POINTED CLOSED CONVEX SETS ARE THE INTERSECTION OF ALL RATIONAL SUPPORTING CLOSED HALFSPACES

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Abstract. We prove that every pointed closed convex set in $\mathbb{R}^n$ is the intersection of all the rational closed halfspaces that contain it. This generalizes a previous result by the authors for compact convex sets.

A *rational closed halfspace* is a subset of $\mathbb{R}^n$ of the form $\{\, x \in \mathbb{R}^n : \langle a, x\rangle \leq \beta\}$ for some $a \in \mathbb{Q}^n \setminus \{0\}$ and $\beta \in \mathbb{Q}$. In [1, Theorem 8] it is proved that every compact convex set is the intersection of all the rational closed halfspaces that contain it. In that paper, this result was a key step in generalizing the polyhedral notion of total dual integrality (see [3, 4]) to more general convex sets. A natural question is whether the same is true for more general families of convex sets. Closedness is an obvious necessary condition for such sets. The statement is clearly false for arbitrary (in fact, even polyhedral) closed convex sets: if $a \in \mathbb{R}^n$ has both rational and irrational entries, then no rational closed halfspace of $\mathbb{R}^n$ contains $\{\, x \in \mathbb{R}^n : \langle a, x\rangle \leq \beta\}$, for any $\beta \in \mathbb{R}$.

In this short note, we generalize the result to pointed closed convex sets, using elementary convex analysis. We use standard notation from [2], and we make extensive use of Minkowski set operations. The *effective domain* of an extended real-valued function $f \colon \mathbb{R}^n \to (-\infty, +\infty]$ is $\operatorname{dom}(f) \coloneqq \{\, x \in \mathbb{R}^n : f(x) < +\infty\}$. Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. The *support function* of C is $\delta^*(a \,|\, C) \coloneqq \sup_{x \in C} \langle a, x\rangle \in (-\infty, +\infty]$ for each $a \in \mathbb{R}^n$, the *barrier cone* of C is $B_C \coloneqq \operatorname{dom}(\delta^*(\cdot \,|\, C))$, the *recession cone* of C is $0^+C \coloneqq \{\, d \in \mathbb{R}^n : \forall x \in C,\ x + \mathbb{R}_+ d \subseteq C\}$, and the *polar* of C is $C^\circ \coloneqq \{\, y \in \mathbb{R}^n : \forall x \in C,\ \langle y, x\rangle \leq 1\}$. The unit ball in $\mathbb{R}^n$ is $\mathbb{B} \coloneqq \{\, x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Lemma 1. Let $C \subseteq \mathbb{R}^n$ be a nonempty pointed closed convex set. Then B_C has nonempty interior.

Proof. Clearly B_C is a convex cone containing the origin. Then $B_C^\circ = 0^+C$ by [2, Corollary 14.2.1] whence $\operatorname{cl}(B_C) = (0^+C)^\circ$. Since 0^+C is pointed, $n = \dim((0^+C)^\circ) = \dim(\operatorname{cl}(B_C))$ whence $\operatorname{int}(B_C) = \operatorname{int}(\operatorname{cl}(B_C))$ is nonempty. ☐

Lemma 2. Let $x_0, d \in \mathbb{R}^n$ and $\varepsilon, \delta > 0$. Then $\operatorname{conv}(\{x_0\} \cup (d + \varepsilon\mathbb{B})) \cap (x_0 + \delta\mathbb{B})$ has nonempty interior.

Proof. Set $\lambda \coloneqq \min\bigl\{\delta/(\|d - x_0\| + \varepsilon), 1\bigr\} \in (0, 1]$. Then $X \coloneqq (1 - \lambda)x_0 + \lambda(d + \varepsilon\mathbb{B}) \subseteq \operatorname{conv}(\{x_0\} \cup (d + \varepsilon\mathbb{B}))$ and the inclusion $X \subseteq x_0 + \delta\mathbb{B}$ is equivalent to $\|\lambda(d + \varepsilon u - x_0)\| \leq \delta$ for each $u \in \mathbb{B}$, which holds by the definition of λ. Since $\lambda > 0$ and $\varepsilon > 0$, it follows that $\operatorname{int}(X) \neq \varnothing$. ☐

Theorem 3. Every pointed closed convex set is the intersection of all rational closed halfspaces that contain it.

