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On classes of set optimization problems which are reducible to vector optimization problems and its impact on numerical test instances

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Abstract

Set optimization with the set approach has recently gained increasing interest due to its practical relevance. In this problem class one studies optimization problems with a set-valued objective map and defines optimality based on a direct comparison of the images of the objective function, which are sets here. Meanwhile, in the literature a wide range of theoretical tools as scalarization approaches and derivative concepts as well as first numerical algorithms are available. These numerical algorithms require on the one hand test instances where the optimal solution sets are known. On the other hand, in most examples and test instances in the literature only set-valued maps with a very simple structure are used. We study in this paper such special set-valued maps and we show that some of them are such simple that they can equivalently be expressed as a vector optimization problem. Thus we try to start drawing a line between simple set-valued problems and such problems which have no representation as multiobjective problems. Those having a representation can be used for defining test instances for numerical algorithms with easy verifiable optimal solution set.

Key Words: Order relations, Set optimization, Set approach, Test instances, Vector optimization

Mathematics subject classifications (MSC 2010): 26E25, 49J53, 54C60, 90C29, 90C30

1 Introduction

Set optimization problems can be considered as a significant generalization and unification of scalar and vector optimization problems and have numerous applications in optimal control, operations research, and economics equilibrium (see for instance [2, 13]). Especially set optimization using the set approach has recently gained increasing interest due to its

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practical relevance. In this problem class one studies optimization problems with a set-valued objective map and defines optimality based on a direct comparison of the images of the objective function, which are sets here. Hence one needs relations for comparing sets. A large number of such relations is proposed in the literature [12]. Examples are the possibly less or the certainly less order relation. However, the l-less order relation, the u-less order relation, and most of all the set less order relation are considered to be of highest interest from the practical point of view. For the latter relation one says that a set A is less or equal than a set B if any element of the set A has an element in the set B which is worse and if to any element of the set B there exists an element of the set A which is better. Therefore one also needs a way to compare the elements of the sets. Thus one assumes in general that the space is equipped with a partial ordering.

It is obvious that it is numerically difficult to compare sets. For convex sets Jahn has proposed in [10] to use a characterization with supporting hyperplanes. For numerical calculations he proposes to do a single comparison based on the optimal values of 4000 linear optimization problems over these convex sets. This illustrates that the development of numerical algorithms for set optimization problems is very challenging. Nevertheless, meanwhile some first algorithms have been proposed in the literature.

The first algorithm for unconstrained set optimization problems and the l-less order relation was presented by Löhne and Schrage [17]. This approach is for linear problems only and requires an objective map F with a polyhedral convex graph and a representation by inequalities of this graph. Recently, a derivative free descent method was proposed by Jahn [11] based on the comparison mentioned above. This was extended for nonconvex sets by Köbis and Köbis [14]. Both methods aim on the determination of one single minimal solution of the set optimization problem.

Clearly, set optimization problems, as vector optimization problems, have in general an infinite number of minimal solutions. Hence, one is not only interested in finding one of these solutions but in finding a representation of the set of all optimal solutions. By varying the starting points in the above methods a representation of the set of optimal solutions can be determined. For such approaches it is important to have also test instances to verify whether such a representation was successfully obtained.

In the following we study set optimization problems which are such simple that they have an equivalent reformulation as a vector optimization problem. They can be used to construct new test instances.

In Section 2 we state the basic definitions and concepts from set optimization which we need in the following. We also recall a result by Jahn which is helpful for some of the following proofs. Set optimization problems being reducible to vector optimization problems are defined and studied in Section 3. Based on this we make in the final Section 4 some suggestions how the results can be used for the construction of set-valued test instances based on vector-valued or scalar-valued optimization problems.

2 Basics of vector and set optimization

Throughout this paper we assume that Y is a locally convex real linear space which is partially ordered by a pointed, convex, and closed nontrivial cone C. Recall that a nonempty subset C of Y is called a cone if $y \in C$ and $\lambda \geq 0$ imply $\lambda y \in C$. A cone C is called pointed, if $C \cap (-C) = \{0_Y\}$. We denote by $Y^\star$ the topological dual space of Y and by

$$C^\star := \{\ell \in Y^\star \mid \ \ell(y) \geq 0 \ \forall y \in C\}$$

the dual cone of C. For the partial ordering introduced by C we write

$$y^1 \leq_C y^2 \Leftrightarrow y^2 - y^1 \in C \text{ for all } y^1, y^2 \in Y.$$

Later, for more specific results, we will often assume that Y is the finite dimensional space $\mathbb{R}^m$ partially ordered by some pointed convex cone or even just by the nonnegative orthant, i.e. $C = \mathbb{R}^m_+$.

We denote an element $\hat{y}$ of a nonempty set $M \subset Y$ nondominated w.r.t. C if

$$(\{\hat{y}\} - C) \cap M = \{\hat{y}\}.$$

This implies the definition of optimal solutions of a vector optimization problem. Let S be a nonempty set and let $f\colon S \to Y$ be a given vector-valued map. Then $\bar{x} \in S$ is called an efficient solution w.r.t. C of the vector optimization problem

$$\min_{x \in S} f(x) \tag{VOP$_{f,S}$}$$

if $f(\bar{x})$ is a nondominated element of $f(S) := \{f(x) \mid \ x \in S\}$ w.r.t. C, i.e., if

$$f(x) \leq_C f(\bar{x}), \ x \in S \Longrightarrow f(\bar{x}) \leq_C f(x) \tag{1}$$

holds. Clearly, since C is pointed, (1) can be formulated as

$$f(x) \leq_C f(\bar{x}), \ x \in S \Longrightarrow f(\bar{x}) = f(x).$$

The main topic of our paper are set optimization problems with the set approach. We use the following three set order relations [5, 15, 18, 19]:

(a) the l-less order relation w.r.t. C is defined by: $A \preccurlyeq^l_C B :\Leftrightarrow B \subset A + C$,

(b) the u-less order relation w.r.t. C is defined by: $A \preccurlyeq^u_C B :\Leftrightarrow A \subset B - C$, and

(c) the set less order relation w.r.t. C is defined by: $A \preccurlyeq^s_C B :\Leftrightarrow A \preccurlyeq^l_C B$ and $A \preccurlyeq^u_C B$.

Obviously it holds by definition

$$A \preccurlyeq^l_C B \quad \Leftrightarrow \quad \forall b \in B \ \exists a \in A: \ a \leq_C b \quad \text{and} \quad A \preccurlyeq^u_C B \quad \Leftrightarrow \quad \forall a \in A \ \exists b \in B: \ a \leq_C b.$$

Based on these set relations we can easily define minimal elements of a family $\mathcal{A}$ of nonempty subsets of Y. A set $\bar{A}$ is a minimal element of $\mathcal{A}$ w.r.t. the order relation $\preccurlyeq^\star_C$ and $\star \in \{l, u, s\}$ if

$$A \preccurlyeq^\star_C \bar{A}, \ A \in \mathcal{A} \Longrightarrow \bar{A} \preccurlyeq^\star_C A.$$

Thus for a set optimization problem

$$\min_{x \in S} F(x). \tag{SOP$_{F,S}$}$$

with feasible set S and set-valued map $F\colon S \rightrightarrows Y$ with $F(x) \neq \emptyset$ and $F(x) \neq Y$ for all $x \in S$, we denote $\bar{x} \in S$ a minimal solution w.r.t. the order relation $\preccurlyeq^\star_C$ with $\star \in \{l, u, s\}$ if

$$F(x) \preccurlyeq^\star_C F(\bar{x}), \ x \in S \Longrightarrow F(\bar{x}) \preccurlyeq^\star_C F(x) \tag{2}$$

holds.

The following theorem gives an important characterization of these set relations by using supporting hyperplanes.

Theorem 2.1. *[10, Theorem 2.1] Let A and B be nonempty subsets of Y.*

(i)

$$A \preccurlyeq_C^l B \;\Rightarrow\; \forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \inf_{y\in A} \ell(y) \le \inf_{y \in B} \ell(y).$$
$$A \preccurlyeq_C^u B \;\Rightarrow\; \forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \sup_{y\in A} \ell(y) \le \sup_{y \in B} \ell(y).$$

(ii) If the set $A+C$ is closed and convex, then

$$A \preccurlyeq_C^l B \;\Leftrightarrow\; \forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \inf_{y\in A} \ell(y) \le \inf_{y \in B} \ell(y).$$

If the set $B-C$ is closed and convex, then

$$A \preccurlyeq_C^u B \;\Leftrightarrow\; \forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \sup_{y\in A} \ell(y) \le \sup_{y \in B} \ell(y).$$

(iii) If the sets $A+C$ and $B-C$ are closed and convex, then

$$A \preccurlyeq_C^s B \;\Leftrightarrow\; \forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \inf_{y\in A} \ell(y) \le \inf_{y \in B} \ell(y) \quad \text{and} \quad \sup_{y\in A} \ell(y) \le \sup_{y \in B} \ell(y).$$

If we apply these characterizations to (2) then we obtain the following corollary:

Corollary 2.2. *[10, Corollary 2.2] Let $(\mathrm{SOP}_{F,S})$ be given and let for all $x \in S$ the sets $F(x)+C$ and $F(x)-C$ be closed and convex. Then $\bar{x} \in S$ is a minimal solution of $(\mathrm{SOP}_{F,S})$ w.r.t. the order relation $\preccurlyeq_C^s$ if and only if there is no $x \in S$ with*

$$\forall \ell \in C^\star \setminus \{0_{Y^\star}\}: \; \inf_{y\in F(x)} \ell(y) \le \inf_{\bar{y} \in F(\bar{x})} \ell(\bar{y}) \quad \text{and} \quad \sup_{y\in F(x)} \ell(y) \le \sup_{\bar{y} \in F(\bar{x})} \ell(\bar{y})$$

and

$$\exists \hat{\ell} \in C^\star \setminus \{0_{Y^\star}\}: \; \inf_{y\in F(x)} \hat{\ell}(y) < \inf_{\bar{y} \in F(\bar{x})} \hat{\ell}(\bar{y}) \quad \text{or} \quad \sup_{y\in F(x)} \hat{\ell}(y) < \sup_{\bar{y} \in F(\bar{x})} \hat{\ell}(\bar{y}).$$

Based on this corollary Jahn proposed in [10] a vector optimization problem which is equivalent to the set optimization problem $(\mathrm{SOP}_{F,S})$. This vector optimization problem requires the introduction of a new linear space which is defined by maps. This special space is partially ordered by a pointwise ordering, see the next theorem.

