bounds on all variables and those imposing upper bounds on variables x_1, x_2 and y_1 are facet-defining for $\widehat{\mathcal{PS}}^{3,3}$. These inequalities are labeled $(A1)$-$(A10)$ in Table 7. Proposition 12*(iii)* also shows that $x_3 \leq 13$, $y_2 \leq 11$ and $y_3 \leq 13$ are not facet-defining for $\widehat{\mathcal{PS}}^{3,3}$.

We next introduce a family of non-trivial valid inequalities for $\mathcal{PS}^{n,m}$ and give sufficient conditions for them to be facet-defining. A graphical representation of how the parameters of these inequalities relate to each other is given in Figure 3.

Proposition 13. *Let $j \in M$ and $i \in N \setminus \{1\}$ be such that $u_{i-1} \leq v_j \leq u_i$. Select $k \in \{1, \ldots, i-1\}$, and set $L = \emptyset$ or $L = \{l\}$ for some $l \in \{1, \ldots, j-1\}$ such that $u_k \leq v_l$. Also, select $P = \{p_1, \ldots, p_t\} \subseteq \{i, \ldots, n\}$ such that $p_1 < p_2 < \ldots < p_t$. If $P \neq \emptyset$, select $h \in \{j+1, \ldots, m\}$,* *otherwise select $h = m$. Then, the NSNM inequality*

$$\sum_{r\in R}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r + \sum_{r\in P}\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(x_r - w_{r,h})$$
$$+\sum_{s\in L}\left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,s}}\right)\left(\frac{w_{k+1,s}}{w_{k+1,j}}\right)(y_s - u_k) + \sum_{s\in H}\left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)y_s + y_j \le u_k, \quad (11)$$

where $R = \{k+1,\dots,n\} \setminus P$, $H = \{h+1,\dots,m\}$, and where all denominators are nonzero, is valid for $\mathcal{PS}^{n,m}$. Further, (11) is facet-defining for $\mathcal{PS}^{n,m}$ if any of the following conditions holds:

(i) $P = L = \emptyset$,

(ii) $P = \emptyset$, $L \neq \emptyset$, $k \le n-2$, *and* $m \ge 3$,

(iii) $L = \emptyset$, $P \neq \emptyset$, *and* $w_{p_f,h} = w_{p_f,q_f}$ *for* $f \in \{1,\dots,t\}$ *where* $q_f = h - t + f$ *and* $q_1 \ge j+1$,

(iv) $L \neq \emptyset$, $P \neq \emptyset$, $p_1 \ge k+2$, *and* $w_{p_f,h} = w_{p_f,q_f}$ *for* $f \in \{1,\dots,t\}$ *where* $q_f = h - t + f$ *and* $q_1 \ge j+1$. □

We originally obtained the NSNM inequality (11) through an application of the lifting procedures of Section 2.2.3. To give an illustration, we describe the steps for the case where $P = L = \emptyset$. In this case, we use $\left(\frac{u_k - w_{k+1,j}}{w_{k+1,j}}\right)x_{k+1} + y_j \le u_k$ as a seed inequality for the restriction of $\mathcal{S}^{n,m}$ that contains variables x_k, x_{k+1}, y_j and y_g for some $g \in M \setminus \{j\}$. This inequality is given in Proposition 7. Then, we use Proposition 8 to sequentially lift the projected variables. First, we lift x_r for $r \in \{1,\dots,k-1\}$ and obtain coefficients 0. Next, we lift x_r for $r \in \{k+2,\dots,n\}$ and obtain coefficients $\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)$. Finally, we lift y_s for $s \in M \setminus \{j, g\}$ and obtain coefficients 0. The resulting inequality is (11).

The procedure to obtain (11) for cases (ii)–(iv) is similar in that we first derive a seed inequality for a certain restriction, and then we sequentially lift it into the space of the fixed variables according to a certain ordering. While such derivation gives an intuitive understanding to the NSNM inequality (11), its steps are cumbersome. For this reason, the proof that we give utilizes a shorter case-by-case analysis.

The conditions under which Proposition 13 is established are not symmetric with respect to variables $\boldsymbol{x}$ and $\boldsymbol{y}$. Remark 1 then directly implies that another family of facet-defining inequalities can be derived by switching the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$. The resulting inequality, which we refer to as *reverse NSNM inequality* is described in Proposition 15.

Example 4. Consider the set $\widehat{\mathcal{PS}}^{3,3}$ of Example 3. Twenty-two out of the 35 non-trivial inequalities in the description of $\widehat{\mathcal{PS}}^{3,3}$ are NSNM and reverse NSNM inequalities. A complete list of these inequalities together with the choice of indices that explains them can be found in Table 9.

NSNM inequalities are exponential in number because they allow for any subset P of $\{i,\dots,n\}$ to be chosen. However, since indices not included in P are included in R, deciding whether to include an index in P or in R to generate the largest violation value can be done in polynomial time, the following separation result is easily established.

Proposition 14. *Assume that* $(\boldsymbol{x}^*; \boldsymbol{y}^*)$ *violates an inequality of the form* (11) *for some indices* (i, j, k, l, h) *and some subset* P *of* $\{i, \ldots, n\}$ *that satisfy the conditions of Proposition 13. Then, one such inequality can be found in time* $\mathcal{O}\left(n^2 m^2 (n + m)\right)$. □

As discussed before, NSNM inequalities belong to the families of lifted inequalities described in Section 2.2.3. They have the advantages that separation over the entire family is simple and that, according to our computational experiments, they contribute to a significant portion of gap reduction. These characteristics make NSNM inequalities valuable when tackling problems arising in applications where networks are large. We support these assertions computationally in Section 3.

3 Computational Experiments

In this section, we perform a computational study to evaluate, on a model that prominently displays the role of NSNM requirements, the different formulations and cutting planes we studied.

3.1 Problem Description

First we describe a streamlined model that captures the salient features of the unit train scheduling. We refer to this problem as generalized unsplittable flow problem (GUFP). This model generalizes the single-commodity UFP studied in the literature (see Section 1.2) because *(i)* it contains nodes with and without NSNM requirements, *(ii)* demands can be satisfied by several edge-disjoint paths, and *(iii)* objective coefficients associated with arc flows are not restricted to be zero or one. GUFP also generalizes the single-commodity variant of the k-splittable flow problem introduced in [4], which is NP-Hard.

Consider directed graph $G'(V', E')$ that provides a (possibly time-expanded) representation of the railroad, where nodes in V' correspond to yards and arcs in E' correspond to possible movements of trains between them. Nodes $i \in S \subseteq V'$ have available car supplies in an amount we denote by $c_i \in \mathbb{Z}_+$. Similarly, nodes $j \in D \subseteq V'$ have car demands in an amount we denote by $c_j \in \mathbb{Z}_+$. All available supplies need not be shipped whereas all demands must be met. Nodes in the network either correspond to build locations where necessary equipment is available to resize trains or correspond to traditional yards where trains can be transshipped without being resized. We denote the set of nodes that must satisfy the NSNM requirement as $\bar{V}$. Associated to arc $(i, j) \in E'$, we define $u_{i,j} \in \mathbb{Z}_+$ and $l_{i,j} \in \mathbb{Z}_+$ to be the maximum and minimum number of cars that can be assigned to a train on this leg of its trip. Finally, we let $r_{i,j} \in \mathbb{R}$ be the reward associated with sending a car on arc (i, j). Given G', we construct a directed graph $G(V, E)$ for our model as follows. First, we let $V = V' \cup \{s_0, s, t\}$, where s and t are the source and sink nodes, and s_0 is an artificial supply node that we use to guarantee that the model is feasible and all demands are met. All nodes in $\bar{V}$ are subject to NSNM. Arc set E is obtained by adding arcs (s, i) for $i \in S$, and arcs (j, t) and (s_0, j) for $j \in D$ to the arc set E'. We represent the set of nodes that have a directed arc into node j by V_j^-. Similarly, we represent the set of nodes towards which node j has a directed arc by V_j^+. Each arc (i, j) is associated with a triple $(l_{i,j}, u_{i,j}, r_{i,j})$ whose components

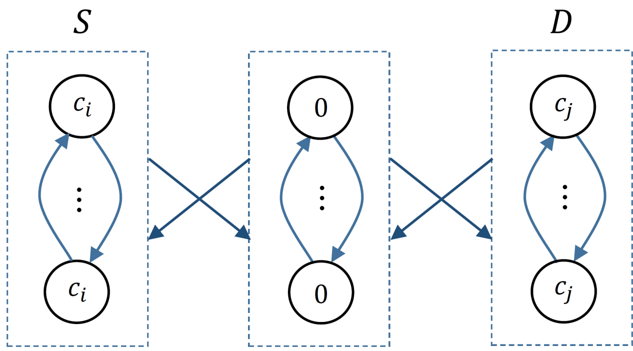

Figure 4: An instance of $G'(V', E')$.

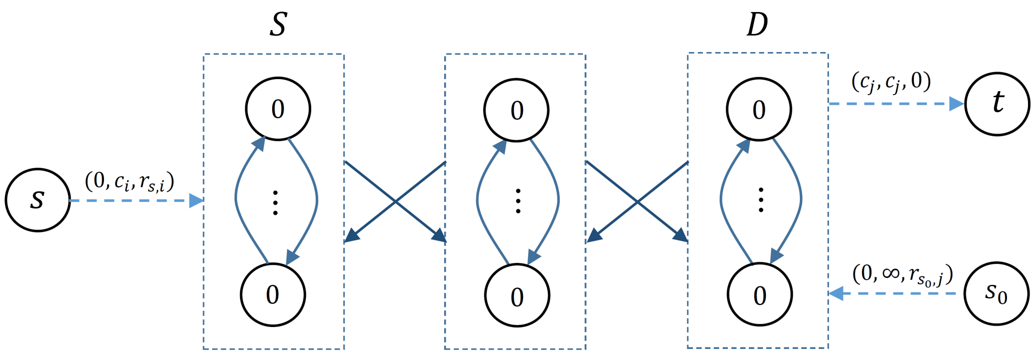

Figure 5: An instance of $G(V, E)$ created from $G'(V', E')$ of Figure 4.

describe lower bound, upper bound and reward. These triples are $(0, c_i, r_{s,i})$ for arcs (s, i) where $i \in S$, $(c_j, c_j, 0)$ for arcs (j, t) where $j \in D$ and $(0, \infty, r_{s_0,j})$ for (s_0, j) where $j \in D$. Rewards $r_{s,i}$ are usually positive to represent incentive for transporting available supplies, while rewards $r_{s_0,j}$ are negative to represent the cost of lost demand. For other arcs, these values are inherited from the original network G'. GUFP is the problem of finding a flow of maximum reward in G that satisfies all NSNM requirements. A graphical illustration of graphs $G'(V', E')$ and $G(V, E)$ is given in Figures 4 and 5, respectively. Observe that situations where multiple trains are allowed to be sent through an arc can be modeled similarly by creating multiple replicas of that arc.

We next present an MIP formulation for GUFP. This model has two sets of decision variables. Integer variables $x_{i,j}$ represent the flow from node i to node j through arc (i, j). Binary variables $z^j_{i_1,i_2}$ equal 1 if the flow entering node $j \in \bar{V}$ through arc (i_1, j) leaves node j through arc (j, i_2).

Using these variables, GUFP can be formulated as:

$$\max \quad \sum_{i \in V} \sum_{j \in V_i^+} r_{i,j} x_{i,j} \tag{12a}$$

$$\text{s.t.} \quad \sum_{i \in V_j^-} x_{i,j} - \sum_{i \in V_j^+} x_{j,i} = 0, \qquad \forall j \in V', \tag{12b}$$

$$l_{i,j} \le x_{i,j} \le u_{i,j}, \qquad \forall i \in V, j \in V_i^+, \tag{12c}$$

$$x_{i_1,j} - x_{j,i_2} \le u_{i_1,j}(1 - z^j_{i_1,i_2}), \qquad \forall j \in \bar{V}, (i_1, i_2) \in V_j^- \times V_j^+, \tag{12d}$$

$$x_{j,i_2} - x_{i_1,j} \le u_{j,i_2}(1 - z^j_{i_1,i_2}), \qquad \forall j \in \bar{V}, (i_1, i_2) \in V_j^- \times V_j^+, \tag{12e}$$

$$x_{i_1,j} \le u_{i_1,j} \sum_{i_2 \in V_j^+} z^j_{i_1,i_2}, \qquad \forall j \in \bar{V}, i_1 \in V_j^-, \tag{12f}$$

$$x_{j,i_2} \le u_{j,i_2} \sum_{i_1 \in V_j^-} z^j_{i_1,i_2}, \qquad \forall j \in \bar{V}, i_2 \in V_j^+, \tag{12g}$$

$$\sum_{i_2 \in V_j^+} z^j_{i_1,i_2} \le 1, \qquad \forall j \in \bar{V}, i_1 \in V_j^-, \tag{12h}$$

$$\sum_{i_1 \in V_j^-} z^j_{i_1,i_2} \le 1, \qquad \forall j \in \bar{V}, i_2 \in V_j^+, \tag{12i}$$

$$z^j_{i_1,i_2} \in \{0, 1\}, \qquad \forall j \in \bar{V}, (i_1, i_2) \in V_j^- \times V_j^+. \tag{12j}$$

In the above formulation, (12b) is the flow balance equation at each node of V'. This constraint implicitly satisfies the supply/demand requirements because of the addition of arcs connecting the supply/demand nodes to the source and sink, and because of the capacity of these arcs. Constraint (12c) limits the number of cars that can be sent through trains on each arc $(i, j) \in E$. Constraints (12d)–(12j) model the NSNM requirement analogously to inequalities (1a)–(1i) in Section 2.1.

3.2 Test Instances

We next describe the families of randomly generated instances of model (12a)–(12j) on which we perform our computational experiments. We consider networks $G(V \cup \{s_0, s, t\}, E)$, where s and t are source and sink nodes, and where s_0 is an artificial supply node. The three families of instances we consider have 500, 1000 and 2000 nodes, respectively. Each family is labeled with its associated number of nodes, and contains 5 different instances. We generate the instance specifics in a way that is reminiscent of the unit train network structure of existing railroads. In particular, we select supply and demand node counts to be 10% and 50%, respectively, of the problem size. Supply nodes correspond to car pool locations where required machinery and manpower to form a train is available. The NSNM property does not have to be satisfied at these nodes. Demand nodes, however, must satisfy the NSNM requirement. Further, we randomly generate hub nodes (build locations) throughout the network that do not need to respect the NSNM requirement. We restrict the number of these nodes to be between 5% and 10% of the problem size. All other nodes are subject to NSNM requirements.

For each node in V, we randomly generate a number of outgoing arcs between 5 and 10. We then assign car capacities to each of these arcs to reflect actual train sizes. In particular, each arc is given a car capacity that is an integer value randomly selected from the discrete uniform distribution $[5, 15]*20$. The supply at supply nodes is randomly generated from the discrete uniform distribution $[5, 30]*20$, and is imposed as the car capacity of arcs connecting the source to these nodes. Similarly, the demand at demand nodes is randomly selected from the discrete uniform distribution $[5, 10]*20$, and is imposed as the lower and upper car capacities of arcs connecting these nodes to the sink. The reward associated with arcs connecting the source to supply nodes is randomly selected from the discrete uniform distribution $[5, 10]$. This represents the gain for using supply nodes to meet unit car demands. The reward associated with arcs connecting the artificial supply node to demand nodes is randomly generated from the discrete uniform distribution $[-10, -5]$. This represents the cost of not satisfying the demand for a car at that particular node. The arcs connecting the demand nodes to the sink have zero reward as the flow on such arcs is fixed. All remaining nodes have rewards selected randomly from the discrete uniform distribution $[-2, 2]$. Positive values represent incentives for transporting flows through arcs experiencing light traffic, while negative values represent costs associated with using congested arcs.

