Robust Optimal Discrete Arc Sizing for Tree-Shaped Potential Networks

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Abstract. We consider the problem of discrete arc sizing for tree-shaped potential networks with respect to infinitely many demand scenarios. This means that the arc sizes need to be feasible for an infinite set of scenarios. The problem can be seen as a strictly robust counterpart of a single-scenario network design problem, which is shown to be NP-complete even on trees. In order to obtain a tractable problem, we introduce a method for generating a finite scenario set such that optimality of a sizing for this finite set implies the sizing's optimality for the originally given infinite set of scenarios. We further prove that the size of the finite scenario set is quadratically bounded above in the number of nodes of the underlying tree and that it can be computed in polynomial time. The resulting problem can then be solved as a standard mixed-integer linear optimization problem. Finally, we show the applicability of our theoretical results by computing globally optimal arc sizes for a realistic hydrogen transport network of Eastern Germany.

1. Introduction

Potential-based flows are an extension of classical network flows because these flows depend on the potential gradients at the incident nodes of the underlying graph. They play an important role in the field of energy transport networks such as hydrogen, gas, water, or power networks. In such networks, the interaction between flows and potentials is determined by physical laws. On the one hand, these physical laws typically make the potential-based flows unique—but on the other hand, they often lead to nonlinear optimization problems. In the literature, potential networks have mainly been studied regarding the uniqueness of solutions and in the light of classical algorithmic flow problems; see the recent paper Groß et al. (2017) and the references therein for an overview.

In classical problems for potential networks, flows and potentials are to be optimized. Here, we consider an extension of this classical potential-based flow setting in which we also optimize over a finite set of possible capacities, i.e., the arc sizes of the network, that influence the physics models that describe the relation between arc flows and potentials on incident nodes. Thus, we consider a special discrete network design problem for potential networks and call it the problem of discrete arc sizing in potential networks. This problem is, in general, a very challenging mixed-integer nonlinear optimization problem (MINLP). In order to arrive at a more accessible problem, both in theory and in practice, we restrict ourselves to the case of tree-shaped potential networks.

The topic of optimally choosing arc capacities plays an important role in many real-world applications such as in hydrogen, gas, water, or power networks. For example, in Hansen et al. (1991), discrete pipe-sizing of pipeline networks with

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respect to (w.r.t.) a single scenario is discussed and linear programming techniques are applied. To this end, linear subproblems are set up and solved to adjust the discrete selection of diameters. In Shiono and Suzuki (2016), continuous and discrete pipe-sizing of a tree-shaped gas distribution network with a single source and a single scenario is studied. For the case of continuous pipe-sizing, an analytic approach is presented, which is then used to approximate the discrete case. Furthermore, the problem of discrete arc sizing is considered as a part of capacity expansion planning for gas transportation networks in Andre et al. (2009). There, the authors first identify a set of pipelines by using mixed-integer nonlinear models in order to afterward solve the discrete arc sizing problem for this selection of pipes. The latter is done using a branch-and-bound approach. Again, only the single-scenario case is considered. A more general nonlinear nonconvex network design problem is considered in Humpola and Fügenschuh (2015). It is solved w.r.t. one scenario with the help of branch-and-bound, spatial branching, and special relaxations of the occurring subproblems. In Vuffray et al. (2015) the robust minimum loss problem in a dissipative flow network w.r.t. infinitely many scenarios is studied. This problem consists of minimizing costs w.r.t. uncertain demand at pre-specified exits, whereas there is no limitation for the supply of entry nodes. Mainly due to the latter, the authors can reduce the considered infinite set of scenarios to two scenarios. In Bragalli et al. (2008), Bragalli, D'Ambrosio, et al. (2006), and Bragalli et al. (2012), the problem of choosing optimal diameters for water distribution networks is studied. The problem is solved using mixed-integer nonlinear programming techniques. Multiple-scenario cases are not considered. In Afshar and Mariño (2007), a self-adapting genetic algorithm for pipe network optimization is presented and the rehabilitation of a water supply network w.r.t. one scenario is considered. A more general approach to the problem is given in Raghunathan (2013), where global optimization of nonlinear network design problems is addressed. In this paper, an MINLP model is studied and a linearization-based approach is used to solve the problem for the single-scenario case.

The papers cited so far all have the following in common: They consider a physical model that relates arc flows to node potentials (like gas pressure or water heights) at nodes. Moreover, this relation is typically nonlinear. All these models are captured in the general framework of potential-based flows considered in this paper. An additional similarity is that all discussed publications, except of Vuffray et al. (2015), consider the single-scenario case.

We study the optimal choice of discrete arc sizes for the multi-scenario case, where the uncertainty of our scenarios possibly affects every entry and exit node. Consequently, none of the results in Vuffray et al. (2015) is applicable to our case. Moreover, the considered scenario set is of infinite size. This is also the reason that our model and the presented solution strategy can be seen in a completely different light—namely in the light of optimization under uncertainty. In fact, the model that we consider is a robust counterpart of a single-scenario network design MINLP, where the uncertain aspect is robustified in the way of strict robust optimization; cf., e.g., Ben-Tal et al. (2009). To the best of our knowledge, no theoretical results or solution methods for the problem of discrete arc sizing exists in the case of infinitely many scenarios, affecting entries and exits, if the underlying nonlinear physical constraints are present. Neglecting the latter, however, there are some related approaches from the field of robust network design. In this field, the computation of minimum-cost edge capacities that yield the feasibility of routed flows through a network for different scenarios is considered. A popular approach that is discussed in Ben-Ameur and Kerivin (2005) describes the set of infinitely many scenarios as a polyhedron. Many other approaches use the so-called hose model, introduced in Duffield et al. (1999) and Fingerhut et al. (1997), to cope with infinitely many scenarios; cf., e.g., Cacchiani et al. (2016). Both the polyhedral and the hose model consider the demand between to given nodes that determines how much flow must be routed between these two nodes. Our approach only considers the demand at the single nodes separately without imposing dependencies of demands between different nodes. The methods presented in the cited papers on robust network design usually rely on constraint generation and mixed-integer linear programming techniques; cf., e.g., Altın et al. (2007), Ben-Ameur and Kerivin (2005), and Erlebach and Ruegg (2004). All the discussed approaches of robust network design exploit the property that the "capacity" of an edge does not depend on its incident edges and their capacities. In contrast to this, arc sizing needs to take these dependencies into account in the context of potential-based flows. Thus, the methods from the literature cannot be used for the problem of discrete arc sizing for potential networks as it is discussed in this paper.

We consider tree-shaped potential networks in which node demands are bounded by lower and upper node capacities. A balanced set of nodal supplies and demands that satisfies these capacities is called a scenario. The problem that we solve is the computation of a minimum-cost arc sizing that is feasible for all possible scenarios. At a first glance, this can be seen as the strictly robust counterpart of a single-scenario mixed-integer nonlinear and nonconvex problem, which is—without applying any further techniques—not tractable. Our contribution is the following: We present a method for generating a finite set of scenarios (the so-called finite feasible design set) such that optimality of a sizing w.r.t. this finite set is shown to imply the optimality w.r.t. the originally given infinite set of scenarios. Moreover, the number of scenarios in this finite set is shown to be bounded above by $|V|^2$, i.e., by the square of the number of nodes of the considered tree. Thus, we prove that it is possible to construct an equivalent and finite-dimensional counterpart of the underlying robust optimization problem. Moreover, we show that this finite counterpart can be generated in polynomial time. The assumptions that we need for proving these results are mild and are satisfied in typical applications like in hydrogen networks. The latter will serve as a running example throughout this paper. Lastly, we show that the resulting problem can be solved as a standard mixed-integer linear optimization problem and we apply our techniques for computing optimal pipe diameters for a realistic hydrogen network of Eastern Germany. For more details about this application we refer to Reuss et al. (2017), Robinius (2015), Robinius et al. (2017), Syranidis et al. (2018), and Welder et al. (2017).

The remainder of the paper is structured as follows: We formally define potential networks, collect preliminary results, and introduce the considered problem of discrete arc sizing for tree-shaped potential networks in Section 2. In Section 3 we prove the existence of a finite scenario set with the property that optimality of arc sizes w.r.t. this finite set implies optimality for the originally given infinite set of scenarios. The proof of this property mainly relies on solving properly chosen linear programs. In contrast to that, a purely combinatorial algorithm for computing these finite sets, including a complexity analysis, is presented in Section 4. In particular, we prove that the problem of discrete arc sizing is NP-complete even for tree-shaped potential networks. In Section 5 we illustrate the applicability of our approach by computing optimal pipe diameters for a realistic hydrogen transport network of Eastern Germany. The paper ends with some concluding remarks in Section 6.

2. Modeling and Basic Results

In this section, we introduce potential networks and present first results. Furthermore, we illustrate these results by an application in hydrogen pipeline networks.

At the end of this section, we give the specific definition and a mixed-integer model of the problem of discrete arc sizing in potential networks.

2.1. **Potential Networks.** First, we formally introduce potential networks.

Definition 2.1. A *potential network* consists of a connected digraph $G = (V, A)$ with node set V and arc set A. To each node $v \in V$, a potential π_v is assigned with lower and upper bounds $0 \leq \underline{\pi}_v \leq \bar{\pi}_v \in \mathbb{R}$. Moreover, to every arc $a \in A$, a flow q_a is assigned.

Note that the flow q_a on an arc $a = (u, v)$ can be negative with the interpretation of flow against the direction of the arc, whereas flow from u to v is positive. For an arc $a = (u, v) \in A$, the potential difference at its incident nodes u and v depends on the flow q_a and the arc size d_a. We model this relation as

$$\pi_u - \pi_v = \psi_a(d_a, q_a).$$

Here, $\psi_a(d_a, \cdot) : \mathbb{R} \to \mathbb{R}$ is a *potential function* that satisfies the following properties:

(1) $\psi_a(d_a, \cdot)$ is continuous,
(2) $\psi_a(d_a, \cdot)$ is strictly increasing, and
(3) $\psi_a(d_a, \cdot)$ is odd w.r.t. the second argument q_a, i.e., $\psi_a(d_a, -q_a) = -\psi_a(d_a, q_a)$.

The potential function ψ_a varies for different applications. In the following example we describe a typical choice for ψ_a in hydrogen networks. The assumptions that $\psi_a(d_a, \cdot)$ is continuous and strictly increasing are natural in the physical contexts of utility networks like gas, power, water, or hydrogen networks. Moreover, the condition that $\psi_a(d_a, \cdot)$ is odd makes the situation symmetric w.r.t. the direction of flow.

Example 2.2 (Hydrogen Transport Networks)**.** Stationary and isothermal models of hydrogen transport networks often use the Weymouth equation; see Mischner et al. (2011), Ríos-Mercado and Borraz-Sánchez (2015), and Weymouth (1912), and the chapter Fügenschuh et al. (2015) in the the recent book Koch et al. (2015). The latter describes a detailed modeling of stationary and isothermal gas flow in pipeline networks. In this application, lines correspond to pipes, potentials to squared pressure levels at nodes, and flows are physical mass flows. Furthermore, discrete arc sizing translates to the discrete selection of pipe diameters. For a pipe $a = (u, v) \in A$, the relation between mass flow, diameter, and potentials can be approximated by the Weymouth-type pressure loss equation

$$\pi_u - \pi_v = \psi_a(d_a, q_a), \quad \psi_a(d_a, q_a) = \beta_a \frac{\lambda(d_a, q_a)}{d_a^5} q_a |q_a|. \tag{1}$$

The arc-specific constant $\beta_a \geq 0$ depends on technical parameters of the pipe such as length, temperature, and specific properties of hydrogen. Additionally, the so-called friction factor λ depends on the chosen diameter of the pipe and the mass flow through the pipe. We refer to the above mentioned papers and books for a more detailed description of the pressure loss function (1). For a fixed mass flow, fixed pipe diameters, and a fixed initial pressure at the beginning of a pipe, we illustrate the pressure loss in Figure 1. As we can see, the size of the diameter strongly influences the pressure drop in the hydrogen pipe. Hence, the selection of diameters plays an important role for constructing new pipeline networks.