Theorem 2.3. *[10, Theorem 3.1] Let $(\mathrm{SOP}_{F,S})$ be given and let for all $x \in S$ the sets $F(x)+C$ and $F(x)-C$ be closed and convex. Then $\bar{x} \in S$ is a minimal solution of $(\mathrm{SOP}_{F,S})$ w.r.t. the order relation $\preccurlyeq_C^s$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem*

$$\min_{x \in S} v(F(x)) \tag{3}$$

where $v(F(x))$ is for each set $F(x)$ a map on $C^\star \setminus \{0_{Y^\star}\}$ which is defined pointwise for each $\ell \in C^\star \setminus \{0_{Y^\star}\}$ by

$$v(F(x))(\ell) := \begin{pmatrix} \inf\limits_{y \in F(x)} \ell(y) \\ \sup\limits_{y \in F(x)} \ell(y) \end{pmatrix}$$

and the partial ordering in the image space of the vector optimization problem (3) is defined by

$$v(F(x^1)) \le v(F(x^2)) \;:\Leftrightarrow\; v(F(x^1))(\ell) \le_{\mathbb{R}^2_+} v(F(x^2))(\ell) \;\forall \ell \in C^\star \setminus \{0_{Y^\star}\}.$$

Note that even if the original set optimization problem ($\text{SOP}_{F,S}$) is a finite dimensional problem (for instance if $S \subset \mathbb{R}^n$ and $Y = \mathbb{R}^m$) the associated vector optimization problem (3) is an infinite dimensional problem. It can then also be interpreted as a multiobjective optimization problem with an infinite number of objectives (two for each $\ell \in C^\star \setminus \{0_{Y^\star}\}$). Moreover, for solving (3) an infinite number of scalar-valued optimization problems would have to be solved. This is due to the fact that according to Theorem 2.1 (*iii*) in general already for comparing just two sets $F(x^1)$ and $F(x^2)$ with $x^1, x^2 \in S$ using the set less order relation $\preccurlyeq_C^s$ all (normed) elements ℓ of $C^\star \setminus \{0_{Y^\star}\}$ are needed. In [11] Jahn showed that (under additional assumptions) this is not the case if $Y = \mathbb{R}^m$ and the sets $F(x)$ are polyhedral for all $x \in S$:

Theorem 2.4. *[11, Theorem 2.2] Let S be a nonempty subset of $\mathbb{R}^n$, $Y = \mathbb{R}^m$, C be polyhedral, and let $F\colon S \rightrightarrows Y$ be a set-valued map defined by*

$$F(x) := \{y \in \mathbb{R}^m \mid A(x) \cdot y \leq b(x)\} \text{ for all } x \in S$$

where $A\colon S \to \mathbb{R}^{p\times m}$ and $b\colon S \to \mathbb{R}^p$ are given maps such that $F(x)$ is compact and nonempty for all $x \in S$. Moreover let for all $x \in S$ and $i \in \{1, \dots, p\}$ the i-th row of the matrix $A(x)$ be denoted by $a^i(x)$ and let $\mathcal{L}$ be the finite set of all normed extremal directions of $C^\star$. Then for all $x^1, x^2 \in S$ it holds

$$\begin{aligned}
& F(x^1) \preccurlyeq_C^s F(x^2) \\
\Leftrightarrow\ & \forall i \in \{1, \dots, p\} \text{ with } \ell^i := -a^i(x^1) \in C^\star \setminus \{0_{\mathbb{R}^m}\} :\ b_i(x^1) \geq \max_{y \in F(x^2)} -(\ell^i)^\top y, \\
& \forall j \in \{1, \dots, p\} \text{ with } \ell^j := -a^j(x^2) \in C^\star \setminus \{0_{\mathbb{R}^m}\} :\ \max_{y \in F(x^1)} (\ell^j)^\top y \leq \max_{y \in F(x^2)} (\ell^j)^\top y, \text{ and} \\
& \forall \ell \in \mathcal{L} :\ \min_{y \in F(x^1)} \ell^\top y \leq \min_{y \in F(x^2)} \ell^\top y \text{ and } \max_{y \in F(x^1)} \ell^\top y \leq \max_{y \in F(x^2)} \ell^\top y.
\end{aligned}$$

Note that in case the cardinality of the set of all rows of the matrices $A(x)$ with $x \in S$ is infinite, still an infinite number of functionals $\ell \in C^\star$ and their associated scalar-valued optimization problems have to be considered when all elements $F(x)$ should be comparable by using just the optimal values of these functionals over the sets. For that reason Jahn proposes in [11] a descent method which uses direct comparisons of two polyhedral sets only.

In contrast to the previous results we aim in the following on suitable "simple" set optimization problems, such that equivalent reformulations as vector optimization problems in the same or a similar image space or even (for some special cases) as multiobjective optimization problems with a finite number of objectives can be given. In the latter case this allows to make the problems numerically tractable as known techniques from multiobjective optimization can then be applied.

3 Set optimization problems being reducible to vector optimization problems

In this section we discuss several classes of set optimization problems for which we can show that they are equivalent to vector optimization problems. In all cases we can show that we can reduce the problems to comparatively simple vector optimization problems.

3.1 Set-valued maps based on a fixed set

We start by examining set-valued maps with the most simple structure: those where the images $F(x)$ are determined by a constant nonempty set $H \subset Y$ which is only moved around in the space by adding $f(x)$ with $f\colon S \to Y$ some vector-valued map. This means we are interested in set-valued maps with the structure

$$F(x) := \{f(x)\} + H = \{f(x) + h \mid h \in H\} \text{ for all } x \in S \tag{4}$$

for some nonempty set S, a nonempty subset H of Y, and $f\colon S \to Y$.

Such a map was for instance studied as a test instance by Köbis and Köbis in [14] for evaluating the properties of their proposed numerical algorithm and was motivated as the basis of the Markowitz stock model.

Example 3.1. *[14, Example 4.7] Let $S = Y = \mathbb{R}^2$, the vector-valued map f be defined by*

$$f\colon S \to \mathbb{R}^2 \text{ with } f(x) := \begin{pmatrix} x_1^2 + x_2^2 \\ 2(x_1 + x_2) \end{pmatrix} \text{ for all } x \in S,$$

the set H be given by

$$H := \left\{ \frac{1}{4} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t \in \left\{0, \frac{1}{7}\pi, \ldots, \frac{13}{7}\pi\right\} \right\},$$

and the set-valued map $F\colon S \rightrightarrows \mathbb{R}^2$ be defined according to (4)*. Using the forthcoming Theorem 3.4 the set of all minimal solutions of the corresponding set optimization problem* ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^{\star}_{\mathbb{R}^2_+}$ with $\star \in \{l, u, s\}$ is given by $\{\bar{x} \in \mathbb{R}^2 \mid \bar{x}_1 = \bar{x}_2 \leq 0\}$ (see the forthcoming Example 3.5).*

Also Hernández and López have studied in [8] special set-valued maps of the type (4). To be more concrete they have considered the following special cases:

- $F_1\colon S \rightrightarrows \mathbb{R}^m$ with
$$F_1(x) \;:=\; \{f(x)\} + H$$
for a given nonempty set $H \subset \mathbb{R}^m$ and a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

- $F_2\colon S \rightrightarrows \mathbb{R}^m$ with
$$\begin{aligned} F_2(x) &:= \{f(x)\} + \{\lambda\, q \in \mathbb{R}^m \mid \lambda \in [0,1]\} \\ &= \{\lambda\, f(x) + (1-\lambda)\,(f(x) + q) \in \mathbb{R}^m \mid \lambda \in [0,1]\} \end{aligned}$$
for a given map $f\colon S \to \mathbb{R}^m$ and $q \in \mathbb{R}^m$.

- $F_3\colon S \rightrightarrows \mathbb{R}^m$ with
$$F_3(x) \;:=\; \{f(x)\} + \{y \in \mathbb{R}^m \mid \|y\|_2 \leq r\}$$
for a given map $f\colon S \to \mathbb{R}^m$ and $r \in \mathbb{R}_+$.

For these special classes they have studied basic properties of the set-valued maps as semicontinuity and convexity in a finite dimensional setting. They also mention that for instance F_1 appears in the literature in the context of Hahn-Banach theorems or in subgradient theory and F_2 is of course related to interval optimization. The map F_3 appears in approximation theory and viability theory (cf. [8]).

We first state a result on the direct comparison of two sets $\{y^1\}+H$ and $\{y^2\}+H$ with $y^1, y^2 \in Y$.