3.3 Numerical Results

We evaluate the quality of bounds obtained from the different formulations, the CGLP and the cutting planes described in Section 2. These results are obtained on a Windows 8 (64-bit) operating system, 8 GB RAM, 2.20 GHz Core i7 CPU. Codes are written in VC++ and optimization models are solved with CPLEX 12.5. For all instances where cutting planes are implemented, the LPs are resolved each time a cut is added. Further, we impose a time limit of 5000 seconds for all computations. When the time limit is reached without achieving optimality, the process is aborted and the best bound at termination is reported. We indicate that the time limit is reached in the tables using the symbol *. Tables 1–3 show the relative strength of the CGLP and the NSNM inequalities with respect to various formulations of the NSNM model. Table 4 shows the impact of the NSNM inequalities in improving the bounds achieved by CPLEX.

We also implemented a solution approach to GUFP that is based on a path formulation of the problem. A detailed description of the model and of our implementation is given in Section C. We use the same computational set-up as above, and impose a time limit of 4500 seconds for the solution of the LP relaxation of the path formulation. Because this model has a large number of variables, we use column generation. At termination of the column generation algorithm, we construct an MIP model of GUFP based on the obtained columns and we run CPLEX for a maximum of 500 seconds to obtain a heuristic feasible solution. Results are reported in Table 6. We observe that the column generation approach produces good-quality feasible solutions, especially for smaller networks. However, for our instances, it fails to terminate within the alloted time and therefore does not provide directly usable upper-bounds.

The first column of Table 1 contains the number of nodes in the considered network. The column "LP" contains the optimal value of the LP relaxation of the GUFP formulation in the space of variables $\boldsymbol{x}$, described by (12a)–(12c). The next column shows the time (in seconds) to solve this relaxation. The column "Ideal" contains the bound obtained using $\mathcal{LD}^{n,m}$ for each node with NSNM requirement. This formulation is obtained by introducing nm variables for each node having n incoming and m outgoing arcs. As described in Section 2.1, $\mathcal{LD}^{n,m}$ gives the convex hull of $\mathcal{S}^{n,m}$ in a higher dimension, and therefore improves the bound of the column "LP" by convexifying NSNM requirements for all single node substructures. We refer to the difference between the values in column "LP" and those in column "Ideal" as the total gap $\Delta := v^{\mathrm{LP}} - v^{\mathrm{Ideal}}$. The column labeled "Ext" gives the percentage of the total gap Δ that is closed by the LP relaxation of the natural extended formulation of NSNM described by (12a)–(12i). This percentage is calculated as $(v^{\mathrm{LP}} - v^{\mathrm{Ext}})/\Delta$. The next column presents the time to solve this model. The column labeled "CGLP" shows the percentage of the total gap Δ that is closed by using CGLP for nodes with NSNM requirements; see Section 2.2. This percentage is calculated as $(v^{\mathrm{LP}} - v^{\mathrm{CGLP}})/\Delta$. CGLP cuts are added one at a time to the LP relaxation (12a)–(12c) for each node individually (in the node index order) until the bound improvement between iterations becomes negligible. Once all nodes are visited, we reset the cut-generating procedure starting from the first node. We repeat this procedure until the bound improvement over all nodes is negligible. Note that the gap closure is 100% for these cuts, as they are obtained from the projection of the model $\mathcal{LD}^{n,m}$ of column "Ideal". The next column shows the time needed to generate these cuts. The column labeled "NSNM" contains the percentage of the total gap Δ that is closed by adding violated NSNM and reverse NSNM inequalities to the LP relaxation (12a)–(12c). This percentage is calculated as $(v^{\mathrm{LP}} - v^{\mathrm{NSNM}})/\Delta$. The NSNM inequalities are generated as follows. For each node with NSNM requirement, we add the most violated NSNM inequality using the separation method of Proposition 14. We generate these inequalities for $P = \emptyset$. As a result, the separation can be performed in $\mathcal{O}(m^2 n)$ for NSNM inequalities and $\mathcal{O}(n^2 m)$ for reverse NSNM inequalities. For each given node, we repeat adding violated cuts until the bound improvement between iterations becomes negligible. Once all nodes are visited, we reset the cut-generating procedure starting from the first node. We repeat this procedure until the bound improvement over all nodes is negligible. The next column represents the time required to generate these cuts.

Next, we investigate improvement in bounds that can be obtained by aggregating multiple nodes with NSNM requirements. Table 2 is defined similarly to Table 1 with the difference that it considers models where NSNM requirements are imposed for each aggregation of pairs of adjacent nodes with NSNM requirements. Similarly, Table 3 evaluates the bounds obtained by considering all aggregations of triples of connected nodes that satisfy the NSNM requirement. Blank values in this table indicate that the model could not be solved because of memory errors. For this reason, we replace the column "Ideal" with the column "CGLP" as lower bound on the total gap.

We observe in Tables 1–3 that considering different aggregations of nodes with NSNM property can significantly improve the bound obtained from LP relaxations. Among the formulations we introduced in this paper, we observe that the column "Ideal" provides the best bound at the additional cost of introducing new variables. As a result, for families of larger sizes and for more complicated aggregation of nodes, this formulation requires large solution times. In addition, this formulation cannot be used in branch-and-bound. In contrast, using CGLP and/or NSNM inequalities has the advantage of keeping the model in the space of original variables $\boldsymbol{x}$, and would allow for the use of traditional branch-and-cut techniques. The CGLP approach can be very expensive as demonstrated in Tables 1–3 for problems of larger sizes. In contrast, NSNM inequalities provide a bound that can be generated fast and still achieves a substantial gap reduction. We conclude that such inequalities are promising for use inside branch-and-cut for large problem instances and provide the best trade-off between bound improvement and solution time. It is also clear that both approaches could be used in combination.

In Table 4, we evaluate the strength of the NSNM inequalities by comparing the bound they yield with that of CPLEX at the root node. The column "Best LB" contains the best lower bound (integer solution) obtained by CPLEX after 5000 seconds running time, or the value of the heuristic solution obtained through column generation within the same time, whichever is largest; see Table 6 for a comparison. The column "CPX UB" represents the best upper bound obtained by CPLEX at the root node in its default setting. The next column shows the number of cuts added by CPLEX at the root node before the branch-and-bound is started. The following column represents the time CPLEX takes at root node to compute the bound. The column "NSNM" denotes the bound obtained by adding NSNM inequalities to the LP relaxation as follows. First, we add the violated NSNM inequalities for single nodes with NSNM property in a manner similar to that described in Table 1. When the total bound improvement becomes negligible, we add violated NSNM inequalities for aggregation of two nodes with the NSNM property as described in Table 2. Finally, we add the violated NSNM inequalities for aggregation of three nodes with the NSNM property as described in Table 3. Confirming the discussion of Example 1, we observe that adding NSNM cuts for different aggregation of nodes leads to improved bounds compared to each of the individual aggregations. The next two columns show the number of added NSNM inequalities and the time to generate these cuts, respectively. The column "Gap 1" contains a lower bound on the percentage of the total gap $\Delta_1 := v^{\text{LP}} - v^{\text{Best LB}}$ that is closed by NSNM inequalities. This value is calculated as $(v^{\text{LP}} - v^{\text{NSNM}})/\Delta_1$. The column "Gap 2" contains a lower bound on the percentage of the total gap $\Delta_2 := v^{\text{CPX UB}} - v^{\text{Best LB}}$ that is closed by NSNM inequalities. This value is calculated as $(v^{\text{CPX UB}} - v^{\text{NSNM}})/\Delta_2$. As observed in this table, NSNM inequalities provide better bounds than CPLEX at the root node, are generated in smaller number, and are separated faster than CPLEX cuts. This indicates the computational potential of these cuts in the solution of practical problems with NSNM requirements. For instance, the model tightened by adding NSNM inequalities is much smaller in size than that obtained by adding CPLEX cuts at the root node, which can lead to faster branch-and-bound.

Our computational results suggest that NSNM inequalities obtained for different node aggregations contribute to a significant gap reduction. As future work, it should be useful to investigate multi-node structures beyond those we considered. Further, since NSNM inequalities are defined in the space of variables $\boldsymbol{x}$, they can be directly added in the space of original variables without the need to introduce binary variables $\boldsymbol{z}$ in the model described by (12a)–(12j). To take full advantage of this feature inside of a branch-and-bound framework, we need to develop a mechanism to impose and refine the NSNM requirement through branching. NSNM inequalities could also be applied down the branch-and-bound tree as local cuts. In particular, if it is known for a node with NSNM property that incoming flow x_i is matched to outgoing flow y_j, then the variables x_i and y_j can be removed to reduce the size of the corresponding single node problem. NSNM inequalities can be generated for this modified structure. Further, the generated local cuts can be lifted using the procedures of Section 2.2.3 to obtain globally valid inequalities.

Table 1: Evaluating formulations for single node relaxations.

N	#	LP	T	Ideal	T	Ext %	T	CGLP %	T	NSNM %	T
500	1	86400	0.1	76134.5	0.4	45.0	2.8	100.0	40	95.4	1.1
	2	107520	0.1	97536.8	0.6	37.2	5.1	100.0	48.1	92.6	1.1
	3	104620	0.1	92679.3	0.5	53.5	1.6	100.0	40.1	95.6	1.2
	4	136520	0.1	123221.3	1	42.3	2.1	100.0	41.2	95.1	1.3
	5	133040	0.1	118552.5	0.5	49.0	1.7	100.0	41.6	93.1	1.1
1000	1	234100	0.1	209655	1.4	44.0	6.7	100.0	104.3	96.4	14.9
	2	203620	0.2	180201.7	1.7	36.4	10.8	100.0	121.4	94.0	13.5
	3	226860	0.2	205725	1.6	48.2	9.4	100.0	86.2	95.0	11.9
	4	226680	0.1	204504.3	1.3	43.3	7.3	100.0	113.5	93.5	9.7
	5	222900	0.1	204350.4	1.2	30.1	6	100.0	101.4	93.3	9.6
2000	1	387300	0.3	341291.9	5.2	46.7	25.4	100.0	254.6	94.7	41.9
	2	387820	0.3	346400.8	5.4	45.6	23.9	100.0	274.9	95.5	38.3
	3	396480	0.3	348775.8	3.8	43.9	12.2	100.0	224.1	95.0	30.6
	4	414000	0.2	363884.1	3.5	41.8	20.3	100.0	223.1	94.2	29.3
	5	397260	0.3	356014.8	3.9	38.7	24.3	100.0	1261.9	95.2	26.8

Table 2: Evaluating formulations for two-node relaxations.

N	#	LP	T	Ideal	T	Ext %	T	CGLP %	T	NSNM %	T
500	1	86400	0.1	72332	18.5	12.0	139.8	100.0	136.9	91.9	3.6
	2	107520	0.1	94938.7	27.2	8.3	166.7	100.0	122.4	91.7	4.8
	3	104620	0.1	91832.1	13.9	13.4	88.5	100.0	93.1	92.2	3.8
	4	136520	0.1	121581.1	21.2	10.2	141.2	100.0	123.1	90.9	4.5
	5	133040	0.1	117621.8	26.3	13.0	155.3	100.0	130.4	90.3	4.1
1000	1	234100	0.1	204502.6	121.3	9.8	3342.4	100.0	414	88.7	28.8
	2	203620	0.1	174309.3	139.4	11.8	3847.1	100.0	468.8	90.0	32
	3	226860	0.1	202049.5	74.4	12.5	2358.5	100.0	396.4	88.0	30
	4	226680	0.1	200427.6	116	13.8	2241.4	100.0	416.9	91.2	30
	5	222900	0.1	197916.6	63.3	10.6	1603.5	100.0	381.4	88.7	27
2000	1	387300	0.4	336595	445.7	10.6	$5000^\star$	100.0	1237.6	88.8	199
	2	387820	0.3	334288	773	13.6	$5000^\star$	100.0	1325.4	89.8	218.8
	3	396480	0.3	338251.1	1426.6	12.2	$5000^\star$	100.0	1467.5	88.2	221.9
	4	414000	0.2	355732.2	979.7	10.6	$5000^\star$	100.0	1174.1	88.5	134.1
	5	397260	0.3	343455.6	692.8	10.9	$5000^\star$	100.0	864.9	89.9	123.6

Table 3: Evaluating formulations for three-node relaxations.

N	#	LP	T	CGLP	T	Ideal %	T	Ext %	T	NSNM %	T
500	1	86400	0.1	70459.2	1902.5	100.0	2718	–	–	88.6	34.5
	2	107520	0.1	92208.8	962.8	100.0	1792.3	–	–	87.2	43.7
	3	104620	0.1	89985.9	923.5	100.0	620.1	–	–	84.0	25.2
	4	136520	0.1	120045.5	1456.2	100.0	1371	–	–	88.7	39.1
	5	133040	0.1	115016.4	1032.7	100.0	2106.7	–	–	82.7	30.3
1000	1	234100	0.2	199771	4072.8	–	–	–	–	86.0	133.7
	2	203620	0.2	168691.7	5000*	–	–	–	–	85.4	267.8
	3	226860	0.2	197109.5	5000*	–	–	–	–	82.6	229.5
	4	226680	0.2	195703	4633.4	–	–	–	–	87.0	171.8
	5	222900	0.2	193793.1	3947	–	–	–	–	84.5	150.2
2000	1	387300	0.3	329545	5000*	–	–	–	–	85.0	629.2
	2	387820	0.4	326936.2	5000*	–	–	–	–	88.2	863.1
	3	396480	0.3	329950.9	5000*	–	–	–	–	86.8	1042.4
	4	414000	0.3	346372.5	5000*	–	–	–	–	87.1	897.7
	5	397260	0.3	335167.2	5000*	–	–	–	–	86.3	693.4

Table 4: Evaluating the NSNM inequalities.

N	#	LP	Best LB	CPX UB	Cut	T	NSNM	Cut	T	Gap 1 %	Gap 2 %
500	1	86400	67800	74103.8	6422	24.3	71403.6	443	14.5	80.6	42.8
	2	107520	89400	97238.6	6800	30	93403.3	543	20.5	77.9	48.9
	3	104620	87580	92171.4	4822	21.1	89579.9	489	17	88.3	56.4
	4	136520	113800	123588.5	5458	31	119074.9	518	16.5	76.8	46.1
	5	133040	112820	117003.6	5606	28.7	114469.9	465	16	91.8	60.6
1000	1	234100	185760	208229	12226	114.2	199885.9	1075	87.6	70.8	37.1
	2	203620	136640	178081.2	13014	152.7	169141.4	1097	101.3	51.5	21.6
	3	226860	185980	202714.8	12082	111.9	197571	977	103.7	71.6	30.7
	4	226680	176960	203195.1	12418	129.8	195572.8	1091	107.5	62.6	29.1
	5	222900	182360	202379.2	11476	99.3	196266.2	906	64.4	65.7	30.5
2000	1	387300	291940	339348.3	22284	514.1	328324.7	1888	157.2	61.8	23.3
	2	387820	259040	339411.1	23760	654.9	326491	2053	170.6	47.6	16.1
	3	396480	242300	344351.3	26374	801.6	329538.5	2246	187.7	43.4	14.5
	4	414000	253780	360231	25232	720.5	344749.3	2198	197.8	43.2	14.5
	5	397260	290900	351831.9	22494	621	336964.7	2007	170.4	56.7	24.4

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Appendices

A Proofs

In this section, we provide proofs omitted from the paper.