Further applications of potential networks in natural gas, water, and lossless DC power flow networks are described in Groß et al. (2017).

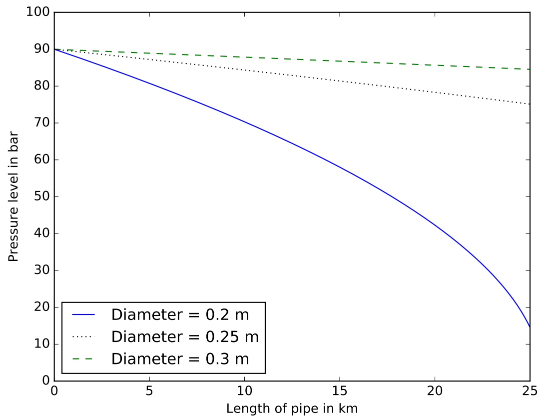

FIGURE 1. Pressure levels in bar (y-axis) for a mass flow of 5 kg/s and an initial pressure of 90 bar

2.2. **Potential-Based b-Flows.** Now, we consider node balances $b \in \mathbb{R}^{|V|}$ with $\sum_{v \in V} b_v = 0$, which we also call *scenarios*. They describe a specific supply and demand situation at the nodes. In a potential network $G = (V, A)$ with arc sizes $d \in \mathbb{R}^{|A|}$, a *(feasible) potential-based b-flow* (q, π) related to a given scenario $b \in \mathbb{R}^{|V|}$ consists of flows $q \in \mathbb{R}^{|A|}$ and potentials $\pi \in \mathbb{R}^{|V|}$ that satisfy the conditions

$$\sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = b_v, \qquad v \in V, \tag{2a}$$

$$\pi_u - \pi_v = \psi_a(d_a, q_a), \qquad a = (u, v) \in A, \tag{2b}$$

$$\underline{\pi}_v \le \pi_v \le \bar{\pi}_v, \qquad v \in V, \tag{2c}$$

where we used the standard δ-notation for the sets of in- and outgoing arcs, i.e.,

$$\delta^{\text{in}}(v) = \{a \in A \colon a = (u, v)\}, \quad \delta^{\text{out}}(v) = \{a \in A \colon a = (v, u)\}.$$

Furthermore, we say that a (feasible) potential-based b-flow (q, π) is in *standard form* if it satisfies the condition

$$\exists v \in V \colon \pi_v = \bar{\pi}_v. \tag{3}$$

A known property of potential networks is that, neglecting the potential bounds (2c), every balanced set of supplies and demands can be transported.

Now, we present some results about the relation between scenarios and potential-based b-flows. These results allow us to focus on potential-based b-flows in standard form. Using the next three lemmas, we can show that for given arc sizes and a given scenario, a corresponding feasible potential-based b-flow exists if and only if a corresponding potential-based b-flow in standard form exists. The first lemma shows that in tree-shaped networks there is a 1-1 correspondence between scenarios b and flows q that satisfy Condition (2a). Furthermore, we can compute the flows for a given scenario in linear time.

Lemma 2.3. *Let $G = (V, A)$ be a tree-shaped potential network. Then, for a given scenario $b \in \mathbb{R}^{|V|}$ we can compute unique flows $q \in \mathbb{R}^{|A|}$ that satisfy Condition (2a) in $O(|V|)$. Additionally, for given flows $q \in \mathbb{R}^{|A|}$ satisfying Condition (2a) we can compute the unique corresponding scenario b in $O(|V|)$.*

Proof. Let $V(u) := \{v \in V \colon (u, v) \in A \vee (v, u) \in A\}$ contain the neighbors of a node $u \in V$ and let $L := \{u \in V \colon |V(u)| = 1\}$ be the set of leaf nodes. Additionally, we assume w.l.o.g. that G is an out-tree. For a given scenario $b \in \mathbb{R}^{|V|}$, we compute the corresponding unique flows as follows. As long as the arc set A is nonempty we do the following: For each leaf node $u \in L$ and unique arc $a \in \delta^{\text{in}}(u)$ we set $q_a = -b_u$ and then update $b_v = b_v - q_a$ for the unique neighbor $v \in V(u)$. Then, we set $V = V \setminus L,\ A = A \setminus \bigcup_{u \in L} \delta^{\text{in}}(u)$, and iterate. This procedure computes unique flows for a given scenario and terminates after $|A| = |V| - 1$ iterations. For given flows satisfying Condition (2a), the computation of the scenario is trivial. □

Given a scenario, a potential-based b-flow is uniquely determined if one of its potentials is fixed. This is formally stated in the next lemma, which is a slightly adapted version of the results in Collins et al. (1978) and Ríos-Mercado, Wu, et al. (2002).

Lemma 2.4. *Let $G = (V, A)$ be a potential network with fixed arc sizes $d \in \mathbb{R}^{|A|}$ and let $b \in \mathbb{R}^{|V|}$ be a scenario. Furthermore, assume that no potential bounds are given and that for a given node $v \in V$, the potential π_v is fixed. Then, there exists a unique feasible potential-based b-flow (q, π).*

With the help of the next lemma, which is an adaption of Szabó (2012), we see that shifting the potentials of a potential-based b-flow again leads to a potential-based b-flow.

Lemma 2.5. *Let $G = (V, A)$ be a potential network with fixed arc sizes $d \in \mathbb{R}^{|A|}$ and let $b \in \mathbb{R}^{|V|}$ be a scenario. Furthermore, assume that no potential bounds are given. If (q, π) is a potential-based b-flow, then $(q, \pi + c\mathbb{1})$ is a potential-based b-flow for every $c \in \mathbb{R}$, where $\mathbb{1}$ is the vector of ones in appropriate dimension.*

In contrast to the last lemma, we cannot shift the potentials of a potential-based b-flow in standard form upwards without violating an upper potential bound $\bar{\pi}_v$. Hence, we can prove the next lemma, which allows us to focus on potential-based b-flow in standard form in the following.

Lemma 2.6. *Let $G = (V, A)$ be a potential network with arc sizes $d \in \mathbb{R}^{|A|}$ and let $b \in \mathbb{R}^{|V|}$ be a scenario. Then, a feasible potential-based b-flow exists if and only if a corresponding potential-based b-flow in standard form exists.*

Proof. The lemma follows directly by applying Lemma 2.4 and 2.5. □

Furthermore, it is easy to see that if a potential-based b-flow in standard form exists, then it is unique.

Now, we introduce a scenario space for which we subsequently define the problem of discrete arc sizing. In what follows, we assume lower and upper node capacities $\underline{b}_v \le 0 \le \bar{b}_v$ that restrict the supply or demand b_v of a node $v \in V$ in all considered scenarios. With this, we obtain the scenario space

$$B := \left\{ b \in \mathbb{R}^{|V|} \colon \sum_{v \in V} b_v = 0 \text{ and } \underline{b}_v \le b_v \le \bar{b}_v,\ v \in V \right\}. \tag{4}$$

For each arc $a \in A$, we consider a finite set of arc sizes D_a with $|D_a| < \infty$. Additionally, $c_{a,d} \in \mathbb{R}$ describes the cost for arc size $d_a \in D_a$. Obviously, computing optimal arc sizes for a potential network leads to a discrete optimization problem that we derive in the following. In words, the problem of discrete arc sizing for a tree-shaped potential network $G = (V, A)$ consists of selecting the cheapest arc sizes $d_a \in D_a$ for all $a \in A$ so that for each scenario $b \in B$, its potential-based b-flow in standard form exists. This problem can be modeled as

$$\min_{x,q,\pi} \quad \sum_{a\in A}\sum_{d\in D_a} c_{a,d}\, x_{a,d} \tag{5a}$$

$$\text{s.t.} \quad \pi_{b_u} - \pi_{b_v} = \psi_a\left(\sum_{d\in D_a} d\, x_{a,d}, q_{b_a}\right), \qquad a=(u,v)\in A,\ b\in B, \tag{5b}$$

$$\sum_{a\in\delta^{\text{out}}(v)} q_{b_a} - \sum_{a\in\delta^{\text{in}}(v)} q_{b_a} = b_v, \qquad v\in V,\ b\in B, \tag{5c}$$

$$\underline{\pi}_v \le \pi_{b_v} \le \bar{\pi}_v, \qquad v\in V,\ b\in B, \tag{5d}$$

$$\sum_{d\in D_a} x_{a,d} = 1, \qquad a\in A, \tag{5e}$$

$$x_{a,d}\in\{0,1\}, \qquad a\in A,\ d\in D_a. \tag{5f}$$

Problem (5) is an infinite-dimensional mixed-integer nonlinear program. The nonlinearity appears in the potential function ψ_a in Constraint (5b). The problem consists of infinitely many flow $(q_b)_{b\in B}$ and potential $(\pi_b)_{b\in B}$ variable vectors and infinitely many constraints (5b)–(5d) because B contains infinitely many scenarios. Thus, Problem (5) is not tractable in its current form.

A 1-1 correspondence between a scenario $b\in B$ and flows $q\in\mathbb{R}^{|A|}$ satisfying Constraint (2a) is guaranteed by Lemma 2.3. This makes it possible to compute the unique values of the flow variables (for a given scenario) before optimizing (5). Hence, we collect the unique flows corresponding to a subset of scenarios $K\subseteq B$ in the set

$$F(K) := \left\{ q\in\mathbb{R}^{|A|} \colon \exists b\in K \text{ with} \sum_{a\in\delta^{\text{out}}(v)} q_a - \sum_{a\in\delta^{\text{in}}(v)} q_a = b_v,\ v\in V \right\}. \tag{6}$$

With this notation at hand, we can give an infinite-dimensional mixed-integer nonlinear program that is equivalent to (5) and that only includes known flow parameters instead of flow variables. The equivalent formulation is very similar to the one in (5) except of replacing Constraint (5b) with

$$\pi_{b_u} - \pi_{b_v} = \psi_a\left(\sum_{d\in D_a} d\, x_{a,d}, q_{b_a}\right), \quad a=(u,v)\in A,\ q_b\in F(B). \tag{7}$$

Moreover, we now may neglect Constraint (5c) because it is implicitly satisfied due to (6) and (7).

Finally, we can also get rid of the nonlinearity by reformulating the nonlinear Constraint (7) to a linear constraint via

$$\pi_{b_u} - \pi_{b_v} = \sum_{d\in D_a} x_{a,d}\psi_a\left(d, q_{b_a}\right), \quad a=(u,v)\in A,\ q_b\in F(B). \tag{8}$$

This reformulation is linear because the potential function ψ_a depends only on known parameters and not on variables as before. Hence, it is a constant value in each constraint.

Putting all together, we can model the problem of discrete arc sizing with the help of the infinite-dimensional linear mixed-integer problem (MIP)

$$\min_{\pi,x} \quad \sum_{a\in A}\sum_{d\in D_a} c_{a,d}\, x_{a,d} \quad \text{s.t.} \quad (8), (5d)\text{–}(5f). \tag{9}$$

Of course, (9) is still an infinite-dimensional problem because we still consider infinitely many potential-based b-flows. Thus, a reduction of the considered scenario set is necessary to solve the problem.