Lemma 3.2. *Let H be a bounded nonempty subset of Y. If $y^1, y^2 \in Y$ and the sets A and B are defined by*

$$A := \{y^1\} + H \text{ and } B := \{y^2\} + H.$$

Then it holds

$$A \preccurlyeq^s_C B \;\Leftrightarrow\; A \preccurlyeq^l_C B \;\Leftrightarrow\; A \preccurlyeq^u_C B \;\Leftrightarrow\; y^1 \leq_C y^2.$$

Proof. It is easy to see that the assumption $y^1 \leq_C y^2$, i.e. $y^2 \in \{y^1\}+C$ and $y^1 \in \{y^2\}-C$, is sufficient for the other statements. Thus we now assume that $A \preccurlyeq^l_C B$. By Theorem 2.1.(i) it follows

$$\inf_{y\in A} \ell(y) \leq \inf_{y\in B} \ell(y) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}$$

which is equivalent to

$$\ell(y^1) + \inf_{h\in H} \ell(h) \leq \ell(y^2) + \inf_{h\in H} \ell(h) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}.$$

As H is nonempty and bounded it holds

$$-\infty < \inf_{h\in H} \ell(h) \leq \sup_{h\in H} \ell(h) < \infty \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}.$$

Hence, we obtain

$$\ell(y^2 - y^1) \geq 0 \;\text{ for all }\; \ell \in C^\star$$

and thus, since C is closed, by [9, Lemma 3.21 (a)] $y^1 \leq_C y^2$.
Using similar arguments we obtain

$$A \preccurlyeq^u_C B \Rightarrow y^1 \leq_C y^2,$$

and we are done. □

For the proof we need that

$$\inf_{h\in H} \ell(h) \neq -\infty \;\text{ and }\; \sup_{h\in H} \ell(h) \neq \infty$$

holds for all $\ell \in C^\star \setminus \{0_{Y^\star}\}$. Clearly, this is true in the case H is bounded. Note that for an unbounded set the result of Lemma 3.2 might not be true. For instance for $H = Y$ this can trivially be seen. Additional examples with sets H unequal to the whole space Y are given in the following.

Example 3.3. *Let $Y = \mathbb{R}^2$, $H^1 := \{y \in \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 = 0\}$, $H^2 := \{y \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 = 0\}$, and $H^3 := \{y \in \mathbb{R}^2 \mid y_2 = 0\}$. Then for $y^1 = (2,1)^\top$ and $y^2 = (1,2)^\top$ it holds $y^1 \not\leq_{\mathbb{R}^2_+} y^2$, $\{y^1\} + H^1 \preccurlyeq^l_{\mathbb{R}^2_+} \{y^2\} + H^1$, $\{y^1\} + H^2 \preccurlyeq^u_{\mathbb{R}^2_+} \{y^2\} + H^2$, and $\{y^1\} + H^3 \preccurlyeq^s_{\mathbb{R}^2_+} \{y^2\} + H^3$.*

Lemma 3.2 directly implies that set optimization problems with set-valued maps defined as in (4) are equivalent to vector optimization problems.

Theorem 3.4. *Let the set optimization problem* ($\mathrm{SOP}_{F,S}$) *be given with an objective map* F *as defined in* (4)*, i.e.*

$$F(x) = \{f(x)\} + H \text{ for all } x \in S,$$

and let the set H *be a bounded and nonempty subset of* Y*. Then* $\bar{x} \in S$ *is a minimal solution of the set optimization problem* ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation* $\preccurlyeq_C^\star$ *and* $\star \in \{l, u, s\}$ *if and only if* $\bar{x}$ *is an efficient solution of the vector optimization problem* ($\mathrm{VOP}_{f,S}$) *w.r.t.* C*, i.e. of*

$$\min_{x \in S} f(x)$$

w.r.t. the ordering cone C*.*

Using Theorem 3.4 we can verify the set of all minimal solutions of the set optimization problem stated in Example 3.1.

Example 3.5. *We consider again the set optimization problem* ($\mathrm{SOP}_{F,S}$) *defined in Example 3.1. Let* $\bar{x} \in S = \mathbb{R}^2$ *be a minimal solution of* ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation* $\preccurlyeq_{\mathbb{R}^2_+}^\star$ *and* $\star \in \{l, u, s\}$*. By Theorem 3.4 this is equivalent to that* $\bar{x}$ *is an efficient solution of the vector optimization problem*

$$\min_{x \in \mathbb{R}^2} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \tag{5}$$

w.r.t. $\mathbb{R}^2_+$ *and* $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ *with* $f_1(x) := x_1^2 + x_2^2$ *and* $f_2(x) := 2(x_1 + x_2)$ *for all* $x \in \mathbb{R}^2$*. If* $\bar{x}_1 > 0$ *or* $\bar{x}_2 > 0$ *then it is easy to see that for* $\hat{x} := (-|\bar{x}_1|, -|\bar{x}_2|)^\top \neq \bar{x}$ *it holds* $f_1(\hat{x}) = f_1(\bar{x})$ *and* $f_2(\hat{x}) < f_2(\bar{x})$ *− contradicting that* $\bar{x}$ *is an efficient solution of* (5)*. Hence, we derive* $\bar{x}_1 \leq 0$ *and* $\bar{x}_2 \leq 0$ *for any efficient solution of* (5)*. If additionally* $\bar{x}_1 \neq \bar{x}_2$ *holds, then it follows for*

$$\check{x} := \frac{1}{\sqrt{2}} \left(-\sqrt{\bar{x}_1^2 + \bar{x}_2^2}, -\sqrt{\bar{x}_1^2 + \bar{x}_2^2} \right)^\top \neq \bar{x}$$

also $f_1(\check{x}) = f_1(\bar{x})$ *and* $f_2(\check{x}) < f_2(\bar{x})$*. Thus we obtain* $\bar{x}_1 = \bar{x}_2 \leq 0$*. Let finally* $\bar{x}' = (x', x')^\top$*,* $\bar{x}'' = (x'', x'')^\top$*, and w.l.o.g.* $x' < x'' \leq 0$*. Then it holds* $f_1(\bar{x}') > f_1(\bar{x}'')$ *and* $f_2(\bar{x}') < f_2(\bar{x}'')$*. Hence, the set of all minimal solution of* ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation* $\preccurlyeq_{\mathbb{R}^2_+}^\star$ *and* $\star \in \{l, u, s\}$ *and the set of all efficient solutions of the corresponding vector optimization problem defined by* (5) *w.r.t.* $\mathbb{R}^2_+$ *is given by* $\{\bar{x} \in \mathbb{R}^2 \mid \bar{x}_1 = \bar{x}_2 \leq 0\}$*.*

Finally we relate the result of Theorem 3.4 to some other results from the literature. In [6] the following set-valued optimization problems have been studied: Let $F \colon S \rightrightarrows \mathbb{R}^m$ be defined by

$$F(x) := \{f(x + z) \in \mathbb{R}^m \mid z \in Z\}$$

where $S \subset \mathbb{R}^n$ is some nonempty set, $Z \subset \mathbb{R}^n$ is a compact set with $0 \in Z$, and $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is a given vector-valued map. It was assumed that the linear space $\mathbb{R}^m$ is partially ordered by some pointed convex closed cone C with nonempty interior. These problems arise in the study of multiobjective optimization problems which have some uncertainties in the realization of solutions and when a robust approach is chosen. For the set optimization problem the u-less order relation was used. In [6, Section 5.1] linear objective functions f have been studied. Then one obtains $F(x) = \{f(x)\} + H$ with $H := \{f(z) \in \mathbb{R}^m \mid z \in Z\}$, which is a bounded nonempty set and Theorem 3.4 can be applied.

Other set-valued maps with a simple structure using a constant nonempty set can be defined by multiplication with a scalar-valued function. Thus we are interested now in set-valued maps with the structure

$$F(x) := \varphi(x)H = \{\varphi(x)h \mid h \in H\} \text{ for all } x \in S \tag{6}$$

for some nonempty set S, a nonempty subset H of Y with $H \neq \{0_Y\}$, and $\varphi\colon S \to \mathbb{R}$. We show that such set optimization problems (under strict additional assumptions on the set H) can be formulated equivalently even as scalar-valued optimization problems.

In analogy to Lemma 3.2 we want to formulate results on the direct comparison of two sets αH and βH with $\alpha, \beta \in \mathbb{R}$. However, the following example shows that

$$\alpha H \preccurlyeq_C^\star \beta H \Leftrightarrow \alpha \leq \beta \tag{7}$$

with $\star \in \{l, u, s\}$ does not hold in general even for a compact set.

Example 3.6. *Let $Y = \mathbb{R}$ and $H := [-1, 1]$. Then it holds $\gamma H = [-|\gamma|, |\gamma|]$ for all $\gamma \in \mathbb{R}$ and we obtain $\alpha H \preccurlyeq_{\mathbb{R}_+}^l \beta H \Leftrightarrow |\alpha| \geq |\beta|$, $\alpha H \preccurlyeq_{\mathbb{R}_+}^u \beta H \Leftrightarrow |\alpha| \leq |\beta|$, and $\alpha H \preccurlyeq_{\mathbb{R}_+}^s \beta H \Leftrightarrow |\alpha| = |\beta|$ for all $\alpha, \beta \in \mathbb{R}$. See also the forthcoming Lemma 3.12 for additional results for this type of set-valued maps.*

The following lemma formulates additional assumptions under which (7) can be guaranteed.