A.1 Proof of Proposition 1

Proof. Because the model is symmetric with respect to variables $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$, we show the result only for $\bar{x}_i$ where $i \in N$. Let $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}})$ be an extreme point of $\mathcal{PS}^{n,m}$. We may assume that $\bar{x}_i > 0$, otherwise the result holds trivially. Since $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) \in \mathcal{S}^{n,m}$ as it is an extreme point of $\mathcal{PS}^{n,m}$, the NSNM requirement implies that $\bar{x}_i = \bar{y}_j > 0$ for some $j \in M$. We next show that $\bar{x}_i = \bar{y}_j = w_{i,j}$. Clearly, $\bar{x}_i = \bar{y}_j \leq w_{i,j}$. Assume by contradiction that $0 < \bar{x}_i = \bar{y}_j < w_{i,j}$. Then, for a sufficiently small but positive ϵ, the points $(\hat{\boldsymbol{x}};\hat{\boldsymbol{y}}) = (\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) + (\epsilon \boldsymbol{e}^i;\epsilon \boldsymbol{e}^j)$ and $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}}) = (\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) - (\epsilon \boldsymbol{e}^i;\epsilon \boldsymbol{e}^j)$ belong to $\mathcal{S}^{n,m}$. This is a contradiction to the fact that $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}})$ is an extreme point of $\mathcal{PS}^{n,m}$ as $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) = \frac{1}{2}(\hat{\boldsymbol{x}};\hat{\boldsymbol{y}}) + \frac{1}{2}(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$. □

A.2 Proof of Proposition 3

Proof. We obtain the result in 3 steps.

(1) We argue that $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\,\mathcal{PD}^{n,m} = \text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\,\mathcal{PT}^{n,m}$. We show the first inclusion by establishing that $\mathcal{D}^{n,m} \subseteq \mathcal{T}^{n,m}$. Consider $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{z}) \in \mathcal{D}^{n,m}$. This point satisfies the flow balance constraint as this constraint is redundant in the description of $\mathcal{D}^{n,m}$. This point also satisfies (1e)–(1i) as these inequalities are exactly (2c)–(2f) in the description of $\mathcal{D}^{n,m}$. Further, it follows from (2a) that $x_i \geq 0$ as $w_{i,j} \geq 0$ and $z_{i,j} \geq 0$, for each $(i,j) \in N \times M$. Also, (2a) implies that $x_i \leq u_i \sum_{j \in M} z_{ij}$ since $0 \leq w_{i,j} \leq u_i$, for each $(i,j) \in N \times M$. A similar argument for y_j shows that $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{z})$ satisfies (1c) and (1d). Finally, for each $(k,l) \in N \times M$, a linear combination of (2a) and (2b) with weights 1 and -1, respectively, yields $x_k - y_l = \sum_{j \in M} w_{k,j} z_{k,j} - \sum_{i \in N} w_{i,l} z_{i,l}$. We then write that

$$x_k - y_l = \sum_{j \in M \setminus \{l\}} w_{k,j} z_{k,j} - \sum_{i \in N \setminus \{k\}} w_{i,l} z_{i,l} \leq \sum_{j \in M \setminus \{l\}} w_{k,j} z_{k,j} \leq u_k \sum_{j \in M \setminus \{l\}} z_{k,j} \leq u_k (1 - z_{k,l}),$$

where the equality holds because $w_{k,l} z_{k,l}$ cancels out between the two sums, the first inequality follows from the fact that $w_{i,l}$ and $z_{i,l}$ are nonnegative, for each $i \in N$, the second inequality holds since $w_{k,j} \leq u_k$, and the last inequality follows from (2c). This shows that (1a) is satisfied. A similar argument shows that $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{z})$ satisfies (1b). For the reverse inclusion, we show that any extreme solution of $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\,\mathcal{PT}^{n,m}$ belongs to $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\,\mathcal{PD}^{n,m}$. This is sufficient since $\mathcal{PT}^{n,m}$ is a polytope. Select an extreme point $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$ of $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})}\,\mathcal{PT}^{n,m}$. Then, there exists $\tilde{\boldsymbol{z}} \in \mathbb{R}^{nm}$ such that $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}};\tilde{\boldsymbol{z}})$ is an extreme point of $\mathcal{PT}^{n,m}$. It follows that $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}};\tilde{\boldsymbol{z}}) \in \mathcal{T}^{n,m}$. Next, construct the point $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}};\hat{\boldsymbol{z}})$ where, for any $(i,j) \in N \times M$, $\hat{z}_{i,j} = 0$

if either $\tilde{x}_i = 0$ or $\tilde{y}_j = 0$, and $\hat{z}_{i,j} = \tilde{z}_{i,j}$ otherwise. We next show that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \hat{\boldsymbol{z}}) \in \mathcal{D}^{n,m}$. This point satisfies (2c)–(2f) as $\tilde{\boldsymbol{z}}$ satisfies these inequalities, and $0 \le \hat{z}_{i,j} \le \tilde{z}_{i,j}$ for any $(i,j) \in N \times M$. It remains to show that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ satisfies (2a) as the argument for (2b) is similar. Select any $k \in N$. There are two cases. Assume first that $\tilde{z}_{k,j} = 0$ for all $j \in M$. Then, (1c) implies that $\tilde{x}_k = 0$, which yields $\hat{z}_{k,j} = 0$ for all $j \in M$. Therefore, $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ satisfies (2a) for k. Assume second that $\tilde{z}_{k,l} = 1$ for some $l \in M$, and $\tilde{z}_{k,j} = 0$ for all $j \in M \backslash \{l\}$. Then, it follows from (1a)–(1d) in the description of $\mathcal{T}^{n,m}$ that $0 \le \tilde{x}_k = \tilde{y}_l \le w_{k,l}$. We next argue that either *(i)* $\tilde{x}_k = \tilde{y}_l = w_{k,l}$, or *(ii)* $\tilde{x}_k = \tilde{y}_l = 0$. Assume by contradiction that $0 < \tilde{x}_k = \tilde{y}_l < w_{k,l}$. Construct the points of $\mathcal{T}^{n,m}$, $(\dot{\boldsymbol{x}}; \dot{\boldsymbol{y}}; \dot{\boldsymbol{z}}) = (\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \tilde{\boldsymbol{z}}) + (\epsilon \boldsymbol{e}^k; \epsilon \boldsymbol{e}^l; \boldsymbol{0})$ and $(\ddot{\boldsymbol{x}}; \ddot{\boldsymbol{y}}; \ddot{\boldsymbol{z}}) = (\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \tilde{\boldsymbol{z}}) - (\epsilon \boldsymbol{e}^k; \epsilon \boldsymbol{e}^l; \boldsymbol{0})$ for a sufficiently small but positive ϵ. The fact that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \tilde{\boldsymbol{z}}) = \frac{1}{2}(\dot{\boldsymbol{x}}; \dot{\boldsymbol{y}}; \dot{\boldsymbol{z}}) + \frac{1}{2}(\ddot{\boldsymbol{x}}; \ddot{\boldsymbol{y}}; \ddot{\boldsymbol{z}})$ contradicts the assumption that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \tilde{\boldsymbol{z}})$ is an extreme point of $\mathcal{PT}^{n,m}$. For *(i)* it follows from construction that $\tilde{z}_{k,j} = \hat{z}_{k,j}$ for all $j \in M$. As a result, $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ satisfies (2a) for k. For *(ii)* it follows from construction that $\hat{z}_{k,j} = 0$ for all $j \in M$, which shows that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}}; \hat{\boldsymbol{z}})$ satisfies (2a) for k.

(2) We argue that $\mathcal{LD}^{n,m} = \mathcal{PD}^{n,m}$. Let $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{z})$ be an extreme point of $\mathcal{LD}^{n,m}$. We show that $\boldsymbol{z} \in \{0,1\}^{nm}$. Since $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{z})$ is an extreme point of $\mathcal{LD}^{n,m} \subseteq \mathbb{R}^{n+m+nm}$ at least nm inequalities among (2c)–(2e) are binding at $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{z})$ as the $n+m$ constraints (2a) and (2b) are tight. As a result, $\boldsymbol{z}$ is an extreme solution to (2c)–(2e). It is easily verified that $\boldsymbol{z} \in \{0,1\}^{nm}$ as the coefficient matrix associated with constraints (2c)–(2e) is totally unimodular.

(3) We write that $\text{proj}_{(\boldsymbol{x};\boldsymbol{y})} \mathcal{LD}^{n,m} = \text{proj}_{(\boldsymbol{x};\boldsymbol{y})} \mathcal{PD}^{n,m} = \text{proj}_{(\boldsymbol{x};\boldsymbol{y})} \mathcal{PT}^{n,m} = \mathcal{PS}^{n,m}$, where the first equality follows form part (2), the second equality holds because of part (1) and the last equality follows from Proposition 2.

□

A.3 Proof of Proposition 5

Proof. The direct inclusion is clear as $\mathcal{PS}^{n,m} \subseteq \mathcal{RS}^{n,m}$. For the reverse inclusion, consider without loss of generality an instance where all bounds on variables equal 1. This set is a unit hypercube intersected with the balance equation. Its extreme points have 0 or 1 components. Therefore, the balance equation implies that these extreme points satisfy the NSNM requirement. We conclude that $\mathcal{RS}^{n,m} \subseteq \mathcal{PS}^{n,m}$. □

A.4 Proof of Proposition 6

Proof. It follows from Proposition 1 that the extreme points of $\mathcal{PS}^{n,1}$ are $(\boldsymbol{x}; y_1) = (\boldsymbol{0}; 0)$ together with $(u_i \boldsymbol{e}^i, u_i)$ for $i \in N$. Projecting $\mathcal{PS}^{n,1}$ onto the space of variables $\boldsymbol{x}$ (variable y_1 can be eliminated as $y_1 = \sum_{i \in N} x_i$), we obtain a collection of $n+1$ points composed of $\boldsymbol{0}$ together with $u_i \boldsymbol{e}^i$ for $i \in N$. Since these points are affinely independent, their convex hull is the simplex $\Delta = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \sum_{i \in N} \frac{x_i}{u_i} \le 1 \right\}$. We conclude that $\mathcal{PS}^{n,1}$ is described by equality $y_1 = \sum_{i \in N} x_i$ together with the inequalities defining Δ. □

A.5 Proof of Proposition 7

Proof. We prove this result using disjunctive programming; see [5]. In fact $\mathcal{S}^{2,2} = \mathcal{S}_1 \cup \mathcal{S}_2$ where $\mathcal{S}_1 = \{(\boldsymbol{x};\boldsymbol{y}) \in \mathbb{R}^4 \,|\, x_1 = y_1, x_2 = y_2, \mathbf{0} \le \boldsymbol{x} \le \boldsymbol{u}, \mathbf{0} \le \boldsymbol{y} \le \boldsymbol{v}\}$ and $\mathcal{S}_2 = \{(\boldsymbol{x};\boldsymbol{y}) \in \mathbb{R}^4 \,|\, x_1 = y_2, x_2 = y_1, \mathbf{0} \le \boldsymbol{x} \le \boldsymbol{u}, \mathbf{0} \le \boldsymbol{y} \le \boldsymbol{v}\}$. We obtain that $\mathcal{P}\left(\mathcal{S}_1 \cup \mathcal{S}_2\right) = \mathrm{proj}_{(\boldsymbol{x};\boldsymbol{y})}\, \bar{\mathcal{S}}$ where

$$\bar{\mathcal{S}} = \left\{ (\boldsymbol{x};\boldsymbol{y};\bar{\boldsymbol{x}};\bar{\boldsymbol{y}};\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}};\lambda) \in \mathbb{R}^{13} \,\middle|\, \begin{array}{l} \boldsymbol{x} = \bar{\boldsymbol{x}} + \tilde{\boldsymbol{x}}, \quad \boldsymbol{y} = \bar{\boldsymbol{y}} + \tilde{\boldsymbol{y}} \\ \bar{x}_1 = \bar{y}_1, \quad \bar{x}_2 = \bar{y}_2 \\ \tilde{x}_1 = \tilde{y}_2, \quad \tilde{x}_2 = \tilde{y}_1 \\ \mathbf{0} \le \bar{\boldsymbol{x}} \le \lambda \boldsymbol{u}, \quad \mathbf{0} \le \bar{\boldsymbol{y}} \le \lambda \boldsymbol{v} \\ \mathbf{0} \le \tilde{\boldsymbol{x}} \le (1-\lambda)\boldsymbol{u}, \quad \mathbf{0} \le \tilde{\boldsymbol{y}} \le (1-\lambda)\boldsymbol{v} \\ 0 \le \lambda \le 1 \end{array} \right\},$$

as $\mathcal{S}_1$ and $\mathcal{S}_2$ are bounded. We next project $\bar{\mathcal{S}}$ onto the space of $(\boldsymbol{x};\boldsymbol{y})$ variables using Fourier-Motzkin elimination; see Theorem 1.4 in [32]. Using equalities $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \bar{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}} = \boldsymbol{y} - \bar{\boldsymbol{y}}$, $\bar{y}_1 = \bar{x}_1$, and $\bar{y}_2 = \bar{x}_2$, we eliminate $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, and $\bar{\boldsymbol{y}}$ from the description of $\bar{\mathcal{S}}$ to obtain

$$\mathop{\mathrm{proj}}_{(\boldsymbol{x};\boldsymbol{y};\bar{\boldsymbol{x}};\lambda)} \bar{\mathcal{S}} = \left\{ (\boldsymbol{x};\boldsymbol{y};\bar{\boldsymbol{x}};\lambda) \in \mathbb{R}^{7} \,\middle|\, \begin{array}{l} x_1 - \bar{x}_1 = y_2 - \bar{x}_2 \\ x_2 - \bar{x}_2 = y_1 - \bar{x}_1 \\ 0 \le \bar{x}_1 \le \lambda u_1 \\ 0 \le \bar{x}_2 \le \lambda u_2 \\ 0 \le y_2 - \bar{x}_2 \le (1-\lambda) u_1 \\ 0 \le y_1 - \bar{x}_1 \le (1-\lambda) v_1 \\ 0 \le \lambda \le 1 \end{array} \right\},$$

where we use the assumptions that $u_1 \le v_1$ and $u_2 = v_2$, and we replace $x_1 - \bar{x}_1$ with $y_2 - \bar{x}_2$ and $x_2 - \bar{x}_2$ with $y_1 - \bar{x}_1$ whenever possible. After projecting out variables $\bar{\boldsymbol{x}}$ by replacing $\bar{x}_1$ with $x_1 - y_2 + \bar{x}_2$, we obtain