3. Existence of a Finite Feasible Design Set

In this section, we prove that we can reduce the infinite-dimensional model (9) to a MIP of polynomial size without changing the optimal solution. To this end, we prove that a finite subset $B^* \subset B$ of scenarios exists so that an optimal discrete arc sizing w.r.t. B^* is also optimal to all infinitely many scenarios of B. We call a scenario set with this property a finite *feasible design set*. At the end of this section, we prove that a finite feasible design set B^* exists for which $|B^*|$ is quadratically bounded above in terms of the tree's nodes.

First, we introduce some terms and results that we need for proving the existence of a finite feasible design set. Due to the fact that flow is not influenced by the direction of the arcs, we define for two nodes $u, v \in V$ the so-called *flow-path* $P(u,v) = (V_{P(u,v)}, A_{P(u,v)})$ in which $V_{P(u,v)} \subset V$ contains the nodes of the path from u to v in the undirected version of the tree G and $A_{P(u,v)} \subset A$ contains the corresponding arcs of the path. This flow-path is unique in trees and it is symmetric, i.e., $P(u,v) = P(v,u)$ holds. For flows $q \in F(B)$, we say that node $u \in V$ *supplies* node $v \in V \setminus \{u\}$ if for each arc $a = (i,j) \in A_{P(u,v)}$, the conditions

$$V_{P(i,v)} \subset V_{P(j,v)} \implies q_a < 0, \quad V_{P(j,v)} \subset V_{P(i,v)} \implies q_a > 0$$

hold. For a scenario $b \in B$, we call a node $u \in V$ *active* if $b_u \neq 0$ is satisfied. The set of nodes V is partitioned into entry nodes $V_{\text{entries}} := \{v \in V : \bar{b}_v > 0, \underline{b}_v = 0\}$, exits nodes $V_{\text{exits}} := \{v \in V : \bar{b}_v = 0, \underline{b}_v < 0\}$, storage nodes $V_{\text{storages}} := \{v \in V : \bar{b}_v > 0, \underline{b}_v < 0\}$, and inner nodes $V_{\text{inner}} := \{v \in V : \bar{b}_v = \underline{b}_v = 0\}$. Additionally, we define the subset of nodes that can supply flow as $V_+ := V_{\text{entries}} \cup V_{\text{storages}}$ and the subset of nodes at which flow can be withdrawn as $V_- := V_{\text{exits}} \cup V_{\text{storages}}$. For simplification we call a node in the set V_+ an entry and a node in the set V_- an exit in the remainder of the paper.

Example 3.1 (Hydrogen Transportation Network)**.** In hydrogen networks, elements of V_+ are hydrogen suppliers, elements of V_- are hydrogen consumers, and elements of the intersection $V_+ \cap V_-$ are hydrogen storages. A scenario $b \in \mathbb{R}^{|V|}$ represents the supply and demand of suppliers, consumers, and storages. Considering sizing as the choice of diameters for pipes, the mixed-integer problem (9) corresponds to selecting the cheapest diameter for each pipe so that all possible scenarios are feasible w.r.t. given pressure conditions. Note that, in general, examples can be easily constructed in which more than one scenario has to be considered in order to guarantee the feasibility of a sizing for all scenarios in B. Thus, the reduction to a single-scenario case is not possible in general.

Next, we define a special type of scenarios, which constitutes the basis for proving the existence of a finite feasible design set. For a given entry and exit, the corresponding scenarios of this type have the property that the pressure drop on the flow-path between these two nodes is maximal. For a formal definition of these scenarios, we need a special node set depending on two different nodes u and v. This node set corresponds to the nodes of the connected component that includes the flow-path $P(u,v)$ after deleting all arcs that include u and are not part of $P(u,v)$. We define this node set with the help of the following equivalence relation: For a given node $u \in V$, we define the relation $\sim_u$ on two nodes $v, w \in V \setminus \{u\}$ by $v \sim_u w$ if and only if the flow-paths $P(u,v)$ and $P(u,w)$ share more than one node, i.e., $(V_{P(u,v)} \cap V_{P(u,w)}) \supset \{u\}$ holds. Then, for two different nodes $u, v \in V$, the above mentioned node set is described by $V^{\sim_u}(v) := \{w \in V : v \sim_u w\} \cup \{u\}$. If all nodes have the same upper pressure bounds and if the potential of an entry u in a given scenario is equal to this pressure bound, then only nodes of set $V^{\sim_u}(v)$ can supply

v in this scenario. This is the reason why this node set plays an important role for our special type of scenarios.

Definition 3.2. Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. The set $B_{u,v}$ contains all scenarios $b \in B$ for which the corresponding unique flows $q \in \mathbb{R}^{|A|}$ satisfy the following conditions:

(a) No node in $V \setminus V^{\sim_u}(v)$ supplies u.
(b) Node u supplies node v.
(c) The absolute flow $|q_a|$ on each arc $a \in A_{P(u,v)}$ is maximal w.r.t. all flows $\tilde{q} \in F(B)$ satisfying (a) and (b).

Next, we show the existence of at least one of the previously defined scenarios for each entry and exit node with the help of a linear program. At a first glance, one would expect that we have to solve $|A|$ many different problems to compute a scenario satisfying Definition 3.2 because of Property (c). We prove that it is sufficient to solve a single linear program to obtain one of these scenarios. To this end, we partition the set of arcs of an flow-path $A_{P(u,v)}$ into two sets: The arc set $A^+_{u,v}$ contains all arcs a of flow-path $P(u, v)$ for which q_a is positive if u supplies v and the set $A^-_{u,v}$ contains the remaining arcs:

$$A^+_{u,v} := \{a = (i, j) \in A_{P(u,v)} \colon V_{P(j,v)} \subset V_{P(i,v)}\},$$
$$A^-_{u,v} := A_{P(u,v)} \setminus A^+_{u,v}.$$

The above mentioned linear program is defined as follows

$$\max_{b,q} \quad \sum_{a \in A^+_{u,v}} q_a - \sum_{a \in A^-_{u,v}} q_a \tag{10a}$$
$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(w)} q_a - \sum_{a \in \delta^{\text{in}}(w)} q_a = b_w, \qquad w \in V, \tag{10b}$$
$$b_w = 0, \qquad w \in V \setminus V^{\sim_u}(v), \tag{10c}$$
$$\underline{b}_w \le b_w \le \bar{b}_w, \qquad w \in V. \tag{10d}$$

The linear program (10) has an optimal solution because the zero scenario together with its corresponding flows are feasible and because each feasible solution is bounded and an element of set B due to Constraints (10b) and (10d). Additionally, q are the corresponding flows to scenario b in every feasible solution (b, q).

The next theorem proves that an optimal solution of the linear program (10) provides a scenario of set $B_{u,v}$. To this end, we first show the following technical lemma.

Lemma 3.3. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. Additionally, let the orientation of G satisfy that the flow-path $P(u, v)$ is a directed path from u to v. If (b, q) is an optimal solution of (10), then $q_a \ge 0$ holds for all arcs $a \in A_{P(u,v)}$.*

Proof. Let (b, q) be an optimal solution of the linear program (10) and let $P_b := \{P(w_1, w_2) \colon b_{w_1} > 0,\ b_{w_2} < 0\}$ be the set of flow-paths in G that have an active entry node of b as start node and an active exit node of b as end node. We consider a decomposition of the given arc flows q into the following path flows

$$q_{(v_1,v_2)} = \sum_{P(w_1,w_2) \in P_b} \chi_{(v_1,v_2)}(P(w_1, w_2))\, f(P(w_1, w_2)), \quad (v_1, v_2) \in A,$$

in which $f(P(w_1, w_2))$ is the non-negative flow on flow-path $P(w_1, w_2)$. The function $\chi_{(v_1,v_2)}(P(w_1, w_2))$ evaluates to zero if (v_1, v_2) is not part of $P(w_1, w_2)$, to -1 if

(v_1, v_2) is part of $P(w_1, w_2)$ and if $P(v_1, w_2) \subset P(v_2, w_2)$ is satisfied, and otherwise to 1. Additionally, for each active entry w_1 the condition

$$\sum_{w_2 \in V :\, b_{w_2} < 0} f(P(w_1, w_2)) = b_{w_1}$$

and for each active exit w_2 the condition

$$\sum_{w_1 \in V :\, b_{w_1} > 0} f(P(w_1, w_2)) = |b_{w_2}|$$

is satisfied. For more detailed information about flow decompositions see Chapter 3.5 of the book by Ahuja et al. (1993).

Now, we assume that $\tilde{a} \in A_{P(u,v)}$ with $q_{\tilde{a}} < 0$ exists. For all flow-paths $P(w_1, w_2) \in P_b$ that satisfy $f(P(w_1, w_2)) > 0$ and contain arc $\tilde{a}$ with $\chi_{\tilde{a}}(P(w_1, w_2)) = -1$, we set $f(P(w_1, w_2)) = 0$. At least one of these flow-paths exists by assumption. Let b' and q' be the corresponding scenario and flow after these modifications. Due to the tree structure of G and by construction, $q'_a \geq q_a,\ a \in A_{P(u,v)}$, and $q'_{\tilde{a}} > q_{\tilde{a}}$ hold. Furthermore, (b', q') is a feasible solution for the linear program (10). Note that $A^-_{u,v}$ is empty because the flow-path $P(u, v)$ is a directed path from u to v. Hence, the inequality

$$\sum_{a \in A^+_{u,v}} q_a - \sum_{a \in A^-_{u,v}} q_a < \sum_{a \in A^+_{u,v}} q'_a - \sum_{a \in A^-_{u,v}} q'_a$$

is satisfied, which is a contradiction to the optimality of solution (b, q). □

Theorem 3.4. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. If (b, q) is an optimal solution of the linear program (10), then $b \in B_{u,v}$ holds.*

Proof. We choose the orientation of G such that the flow-path $P(u, v)$ is a directed path from u to v and, thus, the set $A^-_{u,v}$ is empty. Let (b, q) be an optimal solution of the linear program (10). We now show that (b, q) satisfies the Properties (a)–(c) of Definition 3.2. The first property follows from Condition (10c). Next, we prove that Condition (c) of Definition 3.2 is satisfied for (b, q). We assume that a scenario $b' \in B$ with corresponding flows q' exists that satisfies Properties (a) and (b). Additionally, we assume that an arc $\tilde{a} = (h_1, h_2) \in A_{P(u,v)}$ exists with $q'_{\tilde{a}} > q_{\tilde{a}}$. The flow $q_{\tilde{a}}$ is non-negative due to Lemma 3.3. Consequently, the flow $q'_{\tilde{a}}$ is positive. Because of Property (a) of Definition 3.2 and the tree structure of G, the flows on the arcs of the flow-path $P(u, v)$ are not influenced by the nodes in $V \setminus V^{\sim_u}(v)$. Hence, we can assume that scenario b' satisfies Condition (10c). Thus, scenario b' and its flows q' are feasible for the linear program (10). In analogy to the proof of Lemma 3.3, we consider the following flow decompositions for arc flows q and q':

$$q_{(v_1,v_2)} = \sum_{P(w_1,w_2) \in P_b} \chi_{(v_1,v_2)}(P(w_1, w_2))\, f(P(w_1, w_2)), \qquad (v_1, v_2) \in A,$$

$$q'_{(v_1,v_2)} = \sum_{P(w_1,w_2) \in P_{b'}} \chi_{(v_1,v_2)}(P(w_1, w_2))\, f'(P(w_1, w_2)), \qquad (v_1, v_2) \in A.$$

