Extensions of Yuan's Lemma to fourth-order tensor system with applications

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Abstract

Yuan's lemma is a basic proposition on homogeneous quadratic function system. In this paper, we extend Yuan's lemma to 4th-order tensor system. We first give two generalized definitions of positive semidefinite of 4th-order tensor, and based on them, two extensions of Yuan's lemma are proposed. We illustrate the difference between our extensions and existing another extension of Yuan's lemma. We also put forward several 4th-order tensor optimization problems and show extended Yuan's lemma how to be applied.

Key words: Yuan's lemma, S-lemma, 4th-order tensor, symmetric, Positive semidefinite, Semidefinite programming relaxation

1 Introduction

Having been studied for many years, nonlinear optimization and related solution methods play a central role in the mathematical programming community, among which, the well-known trust-region method is admitted as one of the most efficient methods for nonlinear optimization problems. The convergence analysis of the trust-region method heavily relies on

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a lemma proved in [35], which is known to as the Yuan's lemma. Yuan's lemma builds the equivalence between the nonnegativity of the maximum of two homogeneous quadratic forms and the positive definiteness of a convex combination of the associated Hessian matrices of the quadratic functions, and is usually referred to as the Yuan's theorem of alternative. As an alternative theorem, it finds applications in the development of optimality conditions of optimization problems [6] as well.

Closely related to Yuan's lemma is the so-called S-lemma established by Yakubovich [33] half a century ago. S-lemma concerns the solvability of a system of two quadratic inequalities. Most of its applications may be found in control theory, but its theoretical development has been popularized by the optimization community, as it serves as a fundamental tool for linear matrix inequalities, and is crucial in certain nonconvex quadratic programming [1]. The close relationship with Yuan's lemma lies in that its validity can be deduced from Yuan's lemma; see the excellent survey on the S-lemma [23]; [34] established the equivalence between the two lemmas under certain assumptions. Other related theorems of alternative can be found in [3, 7–9, 11, 21, 22, 29] and so on.

As the above two lemmas hold for only two quadratic forms, people naturally wish to extend them to three or more than three quadratic forms. However, such extensions are in general hopeless; counterexample can be found in [23]. Nevertheless, effects have been devoted to the generalizations under certain conditions. For example, [8] extends Yuan's lemma to the system of three quadratic inequalities, by allowing the convex combination of the associated Hessian matrices contains at most one negative eigenvalue. [24] and [13] study the setting of three quadratic forms, where the former assumes that certain linear combination of the quadratic functions is a nonnegative form, while the latter holds by assuming that at least one of the Hessian is negative definite. [21] extends Yuan's lemma to finitely many quadratic forms under the criterion that all the Hessian matrices are essentially nonpositive, namely, the off-diagonal entries of the matrices are nonpositive; also known as Z-matrices. [17] shows that several theorems of alternative are still valid with a regular cone. In the complex case, the S-lemma holds for three quadratic forms [11], while Yuan's lemma has been generalized to the setting of four complex quadratic forms, under certain positive definiteness regularity [2]. Note that these results have been applied to study the strong duality of quadratic programs with two nonconvex constraints, both in real and complex fields [2, 4]. More generalizations and extensions can be found in [23].

Another way of generalizations considers going beyond quadratic forms, namely, trying to extend the theorems of alternative to higher-degree polynomials. It should be mentioned that, such kind of generalizations does not hold generally even when the system consists of two polynomials only. Nonetheless again, theorems of alternative of higher-degree polynomials are still valid in some situations. [13] shows that S-lemma is true for the system of two even-degree polynomials under certain positive semidefiniteness regularities. For system of finitely many polynomials, a type of Yuan's lemma has been established, provided all the associated symmetric tensors[1] of the polynomials are essentially nonpositive. [12] obtains an S-lemma for higher-degree polynomials in the sense that the combination parameters have been replaced by certain polynomials. Higher-degree S-lemma has also been studied in the context of homogeneous polynomials with respect to dilation [36]. Last but not least, Hilbert's Nullstellensatz is also a theorem of alternative for polynomials; see e.g., [5, 23].

This work is focused on the generalizations of Yuan's lemma to a special system of quadratic forms characterized by 4-th order tensors. Its specificity lies in that the variable of each quadratic form is restricted onto the positive semidefinite cone, which is different from the existing work. We show that Yuan's lemma holds for this type of systems consisting of finite many inequalities, provided the associated 4-th order tensors are essentially nonpositive. Based on this result, we are able to show the semidefinite relaxation of a class of tensor optimization problems is tight.