$$\mathop{\mathrm{proj}}_{(\boldsymbol{x};\boldsymbol{y};\lambda)} \bar{\mathcal{S}} = \left\{ (\boldsymbol{x};\boldsymbol{y};\lambda) \in \mathbb{R}^{5} \,\middle|\, \begin{array}{l} x_1 + x_2 = y_1 + y_2 \\ 0 \le y_1 + y_2 - x_2 \le u_1 \\ 0 \le x_2 \le \lambda u_2 + (1-\lambda) v_1 \\ 0 \le y_1 \le \lambda u_1 + (1-\lambda) v_1 \\ 0 \le y_2 \le \lambda u_2 + (1-\lambda) u_1 \\ x_2 - y_1 \le \lambda u_2 \\ y_1 - x_2 \le \lambda u_1 \\ x_2 - y_2 \le (1-\lambda) v_1 \\ y_2 - x_2 \le (1-\lambda) u_1 \\ 0 \le \lambda \le 1 \end{array} \right\},$$

where the first equality is used to eliminate x_1 from the other inequalities. Finally, we project out variable λ to obtain the following system:

$$
\begin{aligned}
& x_1 + x_2 = y_1 + y_2 && \text{(13a)}\\
& x_2 - y_1 - y_2 \le 0 && \text{(13b)}\\
& y_1 + y_2 - x_2 \le u_1 && \text{(13c)}\\
& -x_2 \le 0 && \text{(13d)}\\
& x_2 \le u_2 && \text{(13e)}\\
& -y_1 \le 0 && \text{(13f)}\\
& y_1 \le v_1 && \text{(13g)}\\
& -y_2 \le 0 && \text{(13h)}\\
& y_2 \le u_2 && \text{(13i)}\\
& (u_1 - u_2)x_2 + u_2 y_2 \le u_1 u_2 && \text{(13j)}\\
& (u_1 - u_2 + v_1)x_2 + (u_2 - v_1)y_2 \le u_1 u_2 && \text{(13k)}\\
& (u_1 - u_2)x_2 - u_1 y_1 + u_2 y_2 \le u_1 u_2 && \text{(13l)}\\
& -2x_2 + y_1 + y_2 \le u_1 && \text{(13m)}\\
& -x_2 + y_2 \le u_1 && \text{(13n)}\\
& x_2 - y_1 \le u_2 && \text{(13o)}\\
& y_1 - x_2 \le u_1 && \text{(13p)}\\
& x_2 - y_2 \le v_1 && \text{(13q)}\\
& (u_2 - u_1)x_2 + (v_1 - u_2 + u_1)y_2 \le u_2 v_1 && \text{(13r)}\\
& u_2 x_2 + (v_1 - u_2)y_2 \le u_2 v_1 && \text{(13s)}\\
& (u_2 + v_1)x_2 - v_1 y_1 - u_2 y_2 \le u_2 v_1 && \text{(13t)}\\
& (u_1 - v_1)x_2 + v_1 y_1 - u_1 y_2 \le u_1 v_1 && \text{(13u)}\\
& (u_2 - u_1)y_1 + (v_1 - u_1)y_2 \le u_2 v_1 - u_1^2 && \text{(13v)}\\
& (v_1 - u_1)x_2 + (u_2 - v_1)y_1 \le v_1(u_2 - u_1) && \text{(13w)}\\
& (v_1 - u_1)x_2 + (u_2 - v_1 + u_1)y_1 \le u_2 v_1 && \text{(13x)}\\
& (u_1 - v_1)x_2 + v_1 y_1 \le v_1 u_1. && \text{(13y)}
\end{aligned}
$$

We claim that

$$
\operatorname*{proj}_{(\boldsymbol{x};\boldsymbol{y})} \bar{\mathcal{S}} = \left\{ (\boldsymbol{x};\boldsymbol{y}) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \text{(13a)} - \text{(13d)}, \text{(13f)}, \text{(13h)}, \text{(13j)} \\ \text{(13k)}, \text{(13r)}, \text{(13s)}, \text{(13w)}, \text{(13y)} \end{array} \right\}.
$$

To verify this claim, we show how the eliminated inequalities are obtained as conic combinations of those included in the given description. In particular, (13e) is a conic combination of (13k) and (13s) with weights 1. Inequality (13g) is a conic combination of (13w) and (13y) with weights 1. Inequality (13i) is a conic combination of (13j) and (13r) with weights 1. Inequality (13l) is a conic combination of (13f) and (13j) with weights u_1 and 1, respectively. Inequality (13m) is a conic combination of (13c) and (13d) with weights 1. Inequality (13n) is a conic combination of (13c) and (13f) with weights 1. Inequality (13o) is a conic combination of (13f), (13k) and (13s) with weights $(u_1 + v_1)$, 1 and 1, respectively. Inequality (13p) is a conic combination of (13c) and (13h) with weights 1. Inequality (13q) is a conic combination of (13b), (13w) and (13y) with weights u_2, 1 and 1, respectively. A conic combination of (13b) and (13s) with weights v_1 and 1, respectively, yields (13t). A conic combination of (13h) and (13y) with weights u_1 and 1, respectively, yields (13u). Inequality (13v) is a conic combination of (13c) and (13w) with weights $(v_1 - u_1)$ and 1, respectively. Finally, (13x) is a conic combination of (13g) and (13w) with weights u_1 and 1, respectively. □

A.6 Proof of Proposition 8

Proof. We first show that (7) is valid for $\mathcal{PS}^{n,m}_{X\cup\{k\},Y}$. It follows from Proposition 1 that extreme points of $\mathcal{PS}^{n,m}_{X\cup\{k\},Y}$ satisfy $x_k = 0$ or $x_k = y_j = w_{k,j}$ for some $j \in Y$. Therefore, we write

$$\mathcal{PS}^{n,m}_{X\cup\{k\},Y} = \mathcal{P}\left(\bigcup_{j\in Y\cup\{0\}} F_j\right),$$

where

$$F_0 = \left\{(x_k; \boldsymbol{x}; \boldsymbol{y}) \,\middle|\, (\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X,Y},\ x_k = 0\right\},$$
$$F_j = \left\{(x_k; \boldsymbol{x}; \boldsymbol{y}) \,\middle|\, (\boldsymbol{x}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X,Y\setminus\{j\}},\ x_k = y_j = w_{k,j}\right\},$$

for $j \in Y$. We conclude that it is sufficient to show that (7) is valid for F_j for all $j \in Y \cup \{0\}$. The result holds for F_0 since $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ is valid for $\mathcal{S}^{n,m}_{X,Y}$ and since this inequality does not contain x_k. Consider now $j \in Y$. It follows from (5) and (6) that $\mu^k \le Z^k_j \le \frac{\gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y}}{w_{k,j}}$ for any $(\boldsymbol{x}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X,Y\setminus\{j\}}$ and $y_j = w_{k,j}$. This shows that (7) is valid for F_j as x_k equals $w_{k,j}$.

We next show that there exists $d+2$ affinely independent points of $\mathcal{PS}^{n,m}_{X\cup\{k\},Y}$ that satisfy (7) at equality. It follows from the assumption that there are $d+1$ affinely independent points of $\mathcal{PS}^{n,m}_{X,Y}$ that satisfy $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ at equality, call them $(\boldsymbol{x}^l; \boldsymbol{y}^l)$ for $l \in \{1, \ldots, d+1\}$. Assume that $\mu^k = Z^k_{j^*}$, for some $j^* \in Y$. Let $(\boldsymbol{x}^*; \boldsymbol{y}^*)$ be an optimal solution of (6) for index j^*. It is then easy to verify that the points $(x_k; \boldsymbol{x}; \boldsymbol{y}) = (w_{k,j^*}; \boldsymbol{x}^*; \boldsymbol{y}^*)$, and $(0; \boldsymbol{x}^l; \boldsymbol{y}^l)$ for $l \in \{1, \ldots, d+1\}$ belong to $S^{n,m}_{X\cup\{k\},Y}$, are affinely independent as $w_{k,j^*} > 0$, and satisfy (7) at equality. □

A.7 Proof of Proposition 9

Proof. Let $(\bar{\lambda}, \bar{\mu})$ be an extreme point of $\mathcal{X}$. We first show that (10) is valid for $\mathcal{P}\mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}}$. To this end, we introduce for any $i \in X$ and $j \in Y$

$$
\begin{aligned}
F_{i,j} &= \left\{ (x_k, y_l; \boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}} \,\middle|\, x_k = y_j = w_{k,j},\, y_l = x_i = w_{i,l} \right\}, \\
F_{i,0} &= \left\{ (x_k, y_l; \boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}} \,\middle|\, x_k = 0,\, y_l = x_i = w_{i,l} \right\}, \\
F_{0,j} &= \left\{ (x_k, y_l; \boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}} \,\middle|\, x_k = y_j = w_{k,j},\, y_l = 0 \right\}, \\
F_{0,0} &= \left\{ (x_k, y_l; \boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}} \,\middle|\, x_k = y_l = 0 \right\}.
\end{aligned}
$$

Similar to the argument given in Proposition 8, it follows from Proposition 1 that

$$
\mathcal{P}\mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}} = \mathcal{P}\left(\bigcup_{i\in X\cup\{0\}, j\in Y\cup\{0\}} F_{i,j} \right).
$$

Therefore, it is sufficient to show that (10) is valid for $F_{i,j}$ for all $i \in X \cup \{0\}$ and $j \in Y \cup \{0\}$. There are four cases. First, assume that $i = j = 0$. As implied from the definition of $F_{0,0}$, we need to show that $\bar{\lambda} w_{k,l} + \bar{\mu} w_{k,l} + \boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ for any $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X,Y}$. We write that $\gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y} \ge 0 \ge w_{k,l}(\bar{\lambda} + \bar{\mu})$, where the first inequality follows from the assumption that $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ is valid for any $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}^{n,m}_{X,Y}$, and the second inequality holds because $w_{k,l} > 0$ and $(\bar{\lambda}, \bar{\mu})$ satisfies (9a). Second, assume that $i \in X$ and $j \in Y$. As implied from the definition of $F_{i,j}$, we need to show that $(w_{k,l} - w_{k,j})\bar{\lambda} + (w_{k,l} - w_{i,l})\bar{\mu} + \boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ for any $(\boldsymbol{x}; \boldsymbol{y})$ such that $(\boldsymbol{x}_{-i}; \boldsymbol{y}_{-j}) \in \mathcal{S}^{n,m}_{X\setminus\{i\},Y\setminus\{j\}}$, $x_i = w_{i,l}$ and $y_j = w_{k,j}$. We write that $\gamma - \boldsymbol{\alpha}^\intercal \boldsymbol{x} - \boldsymbol{\beta}^\intercal \boldsymbol{y} \ge Z^{k,l}_{i,j} \ge (w_{k,l} - w_{k,j})\bar{\lambda} + (w_{k,l} - w_{i,l})\bar{\mu}$, where the first inequality follows from (8a), and the second inequality holds because $(\bar{\lambda}, \bar{\mu})$ satisfies (9d). Similar arguments yield the result for cases where $i \in X$ and $j = 0$, and where $i = 0$ and $j \in Y$.

We now show that there exists $d+3$ affinely independent points of $\mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}}$ that satisfy (10) at equality. It follows from the assumption that there exists $d+1$ affinely independent points of $\mathcal{S}^{n,m}_{X,Y}$ that satisfy $\boldsymbol{\alpha}^\intercal \boldsymbol{x} + \boldsymbol{\beta}^\intercal \boldsymbol{y} \le \gamma$ at equality, call them $(\boldsymbol{x}^t; \boldsymbol{y}^t)$ for $t \in \{1, \ldots, d+1\}$. Since $(\bar{\lambda}, \bar{\mu})$ is an extreme point of $\mathcal{X}$, there are at least two linearly independent constraints in the description of $\mathcal{X}$ that are binding at this point. Denote these two binding constraints by $(w_{k,l} - \hat{\rho})\lambda + (w_{k,l} - \hat{\eta})\mu \le \hat{\zeta}$ and $(w_{k,l} - \tilde{\rho})\lambda + (w_{k,l} - \tilde{\eta})\mu \le \tilde{\zeta}$. In the above inequalities, we select $\hat{\zeta} = 0$ if (9a) is binding. In this case, we define $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}}) = (\boldsymbol{x}^1; \boldsymbol{y}^1)$. Otherwise, we let $\hat{\zeta}$ and $\tilde{\zeta}$ be the optimal values of (8a)–(8c) corresponding to inequalities (9b)–(9d), and let $(\hat{\boldsymbol{x}}; \hat{\boldsymbol{y}})$ and $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ be corresponding optimal solutions that yield $\hat{\zeta}$ and $\tilde{\zeta}$, respectively. Next, construct points $\dot{\boldsymbol{\omega}}^1 = (x_k, y_l, \boldsymbol{x}; \boldsymbol{y}) = (\hat{\rho}, \hat{\eta}, \hat{\boldsymbol{x}}; \hat{\boldsymbol{y}})$, $\dot{\boldsymbol{\omega}}^2 = (\tilde{\rho}, \tilde{\eta}, \tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ and $\boldsymbol{\omega}^t = (w_{k,l}, w_{k,l}, \boldsymbol{x}^t; \boldsymbol{y}^t)$ for $t \in \{1, \ldots, d+1\}$. Since the two binding constraints described above are linearly independent, we must have that the vectors $(\hat{\rho}, \hat{\eta})$, $(\tilde{\rho}, \tilde{\eta})$ and $(w_{k,l}, w_{k,l})$ are affinely independent. This implies that $\dot{\boldsymbol{\omega}}^1$, $\dot{\boldsymbol{\omega}}^2$ and $\boldsymbol{\omega}^t$ for $t \in \{1, \ldots, d+1\}$ are affinely independent. Further, it is easy to verify from construction that these points belong to $\mathcal{S}^{n,m}_{X\cup\{k\},Y\cup\{l\}}$, and satisfy (10) at equality. □

A.8 Proof of Proposition 10

Proof. It is clear that both problems ($\mathtt{Q}$) and ($\bar{\mathtt{Q}}$) have optimal solutions. First, it is easy to verify that if $\bar{\boldsymbol{z}}$ is a feasible solution of ($\bar{\mathtt{Q}}$) with objective value $\bar{\psi}$, then $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}})$ is a feasible solution of ($\mathtt{Q}$) with objective value $\bar{\psi}$ where $\bar{x}_i = \sum_{j\in M} w_{i,j}\bar{z}_{i,j}$ for $i \in N$, and $\bar{y}_j = \sum_{i\in N} w_{i,j}\bar{z}_{i,j}$ for $j \in M$. This shows that $\psi^* \leq \bar{\psi}^*$. Second, since $\mathcal{PS}^{n,m}$ is a polytope, ($\mathtt{Q}$) has an optimal extreme point. If $(\bar{\boldsymbol{x}};\bar{\boldsymbol{y}})$ is an extreme solution of ($\mathtt{Q}$) with objective value $\bar{\psi}$ and matching $\bar{\mathcal{M}}$, then $\bar{\boldsymbol{z}}$ is a feasible solution of ($\bar{\mathtt{Q}}$) with objective value $\bar{\psi}$ where $\bar{z}_{i,j} = 1$ for $(i,j) \in \bar{\mathcal{M}}$, and $\bar{z}_{i,j} = 0$ for $(i,j) \notin \bar{\mathcal{M}}$. This result follows from the properties of extreme points of $\mathcal{PS}^{n,m}$ given in Proposition 1. Therefore, we conclude that $\psi^* \geq \bar{\psi}^*$. □