Furthermore, we know that for all flow-paths $P(w_1, w_2) \in P_b$ that contain arc $\tilde{a}$ with $\chi_{\tilde{a}}(P(w_1, w_2)) = -1$, the condition $f(P(w_1, w_2)) = 0$ holds. Due to this, the assumption $q'_{\tilde{a}} > q_{\tilde{a}}$, the tree structure of G, and Condition (10c), an entry $w_1 \in V^{\sim_u}(v)$ with $P(w_1, h_1) \subset P(w_1, h_2)$ and an exit $w_2 \in V^{\sim_u}(v)$ with $P(w_2, h_2) \subset P(w_2, h_1)$ exist so that

$$\sum_{\substack{P(w_1,w) \in P_b \\ \tilde{a} \in A_{P(w_1,w)}}} f(P(w_1, w)) < \bar{b}_{w_1}, \qquad \sum_{\substack{P(w,w_2) \in P_b \\ \tilde{a} \in A_{P(w,w_2)}}} f(P(w, w_2)) < |\underline{b}_{w_2}| \tag{11}$$

hold. Additionally, the flow-path $P(w_1, w_2)$ includes arc $\tilde{a}$. We now increase the flow on $P(w_1, w_2)$ and maintain the feasibility of (10). To this end, we define the following parameters:

$$\varepsilon_1 := \begin{cases} \beta_1 = f(P(w_1, w_3)), & \text{if } \bar{b}_{w_1} = b_{w_1}, \\ \bar{b}_{w_1} - b_{w_1}, & \text{if } \bar{b}_{w_1} > b_{w_1}, \end{cases}$$

$$\varepsilon_2 := \begin{cases} \beta_2 = f(P(w_4, w_2)), & \text{if } |\underline{b}_{w_2}| = |b_{w_2}|, \\ |\underline{b}_{w_2}| - |b_{w_2}|, & \text{if } |\underline{b}_{w_2}| > |b_{w_2}|, \end{cases}$$

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2\},$$

with

$$\beta_1 := \min_{\substack{P(w_1,w) \in P_b \\ \tilde{a} \notin A_{P(w_1,w)}}} \{f(P(w_1, w)) \colon f(P(w_1, w)) > 0\},$$

$$\beta_2 := \min_{\substack{P(w,w_2) \in P_b \\ \tilde{a} \notin A_{P(w,w_2)}}} \{f(P(w, w_2)) \colon f(P(w, w_2)) > 0\}.$$

Note that ε is positive due to the construction as well as (11). If $\bar{b}_{w_1} = b_{w_1}$, we decrease $P(w_1, w_3)$ by ε because w_1 is not able to supply more flow. If $|\underline{b}_{w_2}| = |b_{w_2}|$, we decrease $P(w_4, w_2)$ by ε because w_2 is not able to discharge more flow. Now, we increase $f(P(w_1, w_2))$ by ε. Due to the tree structure of G, these modifications do not decrease the flow on arcs of the flow-path $P(u, v)$. We call the obtained scenario $\tilde{b}$ and the corresponding flow is denoted by $\tilde{q}$. By construction $(\tilde{b}, \tilde{q})$ is a feasible solution for (10) and satisfies

$$\sum_{a \in A^+_{u,v}} q_a - \sum_{a \in A^-_{u,v}} q_a < \sum_{a \in A^+_{u,v}} \tilde{q}_a - \sum_{a \in A^-_{u,v}} \tilde{q}_a.$$

This is a contradiction to the optimality of (b, q). Hence, Property (c) is satisfied by (b, q).

Finally, we show that Property (b) is satisfied by (b, q). For an entry u and an exit v there exists a scenario in which u supplies v. This together with Lemma 3.3 and Property (c) leads to the result that (b, q) satisfies Property (b). In total, (b, q) satisfies Definition 3.2 and hence, b is an element of $B_{u,v}$. □

The next lemma directly follows from the previously shown Theorem 3.4 and the existence of solutions for (10).

Lemma 3.5. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. Then, the set $B_{u,v}$ is non-empty.*

Before we finally prove the existence of a finite feasible design set, we show two important properties of the scenarios of Definition 3.2. To this end, we define a special type of entry nodes and show their existence. We also note that all results that we discussed so far in this section do not depend on the node potentials in the networks and that they also hold for general flow networks. However, from now on we also make use of the presence of node potentials.

Definition 3.6. Let $G = (V, A)$ be a potential network with fixed arc sizes $d_a \in D_a$ for all $a \in A$. Let $b \in B$ be a scenario and (q, π) the corresponding potential-based b-flow in standard form. An active entry $u \in V_+$ satisfying $\pi_u = \bar{\pi}_u$ is called *potential indicating entry* for b. We denote the set of potential indicating entries for scenario b and arc sizes d by $I_{b,d}$.

The following lemma shows that for each non-zero scenario, a potential indicating entry exists if we consider the same upper pressure bound for every node.

Lemma 3.7. *Let $G = (V, A)$ be a potential network with fixed arc sizes $d_a \in D_a$ for all arcs $a \in A$ and $b \in B$ a non-zero scenario. Additionally, all nodes have the same upper pressure bound, i.e., $\bar{\pi}_v = \bar{\pi}$ for all $v \in V$. Then, a potential indicating entry $u \in I_{b,d}$ exists.*

Proof. Let (q, π) be the unique potential-based b-flow in standard form corresponding to b. At least one active entry exists in b because b is non-zero and (2a). Furthermore, an active entry $u \in V_+$ with

$$\pi_u = \max_{v \in V} \pi_v$$

exists due to Condition (2b) and due to the properties of potential functions. The latter mainly consists of the fact that if one node supplies another node, this always causes a potential drop between these nodes. Hence, the maximal potential level is obtained at an active entry. This together with Condition (3) of a potential-based b-flow in standard form shows that the entry u is a potential indicating entry if $\bar{\pi}_v = \bar{\pi}$ holds for all $v \in V$. □

For the remainder of this section we make the additional assumption of Lemma 3.7 that the upper pressure bounds are equal for all nodes, i.e., $\bar{\pi}_v = \bar{\pi}$ for all $v \in V$.

Now, we prove that the flows corresponding to scenarios in $B_{u,v}$ "dominate" the flows of other scenarios on all arcs of the flow-path from u to v. Note that we can choose any orientation of the arcs of the considered tree-shaped graph because the orientation does not restrict the flow on the arcs.

Lemma 3.8. *Let $G = (V, A)$ be a tree-shaped potential network with fixed arc sizes $d_a \in D_a$ for all arcs $a \in A$, $b \in B$ a scenario with corresponding unique flows $q \in \mathbb{R}^{|A|}$, $u \in I_{b,d}$ a potential indicating entry, $v \in V_- \setminus \{u\}$ an exit, and $\tilde{b} \in B_{u,v}$ another scenario with corresponding unique flows $\tilde{q} \in \mathbb{R}^{|A|}$. Additionally, let the orientation of G satisfy that the flow-path $P(u, v)$ is a directed path from u to v. Then, for every arc $a \in P(u, v)$, the relation $q_a \leq \tilde{q}_a$ holds.*

Proof. Due to Property (b) of Definition 3.2 and the orientation of the flow-path $P(u, v)$ we know that $\tilde{q}_a$ is positive for each arc $a \in A_{P(u,v)}$. Hence, the claim is obviously true if $q_a \leq 0$ holds. Thus, assume that q_a is positive. We know that scenario b satisfies Property (a) of Definition 3.2 because node u is a potential indicating entry and all nodes have the same upper pressure bound. Now, we distinguish two different cases.

If u supplies v in scenario b, then scenario b also satisfies Property (b) of Definition 3.2. Hence, the claim follows from Property (c) of Definition 3.2, which is satisfied by $\tilde{b}$.

Now, we assume that u does not supply v in b. We modify the scenario b and the corresponding flows q such that it satisfies Property (b) of Definition 3.2 and such that the modified flow on every arc $a \in A_{P(u,v)}$ is at least as large as the original value q_a. In analogy to the proof of Lemma 3.3, we consider a decomposition of the given arc flows q into the following path flows

$$q_{(v_1,v_2)} = \sum_{P(w_1,w_2) \in P_b} \chi_{(v_1,v_2)}(P(w_1, w_2))\, f(P(w_1, w_2)), \quad (v_1, v_2) \in A. \tag{12}$$

Now, we start modifying scenario b and its corresponding flows q. For each flow-path $P(w_1, w_2) \in P_b$ with $A_{P(w_1,w_2)} \cap A_{P(u,v)} = \emptyset$, we set $f(P(w_1, w_2)) = 0$. This does not affect the flow q_a due to the chosen flow decomposition. For all flow-paths $P(w_1, w_2) \in P_b$ that satisfy $f(P(w_1, w_2)) > 0$ and contain an arc $\tilde{a} \in A_{P(w_1,w_2)} \cap A_{P(u,v)}$ with $\chi_{\tilde{a}}(P(w_1, w_2)) = -1$, we set $f(P(w_1, w_2)) = 0$. This does not decrease the flow value q_a because flow-path $P(u, v)$ is a directed path from u to v. Due to the flow decomposition and these modifications, we know that the modified flows q satisfy that the flow on each arc of flow-path $P(u,v)$ is non-negative and that the value q_a is at least as large as before the modification. If u still does not supply v, we modify the scenario further such that u supplies v. To this end, we define the following parameters

$$\varepsilon_1 = \begin{cases} \beta_1 = f(P(u,w_2)), & \text{if } \bar{b}_u = b_u, \\ \bar{b}_u - b_u, & \text{if } \bar{b}_u > b_u, \end{cases}$$
$$\varepsilon_2 = \begin{cases} \beta_2 = f(P(w_1,v)), & \text{if } |\underline{b}_v| = |b_v|, \\ |\underline{b}_v| - |b_v|, & \text{if } |\underline{b}_v| > |b_v|, \end{cases}$$
$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\},$$

with

$$\beta_1 := \min_{P(u,w)\in P_b} \{f(P(u,w))\colon f(P(u,w)) > 0\},$$
$$\beta_2 := \min_{P(w,v)\in P_b} \{f(P(w,v))\colon f(P(w,v)) > 0\}.$$

If $\bar{b}_u = b_u$, we decrease $P(u,w_2)$ by $\varepsilon > 0$ because u is not able supply more flow. If $|\underline{b}_v| = |b_v|$, we decrease $P(w_1,v)$ by ε because v is not able to discharge more flow. Note that flow-paths $P(u,w_2)$ and $P(w_1,v)$ are disjoint because otherwise, u would already supply v because of the flow decomposition and the applied modifications. We now set $f(P(u,v)) = \min\{\varepsilon_1,\varepsilon_2\} = \varepsilon$. By construction,

$$\underline{b}_w \le b_w \le \bar{b}_w,\ w\in V, \quad b_u > 0, \quad b_v < 0.$$

holds and u supplies v.

We call the modified scenario b' and its corresponding flows q'. These modifications do not decrease the flow on an arc $a \in P(u,v)$, i.e., $q'_a \ge q_a$ holds. Additionally, scenario b' and its flow q' satisfy Property (a) and (b) of Definition 3.2. The claim then follows from Property (c) of Definition 3.2, which is satisfied by scenario $\tilde{b}$. □

Now, we show that by considering all potential-based b-flows in standard form w.r.t. scenarios in B, the lowest potential value at all exit nodes corresponds to a scenario that satisfies Definition 3.2.