Lemma 3.7. *Let H be a bounded nonempty subset of Y with $H \neq \{0_Y\}$, let $\alpha, \beta \in \mathbb{R}$, and let the sets A and B are defined by*

$$A := \alpha H \text{ and } B := \beta H.$$

Then the following holds:

(i) If $H \subset C$ and $\alpha \leq \beta$, then it holds $A \preccurlyeq_C^\star B$ with $\star \in \{l, u, s\}$.

(ii) If $H \subset C$ and there exists an $\hat{\ell} \in C^\star \setminus \{0_{Y^\star}\}$ such that $\inf_{h \in H} \hat{\ell}(h) > 0$, then it holds

$$A \preccurlyeq_C^s B \;\Leftrightarrow\; A \preccurlyeq_C^l B \;\Leftrightarrow\; A \preccurlyeq_C^u B \;\Leftrightarrow\; \alpha \leq \beta.$$

(iii) If $H \subset -C$ and $\alpha \geq \beta$, then it holds $A \preccurlyeq_C^\star B$ with $\star \in \{l, u, s\}$.

(iv) If $H \subset -C$ and there exists an $\bar{\ell} \in C^\star \setminus \{0_{Y^\star}\}$ such that $\sup_{h \in H} \bar{\ell}(h) < 0$, then it holds

$$A \preccurlyeq_C^s B \;\Leftrightarrow\; A \preccurlyeq_C^l B \;\Leftrightarrow\; A \preccurlyeq_C^u B \;\Leftrightarrow\; \alpha \geq \beta.$$

Proof. We restrict ourselves to the proofs of *(i)* and *(ii)*. Hence $H \subset C$ and note that

$$0 \leq \inf_{h \in H} \ell(h) \leq \sup_{h \in H} \ell(h) < \infty \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}. \tag{8}$$

For the proof of *(i)* let $\alpha \leq \beta$. For any $b \in B$ there exists $h \in H \subset C$ with $b = \beta h$. For $a := \alpha h \in A$ it holds $b - a = (\beta - \alpha)h \in C$. Hence $a \leq_C b$ and $A \preccurlyeq_C^l B$ is shown. Using similar arguments we obtain $A \preccurlyeq_C^u B$ which proves the assertion.
For *(ii)* it remains to show that $\alpha \leq \beta$ is also necessary for $A \preccurlyeq_C^\star B$ with $\star \in \{l, u, s\}$. If $A \preccurlyeq_C^l B$ then it holds by Theorem 2.1.(i)

$$\inf_{h \in H} \alpha\ell(h) \leq \inf_{h \in H} \beta\ell(h) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}. \tag{9}$$

For $\alpha \geq 0$ and $\beta \geq 0$ it is easy to see that (9) is equivalent to

$$\alpha \inf_{h \in H} \ell(h) \leq \beta \inf_{h \in H} \ell(h) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}$$

and we obtain $\alpha \inf_{h \in H} \hat{\ell}(h) \leq \beta \inf_{h \in H} \hat{\ell}(h)$. By using $\inf_{h \in H} \hat{\ell}(h) > 0$ and (8) it follows $\alpha \leq \beta$. If $\alpha \geq 0$ and $\beta < 0$ then (9) is equivalent to

$$\alpha \inf_{h \in H} \ell(h) \leq \beta \sup_{h \in H} \ell(h) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}$$

and it follows by (8)

$$\alpha \inf_{h \in H} \ell(h) = \beta \sup_{h \in H} \ell(h) = 0 \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}$$

which contradicts $\beta \sup_{h \in H} \hat{\ell}(h) \leq \beta \inf_{h \in H} \hat{\ell}(h) < 0$. In case $\alpha < 0$ and $\beta \geq 0$ then $\alpha \leq \beta$ holds trivially. For $\alpha < 0$ and $\beta < 0$ we obtain as an equivalent formulation of (9)

$$\alpha \sup_{h \in H} \ell(h) \leq \beta \sup_{h \in H} \ell(h) \text{ for all } \ell \in C^\star \setminus \{0_{Y^\star}\}$$

and $\alpha \leq \beta$ follows immediately by using $\sup_{h \in H} \hat{\ell}(h) \geq \inf_{h \in H} \hat{\ell}(h) > 0$ and (8) again. Using similar arguments we obtain

$$A \preccurlyeq_C^u B \Rightarrow \alpha \leq \beta,$$

and we are done. □

Remark 3.8. *The existence of an* $\hat{\ell} \in C^\star \setminus \{0_{Y^\star}\}$ *with* $\inf_{h \in H} \hat{\ell}(h) > 0$ *can be guaranteed for instance if* H *is a nonempty, compact, and convex subset of* C *with* $0_Y \notin H$. *Since* $-C$ *is closed and convex there exists in this case by a suitable separation theorem (cf. [9, Theorem 3.20]) an* $\ell \in Y^\star$ *such that*

$$0 = \ell(0_Y) \leq \sup_{y \in -C} \ell(y) < \inf_{h \in H} \ell(h) < \infty$$

and $\ell \in C^\star$ *follows by standard arguments.*

We note that for an unbounded set the result of Lemma 3.7 might not be true.

Example 3.9. *Let $Y = \mathbb{R}^2$ and $H := \{y \in \mathbb{R}^2 \mid y = (1+r, 1+r)^\top,\ r \in \mathbb{R}_+\} \subset \mathbb{R}^2_+$. Then it holds $\alpha H \preccurlyeq^l_{\mathbb{R}^2_+} \beta H$ for all $\alpha, \beta < 0$ and $\alpha H \preccurlyeq^u_{\mathbb{R}^2_+} \beta H$ for all $\alpha, \beta > 0$.*

Lemma 3.7 directly implies that under the mentioned strict additional assumptions set optimization problems with set-valued maps defined as in (6) can be formulated equivalently as scalar-valued optimization problems. In the following theorem we restrict ourselves to the case $H \subset C$.

Theorem 3.10. *Let the set optimization problem* ($\mathrm{SOP}_{F,S}$) *be given with an objective map F as defined in* (6)*, i.e.*

$$F(x) = \varphi(x)H \text{ for all } x \in S,$$

and let the set H be a bounded nonempty subset of C with $H \neq \{0_Y\}$. If there exists an $\hat{\ell} \in C^\star \setminus \{0_{Y^\star}\}$ such that $\inf_{h \in H} \hat{\ell}(h) > 0$, then $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^\star_C$ and $\star \in \{l, u, s\}$ if and only if $\bar{x}$ is a minimal solution of the scalar-valued optimization problem*

$$\min_{x \in S} \varphi(x).$$

3.2 Box-valued maps

The set-valued maps which we study in this section are assumed to have values $F(x)$ which are boxes in Y. For this purpose let $a, b \in Y$ with $a \leq_C b$ and the corresponding box $[a, b]_C$ be defined by

$$[a, b]_C := (\{a\} + C) \cap (\{b\} - C).$$

Obviously a box is a convex and closed set and we are now interested in set-valued maps with the structure

$$F(x) := [a(x), b(x)]_C \text{ for all } x \in S \tag{10}$$

for some nonempty set S and two vector-valued maps $a, b \colon S \to Y$ with $a(x) \leq_C b(x)$ for all $x \in S$. In a finite dimensional setting, the properties as semicontinuity and convexity of maps as in (10) have already been studied in [8]. If for some $x \in S$ it holds $a(x) = b(x)$ then the set $F(x)$ is a singleton. A simple example for such set-valued maps are the so called interval-valued maps in the case $Y = \mathbb{R}$ and $C = \mathbb{R}_+$. We use for reasons of simplicity in the case $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$ the usual notation $[a, b]$ instead of $[a, b]_{\mathbb{R}^m_+}$. Another example of such set-valued maps is given in the example below and was provided in [3].

Example 3.11. *[3, Example 4.1] Let $S = [0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F \colon S \rightrightarrows \mathbb{R}^2$ be defined by*

$$F(x) := \{y \in \mathbb{R}^2 \mid y_1 = x,\ y_2 \in [x, 2 - x]\}.$$

Then the images are boxes with

$$F(x) = (\{(x, x)^\top\} + \mathbb{R}^2_+) \cap (\{(x, 2-x)^\top\} - \mathbb{R}^2_+) = [(x, x)^\top, (x, 2-x)^\top].$$

It is easy to see (cf. for instance the forthcoming Theorem 3.13 or Corollary 3.15) that $\bar{x} = 0$ is the unique minimal solution of the set optimization problem ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^l_{\mathbb{R}^2_+}$ and the set of all minimal solutions of* ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^\star_{\mathbb{R}^2_+}$ with $\star \in \{u, s\}$ is given by the whole set S.*

The following lemma formulates a result on the direct comparison of two boxes.

Lemma 3.12. *Let two sets A and B in Y be defined by*

$$A := [a^1, b^1]_C \text{ and } B := [a^2, b^2]_C$$

where $a^1, a^2, b^1, b^2 \in Y$ with $a^1 \leq_C b^1$ and $a^2 \leq_C b^2$. Then it holds

$$A \preccurlyeq^l_C B \Leftrightarrow a^1 \leq_C a^2 \quad \text{and} \quad A \preccurlyeq^u_C B \Leftrightarrow b^1 \leq_C b^2.$$

Proof. If $A \preccurlyeq^l_C B$ then there exists $a \in A \subset \{a^1\} + C$ such that $a \leq_C a^2$ and thus $a \in \{a^2\} - C$. Hence, there exist $k^1, k^2 \in C$ with $a = a^1 + k^1 = a^2 - k^2$. It follows $a^2 - a^1 = k^1 + k^2 \in C$ and we obtain $a^1 \leq_C a^2$. Let now $a^1 \leq_C a^2$ and $b \in B \subset \{a^2\} + C$ be arbitrarily chosen. Hence it holds $a^2 \leq_C b$. By the transitivity of $\leq_C$ for $a := a^1 \in A$ it follows $a \leq_C b$ and thus $A \preccurlyeq^l_C B$. Using similar arguments we obtain $A \preccurlyeq^u_C B$ if and only if $b^1 \leq_C b^2$. □

This result is also stated (without proof) in a finite dimensional setting in [8, Remark 1]. Lemma 3.12 directly implies that set optimization problems with set-valued maps defined as in (10) are equivalent to vector optimization problems.