A.9 Proof of Proposition 11

Proof. The first result follows directly from Propositions 8 and 9. For the second result, first note that each assignment problem corresponding to the lifting problems of Propositions 8 and 9 can be solved in $\mathcal{O}(\zeta^3)$; see Proposition 10. For variables with indices in I or J, the lifting is performed sequentially according to Proposition 8, which leads to an $\mathcal{O}((|I|+|J|)\zeta^3)$ running time. For variable pairs with indices in K, the lifting is performed simultaneously according to Proposition 9. In that proposition, each constraint of $\mathcal{X}$ (9a)-(9d) can be derived in $\mathcal{O}(\zeta^3)$, as its right-hand-side is obtained via the solution of (8a), (8b), or (8c). The total number of these constraints is $\mathcal{O}(n+m+nm)$. Extreme points can therefore be found in $\mathcal{O}(n^2m^2)$ which is dominated by $O(nm\zeta^3)$. Since there are $|K|$ such simultaneous lifting procedures, the total running time is $\mathcal{O}(|K|nm\zeta^3)$. Therefore, the total running time to obtain the lifted inequality is $\mathcal{O}(\zeta^3(|I|+|J|+|K|nm))$. □

A.10 Proof of Proposition 12

Proof. *(i)* It is clear that $\dim(\mathcal{PS}^{n,m}) \leq n+m-1$ as all feasible solutions of $\mathcal{S}^{n,m} \subseteq \mathbb{R}^{n+m}$ satisfy $\sum\limits_{i\in N} x_i = \sum\limits_{j\in M} y_j$. Consider now the points $(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{0},\boldsymbol{0})$, $(\epsilon\boldsymbol{e}^i;\epsilon\boldsymbol{e}^1)$ for $i \in N$, and $(\epsilon\boldsymbol{e}^1;\epsilon\boldsymbol{e}^j)$ for $j \in M \setminus \{1\}$. These $m+n$ points belong to $\mathcal{S}^{n,m}$ and are affinely independent. It follows that $\dim(\mathcal{PS}^{n,m}) \geq n+m-1$.

(ii) We show the result for $x_i \geq 0$. The proof for $y_j \geq 0$ is similar. It is clear that $x_i \geq 0$ is valid for $\mathcal{PS}^{n,m}$ since it is valid for $\mathcal{LS}^{n,m}$. Select $k \in N \setminus \{i\}$, and construct the points $\bar{\boldsymbol{\omega}}^j = (\boldsymbol{x};\boldsymbol{y}) = (\epsilon\boldsymbol{e}^k,\epsilon\boldsymbol{e}^j)$ for $j \in M$, and $\hat{\boldsymbol{\omega}}^l = (\epsilon\boldsymbol{e}^l,\epsilon\boldsymbol{e}^1)$ for $l \in N \setminus \{i,k\}$, where ϵ is a sufficiently small but positive real number. These points together with $(\boldsymbol{0};\boldsymbol{0})$ belong to $\mathcal{S}^{n,m}$, are affinely independent and satisfy $x_i \geq 0$ at equality. This shows that $x_i \geq 0$ is facet-defining for $\mathcal{PS}^{n,m}$.

(iii) We show the result for $x_i \leq u_i$ as the proof for $y_j \leq v_j$ is similar. For the direct implication, it is clear that $x_i \leq u_i$ is valid for $\mathcal{PS}^{n,m}$ since it is valid for $\mathcal{LS}^{n,m}$. Construct the points $\boldsymbol{\omega}^l = (\boldsymbol{x};\boldsymbol{y}) = (u_i\boldsymbol{e}^i;u_i\boldsymbol{e}^l)$ for $l \in \{j,\dots,m\}$, and $\bar{\boldsymbol{\omega}}^l = (u_i\boldsymbol{e}^i;u_i\boldsymbol{e}^j) + (\epsilon\boldsymbol{e}^k;\epsilon\boldsymbol{e}^l)$ for $l \in \{1,\dots,j-1\}$ and some $k \in N \setminus \{i\}$. Further, consider $\hat{\boldsymbol{\omega}}^t = (u_i\boldsymbol{e}^i;u_i\boldsymbol{e}^j) + (\epsilon\boldsymbol{e}^t;\epsilon\boldsymbol{e}^m)$ for $t \in N \setminus \{i,k\}$,

and $\tilde{\boldsymbol{\omega}} = (u_i\boldsymbol{e}^i; u_i\boldsymbol{e}^{j+1}) + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^j)$. It is simple to verify that these points belong to $\mathcal{S}^{n,m}$, are affinely independent, and satisfy $x_i \leq u_i$ at equality. For the reverse implication assume that $v_{m-1} < u_i \leq v_m$. All points of $\mathcal{PS}^{n,m}$ that belong to the face $x_i = u_i$ also satisfy $y_m = u_i$ because of the NSNM requirement in the description of $\mathcal{S}^{n,m}$. This implies that the dimension of the face defined by $x_i = u_i$ is no more than $n + m - 3$, showing that $x_i \leq u_i$ is not facet-defining for $\mathcal{PS}^{n,m}$.

□

A.11 Proof of Proposition 13

Proof. We first argue that (11) is valid for $\mathcal{PS}^{n,m}$ by showing that it is satisfied by every extreme point $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ of $\mathcal{PS}^{n,m}$. To this end, we first give a few definitions. It follows from Proposition 1 that $\tilde{x}_i > 0$ if it has a matching counterpart $\tilde{y}_j > 0$ with $\tilde{x}_i = \tilde{y}_j = w_{i,j}$. This result holds for any $\tilde{y}_j > 0$ as well. Consider one possible matching for these nonzero variables, and define $\tilde{K}$ to be the set of matching flows in the selected matching between variables $\tilde{x}_r$ and $\tilde{y}_s$ where $r \in P$ and $s \in H$, *i.e.*, $\tilde{K} = \{(r,s) \subseteq P \times H \,|\, \tilde{x}_r = \tilde{y}_s = w_{r,s}\}$. Define also $\tilde{K}_x \subseteq P$ (*resp.* $\tilde{K}_y \subseteq H$) to be the set of first (*resp.* second) components of the members of $\tilde{K}$. To prove the main result, we show that the following inequalities hold

$$\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)\tilde{x}_r \leq 0, \qquad \forall r \in R \tag{14a}$$

$$\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(\tilde{x}_r - w_{r,h}) \leq 0, \qquad \forall r \in P \setminus \tilde{K}_x \tag{14b}$$

$$\left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)\tilde{y}_s \leq 0, \qquad \forall s \in H \setminus \tilde{K}_y \tag{14c}$$

$$\left[\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(\tilde{x}_r - w_{r,h}) + \left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)\tilde{y}_s\right] \leq 0, \qquad \forall (r,s) \in \tilde{K}. \tag{14d}$$

Inequality (14a) follows from the fact that the coefficient of $\tilde{x}_r$ is nonpositive because $w_{r,j} = \min\{u_r, v_j\} \geq u_k$ as $v_j \geq u_k$ and $u_r \geq u_k$ for all $r \in R$. To show (14b), consider any $r \in P \setminus \tilde{K}_x$. It follows from definition of $\tilde{K}_x$ that $\tilde{x}_r$ is not matched with any variable $\tilde{y}_s$ with $s > h$. Therefore, either $\tilde{x}_r = 0$, or $\tilde{x}_r = \tilde{y}_s = w_{r,s}$ for some $s \leq h$. This shows that $\tilde{x}_r \leq w_{r,h}$. This leads to (14b) because the coefficient of $(\tilde{x}_r - w_{r,h})$ is nonnegative for all $r \in P \setminus \tilde{K}_x$, as $v_j - u_k \geq 0$ and $w_{r,h} - v_j \geq 0$. Inequality (14c) follows from the facts that $v_j - u_k \geq 0$, $w_{p_t,h} - v_j \geq 0$, and $w_{p_t,h} - w_{p_t,s} \leq 0$ for all $s \in H \setminus \tilde{K}_y$. For (14d), we show that the expression inside the bracket is nonpositive for any $(r,s) \in \tilde{K}$. There are two cases. If $u_r \leq v_h$, then the result follows because $\tilde{x}_r \leq u_r = w_{r,h}$, $v_j - u_k \geq 0$, $w_{r,h} - v_j \geq 0$, $w_{p_t,h} - v_j \geq 0$, and $w_{p_t,h} - w_{p_t,s} \leq 0$. If $u_r > v_h$, then we write that

$$\begin{aligned}
\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(\tilde{x}_r - w_{r,h}) + \left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)\tilde{y}_s \\
&= \left(\frac{v_j - u_k}{v_h - v_j}\right)(w_{r,s} - v_h) + \left(\frac{v_j - u_k}{v_h - v_j}\right)\left(\frac{v_h - w_{p_t,s}}{w_{p_t,s}}\right)w_{r,s} \\
&\le \left(\frac{v_j - u_k}{v_h - v_j}\right)(w_{r,s} - v_h) + \left(\frac{v_j - u_k}{v_h - v_j}\right)\left(\frac{v_h - w_{r,s}}{w_{r,s}}\right)w_{r,s} \\
&= 0,
\end{aligned}$$

where the first equality holds because $w_{r,h} = w_{p_t,h} = v_h$ as $u_{p_t} \ge u_r > v_h$, and because $\tilde{x}_r = \tilde{y}_s = w_{r,s}$. The inequality follows from the facts that $\left(\frac{v_j-u_k}{v_h-v_j}\right) \ge 0$, and that $\left(\frac{v_h-w_{p_t,s}}{w_{p_t,s}}\right) \le \left(\frac{v_h-w_{r,s}}{w_{r,s}}\right)$ as $w_{r,s} \le w_{p_t,s}$.

We are now ready to prove the main result by considering two cases for L.

(1) Assume that $L = \emptyset$. Then, (11) reduces to

$$\sum_{r\in R}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r + \sum_{r\in P}\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(x_r - w_{r,h}) + \sum_{s\in H}\left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)y_s + y_j \le u_k. \quad (15)$$

We argue that $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$ satisfies (15) by considering three cases for the value of $\tilde{y}_j$, depending on which variable $\tilde{x}_r$ it is matched to.

(1.a) Assume that $\tilde{y}_j = 0$ or $\tilde{y}_j = \tilde{x}_r = w_{r,j}$ for $r \in \{1,\dots,k\}$. Then, $\tilde{y}_j \le u_k$. Note that aggregating (14a) over all $r \in R$, (14b) over all $r \in P \setminus \tilde{K}_x$, (14c) over all $s \in H \setminus \tilde{K}_y$, (14d) over all $(r,s) \in \tilde{K}$, together with $\tilde{y}_j \le u_k$ yields (15) for $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$.

(1.b) Assume that $\tilde{y}_j = \tilde{x}_{\dot{r}} = w_{\dot{r},j}$ for some $\dot{r} \in R$. Then, $\left(\frac{u_k-w_{\dot{r},j}}{w_{\dot{r},j}}\right)\tilde{x}_{\dot{r}} + \tilde{y}_j = u_k$. Then, aggregating this equality with (14a) over all $r \in R\setminus\{\dot{r}\}$, (14b) over all $r \in P\setminus\tilde{K}_x$, (14c) over all $s \in H \setminus \tilde{K}_y$, (14d) over all $(r,s) \in \tilde{K}$ yields (15) for $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$.

(1.c) Assume that $\tilde{y}_j = \tilde{x}_{\dot{r}} = w_{\dot{r},j}$ for some $\dot{r} \in P$. Then, we write that

$$\left(\frac{v_j - u_k}{w_{\dot{r},h} - v_j}\right)(\tilde{x}_{\dot{r}} - w_{\dot{r},h}) + \tilde{y}_j = \left(\frac{v_j - u_k}{w_{\dot{r},h} - v_j}\right)(v_j - w_{\dot{r},h}) + v_j = u_k,$$

where the first equality follows from the fact that $w_{\dot{r},j} = v_j$ as $u_{\dot{r}} \ge u_i \ge v_j$. Then, aggregating this equality with (14a) over all $r \in R$, (14b) over all $r \in P \setminus (\tilde{K}_x \cup \{\dot{r}\})$, (14c) over all $s \in H \setminus \tilde{K}_y$, (14d) over all $(r,s) \in \tilde{K}$ yields (15) for $(\tilde{\boldsymbol{x}};\tilde{\boldsymbol{y}})$.

(2) Assume that $L \neq \emptyset$, *i.e.*, $L = \{l\}$ for some $l \in \{1,\dots,j-1\}$ such that $u_k \le v_l$. We consider three cases for the value of $\tilde{y}_l$.

(2.a) Assume that $\tilde{y}_l = 0$ or $\tilde{y}_l = \tilde{x}_r = w_{r,l}$ for $r \in \{1, \ldots, k\}$. Then, $\tilde{y}_l \leq u_k$. Then, it follows that $\left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(\tilde{y}_l - u_k) \leq 0$ since $u_k - w_{k+1,l} \leq 0$ and $u_k - w_{k+1,j} \leq 0$. Aggregating this inequality with (15) yields (11), which we have shown in Part (1) satisfies $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$.

(2.b) Assume that $\tilde{y}_l = \tilde{x}_{\dot{r}} = w_{\dot{r},l}$ for some $\dot{r} \in R$. Consider

$$\left(\frac{u_k - w_{\dot{r},j}}{w_{\dot{r},j}}\right)\tilde{x}_{\dot{r}} + \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(\tilde{y}_l - u_k) \leq 0. \tag{16}$$

It is simple to verify that (11) is obtained by aggregating (16) and (15) where the summation for its first term is over $r \in R \setminus \{\dot{r}\}$. The validity of the latter inequality for $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ follows from arguments similar to those of Part (1). Therefore, to prove that (11) is satisfied by $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$, it suffices to show that (16) is satisfied by $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$. There are two cases. First, assume that $v_l \leq u_{k+1}$. We write that

$$\begin{aligned}\left(\frac{u_k - w_{\dot{r},j}}{w_{\dot{r},j}}\right)\tilde{x}_{\dot{r}} &+ \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(\tilde{y}_l - u_k) \\ &\leq \left(\frac{u_k - w_{k+1,j}}{w_{k+1,j}}\right)w_{k+1,l} + \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(w_{k+1,l} - u_k) = 0,\end{aligned}$$

where the inequality follows from the facts that $\left(\frac{u_k - w_{\dot{r},j}}{w_{\dot{r},j}}\right) \leq \left(\frac{u_k - w_{k+1,j}}{w_{k+1,j}}\right)$ as $w_{k+1,j} \leq w_{\dot{r},j}$, and that $\tilde{x}_{\dot{r}} = \tilde{y}_l = w_{\dot{r},l} = v_l = w_{k+1,l}$ as $u_{\dot{r}} \geq u_{k+1} \geq v_l$. This yields (16). Second, assume that $v_l > u_{k+1}$. We write that

$$\begin{aligned}\left(\frac{u_k - w_{\dot{r},j}}{w_{\dot{r},j}}\right)\tilde{x}_{\dot{r}} &+ \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(\tilde{y}_l - u_k) \\ &= \left(\frac{u_k - w_{\dot{r},j}}{w_{\dot{r},j}}\right)w_{\dot{r},l} + (w_{\dot{r},l} - u_k) \leq 0,\end{aligned}$$

where the equality holds since $w_{k+1,j} = w_{k+1,l} = u_{k+1}$ as $v_j \geq v_l > u_{k+1}$ and since $\tilde{x}_{\dot{r}} = \tilde{y}_l = w_{\dot{r},l}$, and the inequality holds because $w_{\dot{r},l} \leq w_{\dot{r},j}$. This shows (16).