Lemma 3.9. *Let $G = (V,A)$ be a tree-shaped potential network with arc sizes $d_a \in D_a$ for all arcs $a \in A$, $b \in B$ a scenario, $u \in I_{b,d}$ a potential indicating entry, $v \in V_- \setminus \{u\}$ an exit, $\tilde{b} \in B_{u,v}$ a scenario and let (q,π) and $(\tilde{q},\tilde{\pi})$ be the potential-based b-flows in standard form corresponding to b and $\tilde{b}$. Then,*

$$\pi_j \ge \tilde{\pi}_j, \quad j \in V_{P(u,v)}$$

holds.

Proof. We choose the orientation of G such that flow-path $P(u,v)$ is a directed path from u to v. Node u is a potential indicating entry in b and this, in combination with Condition (3), leads to $\tilde{\pi}_u \le \pi_u = \bar{\pi}$. Additionally, we know that the pressure drop relation

$$\psi_a(d_a, q_a) \le \psi_a(d_a, \tilde{q}_a), \quad a \in P(u,v),\ d_a \in D_a, \tag{14}$$

follows from Lemma 3.8 and the property that potential functions are strictly increasing w.r.t. their second argument. Thus, the claim follows with the help of Equation (2b) by induction over the number of nodes of the flow-path from u to v. □

For the remainder of this section we additionally assume that the lower pressure bounds of exit nodes are larger than or equal to the lower pressure bounds of the remaining nodes, i.e. $\max_{u\in V\setminus V_-} \underline{\pi}_u \le \min_{v\in V_-} \underline{\pi}_v$. Next, we show that a set of scenarios that includes for each entry and exit node at least one scenario of Definition 3.2 is a feasible design set.

Theorem 3.10. *Let $G = (V, A)$ be a tree-shaped potential network. If a set of scenarios $B^* \subset B \neq \{0\}$ satisfies the property*

$$|B^* \cap B_{u,v}| \geq 1, \quad u \in V_+,\ v \in V_- \setminus \{u\},$$

then B^* is a feasible design set.

Proof. Let $B^* \subset B$ be a set of scenarios that satisfies the assumptions. If no feasible arc sizes $d_a \in D_a$ for all arcs $a \in A$ for potential network G w.r.t. B^* exists, then obviously the same holds for B.

Let $d_a \in D_a, a \in A$, be feasible arc sizes for G w.r.t. B^*. This means that for each scenario $b \in B^*$ a feasible potential-based b-flow in standard form exists. We assume the contrary, i.e., a scenario $\tilde{b} \in B$ exists so that the corresponding potential-based $\tilde{b}$-flow in standard form $(\tilde{q}, \tilde{\pi})$ is infeasible, i.e., (2c) is violated. Furthermore, we can assume that this scenario is non-zero because the potential function $\psi_a(d_a, \cdot)$ is strictly increasing for each arc $a \in A$. Due to the Lemmas 2.3 and 2.4, we can always find a potential-based b-flow that satisfies Conditions (2a), (2b). Furthermore, we can shift the potentials such that inequality $\tilde{\pi}_w \leq \bar{\pi}, w \in V$ is satisfied because of Lemma 2.5 and the equal upper pressure bound for each node. Due to the infeasibility of $(\tilde{q}, \tilde{\pi})$, a node $v \in V$ which violates its lower pressure bound, i.e. $\tilde{\pi}_v < \underline{\pi}_v$, exists. First, we assume that $v \in V \setminus V_-$ and $\tilde{\pi}_w \geq \underline{\pi}_w, w \in V_-$ hold. The smallest potential of $\tilde{\pi}$ is attained by an exit node , i.e. $\exists w \in V_- : \tilde{\pi}_w = \min_{u \in V} \tilde{\pi}_u$, because $\psi_a(d_a, \cdot)$ is strictly increasing and Condition (2b) holds. Consequently, $\tilde{\pi}_w \leq \tilde{\pi}_v < \underline{\pi}_v \leq \underline{\pi}_w$ is satisfied due to $\max_{u \in V \setminus V_-} \underline{\pi}_u \leq \min_{v \in V_-} \underline{\pi}_v$ and this is a contradiction to the assumption. We now assume that an exit $v \in V_-$ with $\tilde{\pi}_v < \underline{\pi}_v$ exists. Applying Lemma 3.7 leads to the result that a potential indicating entry $u \in V_+$ in $\tilde{b}$ exists. For each scenario $b \in B^* \cap B_{u,v}$ and its potential-based b-flow (q, π) in standard form, the inequality $\pi_v \leq \tilde{\pi}_v$ is valid by Lemma 3.9. The intersection $B^* \cap B_{u,v}$ is non-empty by assumption. As a consequence, relation $\pi_v \leq \tilde{\pi}_v < \underline{\pi}_v$ is satisfied. This is a contradiction to the feasibility of each potential-based b-flow in standard form corresponding to a scenario of set B^* in potential network G with arc sizes $d_a, a \in A$. □

Now, we are able to prove that we can reduce the set of scenarios B to a finite feasible design set. Additionally, we prove that the size of our finite feasible design set is maximally quadratic in the number of nodes of the given network.

Theorem 3.11. *Let $G = (V, A)$ be a tree-shaped potential network and B the set of scenarios. Then, a finite feasible design set for B with at most $|V_+| \cdot |V_-|$ elements exists.*

Proof. If set B only contains the scenario corresponding to the zero node balance, it is obviously a finite feasible design set itself. Otherwise, a scenario set $B^* \subset B$ that satisfies

$$B^* \cap B_{u,v} \neq \emptyset, \quad u \in V_+,\ v \in V_- \setminus \{u\} \tag{15}$$

exists by Lemma 3.5. Furthermore, Condition (15) is satisfied if B^* contains at least one scenario of $B_{u,v}$ for every entry $u \in V_+$ and exit $v \in V_- \setminus \{u\}$. Hence, we can find a set B^* with at most $|V_+| \cdot |V_-|$ scenarios that satisfies (15). Finally, it is a finite feasible design set because of Theorem 3.10. □

Theorem 3.11 shows that we can optimize the problem of discrete arc sizing in a tree-shaped potential network w.r.t. a finite feasible design set instead of scenario set B with infinite size. Applying this result reduces the size of the infinite mixed-integer program (9) to polynomial size.

4. A Combinatorial Polynomial Time Algorithm

In the last section, we presented an algorithm that is mainly based on solving tailored linear programs. In this section, we additionally provide a combinatorial algorithm for computing finite feasible design sets. For this algorithm, we can explicitly state its polynomial run time. Afterward, we prove that discrete arc sizing is still NP-complete.

Throughout this section, we choose the orientation of the arcs of G such that $P(u,v)$ is a directed path from u to v. For constructing the combinatorial algorithm we need the following definitions and notational conventions: Let $G=(V,A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. We represent the nodes of $V_{P(u,v)}$ as $\{u = h_0, \ldots, h_n = v\}$ with the property

$$V_{P(h_j,v)} \subset V_{P(h_i,v)}, \quad 0 \le i < j \le n.$$

Moreover, we define a special aggregation of the upper capacities of certain entries by

$$Y_{u,v}^{h_i} := \{\tilde{u} \in V_+ \colon V_{P(\tilde{u},h_i)} \cap V_{P(u,v)} = \{h_i\}\}, \qquad 1 \le i \le n-1, \tag{16a}$$

$$Q_{u,v}^{h_i} := \bar{b}_u + \sum_{j=1}^{i} \sum_{\tilde{v} \in Y_{u,v}^{h_j}} \bar{b}_{\tilde{v}}, \qquad 0 \le i \le n-1. \tag{16b}$$

Finally, we define a special aggregation of the lower capacities of certain exits via

$$\tilde{Y}_{u,v}^{h_i} := \{\tilde{u} \in V_- \colon V_{P(\tilde{u},h_i)} \cap V_{P(u,v)} = \{h_i\}\}, \qquad 1 \le i \le n, \tag{17a}$$

$$\tilde{Q}_{u,v}^{h_i} := \sum_{j=i}^{n} \sum_{\tilde{v} \in \tilde{Y}_{u,v}^{h_j}} |\underline{b}_{\tilde{v}}|, \qquad 1 \le i \le n. \tag{17b}$$

For $1 \le i \le n-1$ the set $Y_{u,v}^{h_i}$ contains all entries $\tilde{u}$ that can supply node v while the corresponding unique flow-path $P(\tilde{u},v)$ contains all nodes $h_j, i \le j \le n$, and these are the only nodes of the flow-path $P(u,v)$ in $P(\tilde{u},v)$. For $1 \le i \le n$ the set $\tilde{Y}_{u,v}^{h_i}$ contains all exits $\tilde{v}$ that can be supplied by u while the corresponding unique flow-path $P(u,\tilde{v})$ contains all nodes $h_j, 0 \le j \le i$, and these are the only nodes of the flow-path $P(u,v)$ in $P(u,\tilde{v})$.

We first present some subroutines that are later used to construct the final algorithm. The first subroutine, Algorithm 1, returns a tuple (h_m, flag) that is used to decide which entries and exits must be active or inactive in the optimal solution of the linear program (10). By construction of Algorithm 1, we know that the maximum supply of all entry nodes in $\{u\} \cup \bigcup_{j=1}^{m} Y_{u,v}^{h_j}$ is sufficient to supply all exit nodes in $\bigcup_{j=m+1}^{n} \tilde{Y}_{u,v}^{h_j}$ at maximum demand. Moreover, m is the minimal index with this property. If we need entries of $Y_{u,v}^{h_m}$ for satisfying this property, it is determined by $\mathsf{flag} = \mathsf{entries}$. Otherwise, we set $\mathsf{flag} = \mathsf{exits}$.

The second subroutine, see Algorithm 2, distributes a total load of M into a given set K of exits in dependence of their lower capacity $\underline{b}$. We note that it is not important in which order we consider the exits in K.

The subroutine DistributeEntryLoads works analogously and we thus refrain from formally stating it. It distributes a total load of M onto a given set K of entries.

Algorithm 3 uses the discussed subroutines and computes a scenario $b \in B_{u,v}$. We only consider entries and exits of set $V^{\sim_u}(v)$ explicitly and leave the remaining entries and exits inactive; see Property (a) of Definition 3.2. Only entries in the set $\bigcup_{j=1}^{n-1} Y_{u,v}^{h_j} \cup \{u\}$ can be activated in Algorithm 3 because activating other entries only has a negative or no effect on the corresponding solution of the linear program (10). To this end, Algorithm 3 distinguishes between four different cases depending on the

Algorithm 1: ComputePosition

Input: $G = (V, A)$, $u \in V_+$, $v \in V_- \setminus \{u\}$.
Output: (h_m, flag)

for $i = 0, \dots, n-1$ **do**
  if $Q^{h_i}_{u,v} \geq \tilde{Q}^{h_{i+1}}_{u,v}$ **then**
    if $h_i = u$ **then return** $(h_i, \mathsf{entries})$
    else if $Q^{h_{i-1}}_{u,v} \geq \tilde{Q}^{h_{i+1}}_{u,v}$ **then return** (h_i, exits)
    else return $(h_i, \mathsf{entries})$
return (h_n, exits)

Algorithm 2: DistributeExitLoads

Input: $b \in \mathbb{R}^{|V|}$, $M \in \mathbb{R}$, $K \subset V_-\colon \sum_{v \in K} |\underline{b}_v| \geq M$
Output: b

while $M > 0$ **do**
  Select $v \in K$.
  Set $b_v \leftarrow -\min\{|\underline{b}_v|, M\}$, $M \leftarrow M - |b_v|$, and $K \leftarrow K \setminus \{v\}$.
return b

output of Algorithm 1. These cases are analyzed in the proof of the next theorem, which states that Algorithm 3 returns a solution of (10).