Proof. Let $X \subseteq \mathbb{R}^n$ be a pointed closed convex set. We may assume that $X \neq \varnothing$. Clearly X is contained in the intersection of all rational closed halfspaces that contain X. Hence, it suffices to prove that for each $\tilde{y} \in \mathbb{R}^n \setminus X$, there are $a \in \mathbb{Q}^n$ and $\beta \in \mathbb{Q}$ such that $\langle a, x\rangle \leq \beta$ for each $x \in X$ and $\langle a, \tilde{y}\rangle > \beta$. So let $\tilde{y} \in \mathbb{R}^n \setminus X$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it suffices to prove that

$$\text{there exists } a \in \mathbb{Q}^n \text{ such that } \delta^*(a \,|\, X) < \langle a, \tilde{y}\rangle. \tag{1}$$

Let $\tilde{z} \in X$ be the metric projection of $\tilde{y}$ in X, i.e., $\{\tilde{z}\} = \arg\min_{z \in X} \|z - \tilde{y}\|$. Set $C \coloneqq X - \tilde{z}$ and $\bar{y} \coloneqq \tilde{y} - \tilde{z} \neq 0$. Then $0 \in C$ and

$$\delta^*(a \,|\, C) \geq 0 \qquad \text{for each } a \in \mathbb{R}^n\text{, with equality if } a = \bar{y}. \tag{2}$$

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By Lemma 1, there are $d \in B_C$ and $\varepsilon > 0$ such that the compact set $d + \varepsilon\mathbb{B}$ is a subset of $\operatorname{int}(B_C)$. Hence, $\delta^*(\cdot \,|\, C)$ is Lipschitz continuous on $d + \varepsilon\mathbb{B}$ with Lipschitz constant at most $L > 0$; see, e.g., [2, Theorem 10.4]. In particular,

$$\delta^*(d + \varepsilon u \,|\, C) \leq \delta^*(d \,|\, C) + L\varepsilon \qquad \forall u \in \mathbb{B}. \tag{3}$$

Set

$$\alpha := \frac{\frac{1}{3}\|\bar{y}\|^2}{\delta^*(d \,|\, C) + L\varepsilon} > 0, \qquad \bar{d} := \alpha d, \qquad \bar{\varepsilon} := \alpha\varepsilon > 0, \qquad \bar{\delta} := \tfrac{1}{3}\|\bar{y}\| > 0,$$
$$A := \operatorname{conv}\big(\{\bar{y}\} \cup (\bar{d} + \bar{\varepsilon}\mathbb{B})\big) \cap (\bar{y} + \bar{\delta}\mathbb{B}).$$

We claim that,

$$\delta^*(a \,|\, C) < \langle a, \bar{y}\rangle \qquad \forall a \in A. \tag{4}$$

Let $a \in A$. So there exist $\lambda \in [0,1]$ and $\bar{u} \in \mathbb{B}$ such that $a = (1-\lambda)\bar{y} + \lambda(\bar{d} + \bar{\varepsilon}\bar{u})$. Then

$$\delta^*(a \,|\, C) \leq (1-\lambda)\delta^*(\bar{y} \,|\, C) + \lambda\delta^*(\bar{d} + \bar{\varepsilon}\bar{u} \,|\, C) = \lambda\alpha\delta^*(d + \varepsilon\bar{u} \,|\, C) \leq \tfrac{1}{3}\|\bar{y}\|^2, \tag{5}$$

where we used (2), (3), and the fact that $\lambda \leq 1$. On the other hand, $a = \bar{y} + \bar{\delta}\bar{v}$ for some $\bar{v} \in \mathbb{B}$, so

$$\langle a, \bar{y}\rangle = \|\bar{y}\|^2 + \bar{\delta}\langle \bar{v}, \bar{y}\rangle \geq \|\bar{y}\|\big(\|\bar{y}\| - \bar{\delta}\big) = \tfrac{2}{3}\|\bar{y}\|^2. \tag{6}$$

Combining (5) and (6) yields (4).

By adding $\langle a, \tilde{z}\rangle$ to both sides of (4), we find that $\delta^*(a \,|\, X) < \langle a, \tilde{y}\rangle$ for each $a \in A$. By Lemma 2, we have $\operatorname{int}(A) \neq \varnothing$. Hence, there exists a rational vector a in A. This proves (1) and the proof of the theorem is complete. ☐

The result is tight due to existence of closed halfspaces that are contained in no rational closed halfspace, as discussed above. Even though Theorem 3 does not directly yield a generalization of the notion of total dual integrality in [1] for pointed closed convex sets (due to limitations of the Gomory-Chvátal closure), the theorem does provide a natural generalization of our previous, foundational result for compact convex sets.

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