(2.c) Assume that $\tilde{y}_l = \tilde{x}_{\dot{r}} = w_{\dot{r},l}$ for some $\dot{r} \in P$. Consider

$$\left(\frac{v_j - u_k}{w_{\dot{r},h} - v_j}\right)(\tilde{x}_{\dot{r}} - w_{\dot{r},h}) + \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(\tilde{y}_l - u_k) \leq 0. \tag{17}$$

It is simple to verify that (11) is obtained by a combination of (17) and (15) where the summation for its second term is over $P \setminus \{\dot{r}\}$. Therefore, to prove that (11) is valid for $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$, it suffices to show that (17) and (15) are satisfied by $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$. Arguments similar to those of Part (1) imply that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ satisfies (15). We next show that $(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{y}})$ satisfies (17). There are two cases. First, assume that $v_l \leq u_{k+1}$. Then, $w_{k+1,l} = v_l$. It is clear that $\tilde{x}_{\dot{r}} = \tilde{y}_l = w_{\dot{r},l} = v_l$ as $v_l \leq v_j \leq u_i \leq u_{\dot{r}}$. Plugging these values in the left-hand-side of (17), we obtain

$$\begin{aligned}\left(\frac{v_j - u_k}{w_{\dot{r},h} - v_j}\right)(v_l - w_{\dot{r},h}) &+ \left(\frac{u_k - w_{k+1,j}}{u_k - v_l}\right)\left(\frac{v_l}{w_{k+1,j}}\right)(v_l - u_k) \\ &\leq (u_k - v_j) + (w_{k+1,j} - u_k) \leq 0,\end{aligned}$$

where the first inequality follows from the facts that *(i)* $\left(\frac{v_j-u_k}{w_{\dot{r},h}-v_j}\right)(v_l - w_{\dot{r},h}) \leq u_k - v_j$ since $\frac{v_l-w_{\dot{r},h}}{w_{\dot{r},h}-v_j} \leq -1$ as $v_l \leq v_j \leq w_{\dot{r},h}$, and since $v_j - u_k \geq 0$, and *(ii)* $(u_k - w_{k+1,j})\left(\frac{v_l}{w_{k+1,j}}\right)\left(\frac{v_l-u_k}{u_k-v_l}\right) \leq w_{k+1,j} - u_k$ since $\frac{v_l}{w_{k+1,j}} \leq 1$ as $v_l \leq v_j$ and $v_l \leq u_{k+1}$, and since $u_k - w_{k+1,j} \leq 0$, while the second inequality holds because $w_{k+1,j} - v_j \leq 0$. Second, assume that $v_l > u_{k+1}$. Then, $w_{k+1,l} = w_{k+1,j} = u_{k+1}$. It is clear that $\tilde{x}_{\dot{r}} = \tilde{y}_l = w_{\dot{r},l} = v_l$. Plugging these values in the left-hand-side of (17), we obtain

$$\left(\frac{v_j - u_k}{w_{\dot{r},h} - v_j}\right)(v_l - w_{\dot{r},h}) + (v_l - u_k) \leq (u_k - v_j) + (v_l - u_k) \leq 0,$$

where the first inequality holds because $\frac{v_l-w_{\dot{r},h}}{w_{\dot{r},h}-v_j} \leq -1$ and $v_j - u_k \geq 0$, while the second inequality holds because $v_l - v_j \leq 0$.

We next show that (11) is facet-defining for $\mathcal{PS}^{n,m}$ if any of the conditions *(i)*–*(iv)* holds. To this end, we provide $m+n-1$ affinely independent points of $\mathcal{S}^{n,m}$ that satisfy (11) at equality.

(i) For this case, (11) reduces to

$$\sum_{r\in R}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r + y_j \leq u_k. \tag{18}$$

Consider points $\bar{\boldsymbol{\omega}}^r = (w_{r,j}\boldsymbol{e}^r; w_{r,j}\boldsymbol{e}^j)$ for $r \in \{k+1,\ldots,n\} = R$. Further, select some $j^* \in M \setminus \{j\}$ and construct $\hat{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^n + (\epsilon\boldsymbol{e}^r; \epsilon\boldsymbol{e}^{j^*})$ for $r \in \{1,\ldots,k\}$. Next, construct $\tilde{\boldsymbol{\omega}}^s = \bar{\boldsymbol{\omega}}^n + (\epsilon\boldsymbol{e}^1; \epsilon\boldsymbol{e}^s)$ for $s \in M \setminus \{j, j^*\}$. Finally, consider $\dot{\boldsymbol{\omega}} = (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^j)$. The above $m+n-1$ points belong to $\mathcal{S}^{n,m}$, are affinely independent, and satisfy (18) at equality.

(ii) For this case, (11) reduces to

$$\sum_{r\in R}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r + \left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,l}}\right)\left(\frac{w_{k+1,l}}{w_{k+1,j}}\right)(y_l - u_k) + y_j \leq u_k. \tag{19}$$

Construct $\boldsymbol{\omega}^1 = (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^j) + (w_{k+1,l}\boldsymbol{e}^{k+1}; w_{k+1,l}\boldsymbol{e}^l)$, and $\boldsymbol{\omega}^2 = (w_{n,j}\boldsymbol{e}^n; w_{n,j}\boldsymbol{e}^j) + (w_{k+1,l}\boldsymbol{e}^{k+1}; w_{k+1,l}\boldsymbol{e}^l)$. Next, consider $\bar{\boldsymbol{\omega}}^r = (w_{r,j}\boldsymbol{e}^r; w_{r,j}\boldsymbol{e}^j) + (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^l)$ for $r \in \{k+1,\ldots,n\}$. The above points are affinely independent as $u_k - w_{k+1,l} \neq 0$ by assumption. Further, select some $j^* \in M\setminus\{j,l\}$ and construct $\hat{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^n + (\epsilon\boldsymbol{e}^r; \epsilon\boldsymbol{e}^{j^*})$ for $r \in \{1,\ldots,k-1\}$. Finally, construct $\tilde{\boldsymbol{\omega}}^s = \boldsymbol{\omega}^2 + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^s)$ for $s \in M \setminus \{j,l\}$. The above $m+n-1$ points belong to $\mathcal{S}^{n,m}$, are affinely independent, and satisfy (19) at equality.

(iii) For this case, (11) reduces to

$$\sum_{r\in R}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r + \sum_{r\in P}\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(x_r - w_{r,h}) + \sum_{s\in H}\left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)y_s + y_j \leq u_k. \tag{20}$$

Note that $j < m$ for this case since otherwise there is no possible choice for h. Define $F = \{1, \ldots, t\}$. Consider $\boldsymbol{\omega}^r = (w_{r,j}\boldsymbol{e}^r; w_{r,j}\boldsymbol{e}^j) + \sum_{f \in F}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f})$ for $r \in R \cup \{k\}$. These points satisfy (20) at equality as $w_{p_f,h} = w_{p_f,q_f}$ by assumption. Next, define $\bar{\boldsymbol{\omega}}^r = \sum_{f \in F \setminus \{r\}}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f}) + (v_j\boldsymbol{e}^{p_r}; v_j\boldsymbol{e}^j)$ for any $r \in F$. These points are affinely independent from $\boldsymbol{\omega}^r$ since $w_{p_f,q_f} - v_j \neq 0$ as our assumption requires that $w_{p_f,h} - v_j \neq 0$ and $w_{p_f,h} = w_{p_f,q_f}$. Similarly, construct $\hat{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^r + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^{q_r})$ for any $r \in F$. Next, consider $\tilde{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^t + (\epsilon\boldsymbol{e}^r; \epsilon\boldsymbol{e}^{q_t})$ for $r \in \{1, \ldots, k-1\}$. Define also $\dot{\boldsymbol{\omega}}^s = \bar{\boldsymbol{\omega}}^t + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^s)$ for $s \in M \setminus \{j, q_1, \ldots, q_t, h+1, \ldots, m\}$. Finally, consider $\ddot{\boldsymbol{\omega}}^s = (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^j) + \sum_{f \in F \setminus \{t\}}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f}) + (w_{p_t,s}\boldsymbol{e}^{p_t}; w_{p_t,s}\boldsymbol{e}^s)$ for $s \in H$. The above $m + n - 1$ points belong to $\mathcal{S}^{n,m}$, are affinely independent, and satisfy (20) at equality.

(iv) Note that for the NSNM inequality to be well-defined, we have that $u_k - w_{k+1,l} \neq 0$, and that $w_{r,h} - v_j \neq 0$ for $r \in P$. Define $F = \{1, \ldots, t\}$. Construct $\ddot{\boldsymbol{\omega}}^1 = (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^j) + (w_{k+1,l}\boldsymbol{e}^{k+1}; w_{k+1,l}\boldsymbol{e}^l) + \sum_{f \in F}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f})$, and $\ddot{\boldsymbol{\omega}}^2 = \sum_{f \in F \setminus \{t\}}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f}) + (v_j\boldsymbol{e}^{p_t}; v_j\boldsymbol{e}^j) + (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^l)$. Consider $\boldsymbol{\omega}^r = (w_{r,j}\boldsymbol{e}^r; w_{r,j}\boldsymbol{e}^j) + (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^l) + \sum_{f \in F}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f})$ for $r \in R$. Next, define $\bar{\boldsymbol{\omega}}^r = \sum_{f \in F \setminus \{r\}}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f}) + (v_j\boldsymbol{e}^{p_r}; v_j\boldsymbol{e}^j) + (w_{k+1,l}\boldsymbol{e}^{k+1}; w_{k+1,l}\boldsymbol{e}^l)$ for any $r \in F$. Similarly, construct $\hat{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^r + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^{q_r})$ for any $r \in F$. Consider $\tilde{\boldsymbol{\omega}}^r = \bar{\boldsymbol{\omega}}^t + (\epsilon\boldsymbol{e}^r; \epsilon\boldsymbol{e}^{q_t})$ for $r \in \{1, \ldots, k-1\}$. Construct $\dot{\boldsymbol{\omega}}^s = (w_{k+1,j}\boldsymbol{e}^{k+1}; w_{k+1,j}\boldsymbol{e}^j) + \sum_{f \in F \setminus \{t\}}(w_{p_f,q_f}\boldsymbol{e}^{p_f}; w_{p_f,q_f}\boldsymbol{e}^{q_f}) + (w_{p_t,s}\boldsymbol{e}^{p_t}; w_{p_t,s}\boldsymbol{e}^s) + (u_k\boldsymbol{e}^k; u_k\boldsymbol{e}^l)$ for $s \in H$. Finally, consider $\bar{\bar{\boldsymbol{\omega}}}^s = \bar{\boldsymbol{\omega}}^t + (\epsilon\boldsymbol{e}^k; \epsilon\boldsymbol{e}^s)$ for $s \in M \setminus \{j, l, q_1, \ldots, q_t, h+1, \ldots, m\}$. The above $m + n - 1$ points belong to $\mathcal{S}^{n,m}$, are affinely independent, and satisfy (11) at equality.

□

A.12 Proof of Proposition 14

Proof. Arranging the terms of (11) so that its right-hand-side is zero, the left-hand-side is

$$\sum_{r=k+1}^{i-1}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r^* + \sum_{s \in L}\left(\frac{u_k - w_{k+1,j}}{u_k - w_{k+1,s}}\right)\left(\frac{w_{k+1,s}}{w_{k+1,j}}\right)(y_s^* - u_k) + y_j^* - u_k$$
$$+ \sum_{r \in \{i,\ldots,n\} \setminus P}\left(\frac{u_k - w_{r,j}}{w_{r,j}}\right)x_r^* + \sum_{r \in P}\left(\frac{v_j - u_k}{w_{r,h} - v_j}\right)(x_r^* - w_{r,h}) + \sum_{s \in H}\left(\frac{v_j - u_k}{w_{p_t,h} - v_j}\right)\left(\frac{w_{p_t,h} - w_{p_t,s}}{w_{p_t,s}}\right)y_s^*. \tag{21}$$

Our goal is to determine a set of indices that maximizes the value of (21). Assume that (i, j, k) is known. There are $\mathcal{O}(nm)$ choices for such triple of indices as explained next. Given j, if there are several choices for i because capacities u_i are equal, it is sufficient to choose the one with the largest index; in fact, other indices can only be included in R and lead to zero value in (21). The maximum value achieved by the first line of (21) can be obtained in $\mathcal{O}(n)$ as it only requires a comparison over valid choices of l. To compute the maximum value for the second line of (21), we consider two cases. First, assume that $P = \emptyset$. In this case, the coefficient of each remaining variable is known exactly and can be computed in $\mathcal{O}(n)$. Second, assume that $P \neq \emptyset$ for some

$P = \{p_1, \ldots, p_t\} \subseteq \{i, \ldots, n\}$. The coefficients of terms involved in the second line of (21) depend on h and p_t. There are $\mathcal{O}(nm)$ choices for (h, p_t). Assume that (h, p_t) is fixed. Then, the summation over H in (21) can be computed in $\mathcal{O}(m)$. For the remaining terms in the second line, consider any $\bar{r} \in \{i, \ldots, p_t - 1\}$: either $\bar{r} \in P$ and the term $\left(\frac{v_j - u_k}{w_{\bar{r},h} - v_j}\right)(x^*_{\bar{r}} - w_{\bar{r},h})$ is included in (21), or $\bar{r} \notin P$ and the term $\left(\frac{u_k - w_{\bar{r},j}}{w_{\bar{r},j}}\right) x^*_{\bar{r}}$ is included in (21). As a result, it is sufficient to compare values $\left(\frac{v_j - u_k}{w_{\bar{r},h} - v_j}\right)(x^*_{\bar{r}} - w_{\bar{r},h})$ and $\left(\frac{u_k - w_{\bar{r},j}}{w_{\bar{r},j}}\right) x^*_{\bar{r}}$ to assign $\bar{r}$ to the proper category. For any $\bar{r} > p_t$, the term $\left(\frac{u_k - w_{\bar{r},j}}{w_{\bar{r},j}}\right) x^*_{\bar{r}}$ is included in (21). This procedure can be carried out in $\mathcal{O}(n+m)$ for any (h, p_t). We conclude that the total running time of this algorithm is $\mathcal{O}\left(n^2 m^2 (n+m)\right)$. □

B Propositions and Examples

B.1 Example 5

Example 5. Consider the set $\widehat{\mathcal{S}}^{3,3}$ described in Example 3. First, we fix variables x_i for some $i \in N$ and y_j for some $j \in M$ both at zero or both at $w_{i,j}$. We then project them out to reduce $\widehat{\mathcal{S}}^{3,3}$ into a single node problem with NSNM requirement on two incoming and two outgoing arcs. A complete description of facet-defining inequalities for this lower-dimensional set is given in Proposition 7. These inequalities can be used as seeds for lifting. We next give a detailed derivation of two of the facet-defining inequalities of $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ given in Table 8, one obtained from lifting $x_i = y_j = 0$, the other from lifting $x_i = y_j = w_{i,j}$.