Algorithm 3: ComputeSolution

Input: $(h_m, \mathsf{flag}) = \mathsf{ComputePosition}(G, u, v)$ for a network G, $u \in V_+$, $v \in V_- \setminus \{u\}$
Output: $b \in B_{u,v}$.

1 Set $b \leftarrow 0 \in \mathbb{R}^{|V|}$.
2 **if** $h_m = u$ **then**
3   Set $b_u \leftarrow \tilde{Q}^{h_1}_{u,v}$ and $b_i \leftarrow \underline{b}_i$ for $i \in \bigcup_{j=1}^{n} \tilde{Y}^{h_j}_{u,v}$.
4 **else if** $h_m = v$ **then**
5   Set $b_i \leftarrow \bar{b}_i$ for $i \in \{u\} \cup \bigcup_{j=1}^{n-1} Y^{h_j}_{u,v}$ and $b_v \leftarrow -\min\{Q^{h_{n-1}}_{u,v}, |\underline{b}_v|\}$.
6   Call $\mathsf{DistributeExitLoads}(b, Q^{h_{n-1}}_{u,v} - |\underline{b}_v|, \tilde{Y}^{h_n}_{u,v} \setminus \{v\})$.
7 **else**
8   Set $b_i \leftarrow \bar{b}_i$ for $i \in \{u\} \cup \bigcup_{j=1}^{m-1} Y^{h_j}_{u,v}$ and $b_i \leftarrow \underline{b}_i$ for $i \in \bigcup_{j=m+1}^{n} \tilde{Y}^{h_j}_{u,v}$.
9   **if** *flag* = *exits* **then** call $\mathsf{DistributeExitLoads}(b, Q^{h_{m-1}}_{u,v} - \tilde{Q}^{h_{m+1}}_{u,v}, \tilde{Y}^{h_m}_{u,v})$
10   **else** call $\mathsf{DistributeEntryLoads}(b, \tilde{Q}^{h_{m+1}}_{u,v} - Q^{h_{m-1}}_{u,v}, Y^{h_m}_{u,v})$.
11 **return** b

Theorem 4.1. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, and $v \in V_- \setminus \{u\}$ an exit. Then, Algorithm 3 returns a vector b such that b and its corresponding flow q are an optimal solution of the linear program* (10).

Proof. The vector b returned in Line 11 of Algorithm 3 is a scenario of set B due to the construction of Algorithm 1 and 3. Additionally, all entries and exits of $V \setminus V^{\sim_u}(v)$ are inactive in b. Thus, Algorithm 3 returns a scenario that, together with its corresponding flows q, is a feasible solution (b, q) of (10). Furthermore, no node of $V \setminus V^{\sim_u}(v)$ supplies u in q. A case analysis yields that u supplies v

in q as well. As a consequence, the tuple (b, q) satisfies Properties (a) and (b) of Definition 3.2.

We now prove that it also fulfills Property (c) of Definition 3.2. Let $(\tilde{b}, \tilde{q})$ be an optimal solution of the linear program (10) and choose the orientation of G such that flow-path $P(u, v)$ is a directed path from u to v. Node u supplies v in $\tilde{q}$ due to Theorem 3.4 and Definition 3.2. Thus, q_a and $\tilde{q}_a$ are non-negative for $a \in A_{P(u,v)}$. Now suppose that

$$\exists a = (h_i, h_{i+1}) \in A_{P(u,v)} \colon \tilde{q}_a > q_a \geq 0$$

holds. Choose $a \in A_{P(u,v)}$ such that no arc in $A_{P(u,h_i)}$ satisfies the assumption but a does. Due to the tree structure of G and Condition (10c), the relation

$$0 \leq \tilde{q}_a \leq \min\{Q^{h_i}_{u,v}, \tilde{Q}^{h_{i+1}}_{u,v}\} \tag{18}$$

is valid. Now, we analyze all possible cases in Algorithm 3.

Case 1: The statement in Line 2 of Algorithm 3 is true. The relation $Q^{h_i}_{u,v} \geq \tilde{Q}^{h_{i+1}}_{u,v}$ holds because of the construction of Algorithm 1. As a consequence of Algorithm 3, only entry nodes in $\{u\} \cup \bigcup_{h_j \in V_{P(u,h_i)} \setminus \{u\}} Y^{h_j}_{u,v}$ supply exit nodes in $\bigcup_{h_j \in P(h_{i+1},v)} \tilde{Y}^{h_j}_{u,v}$. Moreover, each of these exits demands at its lower capacity $\underline{b}$. Thus,

$$q_a = \tilde{Q}^{h_{i+1}}_{u,v} = \min\{Q^{h_i}_{u,v}, \tilde{Q}^{h_{i+1}}_{u,v}\} \tag{19}$$

is satisfied.

Case 2: The statement in Line 4 is true. The relation $Q^{h_i}_{u,v} \leq \tilde{Q}^{h_{i+1}}_{u,v}$ is satisfied because of the construction of Algorithm 1. Additionally, all entries in $\bigcup_{h_j \in P(u,h_i) \setminus \{u\}} Y^{h_j}_{u,v} \cup \{u\}$ are active and supply at their upper capacity in b. Furthermore, only exit nodes in $\bigcup_{h_j \in V_{P(h_m,v)}} \tilde{Y}^{h_j}_{u,v}$ can be active. This and the tree structure of G yields

$$q_a = Q^{h_i}_{u,v} = \min\{Q^{h_i}_{u,v}, \tilde{Q}^{h_{i+1}}_{u,v}\}. \tag{20}$$

Case 3: Algorithm 3 reaches Line 7. If $a \in A_{P(u,h_m)}$ holds, Condition (20) is satisfied as well. This can be shown in analogy to Case 2. If $a \in A_{P(h_m,v)}$ holds, then Condition (19) is satisfied. This can be shown in analogy to Case 1.

From this case analysis, $q_a = \min\{Q^{h_i}_{u,v}, \tilde{Q}^{h_{i+1}}_{u,v}\}$ follows and, in combination with (18), this yields a contradiction to the assumption. As a consequence, the tuple (b, q) is an optimal solution of (10). □

With the help of this theorem, it immediately follows that Algorithm 3 is correct, i.e., it returns a scenario $b \in B_{u,v}$.

Theorem 4.2. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, $v \in V_- \setminus \{u\}$ an exit, and b the output of Algorithm 3. Then, $b \in B_{u,v}$ holds.*

Proof. Applying Theorem 4.1 and 3.4 shows the claim. □

We now analyze the output of Algorithm 3 w.r.t. different inputs. Using the next lemma, we can show that Algorithm 3 returns the same output for different input parameters under certain conditions.

Lemma 4.3. *Let $G = (V, A)$ be a tree-shaped potential network, $u \in V_+$ an entry, $v \in V_- \setminus \{u\}$ an exit, and $(h_m, \textsf{flag})$ the output of Algorithm 1. If $h_m \neq v$, Algorithm 3 returns the same output for each exit in the set $\bigcup_{j=m+1}^{n} \tilde{Y}^{h_j}_{u,v}$.*

Proof. Let w be an exit node in $\bigcup_{j=m+1}^{n} \tilde{Y}_{u,v}^{h_j}$. Since the network is a tree, the flow-path $P(u, h_m)$ is part of the flow-path $P(u, v)$ and of the flow-path $P(u, w)$. Hence, $Q_{u,v}^{h_j} = Q_{u,w}^{h_j}$ for all nodes $h_j \in V_{P(u,h_m)}$. The same holds for $\tilde{Q}_{u,v}^{h_j}$ and $\tilde{Q}_{u,w}^{h_j}$ for all nodes $h_j \in P(u, h_{m+1}) \setminus \{u\}$. Thus, Algorithm 1 returns the same output for the different considered inputs. Then, by construction of Algorithm 3, the claim follows. □

This result enables us to formally state Algorithm 4, which computes a finite feasible design set with at most $|V_+| \cdot |V_-|$ elements. Before we prove the correctness of Algorithm 4, we first analyze the run time of the presented algorithms.

Algorithm 4: ComputeFiniteDesignSet

Input: A potential network $G = (V, A)$.
Output: A finite feasible design set B^*.

```
Set B* ← ∅.
while V+ ≠ ∅ do
    Select u ∈ V+ and set K ← V− \ {u}.
    while K ≠ ∅ do
        Select v ∈ K.
        Compute (h_m, flag) ← ComputePosition(G, u, v).
        Compute b ← ComputeSolution((h_m, flag), G, u, v).
        Set B* ← B* ∪ {b}.
        if h_m ≠ v then set K ← K \ ⋃_{i=m+1}^{n} Ỹ_{u,v}^{h_i}.
        else set K = K \ {v}.
    Set V+ ← V+ \ {u}.
return B*
```

Lemma 4.4. *Let $G = (V, A)$ be a tree-shaped potential network. Then, Algorithm 1 and 3 both have a run time in $O(|V|^2)$.*

Proof. We consider that G is given by its adjacency matrix. Additionally, we know that $|V| - 1 = |A|$ holds. We can compute all parameters defined in (16) and (17) in $O(|V|^2)$ using a modified modified depth-first search. Thus, the run time of Algorithm 1 is in $O(|V|^2)$.

The subroutines DistributeExitLoads and DistributeEntryLoads obviously have a run time in $O(|V|)$. Thus, the run time of Algorithm 3 is in $O(|V|^2)$ because all parameters can be pre-computed in $O(|V|^2)$ and a maximum of $O(|V|)$ additional operations are performed. □

As a consequence of Lemma 4.4 and 2.3, we can solve the linear program (10) in $O(|V|^2)$. Finally, we prove that we can compute a finite feasible design set with at most $O(|V|^2)$ elements in polynomial time.

Lemma 4.5. *Let $G = (V, A)$ be a tree-shaped potential network. Then, Algorithm 4 returns a finite feasible design set with at most $|V_+| \cdot |V_-|$ elements and its run time is in $O(|V|^4)$.*

Proof. The claim directly follows by applying Theorem 4.2, Lemma 4.3 and 4.4, and Theorem 3.11. □

Putting everything together we obtain the following theorem.

Theorem 4.6. *Let $G = (V, A)$ be a tree-shaped potential network and B a set of scenarios. Then, a finite mixed-integer linear model for solving the discrete arc sizing problem w.r.t. B can be setup in polynomial time.*

Theorem 4.6 leads to the question about the hardness of the discrete arc sizing problem for tree-shaped potential networks.

Theorem 4.7. *Let $G = (V, A)$ be a tree-shaped potential network and B a set of scenarios. Additionally, let D be a finite set of arc sizes. Then, the problem discrete arc sizing for G w.r.t. B is NP-complete.*

Proof. We first show that we can verify a given solution in polynomial time. We can compute a finite feasible design set w.r.t. set B in polynomial time due to Lemma 4.5. Furthermore, we can verify if the solution is feasible for all scenarios of the computed feasible design set in polynomial time because the size of the computed feasible design set is at most $|V|^2$. Note that the feasibility of one scenario for given arc sizes can be verified in polynomial time due to the Lemmas 2.3–2.6.

The NP-hardness of the problem is shown in Yates et al. (1984). □

As a consequence of Theorem 4.7, there is no hope for a qualitatively better way of solving the problem than by solving a mixed-integer linear program like (9).

5. Computational Results

The algorithms presented in Section 4 have been implemented in Python 3.4.5 and the MIP (9) has been implemented using the Python based open-source software package Pyomo 4.4.1; see Hart, Watson, et al. (2011) and Hart, Laird, et al. (2012). Furthermore, we used Gurobi 7.0.1 to solve the MIPs; see Gurobi Optimization, Inc. (2017). All computations have been performed on a quad-core Intel$^{\mathrm{R}}$ Core$^{\mathrm{TM}}$ i7-7600U with 16 GB RAM.