Another consideration in this work is the solvability of the BEC problem. Although the BEC problem seems to be nonconvex, it is shown that its semidefinite relaxation is tight, which itself is a convex quadratic semidefinite program; hence it can be solved to its optimality.

This paper is organized as follows. Several notations and definitions are provided in Section 2. Our main results concerning the extension of Yuan's lemma and its applications are presented step by step in Section 3. Section 4 studies the solvability of the BEC problem. Some numerical experiments are conducted in 5. Conclusions are drawn in Section 6.

2 Preliminaries on 4th-order Tensors and Notations

A tensor is a multidimensional array and is the generalization of the concept of matrices. An mth-order n-dimensional tensor $\mathcal{A}$ consists of n^m entries in real number: $\mathcal{A} = (\mathcal{A}_{i_1 i_2 \ldots i_m}), 1 \leq i_j \leq n, j = 1, \ldots, m$. In this paper, we focus on 4th-order tensor and related properties. For a 4th-order tensor $\mathcal{A}$, we call it completely symmetric if its elements $\mathcal{A}_{ijkl}$ are invariant under any permutation of its indices $\{i, j, k, l\}$. We call $\mathcal{A}$ partial-symmetric if

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{ijlk} = .\mathcal{A}_{klij}, \quad 1 \leq i, j, k, l \leq n.$$

In this section, we will generalize the concept of positive semidefinite tensors to partial-symmetric tensors. First we recall some concepts and notations.

Definition 2.1. *Let $\mathcal{A}, \mathcal{B}$ be 4th-order n-dimensional tensors; define the inner product of $\mathcal{A}$ and $\mathcal{B}$ by*

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i,j,k,l=1}^{n} \mathcal{A}_{ijkl} \mathcal{B}_{ijkl}.$$

[1]A tensor is a multi-way array. Analogous the Hessian matrix associated with a homogeneous quadratic form, a homogeneous d-degree polynomial can be uniquely characterized by a d-th order tensor.

Moreover, let $X \in R^{n\times n}$*, and let* $\mathcal{A}X$ *denote a matrix with entries being* $(\mathcal{A}X)_{ij} = \sum_{k,l=1}^{n} \mathcal{A}_{ijkl}X_{kl}$*; then the inner product between* X *and* $\mathcal{A}X$ *is*

$$\langle X, \mathcal{A}X\rangle = \sum_{i,j,k,l=1}^{n} \mathcal{A}_{ijkl}X_{ij}X_{kl}.$$

We recall essentially nonpositive tensors.

Definition 2.2 (Essentially nonpositive tensor)**.** *Define the index set* I *by*

$$I = \{(i, i, \dots, i) \in N^m : 1 \le i \le n\}.$$

We say that an m*th-order* n*-dimensional tensor* $\mathcal{A}$ *is*
(i) essentially nonnegative if $\mathcal{A}_{i_1,\dots,i_m} \ge 0$ *for all* $\{i_1, \dots, i_m\} \notin I$*;*
(ii) essentially nonpositive if $\mathcal{A}_{i_1,\dots,i_m} \le 0$ *for all* $\{i_1, \dots, i_m\} \notin I$*.*

The class of essentially nonpositive tensor was introduced in [15,37]. The positive semidefiniteness of is defined as follows [25].

Definition 2.3. *Let* $\mathcal{A}$ *be an* m*th-order* n*-dimensional symmetric tensor with* m *being even. If* $\langle \mathcal{A}, x^{\otimes m}\rangle \ge 0, \forall x \in R^n$*, we say* $\mathcal{A}$ *is positive semidefinite. Here* $x^{\otimes m}$ *represents the rank-1 tensor* $x \circ x \cdots \circ x$ *where* $\circ$ *denotes the outer product.*

Now we give two generalized definitions of positive semidefinite for partial-symmetric 4th-order tensor.

Definition A1. *Let* $\mathcal{A}$ *be a 4th-order* n*-dimensional partial-symmetric tensor. If* $\langle X, \mathcal{A}X\rangle \ge 0, \forall X \in S^n$*, we say* $\mathcal{A}$ *is positive semidefinite. We denote the set of such tensors as* $PSD_{4,n}$*.*

Definition A2. *Let* $\mathcal{A}$ *be 4th-order* n*-dimensional partial-symmetric tensor. If* $\langle X, \mathcal{A}X\rangle \ge 0, \forall X \in S^n_+$*, we say* $\mathcal{A}$ *is positive semidefinite. Also we denote the set of such tensors as* $PSD^+_{4,n}$*.*

When 4th-order tensor $\mathcal{A}$ is symmetric, Definition 1 is given and relative properties are researched in [18].