Theorem 3.13. *Let the set optimization problem* ($\text{SOP}_{F,S}$) *be given with an objective map F as defined in* (10)*, i.e.*

$$F(x) = [a(x), b(x)]_C \text{ for all } x \in S.$$

Then the following holds:

(i) $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\text{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^l_C$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t. C for $f\colon S \to Y$ and $f(x) := a(x)$.*

(ii) $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\text{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^u_C$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t. C for $f\colon S \to Y$ and $f(x) := b(x)$.*

(iii) $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\text{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^s_C$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t. $C \times C := \{(c^1, c^2) \in Y \times Y \mid \; c^1 \in C,\; c^2 \in C\}$ for $f\colon S \to Y \times Y$ and $f(x) := \begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$, i.e. of*

$$\min_{x \in S} \begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$$

w.r.t. the ordering cone $C \times C$.

As a consequence of Theorem 3.13 we obtain the following corollary, which confirms the statement of Theorem 3.4 for a special case of the set H.

Corollary 3.14. *Let $f\colon S \to Y$ be a given map, let $q \in C$, and let the set optimization problem* ($\mathrm{SOP}_{F,S}$) *be given with the objective map F defined by*

$$F\colon S \rightrightarrows Y \text{ and } F(x) := \{f(x)\} + [0_Y, q]_C = [f(x), f(x)+q]_C \text{ for all } x \in S.$$

Then $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^\star_C$ and $\star \in \{l, u, s\}$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem* ($\mathrm{VOP}_{f,S}$) *w.r.t. C.*

Finally we study boxes which are defined by a multiplication of a scalar-valued function and a fixed box to relate our results to those in Theorem 3.10. Let $a, b \in Y$ with $a \leq_C b$, $H := [a,b]_C$, $\varphi\colon S \to \mathbb{R}$, and the set-valued map $F\colon S \rightrightarrows Y$ be defined according to (6) by $F(x) := \varphi(x)[a,b]_C$ for all $x \in S$. Then it holds for the box-valued map F

$$F(x) = \begin{cases} [\varphi(x)b, \varphi(x)a]_C, & \text{if } \varphi(x) \leq 0 \\ \\ [\varphi(x)a, \varphi(x)b]_C, & \text{if } \varphi(x) \geq 0 \end{cases} \quad \text{for all } x \in S.$$

An application of Theorem 3.13 to the corresponding set optimization problem ($\mathrm{SOP}_{F,S}$) is possible only under the additional assumption that either $\varphi(x) \leq 0$ or $\varphi(x) \geq 0$ holds for all $x \in S$. While an application of Theorem 3.10 is only possible in case we have a compact box $[a,b]_C \subset H$ with $0_Y \notin H$.

We end this subsection by applying Theorem 3.13 to the finite dimensional case with the natural ordering:

Corollary 3.15. *Let S be a nonempty subset of $\mathbb{R}^n$, let $a, b\colon \mathbb{R}^n \to \mathbb{R}^m$ be given maps with $a_i(x) \leq b_i(x)$ for all $i \in \{1, \ldots, m\}$ and $x \in S$, and let the function $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be defined by*

$$F(x) := [a(x), b(x)] \text{ for all } x \in S.$$

Then $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^l_{\mathbb{R}^m_+} \mid \preccurlyeq^u_{\mathbb{R}^m_+} \mid \preccurlyeq^s_{\mathbb{R}^m_+}$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem* ($\mathrm{VOP}_{f,S}$) *w.r.t. $\mathbb{R}^m_+ | \mathbb{R}^m_+ | \mathbb{R}^{2m}_+$ for $f\colon S \to \mathbb{R}^m | \mathbb{R}^m | \mathbb{R}^{2m}$ and $f := a | f := b | f := \begin{pmatrix} a \\ b \end{pmatrix}$.*

3.3 Ball-valued maps

Let in this section $(Y, \langle \cdot, \cdot \rangle)$ be a Hilbert space and we write

$$\ell(y) = \langle \ell, y \rangle$$

for the dual pairing. The set-valued maps which we study next are assumed to have values $F(x)$ which are balls with variable midpoints and with variable radii. Therefore let

$$\mathcal{B}_Y(r) := \{y \in Y \mid \ \|y\| \leq r\}$$

for $r \in \mathbb{R}_+$ with $\|y\| = \sqrt{\langle y, y \rangle}$. Thus, we are interested in set-valued maps with the structure

$$F(x) := \{c(x)\} + \mathcal{B}_Y(r(x)) \text{ for all } x \in S \tag{11}$$

for some nonempty set S, a vector-valued map $c\colon S \to Y$, and a function $r\colon S \to \mathbb{R}_+$. For basic properties of such maps as well as for references to applications where such maps are of interest we refer to [8]. Such a set optimization problem was for instance studied by Jahn:

Example 3.16. *[10, Example 3.1] Let* $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $S = [-1, 1]$, *and* $F : S \rightrightarrows \mathbb{R}^2$ *be defined by*

$$F(x) := \{y \in \mathbb{R}^2 \mid (y_1 - 2x^2)^2 + (y_2 - 2x^2)^2 \leq (x^2 + 1)^2\} \text{ for all } x \in S.$$

Then the images are balls with

$$F(x) = \{(2x^2, 2x^2)^\top\} + \mathcal{B}_{\mathbb{R}^2}(x^2 + 1)$$

and $\bar{x} := 0$ *is the unique minimal solution of the corresponding set optimization problem* $(\mathrm{SOP}_{F,S})$ *w.r.t. the order relation* $\preccurlyeq^s_{\mathbb{R}^2_+}$, *cf. Example 3.24.*

We need the following results:

Lemma 3.17. *Let* $r \in \mathbb{R}_+$ *and* $\ell \in C^\star$ *with* $\|\ell\| = 1$. *Then it holds*

$$\min_{y \in \mathcal{B}_Y(r)} \ell(y) = -r \text{ and } \max_{y \in \mathcal{B}_Y(r)} \ell(y) = r.$$

Proof. Let $r \in \mathbb{R}_+$ and $\ell \in C^\star \setminus \{0_{Y^\star}\}$ with $\|\ell\| = 1$ be arbitrarily chosen. Then it holds by the Cauchy-Schwarz inequality for all y with $\|y\| \leq r$

$$|\ell(y)| = |\langle \ell, y \rangle| \leq \|\ell\| \, \|y\| \leq r$$

and hence

$$-r \leq \langle \ell, y \rangle \leq r.$$

Since for $\underline{y} := -\ell r \in \mathcal{B}_Y(r)$ and $\overline{y} := \ell r \in \mathcal{B}_Y(r)$ it holds

$$\ell(\underline{y}) = \langle \ell, \underline{y} \rangle = -r\langle \ell, \ell \rangle = -r \text{ and } \ell(\overline{y}) = \langle \ell, \overline{y} \rangle = r\langle \ell, \ell \rangle = r$$

we are done. □

Using Theorem 2.1(ii) and Lemma 3.17 one can easily verify the following lemma:

Lemma 3.18. *Let two sets* A *and* B *in* Y *be defined by*

$$A := \{y^1\} + \mathcal{B}_Y(r_1) \text{ and } B := \{y^2\} + \mathcal{B}_Y(r_2) \tag{12}$$

where $y^1, y^2 \in Y$ *and* $r_1, r_2 \in \mathbb{R}_+$. *Then it holds*

$$A \preccurlyeq^l_C B \;\Leftrightarrow\; \forall \ell \in C^\star \text{ with } \|\ell\| = 1 : \ell(y^2 - y^1) = \langle \ell, y^2 - y^1 \rangle \geq r_2 - r_1 \text{ and}$$

$$A \preccurlyeq^u_C B \;\Leftrightarrow\; \forall \ell \in C^\star \text{ with } \|\ell\| = 1 : \ell(y^2 - y^1) = \langle \ell, y^2 - y^1 \rangle \geq -(r_2 - r_1).$$

Using Lemma 3.18 we can now proof the following result on the direct comparison of two balls regarding the set less order relation based on only a finite number of inequalities in case the cone C has a finitely generated dual cone.