In this section, we illustrate the effective generation of a finite feasible design set for a realistic hydrogen network with different nodal capacities by using Algorithm 4. With the resulting finite feasible design sets, we solve the finite counterpart of the MIP (9) for selecting discrete pipe diameters. The main goal of this section is to show the applicability of our approach for real-world instances. A more detailed application specific discussion of the results will be given in a companion paper that is currently in preparation.

The data is taken from Robinius (2015) and Robinius et al. (2017). The considered network models a potential hydrogen pipeline network of Eastern Germany. It contains the geographical position of entries and exits. This node set consists of 1420 nodes including 745 exits, one entry, and one storage, which can act as an entry or exit. Figure 2 and 3 illustrate the network including entries and exits.

The use of the network is to supply the future road traffic of Eastern Germany with hydrogen. Entries represent hydrogen suppliers, exits represent hydrogen refueling stations, and storages stand for hydrogen storages. The network has a tree structure with 1419 pipes. The total length of the network is 3135.59 km. As mentioned before, we compute the cheapest selection of diameters for this hydrogen network that are feasible for all given scenarios. The physical equations and more information about this specific application are given in Example 2.2 and 3.1.

We do not consider regulations of the filling level of hydrogen storages. Instead, we only take their lower and upper capacity into consideration. This procedure is appropriate because we want to compute diameters that are feasible for all considered scenarios. Thus, we can assume that the filling level of the storages matches the required level of the considered scenario.

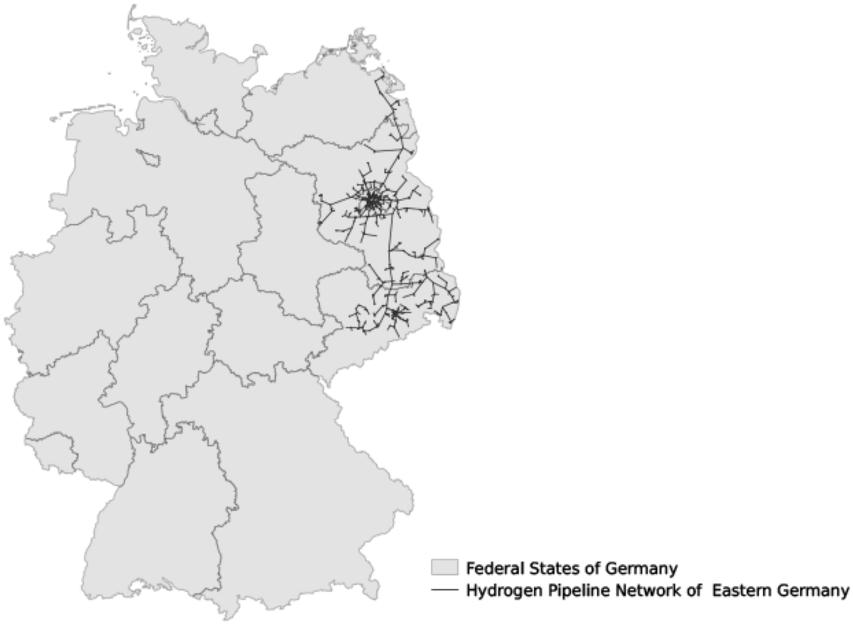

FIGURE 2. Hydrogen pipeline network of Eastern Germany

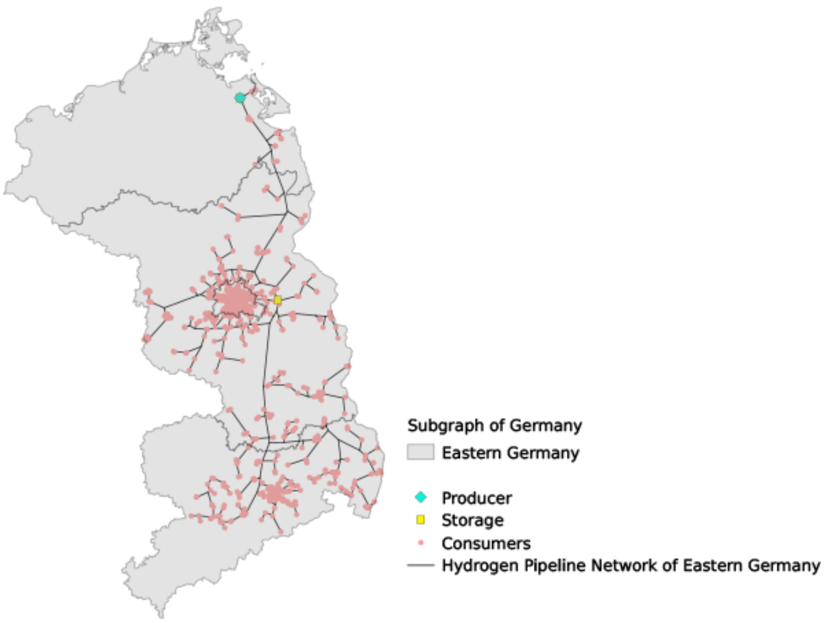

FIGURE 3. Location of entries and exits in the hydrogen pipeline network of Eastern Germany

We consider 28 different diameters in the range of 0.1063 to 1.536 m. The pipe costs are increasing in the size of the diameters. In contrast to this, the pressure drop within the pipes decreases for an increasing size of the pipe and constant flow.

For the capacities of the entry, the storage, and the exits, we distinguish between two cases yielding two different instances; see Table 1. Within both of the two instances, all exits have the same lower and upper capacities because we do not

Table 1. Capacities in instance 1 (local on-site storages; top) and instance 2 (without local on-site storages; bottom)

Type	Lower capacity (kg/s)	Upper capacity (kg/s)
Entry	0.00	7.33
Exits	−0.01	0.00
Storage	0.00	0.00
Entry	0.00	7.74
Exits	−0.02	0.00
Storage	−7.04	9.79

distinguish between different types of exits in an instance. In instance 1, we consider local on-site storages at the exits. This means that each hydrogen refueling station has an integrated local storage that is only used to store and supply hydrogen for this local exit. In this instance, the main grid storage is switched off. Furthermore, the entry is able to supply all exits at maximum demand at once. For this instance, the finite feasible design set computed by Algorithm 4 is reduced to 1 scenario instead of 745 scenarios due to Lemma 4.3.

In instance 2, no local on-site storages exist. Thus, the storage of the grid comes into play. Furthermore, the absolute lower capacities are larger than in instance 1 because no local on-site storage can balance the demand fluctuations. Neither the entry nor the storage is able to supply all exits at maximum demand at once. However, they are able to supply many exits at maximum demand at the same time. As a consequence, the finite feasible design set is reduced to 25 different scenarios instead of 1491 scenarios due to Lemma 4.3 and Theorem 3.11. The run time for the computation of both finite feasible design sets is very low (0.22 s for instance 1 and 0.52 s for instance 2) due to the small number of scenarios.

Next, we solve (9) w.r.t. the finite feasible design sets in order to compute optimal diameters. We consider the potential bounds

$$\underline{\pi}_v = \underline{\pi}^-, \ v \in V, \quad 0 \le \underline{\pi}^- \le \bar{\pi} = 95\,\text{bar}$$

and different lower potential bounds $\underline{\pi}^-$ in our computational study. This means that all nodes have the same lower and upper potential bounds.

The MIP corresponding to instance 1 consists of 2840 constraints and 41 153 variables (thereof 39 732 binaries). The MIP of instance 2 has 36 922 constraints and 75 233 variables (thereof again 39 732 binaries). The number of binary variables is equal in both instances because we consider the same set of possible diameters. In contrast, we have more constraints and continuous variables in instance 2 due to the larger size of its finite feasible design set. The computational results of instance 1 and 2 are given in Table 2. All instances are solved to global optimality. The run times are always below 32 s and show that we can effectively solve real-world instances. As expected, we can see that the run times increase with the size of our finite feasible design set.

Figure 4 illustrates the distribution of the different diameters for instance 1 with pressure bounds $\bar{\pi} = 95\,\text{bar}$ and $\underline{\pi}^- = 70\,\text{bar}$. Furthermore, Figure 5 visualizes a feasible pressure assignment of the corresponding single scenario of the feasible design set.

6. Conclusion

In this paper, we studied the problem of discrete arc sizing for tree-shaped potential networks, where the sizing has to be feasible w.r.t. infinitely many

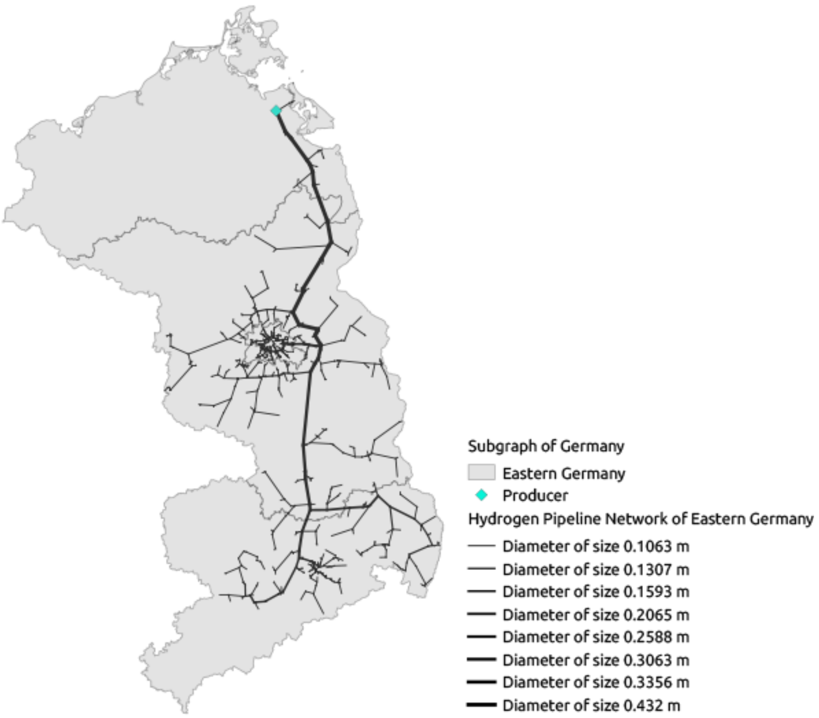

FIGURE 4. Diameter selection for the hydrogen pipeline network of Eastern Germany for instance 1 with pressure bounds $\bar{\pi} = 95$ bar and $\underline{\pi}^{-} = 70$ bar

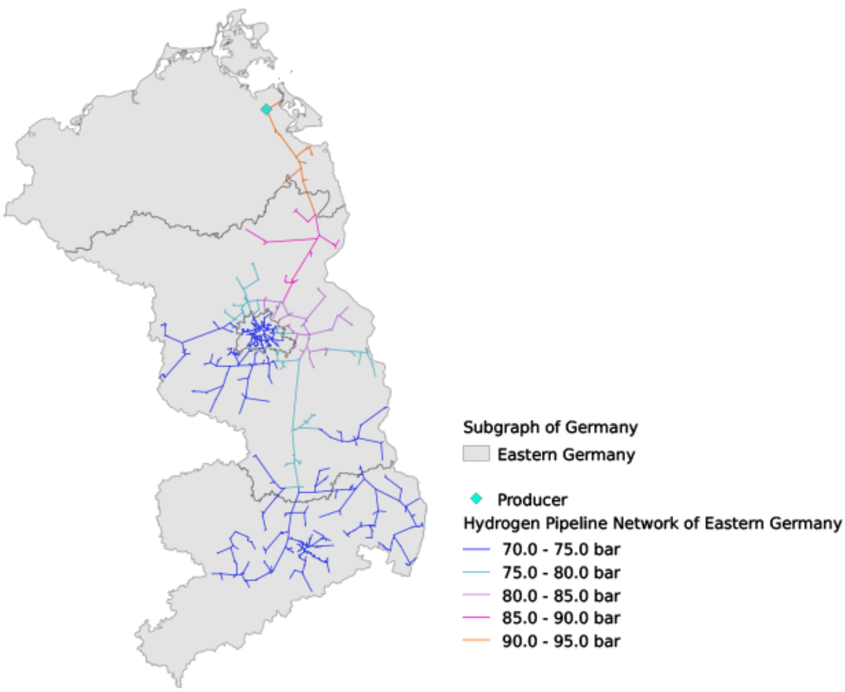

FIGURE 5. Pressure solution in instance 1 with pressure bounds $\bar{\pi} = 95$ bar and $\underline{\pi}^{-} = 70$ bar