First, we derive ($A11$); see Table 7. We fix $x_3 = y_3 = 0$, and define $X = \{1, 2\}$ and $Y = \{1, 2\}$. Inequality $-2x_2 + 5y_1 \leq 15$ is the last inequality in the description of $\mathcal{P}\mathcal{S}^{2,2}$ given in Proposition 7. Further, the affinely independent points $(x_1, x_2, y_1, y_2) = (0, 5, 5, 0)$, $(3, 5, 5, 3)$ and $(3, 0, 3, 0)$ belong to $\widehat{\mathcal{S}}^{3,3}_{X,Y} \subseteq \mathbb{R}^4$, and satisfy this inequality at equality. It is therefore facet-defining for $\mathcal{P}\widehat{\mathcal{S}}^{3,3}_{X,Y}$. Next, we lift variables x_3 and y_3 sequentially using Proposition 8. To lift variable x_3, we compute μ^3 as in (5). This computation requires solution of (6), *i.e.*, $Z^3_1 = \min\left\{\frac{15 + 2x_2 - 5y_1}{5} \,\middle|\, (x_1, x_2, y_2) \in \mathcal{S}^{3,3}_{X, Y\setminus\{1\}},\, y_1 = 5\right\} = -2$. Similarly, we compute that $Z^3_2 = 0$ to obtain $\mu^3 = -2$. The resulting lifted inequality is ($A11$). In a similar manner, we next lift variable y_3. We compute that $Z^3_1 = 0$, $Z^3_2 = 2$, $Z^3_3 = 2$, to obtain $\mu^3 = 0$. The resulting lifted inequality is ($A11$).

Next we derive ($A18$); see Table 7. We fix $x_3 = y_2 = w_{3,2} = 11$, and define $X = \{1, 2\}$ and $Y = \{1, 3\}$. Inequality $-2x_2 + 5y_1 \leq 15$ is facet-defining for $\widehat{\mathcal{S}}^{3,3}_{X,Y}$. Next, we lift variables (x_3, y_2) using Proposition 9. The inequalities (9a)–(9d) in the description of $\mathcal{X}$ can be compactly represented as $\alpha^{3,2}_{i,j}\lambda + \beta^{3,2}_{i,j}\mu \leq \gamma^{3,2}_{i,j}$ for $i \in \{0, 1, 2\}$ and $j \in \{0, 1, 3\}$ where $\gamma^{3,2}_{i,j} = Z^{3,2}_{i,j}$ for $(i, j) \neq (0, 0)$ and $\gamma^{3,2}_{0,0} = 0$ to match (9a). The values of $(\alpha^{3,2}_{i,j}, \beta^{3,2}_{i,j}, \gamma^{3,2}_{i,j})$ are recorded in Table 5. For instance, we can verify that for $i = j = 1$ that $\alpha^{3,2}_{1,1} = w_{3,2} - w_{3,1} = 11 - 5 = 6$ and $\beta^{3,2}_{1,1} = w_{3,2} - w_{1,2} = 11 - 3 = 8$. Further, $\gamma^{3,2}_{1,1} = Z^{3,2}_{1,1}$ where, according to (8c), $Z^{3,2}_{1,1} =$

$$\min\left\{15 + 2x_2 - 5y_1 \,\middle|\, (x_2, y_3) \in \mathcal{S}^{3,3}_{X\setminus\{1\},Y\setminus\{1\}},\, x_1 = w_{1,2} = 3,\, y_1 = w_{3,1} = 5\right\} = -10.$$

Table 5: Parameters $(\alpha^{3,2}_{i,j}, \beta^{3,2}_{i,j}, \gamma^{3,2}_{i,j})$ for inequalities describing $\mathcal{X}$.

i \ j	0	1	3
0	$(1,1,0)$	$(6,11,-10)$	$(-2,11,0)$
1	$(11,8,0)$	$(6,8,-10)$	$(-2,8,0)$
2	$(11,4,14)$	$(6,4,4)$	$(-2,4,14)$

It can be verified that $(\lambda, \mu) = (2, -\frac{11}{4})$ is an extreme point of $\mathcal{X}$ with tight constraints corresponding to $(i,j) = (1,0)$ and $(i,j) = (1,1)$. Plugging these values for λ and μ in (10), we obtain that $2(w_{3,2} - x_3) - \frac{11}{4}(w_{3,2} - y_2) - 2x_2 + 5y_1 \le 15$, *i.e.*, (*A*18), is facet-defining for $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$.

B.2 Proposition 15

Proposition 15. *Assume that $v_{j-1} \le u_i \le v_j$ for some $i \in N$ and $j \in M \setminus \{1\}$. Select $k \in \{1, \dots, j-1\}$, and set $L = \emptyset$ or $L = \{l\}$ for some $l \in \{1, \dots, i-1\}$ and $v_k \le u_l$. Also, select $P = \{p_1, \dots, p_t\} \subseteq \{j, \dots, m\}$ such that $p_1 < p_2 < \dots < p_t$. If $P \ne \emptyset$, select $h \in \{i+1, \dots, n\}$, otherwise set $h = n$. Then, the reverse NSNM inequality*

$$\sum_{s\in R}\left(\frac{v_k - w_{i,s}}{w_{i,s}}\right) y_s + \sum_{s\in P}\left(\frac{u_i - v_k}{w_{h,s} - u_i}\right)(y_s - w_{h,s})$$
$$+ \sum_{r\in L}\left(\frac{v_k - w_{i,k+1}}{v_k - w_{r,k+1}}\right)\left(\frac{w_{r,k+1}}{w_{i,k+1}}\right)(x_r - v_k) + \sum_{r\in H}\left(\frac{u_i - v_k}{w_{h,p_t} - u_i}\right)\left(\frac{w_{h,p_t} - w_{r,p_t}}{w_{r,p_t}}\right) x_r + x_i \le v_k, \quad (22)$$

where $R = \{k+1, \dots, m\} \setminus P$, $H = \{h+1, \dots, n\}$, and where all denominators are nonzero, is valid for $\mathcal{PS}^{n,m}$. Further, (22) is facet-defining for $\mathcal{PS}^{n,m}$ if either of the following conditions holds.

- *(i)* $P = L = \emptyset$.
- *(ii)* $P = \emptyset$, $L \ne \emptyset$, $k \le m - 2$, $n \ge 3$.
- *(iii)* $L = \emptyset$, $P \ne \emptyset$, *and* $w_{h,p_f} = w_{q_f,p_f}$ *for* $f \in \{1, \dots, t\}$ *where* $q_f = h - t + f$ *and* $q_1 \ge i + 1$.
- *(iv)* $L \ne \emptyset$, $P \ne \emptyset$, $p_1 \ge k + 2$, *and and* $w_{h,p_f} = w_{q_f,p_f}$ *for* $f \in \{1, \dots, t\}$ *where* $q_f = h - t + f$ *and* $q_1 \ge i + 1$. □

C Path-Based Model and Column Generation

Instead of using the arc-based model for GUFP described in Section 3.1, we could also seek to obtain solutions and bounds for GUFP using a path-based model. In this section, we describe such a model together with a solution method that uses column generation.

Observe that, in our instances, most flow variables of GUFP have zero lower bounds. The exception comes from arcs (j,t) for $j \in D$ corresponding to demands, for which lower and upper bounds are equal and strictly positive. It is simple to see however that these bounds can be reset to zero by assuming that the corresponding demand has been met a-priori using the artificial supply node, and that flow on the arc now represents the decision of substituting artificial supply for actual supply. After this transformation, the artificial supply node s_0 can be removed, the reward of demand arc (j,t) becomes $-r_{s_0,j}$, the lower bound on the demand arc is reset to zero, and a constant of value $\sum_{j \in D} r_{s_0,j} d_j$ is added to the objective. For this reason, assuming that arc flows all have zero lower bounds is without loss of generality.

C.1 Path-Based Model

In graph $G(V,E)$, define the set of splittable nodes as $\ddot{V} := V \setminus \bar{V}$. Given a path p between two splittable nodes $v, w \in \ddot{V}$, we say that p is *unsplittable* if all nodes in the path except v and w belong to $\bar{V}$. In particular, an arc between two splittable nodes is considered an unsplittable path. We define the capacity $\kappa(p)$ of unsplittable path p to be the minimum capacity of arcs encountered on this path, *i.e.*, $\kappa(p) = \min_{e \in E(p)} u_e$ where $E(p)$ represents the set of arcs of p. Similarly, we define the reward $r(p)$ of unsplittable path p to be the sum of the rewards of its arcs, *i.e.*, $r(p) = \sum_{e \in E(p)} r_e$. Given $v, w \in \ddot{V}$, we denote the set of all unsplittable paths from v to w in $G(V,E)$ by $\mathcal{P}_{v,w}$. For any node $v \in \ddot{V}$, we denote the set of unsplittable paths incoming to v by $\mathcal{P}^+(v)$ and the set of unsplittable paths outgoing from v by $\mathcal{P}^-(v)$. Given an arc e of $G(V,E)$, we refer to the collection of all unsplittable paths using e by $\mathcal{P}(e)$.

Using this notation, GUFP can be formulated as the problem of identifying the nonnegative flow x_p (23d) to assign to each unspittable path p, while maintaining balance equations at each splittable node (23b), ensuring that no more than one unsplittable path is selected for each arc (23e) and enforcing that capacities of arcs/paths are satisfied (23c):

$$\max \quad \sum_{v \in \ddot{V}} \sum_{w \in \ddot{V} \setminus \{v\}} \sum_{p \in \mathcal{P}_{v,w}} r(p) x_p \tag{23a}$$

$$s.t. \quad \sum_{p \in \mathcal{P}^+(v)} x_p - \sum_{p \in \mathcal{P}^-(v)} x_p = 0, \qquad \forall v \in \ddot{V}, \tag{23b}$$

$$\sum_{p \in \mathcal{P}(e)} \frac{x_p}{\kappa(p)} \leq 1, \qquad \forall e \in E, \tag{23c}$$

$$x_p \geq 0, \qquad \forall p \in \bigcup_{v \in \ddot{V}} \bigcup_{w \in \ddot{V} \setminus \{v\}} \mathcal{P}_{v,w}, \tag{23d}$$

$$\text{card}(x_{p | p \in \mathcal{P}(e)}) \leq 1, \qquad \forall e \in E. \tag{23e}$$

The objective (23a) seeks to maximize total flow rewards. The cardinality constraint (23e) has a simple 0-1 MIP reformulation. This formulation requires the addition of a large number of binary variables and yields the same LP relaxation bound as the LP relaxation of (23a)-(23d) obtained by omitting (23e). For these reasons, we investigate next how to solve the (23a)-(23d).

C.2 Pricing problem

Assume that a restriction of (23a)-(23d) has been solved over a subset of variables. Denote the dual variable associated with (23b) for splittable node v by π_v. Denote the dual variable associated with (23c) for arc e by β_e. A crucial problem in the implementation of a column generation approach is to determine whether there exist non-basic variables with positive reduced costs. The reduced cost γ_p associated with variable x_p corresponding to unsplittable path p between splittable nodes v and w is:

$$\gamma_p = r(p) + \pi_v - \pi_w - \frac{1}{\kappa(p)} \sum_{e \in E(p)} \beta_e. \tag{24}$$

Expression (24) leads to a procedure to determine whether there exists a column with positive reduced cost. We consider, in sequence, each pair of distinct splittable nodes v and w in $\ddot{V}$ and each possible path capacity κ from $\underline{u} = \min_{e \in E} u_e$ to $\bar{u} = \max_{e \in E} u_e$. We construct the graph $G_\kappa[v,w]$ with node set $V_\kappa[v,w]$ and arc set $E_\kappa[v,w]$ by removing all of the splittable nodes except v and w from graph G and by removing all arcs with capacity lower than κ. In this network, we then identify a path p that maximizes $r(p) - \frac{1}{\kappa} \sum_{e \in E(p)} \beta_e = \sum_{e \in E(p)} (r_e - \frac{\beta_e}{\kappa})$ as π_v and π_w are now fixed. This corresponds to solving a longest path problem in a graph that potentially contains positive cycles. This problem can be shown to be NP-hard by reduction from the hamiltonian path problem. It has the natural IP formulation:

$$z^*_\kappa[v,w] = \max \sum_{e \in E_\kappa[v,w]} \left(r_e - \frac{\beta_e}{\kappa} \right) x_e \tag{25a}$$

$$s.t. \sum_{e \in \delta^+(u)} x_e - \sum_{e \in \delta^-(u)} x_e = \delta_{vw}(u), \qquad \forall u \in V_\kappa[v,w] \tag{25b}$$

$$\sum_{e \in \delta^+(u)} x_e \leq 1, \qquad \forall u \in V_\kappa[v,w] \tag{25c}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E_\kappa[v,w], \tag{25d}$$

where $\delta_{vw}(u) = 0$ if $u \neq v$ and $u \neq w$, $\delta_{vw}(v) = -1$, and $\delta_{vw}(w) = 1$. It is easy to verify that paths p in $G_\kappa[v,w]$ correspond to those unsplittable paths p of G between splittable nodes v and w with capacity at least κ. Further since $\beta_e \geq 0$ for all $e \in E$, we have that $\sum_{e \in E(p)} \left(r_e - \frac{\beta_e}{\kappa} \right) \leq \sum_{e \in E(p)} \left(r_e - \frac{\beta_e}{\kappa(p)} \right)$. It follows that if a nonbasic column with positive maximum reduced cost corresponds to an unsplittable path between v^* and w^* of capacity κ^*, then this path will be the longest path in network $G_{\kappa^*}[v^*, w^*]$. Further if $z^*_\kappa[v,w] > 0$, the longest path p^* in $G_\kappa[v,w]$ will be such that $\gamma_{p^*} > 0$. If $z^*_\kappa[v,w] \leq 0$, there is no unsplittable path from v to w with capacity κ whose corresponding reduced cost is positive.

C.3 Implementation

We next discuss some choices we made in the implementation of our column generation algorithm.

C.3.1 Initial columns:

We generate an initial collection $\mathcal{P}^*$ of unsplittable paths and their corresponding columns as follows. We first include all arcs from source to supply nodes in $\mathcal{P}^*$. Then we consider every splittable node w in sequence. For each such node, we consider every arc of the form (v, w) in the network. If v is splittable, we include (v, w) in $\mathcal{P}^*$. Otherwise, we include (v, w) to a list $\mathcal{L}$ of partially constructed unsplittable paths. We then iteratively scan through the network for arcs (t, u) such that $\mathcal{L}$ contains a partial path p starting at u not containing t. If t is splittable, we remove p from $\mathcal{L}$ and add unsplittable path $(t, u) - p$ to $\mathcal{P}^*$. Otherwise, we replace p with path $(t, u) - p$ in $\mathcal{L}$. We iterate the process until $\mathcal{L}$ until no arc satisfying the above conditions can be found. At that point, we move on to the next splittable node v. For our instances, this procedure produces a subset of paths for which (23a)-(23d) is feasible.