Lemma 3.19. *Let the two sets* A *and* B *be defined as in* (12). *If there exist* $\ell^1, \ldots, \ell^k \in C^\star$ *with* $\|\ell^i\| = 1$ *for all* $i \in \{1, \ldots, k\}$ *such that* $C^\star = \mathrm{cone}(\mathrm{conv}(\{\ell^i \mid i \in \{1, \ldots, k\}\}))$, *then*

$$A \preccurlyeq^s_C B \Leftrightarrow \langle \ell^i, y^2 - y^1 \rangle \geq |r_2 - r_1| \text{ for all } i \in \{1, \ldots, k\}.$$

Proof. By using Lemma 3.18 it holds

$$\begin{aligned} A \preccurlyeq^s_C B \;&\Leftrightarrow\; \langle \ell, y^2 - y^1 \rangle \geq |r_2 - r_1| \quad \text{for all } \ell \in C^\star \text{ with } \|\ell\| = 1 \\ &\Rightarrow\; \langle \ell^i, y^2 - y^1 \rangle \geq |r_2 - r_1| \quad \text{for all } \ell^i,\ i \in \{1, \ldots, k\}. \end{aligned}$$

Thus we assume now $\langle \ell^i, y^2 - y^1 \rangle \geq |r_2 - r_1|$ for all $i \in \{1, \ldots, k\}$. Let $\ell \in C^\star$ with $\|\ell\| = 1$. Then there exist $\lambda_i \in \mathbb{R}_+$, $i \in \{1, \ldots, k\}$ such that

$$\sum_{i=1}^{k} \lambda_i = 1,\ \ell = \frac{1}{\left\| \sum\limits_{i=1}^{k} \lambda_i \ell^i \right\|} \sum_{i=1}^{k} \lambda_i \ell^i, \text{ and } \left\| \sum_{i=1}^{k} \lambda_i \ell^i \right\| \leq \sum_{i=1}^{k} \lambda_i \, \|\ell^i\| = 1.$$

Finally it follows

$$\langle \ell, y^2 - y^1 \rangle = \frac{1}{\left\| \sum\limits_{i=1}^{k} \lambda_i \ell^i \right\|} \sum_{i=1}^{k} \lambda_i \left\langle \ell^i, y^2 - y^1 \right\rangle \geq \sum_{i=1}^{k} \lambda_i |r_2 - r_1| = |r_2 - r_1|.$$

□

In the case $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$ it holds

$$C^\star = \mathbb{R}^m_+ = \operatorname{cone}(\operatorname{conv}(\{e^i \mid \ i \in \{1, \ldots, m\}\})),$$

where e^i, $i \in \{1, \ldots, m\}$ denotes the i-th unit vector of $\mathbb{R}^m$. Using Lemma 3.19 in this special case it follows:

Lemma 3.20. *Let $y^1, y^2 \in \mathbb{R}^m$ and $r_1, r_2 \in \mathbb{R}_+$. Then the following holds:*

$$\{y^1\} + \mathcal{B}_{\mathbb{R}^m}(r_1) \preccurlyeq^s_{\mathbb{R}^m_+} \{y^2\} + \mathcal{B}_{\mathbb{R}^m}(r_2) \quad \Leftrightarrow \quad y^2_i - y^1_i \geq |r_2 - r_1| \text{ for all } i \in \{1, \ldots, m\}.$$

We use now this result for the formulation of an equivalent vector optimization problem to a set optimization problem with such values of the objective map. For that we need the ordering cone

$$C^{m+1} := \left\{ y \in \mathbb{R}^{m+1} \middle|\ y_i \geq |y_{m+1}| \ \forall\, i \in \{1, \ldots, m\} \right\}. \tag{13}$$

It is easy to see that C^{m+1} is a pointed, convex, and closed nontrivial cone and that $\{y^1\} + \mathcal{B}_{\mathbb{R}^m}(r_1) \preccurlyeq^s_{\mathbb{R}^m_+} \{y^2\} + \mathcal{B}_{\mathbb{R}^m}(r_2)$ if and only if

$$\begin{pmatrix} y^1 \\ r_1 \end{pmatrix} \leq_{C^{m+1}} \begin{pmatrix} y^2 \\ r_2 \end{pmatrix}.$$

Hence we obtain:

Theorem 3.21. *Let S be a nonempty subset of $\mathbb{R}^n$, let the maps $c\colon S \to \mathbb{R}^m$ and $r\colon S \to \mathbb{R}_+$ be given, let the set-valued map $F\colon S \rightrightarrows \mathbb{R}^m$ be defined by*

$$F(x) := \{c(x)\} + \mathcal{B}_{\mathbb{R}^m}(r(x)) \text{ for all } x \in S,$$

and let the pointed convex cone C^{m+1} be defined as in (13). Then $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\text{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^s_{\mathbb{R}^m_+}$ if and only* *if* $\bar{x}$ *is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t.* C^{m+1} *for* $f\colon \mathbb{R}^n \to \mathbb{R}^{m+1}$ *and* $f := \begin{pmatrix} c \\ r \end{pmatrix}$, *i.e. of*

$$\min_{x \in S} \begin{pmatrix} c(x) \\ r(x) \end{pmatrix}$$

w.r.t. the ordering cone C^{m+1}.

The vector optimization problem in Theorem 3.21 is a finite dimensional problem but the ordering cone is not the natural ordering cone. However, it is a finitely generated cone. This can be used to formulate another multiobjective optimization problem to our set optimization problem which is now with respect to the natural (componentwise) ordering. For that let now

$$\bar{K}^{m+1} := \begin{pmatrix} I_m & 1_m \\ I_m & -1_m \end{pmatrix} \in \mathbb{R}^{2m \times (m+1)} \tag{14}$$

where I_m is the m-dimensional identity matrix and 1_m is the m-dimensional all-one vector. It is easy to see that $\text{kernel}(\bar{K}^{m+1}) = \{0_{m+1}\}$ and

$$C^{m+1} = \left\{ y \in \mathbb{R}^{m+1} \middle| \ \bar{K}^{m+1} y \geq 0_{2m} \right\},$$

i.e. C^{m+1} is polyhedral. Using [4, Lemma 1.18] or [16, Lemma 2.3.4] and Theorem 3.21 we obtain our main result of this section, which will also be the main result which we use in Section 4 for the construction of new test instances for set optimization.

Theorem 3.22. *Let* S *be a nonempty subset of* $\mathbb{R}^n$, *let the maps* $c\colon S \to \mathbb{R}^m$ *and* $r\colon S \to \mathbb{R}_+$ *be given, let the set-valued map* $F\colon S \rightrightarrows \mathbb{R}^m$ *be defined by*

$$F(x) := \{c(x)\} + \mathcal{B}_{\mathbb{R}^m}(r(x)) \text{ for all } x \in S,$$

and let the matrix $\bar{K}^{m+1}$ *be defined as in* (14). *Then* $\bar{x} \in S$ *is a minimal solution of the set optimization problem* ($\text{SOP}_{F,S}$) *w.r.t. the order relation* $\preccurlyeq^s_{\mathbb{R}^m_+}$ *if and only if* $\bar{x}$ *is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t.* $\mathbb{R}^{2m}_+$ *for* $f\colon \mathbb{R}^n \to \mathbb{R}^{2m}$ *and* $f := \bar{K}^{m+1} \begin{pmatrix} c \\ r \end{pmatrix}$, *i.e. of*

$$\min_{x \in S} \ \bar{K}^{m+1} \begin{pmatrix} c(x) \\ r(x) \end{pmatrix}$$

w.r.t. the ordering cone $\mathbb{R}^{2m}_+$.

As a consequence of Theorem 3.22 we obtain the following corollary, which confirms the statement of Theorem 3.4 for another special case of the set H.

Corollary 3.23. *Let* S *be a nonempty subset of* $\mathbb{R}^n$, *let the map* $f\colon S \to \mathbb{R}^m$ *be given, let* $r \in \mathbb{R}_+$, *and let the set-valued map* $F\colon S \rightrightarrows \mathbb{R}^m$ *be defined by*

$$F(x) := \{f(x)\} + \mathcal{B}_{\mathbb{R}^m}(r) \text{ for all } x \in S.$$

Then $\bar{x} \in S$ *is a minimal solution of the set optimization problem* ($\text{SOP}_{F,S}$) *w.r.t. the order relation* $\preccurlyeq^s_{\mathbb{R}^m_+}$ *if and only if* $\bar{x}$ *is an efficient solution of the vector optimization problem* ($\text{VOP}_{f,S}$) *w.r.t.* $\mathbb{R}^m_+$.

Proof. By Theorem 3.22 $\bar{x} \in S$ is a minimal solution of the set optimization problem (SOP$_{F,S}$) w.r.t. the order relation $\preccurlyeq^s_{\mathbb{R}^m_+}$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem

$$\min_{x \in S} \begin{pmatrix} f(x) + r\,1_m \\ f(x) - r\,1_m \end{pmatrix}$$

w.r.t. the ordering cone $\mathbb{R}^{2m}_+$. This is equivalent to that $\bar{x}$ is an efficient solution of the vector optimization problem

$$\min_{x \in S} \; f(x)$$

w.r.t. $\mathbb{R}^m_+$, and we are done. □

Finally we use our results to verify the unique minimal solution of the set optimization problem stated in Example 3.16 by using Theorem 3.22.

Example 3.24. *We consider again the set optimization problem* (SOP$_{F,S}$) *defined as in Example 3.16. Using Theorem 3.22 it holds that $\bar{x} \in S$ is a minimal solution of the set optimization problem* (SOP$_{F,S}$) *w.r.t. the order relation $\preccurlyeq^s_{\mathbb{R}^2_+}$ if and only if $\bar{x}$ is an efficient solution of the vector optimization problem*

$$\min_{x \in [-1,1]} \begin{pmatrix} 3x^2 + 1 \\ 3x^2 + 1 \\ x^2 - 1 \\ x^2 - 1 \end{pmatrix} \tag{15}$$

w.r.t. the ordering cone $\mathbb{R}^4_+$. Now it is easy to see that the unique efficient solution of (15) *w.r.t. $\mathbb{R}^4_+$ and thus the unique minimal solution of the corresponding set optimization problem* (SOP$_{F,S}$) *w.r.t. the order relation $\preccurlyeq^s_{\mathbb{R}^2_+}$ is given by $\bar{x} := 0$.*

4 Implication on set-valued test instances

In this section we make some suggestions on how the results of the previous Section 3 can be used for the construction of set-valued test instances based on known vector-valued or scalar-valued optimization problems in the case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $C = \mathbb{R}^m_+$. In the most cases we will restrict ourselves to $m = 2$.