TABLE 2. Results for instance 1 and 2 with pressure bound $\bar{\pi} = 95$ bar

$\underline{\pi}^-$ (bar)	Total time instance 1 (s)	Total time instance 2 (s)
90	6.32	31.11
80	6.53	31.67
70	5.63	31.31
60	6.73	30.74
50	5.64	29.83
40	6.83	31.85
30	6.37	30.84

scenarios. This problem can be seen as a strictly robust counterpart of a single-scenario network design problem and is shown to be NP-complete. In order to obtain a tractable, i.e., finite-dimensional problem, we proposed a method for generating a so-called finite feasible design set such that optimality w.r.t. this finite scenario set implies optimality for the originally given infinite scenario set. For the finite set, we also proved that the number of contained scenarios is bounded above by $|V|^2$ and that this set can be generated in polynomial time. The resulting problem can then be solved as a finite mixed-integer linear problem. Finally, we illustrated the applicability of our approach by solving the real-world problem of sizing a hydrogen transport network of Eastern Germany.

Different extensions of the considered problem may be studied in the future. First, it is interesting to see how the presented approach performs on larger tree-shaped networks and on differently chosen scenario sets. These application-driven questions will be addressed in a companion paper. Moreover, extending the method of generating feasible design sets to potential networks with controllable elements like, e.g., valves, is an interesting topic of future research.

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References

Afshar, M. H. and M. A. Mariño (2007). "A parameter-free self-adapting boundary genetic search for pipe network optimization." In: *Computational Optimization and Applications* 37.1, pp. 83–102. DOI: 10.1007/s10589-007-9016-1.

Ahuja, R. K., T. L. Magnanti, and J. B. Orlin (1993). *Network Flows: Theory, Algorithms, and Applications.* Upper Saddle River, NJ, USA: Prentice-Hall, Inc. ISBN: 0-13-617549-X.

Altın, A., E. Amaldi, P. Belotti, and M. Ç. Pınar (2007). "Provisioning virtual private networks under traffic uncertainty." In: *Networks* 49.1, pp. 100–115. DOI: 10.1002/net.20145.

Andre, J., F. Bonnans, and L. Cornibert (2009). "Optimization of capacity expansion planning for gas transportation networks." In: *European Journal of Operational Research* 197.3, pp. 1019–1027. DOI: 10.1016/j.ejor.2007.12.045.

Ben-Ameur, W. and H. Kerivin (2005). "Routing of Uncertain Traffic Demands." In: *Optimization and Engineering* 6.3, pp. 283–313. DOI: 10.1007/s11081-005-1741-7.

Ben-Tal, A., L. El Ghaoui, and A. Nemirovski (2009). *Robust optimization*. Princeton Series in Applied Mathematics. Princeton University Press. ISBN: 9780691143682. URL: http://press.princeton.edu/titles/9099.html.

Bragalli, C., C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth (2008). *Water Network Design by MINLP*. Tech. rep. IBM Research Division, RC24495 (W0802-056).

Bragalli, C., C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth (2006). "An MINLP Solution Method for a Water Network Problem." In: *Algorithms – ESA 2006*. Ed. by Y. Azar and T. Erlebach. Vol. 4168. Lecture Notes in Computer Science. Springer Berlin Heidelberg, pp. 696–707. ISBN: 978-3-540-38875-3. DOI: 10.1007/11841036_62.

Bragalli, C., C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth (2012). "On the optimal design of water distribution networks: a practical MINLP approach." In: *Optimization and Engineering* 13.2, pp. 219–246. DOI: 10.1007/s11081-011-9141-7.

Cacchiani, V., M. Jünger, F. Liers, A. Lodi, and D. R. Schmidt (2016). "Single-commodity robust network design with finite and Hose demand sets." In: *Mathematical Programming* 157, pp. 297–342. DOI: 10.1007/s10107-016-0991-9.

Collins, M., L. Cooper, R. Helgason, J. Kennington, and L. LeBlanc (1978). "Solving the Pipe Network Analysis Problem Using Optimization Techniques." In: *Management Science* 24.7, pp. 747–760. DOI: 10.1287/mnsc.24.7.747.

Duffield, N. G., P. Goyal, A. Greenberg, P. Mishra, K. K. Ramakrishnan, and J. E. van der Merive (1999). "A Flexible Model for Resource Management in Virtual Private Networks." In: *SIGCOMM Comput. Commun. Rev.* 29.4, pp. 95–108. DOI: 10.1145/316194.316209.

Erlebach, T. and M. Ruegg (2004). "Optimal bandwidth reservation in hose-model VPNs with multi-path routing." In: *INFOCOM 2004. Twenty-third AnnualJoint Conference of the IEEE Computer and Communications Societies*. Vol. 4. IEEE, pp. 2275–2282. DOI: 10.1109/INFCOM.2004.1354650.

Fingerhut, J., S. Suri, and J. S. Turner (1997). "Designing Least-Cost Nonblocking Broadband Networks." In: *J. Algorithms* 24.2, pp. 287–309. DOI: 10.1006/jagm.1997.0866.

Fügenschuh, A., B. Geißler, R. Gollmer, A. Morsi, J. Rövekamp, M. Schmidt, K. Spreckelsen, and M. C. Steinbach (2015). "Chapter 2: Physical and technical fundamentals of gas networks." In: *Evaluating Gas Network Capacities*. Society for Industrial and Applied Mathematics, pp. 17–43. DOI: 10.1137/1.9781611973693.ch2.

Groß, M., P. M. E., L. Schewe, M. Schmidt, and M. Skutella (2017). "Algorithmic Results for Potential-Based Flows: Easy and Hard Cases." URL: http://www.optimization-online.org/DB_HTML/2017/08/6185.html.

Gurobi Optimization, Inc. (2017). *Gurobi Optimizer Reference Manual*. http://www.gurobi.com.

Hansen, C. T., K. Madsen, and H. B. Nielsen (1991). "Optimization of pipe networks." In: *Mathematical Programming* 52.1, pp. 45–58. DOI: 10.1007/BF01582879.

Hart, W. E., C. Laird, J.-P. Watson, and D. L. Woodruff (2012). *Pyomo-Optimization Modeling in Python*. Vol. 67. Springer Science & Business Media. DOI: 10.1007/978-1-4614-3226-5.

Hart, W. E., J.-P. Watson, and D. L. Woodruff (2011). "Pyomo: modeling and solving mathematical programs in Python." In: *Mathematical Programming Computation* 3, pp. 219–260. DOI: 10.1007/s12532-011-0026-8.

Humpola, J. and A. Fügenschuh (2015). "Convex reformulations for solving a nonlinear network design problem." In: *Computational Optimization and Applications* 62.3, pp. 717–759. DOI: 10.1007/s10589-015-9756-2.

Koch, T., B. Hiller, M. Pfetsch, and L. Schewe, eds. (2015). *Evaluating Gas Network Capacities*. Philadelphia, PA: Society for Industrial and Applied Mathematics. DOI: 10.1137/1.9781611973693.

Mischner, J., H. Fasold, and K. Kadner (2011). *gas2energy.net: Systemplanung in der Gasversorgung; gaswirtschaftliche Grundlagen*. Edition gwf, Gas, Erdgas. Oldenbourg Industrieverl. ISBN: 9783835632059.

Raghunathan, A. U. (2013). "Global Optimization of Nonlinear Network Design." In: *SIAM Journal on Optimization* 23.1, pp. 268–295. DOI: 10.1137/110827387.

Reuss, M., T. Grube, M. Robinius, P. Preuster, P. Wasserscheid, and D. Stolten (2017). "Seasonal storage and alternative carriers: A flexible hydrogen supply chain architecture model." In: *Applied Energy* 200, pp. 290–302. DOI: 10.1016/j.apenergy.2017.05.050.

Ríos-Mercado, R. Z. and C. Borraz-Sánchez (2015). "Optimization problems in natural gas transportation systems: A state-of-the-art review." In: *Applied Energy* 147, pp. 536–555. ISSN: 0306-2619. DOI: 10.1016/j.apenergy.2015.03.017.

Ríos-Mercado, R. Z., S. Wu, L. R. Scott, and E. A. Boyd (2002). "A Reduction Technique for Natural Gas Transmission Network Optimization Problems." In: *Annals of Operations Research* 117.1, pp. 217–234. DOI: 10.1023/A:1021529709006.

Robinius, M. (2015). "Strom- und Gasmarktdesign zur Versorgung des deutschen Straßenverkehrs mit Wasserstoff." PhD thesis. RWTH Aachen University. ISBN: 978-3-95806-110-1.

Robinius, M., A. Otto, K. Syranidis, D. Ryberg, P. Heuser, L. Welder, T. Grube, P. Markewitz, V. Tietze, and D. Stolten (2017). "Linking the Power and Transport Sectors—Part 2: Modelling a Sector Coupling Scenario for Germany." In: *Energies* 10.7. DOI: 10.3390/en10070957.

Shiono, N. and H. Suzuki (2016). "Optimal pipe-sizing problem of tree-shaped gas distribution networks." In: *European Journal of Operational Research* 252, pp. 550–560. DOI: 10.1016/j.ejor.2016.01.008.

Syranidis, K., M. Robinius, and D. Stolten (2018). "Control techniques and the modeling of electrical power flow across transmission networks." In: *Renewable and Sustainable Energy Reviews* 82, pp. 3452–3467. DOI: 10.1016/j.rser.2017.10.110.

Szabó, J. (2012). *The set of solutions to nomination validation in passive gas transportation networks with a generalized flow formula*. Tech. rep. 11-44. Takustr.7, 14195 Berlin: ZIB. URL: www.nbn-resolving.de/urn:nbn:de:0297-zib-15151.

Vuffray, M., S. Misra, and M. Chertkov (2015). "Monotonicity of dissipative flow networks renders robust maximum profit problem tractable: General analysis and application to natural gas flows." In: *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 4571–4578. DOI: 10.1109/CDC.2015.7402933.

Welder, L., D. Ryberg, L. Kotzur, T. Grube, M. Robinius, and D. Stolten (2017). "Spatio-Temporal Optimization of a Future Energy System for Power-to-Hydrogen Applications in Germany." In: *SDEWES Conference, Dubrovnik*.

Weymouth, T. R. (1912). "Problems in Natural Gas Engineering." In: *Trans. Amer. Soc. of Mech. Eng.* 34.1349, pp. 185–231.

Yates, D., A. Templeman, and T. Boffey (1984). "The computational complexity of the problem of determining least capital cost designs for water supply networks." In: *Engineering Optimization* 7:2, pp. 143–155. DOI: 10.1080/03052158408960635.

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