C.3.2 Solution of the pricing problem:

We obtain new columns with positive reduced costs by solving formulation (25a)-(25d) using CPLEX for distinct pairs of splittable nodes, and for different capacity thresholds κ. During the initial rounds of column generation, we only add columns whose reduced cost is larger than a pre-set threshold θ. We decrease this threshold if little progress in bound is observed over multiple consecutive iterations. During each round, we consider each node w and limit the number of columns generated that correspond to unsplittable paths incoming to w to be no more than a pre-defined threshold ξ_w. To provide variety in the type of columns generated, we randomize the order in which we consider the nodes v when solving (25a)-(25d). If progress across consecutive iterations is not sufficiently large, we increase this threshold ξ_w and we decrease θ. After all nodes w have been considered and corresponding columns have been generated, these columns are added to the restricted LP which is then resolved. The above procedure is then iterated to generate new columns.

C.3.3 Stabilization:

Stabilization is a technique that has proven very effective in accelerating the convergence of column generation algorithms; see [8] for a description. It is particularly well-suited for problems containing a *convexity constraint*, as in this case, the solution of the pricing problem also allows for the creation of a dual feasible solution from which an upper bound on the full model can be obtained. Although our model does not naturally contain a convexity constraint, we add the redundant inequality $\sum_{e \in E} x_e \leq U$ where $U = \min\{\sum_{e \in E} u_e, \min\{\sum_{i \in S} s_i, \sum_{j \in D} d_j\}(n-1)\}$ as a substitute. Even though the dual bounds we obtain from it are very weak, we found the associated stabilized cut generation technique to converge faster than naïve implementations. In particular, our stabilized implementation uses a piecewise linear penalization with 6 pieces in the primal. The stabilization center is updated (*serious step*) when the dual bound substantially improve. In this situation, penalties are multiplied by a factor 0.9. When the dual bound does not substantially improve, the stabilization center is not updated (*null step*) and the penalties are multiplied by a factor 1.1. We terminate optimization when the largest reduced cost has value no larger than 0.0001 or when the primal and dual bounds are very close. Because these conditions are seldom met in our instances, we also terminate optimization when the total computational time expanded reaches a pre-set time limit. In our computation, optimality was not reached within the alloted time limit. For the networks with 500 nodes, the last iteration of the column generation procedure before termination only added very few columns. Further, the reduced costs of these columns were small. For this reason we believe that for these instances, the optimal value of the restricted LP at termination is close to the optimal value of the full LP. However, for larger networks, optimization was terminated as large number of columns were still generated in each iteration. For these instances, the values we obtained cannot directly be used as upper bounds on the values of GUFP.

C.3.4 Obtaining feasible solutions to GUFP:

At termination of the column generation procedure, we have available a large collection of unsplittable paths and their corresponding columns. We can create an MIP formulation (23a)-(23e) of GUFP over this subset of paths by introducing binary variables. We can then solve this model with CPLEX, imposing a maximum computational time of 500 seconds. It is clear that any mixed integer feasible solution to this model yields a feasible solution to GUFP.

C.4 Results

In Table 6, we report results we obtain with the column generation approach sketched in Section C.1. The first three columns have the same meaning as those of Table 4. The column "CPX LB" contains the best lower bound (integer solution) obtained by CPLEX after 5000 seconds running time. In the column labeled "CG", we report the value of the restricted LP of the column generation approach at termination. Note that because termination occurred before an optimal solution was obtained, this value cannot be guaranteed to be an upper bound on the value of GUFP. We do not report the upper bound obtained from stabilization as it is an order of magnitude larger than that we obtain with other methods. The columns labeled "ITR" and "T" provide the number of iterations performed before termination together with the time used for these computations. The column labeled "CG-MIP" contains the value of the feasible solution of GUFP obtained by solving an IP model created from the restricted LP available at the termination of the column generation procedure. The following column labeled "T" records the computational time CPLEX required to obtain an optimal solution to this model.

We observe that, although we do not obtain guaranteed upper bounds through column generation, the columns generated are useful in identifying good quality solutions. In particular, for all instances with 1000 nodes or less, the feasible solutions obtained were better than those computed by CPLEX. This result however does not seem to extend to larger problems. This can probably be attributed to the fact that our column generation algorithm was not close to optimality at termination in those cases.

Table 6: Evaluating column generation.

N	#	LP	CPX LB	CG	ITR	T	CG-MIP	T
500	1	86400	65220	68086.1	72	4500*	67800	6.2
	2	107520	87160	89708.8	53	4500*	89400	38.8
	3	104620	82440	87769.2	66	4500*	87580	3.7
	4	136520	109400	114004.8	105	4500*	113800	16.2
	5	133040	106000	112993.6	102	4500*	112820	4.2
1000	1	234100	179520	186523.1	67	4500*	185760	336.2
	2	203620	134760	141169.3	165	4500*	136640	500*
	3	226860	178020	187025.7	57	4500*	185980	500*
	4	226680	176960	179112.3	76	4500*	176640	500*
	5	222900	177740	182869.3	63	4500*	182360	420.1
2000	1	387300	285580	295818.6	40	4500*	291940	500*
	2	387820	259040	258781.2	58	4500*	241880	500*
	3	396480	242300	256012.5	65	4500*	234620	500*
	4	414000	253780	270124.7	70	4500*	248760	500*
	5	397260	290900	287666.9	55	4500*	280420	500*

D Tables

In this section, we provide supporting tables omitted from the paper.

Table 7: Convex hull description of $\widehat{S}^{3,3}$ for Example 3.

$(A1)$	x_1	$+x_2$	$+x_3$	$-y_1$	$-y_2$	$-y_3$	$=$	0
$(A2)$		$-x_2$					$\leq$	0
$(A3)$			$-x_3$				$\leq$	0
$(A4)$				$-y_1$			$\leq$	0
$(A5)$					$-y_2$		$\leq$	0
$(A6)$						$-y_3$	$\leq$	0
$(A7)$		x_2	$+x_3$	$-y_1$	$-y_2$	$-y_3$	$\leq$	0
$(A8)$		$-x_2$	$-x_3$	$+y_1$	$+y_2$	$+y_3$	$\leq$	3
$(A9)$				y_1			$\leq$	5
$(A10)$		x_2					$\leq$	7
$(A11)$		$-2x_2$	$-2x_3$	$+5y_1$			$\leq$	15
$(A12)$			$2x_3$		$+y_2$		$\leq$	33
$(A13)$		$7x_2$			$-2y_2$	$-2y_3$	$\leq$	35
$(A14)$		$5x_2$	$-2x_3$	$+5y_1$			$\leq$	50
$(A15)$		$4x_2$	$+x_3$	$+4y_1$			$\leq$	53
$(A16)$			$-4x_3$		$+11y_2$		$\leq$	77
$(A17)$			$-6x_3$			$+13y_3$	$\leq$	91
$(A18)$		$-8x_2$	$-8x_3$	$+20y_1$	$+11y_2$		$\leq$	93
$(A19)$		$8x_2$	$+2x_3$	$+8y_1$	$+y_2$		$\leq$	113
$(A20)$		$-10x_2$	$-10x_3$	$+25y_1$		$+13y_3$	$\leq$	114
$(A21)$		$-2x_2$	$-8x_3$	$+20y_1$	$+11y_2$		$\leq$	123
$(A22)$		$-8x_2$	$+5x_3$	$+20y_1$			$\leq$	125
$(A23)$			$13x_3$			$-2y_3$	$\leq$	143
$(A24)$		$14x_2$			$+7y_2$	$-4y_3$	$\leq$	147
$(A25)$		$-x_2$	$-10x_3$	$+25y_1$		$+13y_3$	$\leq$	159
$(A26)$		$10x_2$	$-4x_3$	$+10y_1$	$+11y_2$		$\leq$	177
$(A27)$		$21x_2$			$-6y_2$	$+7y_3$	$\leq$	196
$(A28)$		$28x_2$	$+13x_3$		$-8y_2$	$-8y_3$	$\leq$	205
$(A29)$		$14x_2$	$+4x_3$	$+16y_1$	$+y_2$		$\leq$	205
$(A30)$		$-44x_2$	$-56x_3$		$+77y_2$		$\leq$	231
$(A31)$		$15x_2$	$-6x_3$	$+15y_1$		$+13y_3$	$\leq$	241
$(A32)$		$-52x_2$	$-70x_3$			$+91y_3$	$\leq$	273
$(A33)$		$-4x_2$	$+28x_3$		$+7y_2$		$\leq$	385
$(A34)$		$-4x_2$	$+28x_3$	$+10y_1$	$+7y_2$		$\leq$	415
$(A35)$		$56x_2$	$+26x_3$		$-11y_2$	$-16y_3$	$\leq$	445
$(A36)$		$-32x_2$	$+20x_3$	$+80y_1$	$+5y_2$		$\leq$	515
$(A37)$		$39x_2$	$+13x_3$	$+39y_1$		$-2y_3$	$\leq$	533
$(A38)$		$-52x_2$	$-70x_3$		$+91y_2$	$+91y_3$	$\leq$	546
$(A39)$		$-44x_2$	$-56x_3$	$+110y_1$	$+77y_2$		$\leq$	561
$(A40)$		$-52x_2$	$-70x_3$	$+130y_1$		$+91y_3$	$\leq$	663
$(A41)$			$143x_3$		$-78y_2$	$-88y_3$	$\leq$	715
$(A42)$		$-104x_2$	$-140x_3$		$+203y_2$	$+182y_3$	$\leq$	1239
$(A43)$		$-78x_2$	$+65x_3$	$+195y_1$		$-10y_3$	$\leq$	1300
$(A44)$		$273x_2$	$+143x_3$		$-78y_2$	$-88y_3$	$\leq$	2080
$(A45)$		$-220x_2$	$-280x_3$	45	$+385y_2$	$+364y_3$	$\leq$	2247

Table 8: Derivation of inequalities for $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ through lifting for Example 5.

Label	Fixing	Seed inequality
$(A11)$	x_3, y_3	$-2x_2 + 5y_1 \leq 15$
$(A12)$	x_1, y_1	$2x_3 + y_2 \leq 33$
$(A13)$	y_3, x_3	$7x_2 - 2y_2 \leq 35$
$(A14)$	(x_2, y_2)	$-2x_3 + 5y_1 \leq 15$
$(A15)$	(x_3, y_3)	$x_2 + y_1 \leq 10$
$(A16)$	x_1, y_3	$-4x_3 + 11y_2 \leq 77$
$(A17)$	x_1, y_1	$-6x_3 + 13y_3 \leq 91$
$(A18)$	(x_3, y_2)	$-2x_2 + 5y_1 \leq 15$
$(A19)$	(x_2, y_2)	$x_3 + 4y_1 \leq 25$
$(A20)$	(x_3, y_3)	$-2x_2 + 5y_1 \leq 15$
$(A21)$	(x_2, y_1)	$-8x_3 + 11y_2 \leq 33$
$(A22)$	(x_3, y_3)	$-2x_2 + 5y_1 \leq 15$
$(A23)$	x_1, y_1	$13x_3 - 2y_3 \leq 143$
$(A24)$	x_1, y_3	$2x_2 + y_2 \leq 21$
$(A25)$	(x_2, y_1)	$-10x_3 + 13y_3 \leq 39$
$(A26)$	(x_2, y_2)	$-2x_3 + 5y_1 \leq 15$
$(A27)$	(x_3, y_3)	$7x_2 - 2y_2 \leq 35$
$(A28)$	(x_3, y_3)	$7x_2 - 2y_2 \leq 35$
$(A29)$	(x_2, y_2)	$x_3 + 4y_1 \leq 25$
$(A30)$	x_2, y_1	$-8x_3 + 11y_2 \leq 33$
$(A31)$	(x_3, y_3)	$x_2 + y_1 \leq 10$
$(A32)$	x_2, y_2	$-10x_3 + 13y_3 \leq 39$
$(A33)$	(x_3, y_3)	$-4x_2 + 7y_3 \leq 21$
$(A34)$	(x_3, y_2)	$-2x_2 + 5y_1 \leq 15$
$(A35)$	(x_3, y_2)	$7x_2 - 2y_3 \leq 35$
$(A36)$	(x_2, y_1)	$4x_3 + y_2 \leq 55$
$(A37)$	(x_3, y_3)	$x_2 + y_1 \leq 10$
$(A38)$	(x_3, y_3)	$-4x_2 + 7y_2 \leq 21$
$(A39)$	(x_3, y_2)	$-2x_2 + 5y_1 \leq 15$
$(A40)$	(x_3, y_3)	$-2x_2 + 5y_1 \leq 15$
$(A41)$	x_2, y_2	$13x_3 - 8y_3 \leq 65$
$(A42)$	(x_3, y_2)	$-4x_2 + 7y_3 \leq 21$
$(A43)$	(x_3, y_3)	$-2x_2 + 5y_1 \leq 15$
$(A44)$	(x_3, y_3)	$7x_2 - 2y_2 \leq 35$
$(A45)$	(x_2, y_2)	$-10x_3 + 13y_3 \leq 39$

Entries of the form x_i, y_j denote that variables x_i and y_j are fixed at zero. Entries of the form (x_i, y_j) denote that x_i and y_j are fixed at $w_{i,j}$.

Table 9: Derivation of inequalities for $\mathcal{P}\widehat{\mathcal{S}}^{3,3}$ for Example 4.

Label	Ref.	i	j	k	L	P	h
$(A11)$	(11)	2	1	1	$\emptyset$	$\emptyset$	3
$(A12)$	(11)	3	2	2	$\emptyset$	$\{3\}$	3
$(A13)$	(22)	2	2	1	$\emptyset$	$\emptyset$	3
$(A14)$	(11)	2	1	1	$\emptyset$	$\{2\}$	3
$(A15)$	(11)	2	1	1	$\emptyset$	$\{2,3\}$	3
$(A16)$	(11)	3	2	2	$\emptyset$	$\emptyset$	3
$(A17)$	(11)	3	3	2	$\emptyset$	$\emptyset$	3
$(A22)$	(11)	2	1	1	$\emptyset$	$\{3\}$	3
$(A23)$	(22)	3	3	2	$\emptyset$	$\emptyset$	3
$(A24)$	(22)	2	2	1	$\emptyset$	$\{2\}$	3
$(A27)$	(22)	2	2	1	$\emptyset$	$\{3\}$	3
$(A30)$	(11)	3	2	1	$\emptyset$	$\emptyset$	3
$(A32)$	(11)	3	3	1	$\emptyset$	$\emptyset$	3
$(A33)$	(11)	3	2	1	$\emptyset$	$\{3\}$	3
$(A34)$	(11)	3	2	1	$\{1\}$	$\{3\}$	3
$(A37)$	(11)	2	1	1	$\emptyset$	$\{2,3\}$	2
$(A38)$	(11)	3	3	1	$\{2\}$	$\emptyset$	3
$(A39)$	(11)	3	2	1	$\{1\}$	$\emptyset$	3
$(A40)$	(11)	3	3	1	$\{1\}$	$\emptyset$	3
$(A41)$	(22)	3	3	1	$\emptyset$	$\emptyset$	3
$(A43)$	(11)	2	1	1	$\emptyset$	$\{3\}$	2
$(A44)$	(22)	3	3	1	$\{2\}$	$\emptyset$	3