Such instances for set optimization problems (SOP$_{F,S}$) using a set-valued map $F\colon S \subset \mathbb{R}^n \to \mathbb{R}^m$ based on a fixed set $H \subset \mathbb{R}^m$ defined as in (4) or (6), i.e.

$$F(x) = \{f(x)\} + H \text{ or } F(x) = \varphi(x)H \text{ for all } x \in S,$$

can easily be established by directly applying Theorem 3.4 or Theorem 3.10, respectively. For this only a suitable set $H \subset \mathbb{R}^m$ in terms of the formulated assumptions in the corresponding theorem and a multi-objective optimization problem $\min_{x \in S} f(x)$ (ideally with known set of all efficient solutions w.r.t. $\mathbb{R}^m_+$) or a scalar-valued optimization problem $\min_{x \in S} \varphi(x)$ (ideally with known set of all minimal solutions) has to be chosen. Using the statements of the mentioned theorems the set of all minimal solutions of the set optimization problem (SOP$_{F,S}$) w.r.t. the order relation $\preccurlyeq^\star_{\mathbb{R}^m_+}$ and $\star \in \{l, u, s\}$ is given by the set of all efficient solutions of the chosen multi-objective optimization problem w.r.t. $\mathbb{R}^m_+$

or by the set of all a minimal solutions of the chosen scalar-valued optimization problem, respectively.

Moreover, test instances for set optimization problems ($\mathrm{SOP}_{F,S}$) using a box-valued map $F\colon S \subset \mathbb{R}^n \to \mathbb{R}^m$ defined according to (10), i.e.

$$F(x) = [a(x), b(x)] \text{ with } a_i(x) \leq b_i(x) \text{ for all } x \in S \text{ and } i \in \{1, \ldots, m\},$$

can be defined by applying Corollary 3.15 and by choosing again a suitable multi-objective optimization problem $\min_{x\in S} f(x)$. For instance, if $f\colon S \subset \mathbb{R}^n \to \mathbb{R}^{2m}$ is a vector-valued function such that $f_i(x) \leq f_{i+m}(x)$ for all $x \in S$ and $i \in \{1, \ldots, m\}$ and we define the vector-valued maps $a, b\colon S \subset \mathbb{R}^n \to \mathbb{R}^m$ by

$$a(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \text{ and } b(x) := \begin{pmatrix} f_{m+1}(x) \\ \vdots \\ f_{2m}(x) \end{pmatrix} \text{ for all } x \in S,$$

then the set of all minimal solutions of the set optimization problem ($\mathrm{SOP}_{F,S}$) w.r.t. the order relation $\preccurlyeq^l_{\mathbb{R}^m_+} \mid \preccurlyeq^u_{\mathbb{R}^m_+} \mid \preccurlyeq^s_{\mathbb{R}^m_+}$ is given by the set of all efficient solutions of the vector optimization problem $\min_{x\in S} a(x) | \min_{x\in S} b(x) | \min_{x\in S} f(x)$ w.r.t. $\mathbb{R}^m_+ | \mathbb{R}^m_+ | \mathbb{R}^{2m}_+$.

It takes more effort to construct test instances for set optimization problems ($\mathrm{SOP}_{F,S}$) with a ball-valued map $F\colon S \subset \mathbb{R}^n \to \mathbb{R}^m$, i.e., see (11), with

$$F(x) := \{c(x)\} + \mathcal{B}_{\mathbb{R}^m}(r(x)) \text{ for all } x \in S$$

with $c\colon S \to \mathbb{R}^m$ and $r\colon S \to \mathbb{R}_+$. We need the following result, which follows with Theorem 3.22 and [1, Theorem 3.1(2)]:

Lemma 4.1. *Let S be a nonempty subset of $\mathbb{R}^n$ and a map $f\colon S \to \mathbb{R}^m$ be given. Moreover, let the matrix $\bar{K}^{m+1}$ be defined as in* (14)*, let $\varrho \in \mathbb{R}^m$ be a vector and $H \in \mathbb{R}^{m\times m}$ be a matrix such that*

$$r(x) := \varrho^\top f(x) \geq 0 \text{ for all } x \in S, \tag{16}$$

and such that the matrix $\bar{H} \in \mathbb{R}^{2m\times m}$ with

$$\bar{H} := \bar{K}^{m+1} \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} \text{ has full rank } m, \tag{17}$$

and such that

$$\{z \in \mathbb{R}^m \mid \bar{H}z \in R^{2m}_+\} = \mathbb{R}^m_+. \tag{18}$$

Then $\bar{x} \in S$ is a minimal solution of the set optimization problem ($\mathrm{SOP}_{F,S}$) *w.r.t. the order relation $\preccurlyeq^s_{\mathbb{R}^m_+}$ and with $F\colon S \rightrightarrows \mathbb{R}^m$ defined by*

$$F(x) := \{Hf(x)\} + \mathcal{B}_{\mathbb{R}^m}(r(x)) \text{ for all } x \in S,$$

if and only if $\bar{x} \in S$ is an efficient solution of the vector optimization problem ($\mathrm{VOP}_{f,S}$) *w.r.t. $\mathbb{R}^m_+$.*

To illustrate how Lemma 4.1 can be used for the construction of test instances we restrict ourselves to the case $m = 2$ and we choose the following matrices $H \in \mathbb{R}^{2\times 2}$ and the following vectors $\varrho \in \mathbb{R}^2$, which guarantee that (17) and (18) are fulfilled:

(i) $H := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\varrho := (\frac{1}{2}, -\frac{1}{2})^\top$, and thus $\bar{H} := \bar{K}^3 \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^\top$.

(ii) $H := \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$, $\varrho := (0, -\frac{1}{2})^\top$, and thus $\bar{H} := \bar{K}^3 \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}^\top$.

(iii) $H := \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$, $\varrho := (\frac{1}{2}, 0)^\top$, and thus $\bar{H} := \bar{K}^3 \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}^\top$.

Furthermore also in all three cases for a map $f\colon S \to \mathbb{R}^2$ it holds for all $x \in S$

$$f_1(x) \geq 0 \text{ and } f_2(x) \leq 0 \Rightarrow r(x) := \varrho^\top f(x) \geq 0.$$

To explain our approach we use in the following two examples which are slight modifications of the test instances from [4, p.145] and [7]. The reason for the slight modifications is to guarantee $r(x) \geq 0$ for all $x \in S$ which is reached by subtracting a suitable constant from the corresponding second objective function. Note that this kind of modification has no influence on the set of all efficient solutions.

We start with an example where the image set of the chosen bicriteria optimization problem is convex.

Example 4.2. *Let $S := \{x \in \mathbb{R}^2_+ \mid x_1^2 - 4x_1 + x_2 + 1.5 \leq 0\}$, $Y = \mathbb{R}^2$, and the vector-valued map $f\colon S \to \mathbb{R}^2$ be defined by*

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} := \begin{pmatrix} \sqrt{1 + x_1^2} \\ x_1^2 - 4x_1 + x_2 \end{pmatrix} \text{ for all } x \in S.$$

For the vector optimization problem $\min_{x \in S} f(x)$ the set of all efficient solutions w.r.t. $\mathbb{R}^2_+$ is given by

$$M := \left\{ x \in \mathbb{R}^2_+ \;\middle|\; x_1 \in \left[2 - \frac{\sqrt{10}}{2}, 2\right],\ x_2 = 0 \right\}. \tag{19}$$

Moreover, $f_1(x) \geq 0$ and $f_2(x) \leq 0$ is satisfied for all $x \in S$.
If now H and ϱ are chosen according to **(i)**, **(ii)**, *and* **(iii)** *above, and the three test instances* **Test 1**, **Test 2**, *and* **Test 3** *are defined by*

$$\min_{x \in S} \{c(x)\} + \mathcal{B}_{\mathbb{R}^2}(r(x))$$

with $\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, *then we obtain for*

(i) Test 1:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\left[\sqrt{1 + x_1^2} + x_1^2 - 4x_1 + x_2\right] \\ \frac{1}{2}\left[\sqrt{1 + x_1^2} + x_1^2 - 4x_1 + x_2\right] \\ \frac{1}{2}\left[\sqrt{1 + x_1^2} - x_1^2 + 4x_1 - x_2\right] \end{pmatrix},$$

(ii) Test 2:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\left[x_1^2 - 4x_1 + x_2\right] \\ \\ \frac{1}{2}\left[2\sqrt{1+x_1^2} + x_1^2 - 4x_1 + x_2\right] \\ \\ -\frac{1}{2}\left[x_1^2 - 4x_1 + x_2\right] \end{pmatrix}, \text{ and}$$

(iii) Test 3:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\left[\sqrt{1+x_1^2} + 2x_1^2 - 8x_1 + 2x_2\right] \\ \\ \frac{1}{2}\sqrt{1+x_1^2} \\ \\ \frac{1}{2}\sqrt{1+x_1^2} \end{pmatrix}.$$

Using Lemma 4.1 for all of the three test instances **Test 1**, **Test 2**, *and* **Test 3** *the set of all minimal solution w.r.t. the order relation* $\preccurlyeq^s_{\mathbb{R}^2_+}$ *is also given by the set* M *defined in* (19)*. In Figure 1 we illustrate by the black circles the boundaries of* $F(x)$ *for some* $x \in M$.

The image set of the chosen bicriteria optimization problem in the following second example is nonconvex. An interesting property of this bicriteria optimization problem is the arbitrary scalability w.r.t. the dimension n of the preimage space $\mathbb{R}^n$.

Example 4.3. *Let* $n \in \mathbb{N}$, $S := [-4,4]^n$, $Y = \mathbb{R}^2$, *and the vector-valued map* $f\colon S \to \mathbb{R}^2$ *be defined by*

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} := \begin{pmatrix} 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) \\ -\exp\left(-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{pmatrix} \text{ for all } x \in S.$$

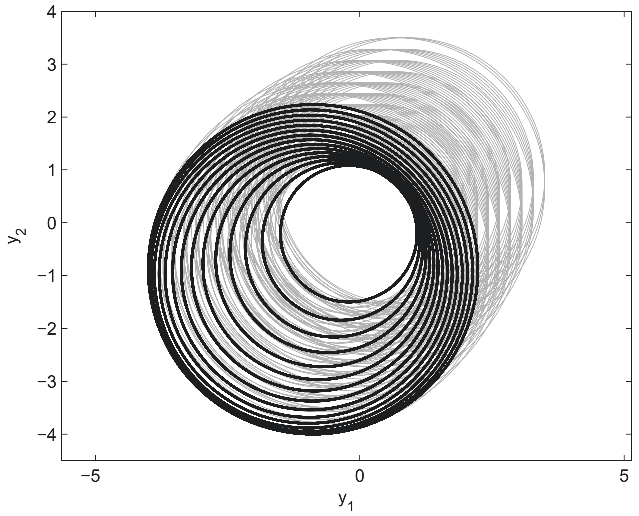
The set of all efficient solutions w.r.t. $\mathbb{R}^2_+$ *for the vector optimization problem* $\min_{x\in S} f(x)$ *is according to [7] given by*

$$M := \left\{ x \in \mathbb{R}^n \,\middle|\, x_1 \in \frac{1}{\sqrt{n}}\left[-1,1\right],\ x_i = x_1,\ i \in \{2,\ldots,n\} \right\}, \tag{20}$$

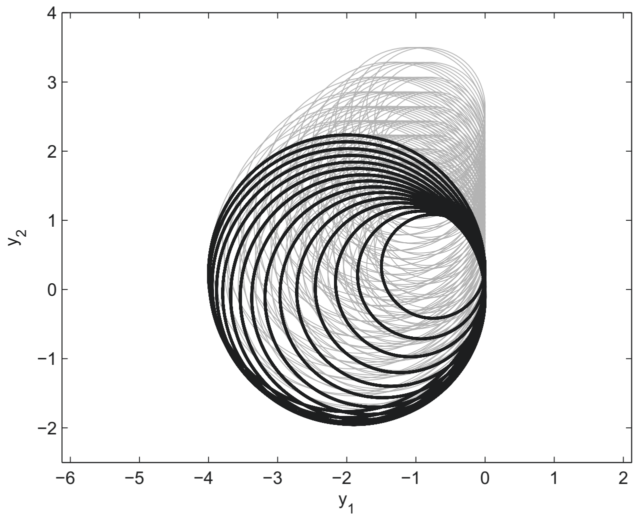
and $f_1(x) \geq 0$ *as well as* $f_2(x) \leq 0$ *is satisfied for all* $x \in S$.
The test instances **Test 4**, **Test 5**, *and* **Test 6** *are defined analogously to Example 4.2 by*

$$\min_{x\in S=[-4,4]^n} \{c(x)\} + \mathcal{B}_{\mathbb{R}^2}(r(x))$$

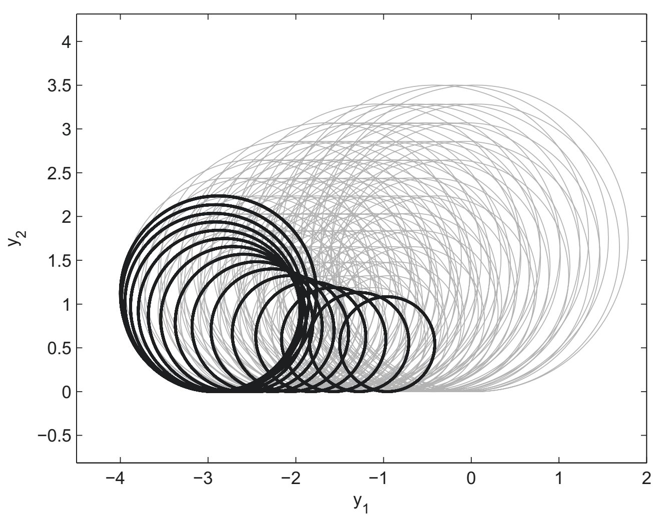
with $\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} H \\ \varrho^\top \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ *and again in consideration of* **(i)**, **(ii)**, *and* **(iii)**. *This leads to*



(a) Test 1



(b) Test 2



(c) Test 3

Figure 1: Boundaries of $F(x)$ for some $x \in M$ (black) and for some $x \in S \setminus M$ (grey) in Example 4.2.

(i) Test 4:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right] \\ \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right] \\ \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) + \exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right] \end{pmatrix},$$

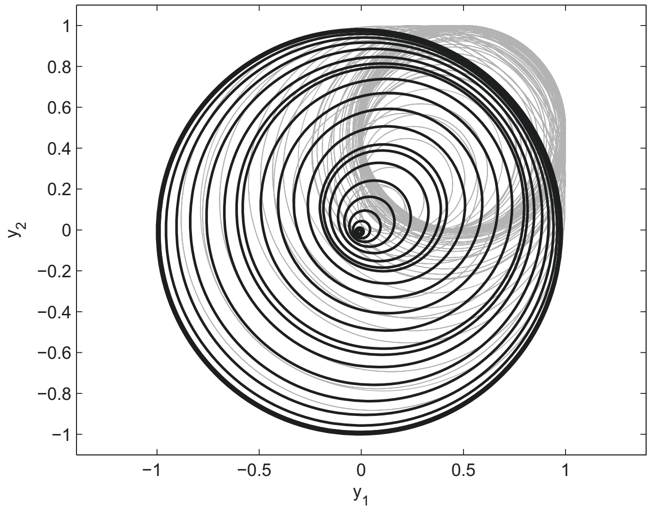
(ii) Test 5:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} -\frac{1}{2}\exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \\ \frac{1}{2}\left[2 - 2\exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right] \\ \frac{1}{2}\exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{pmatrix}, \text{ and}$$

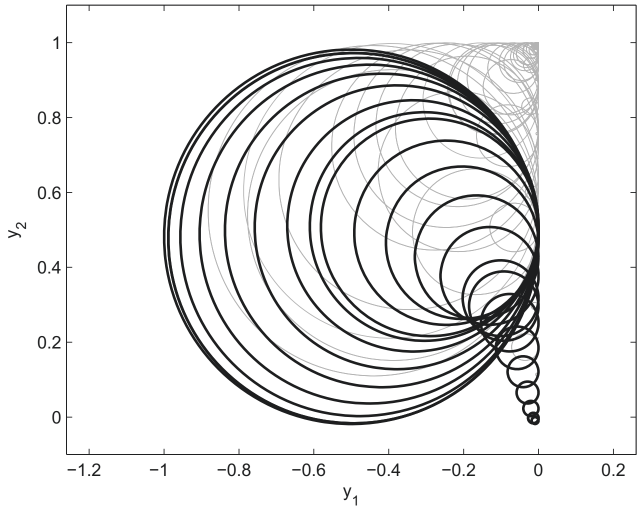
(iii) Test 6:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \\ r(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) - 2\exp\left(-\sum\limits_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right)\right] \\ \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right)\right] \\ \frac{1}{2}\left[1 - \exp\left(-\sum\limits_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right)\right] \end{pmatrix}.$$

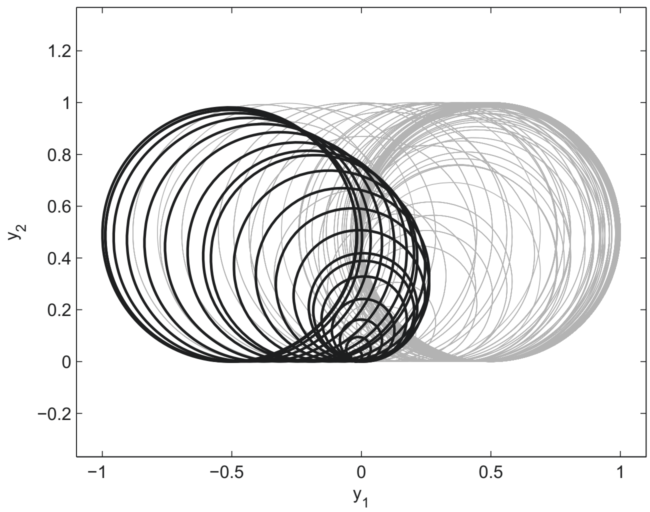
By Lemma 4.1 for **Test 4**, **Test 5**, *and* **Test 5** *the set of all minimal solution w.r.t. the order relation* $\preccurlyeq^s_{\mathbb{R}^2_+}$ *is given by the set* M *defined in* (20)*. In Figure 2 we illustrate for some* $x \in M$ *by the black circles the boundaries of* $F(x)$*.*



(a) Test 4



(b) Test 5



(c) Test 6

Figure 2: Boundaries of $F(x)$ for some $x \in M$ (black) and for some $x \in S \setminus M$ (grey) in Example 4.3.

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