3 Extensions of Yuan's Lemma and Applications

To begin with, we recall Yuan's Lemma stated in [35].

Lemma 3.1 (Yuan's lemma)**.** *Let* $A, B \in S^n$*;* E, F *be two closed sets in* R^n *such that* $E \cup F = R^n$ *and*

$$x^T Ax \ge 0 \quad x \in E; \qquad x^T Bx \ge 0 \quad x \in F.$$

Then there exists a $\lambda \in [0, 1]$ *such that* $\lambda A + (1 - \lambda)B \succeq 0$*.*

Before extending Yuan's Lemma to 4th-order tensor system, we recall a result on rank-one decomposition of positive semidefinite matrices [28] that will be used later.

Lemma 3.2 (c.f. [6])**.** *Let $A, B \in S^n$,$X \succeq 0$. Then there exists an $x \in R^n$ such that*

$$A \circ X = x^T A x, \text{ and } B \circ X = x^T B x.$$

First, we give an extension of Yuan's Lemma to 4th-order tensor system where the tensor is positive semidefinite under Definition A1. Define a set $C_0 = conv\{X \otimes X, X \in S^n\}$, where "*conv*" denotes the convex hull of a set; here $X \otimes X$ denotes a 4th-order tensor given by the out product of X and X, i.e., $(X \otimes X)_{ijkl} = X_{ij} X_{kl},\ i, j, k, l = 1, \dots, n$.

Lemma 3.3. *Let $\mathcal{A}_1, \mathcal{A}_2$ be partial-symmetric 4th-order n-dimensional tensors. Then the following two statements are equivalent:*
(i) $\exists X \in S^n$ such that $\langle X, \mathcal{A}_1 X\rangle < 0, \langle X, \mathcal{A}_2 X\rangle < 0$;
(ii) $\exists \mathcal{X} \in C_0$ such that $\langle \mathcal{X}, \mathcal{A}_1\rangle < 0, \langle \mathcal{X}, \mathcal{A}_2\rangle < 0$.

Proof. (i)⇒(ii) is obvious. We mainly prove (ii)⇒(i). Suppose there exists an $\mathcal{X}^* \in C_0$ such that

$$\langle \mathcal{X}^*, \mathcal{A}_1\rangle < 0, \langle \mathcal{X}^*, \mathcal{A}_2\rangle < 0.$$

Let A_1, A_2, Z be the matricization of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{X}^*$, respectively. Here the matricization A of a 4th-order n-dimensional tensor $\mathcal{A}$ means

$$A_{(i-1)n+j,(k-1)n+l} = \mathcal{A}_{ijkl}, \quad i, j, k, l = 1, \dots, n.$$

Then we have $A_1, A_2, Z \in S^{n^2}$. As $\mathcal{X}^* \in C_0$, there exists $X_i \in S^n, i = 1, \dots, r$ such that $\mathcal{X}^* = \sum_{i=1}^r X_i \otimes X_i$, and correspondingly $Z = \sum_{i=1}^r x_i x_i^T$, where x_i is the associated vectorization of X_i. Clearly, $Z \succeq 0$. From Lemma 3.2, there exists an $\bar{x} \in R^{n^2}$ such that

$$\langle \mathcal{A}_1, \mathcal{X}^*\rangle = \langle A_1, Z\rangle = \bar{x}^T A_1 \bar{x}, \text{ and}$$
$$\langle \mathcal{A}_2, \mathcal{X}^*\rangle = \langle A_2, Z\rangle = \bar{x}^T A_2 \bar{x}.$$

Denote $\bar{X}$ as the folding matricization of $\bar{x}$, namely, $\bar{X}_{ij} = \bar{x}_{(i-1)n+j}, i, j = 1, ...n$.

Let $X = \frac{\bar{X}+\bar{X}^T}{2}$; then $X \in S^n$ and it holds that

$$\begin{aligned}\langle X, \mathcal{A}_1 X\rangle &= \sum_{ijkl} a_{ijkl} \frac{\bar{X}_{ij} + \bar{X}_{ji}}{2} \frac{\bar{X}_{kl} + \bar{X}_{lk}}{2} \\ &= \frac{1}{4} \sum_{ijkl} a_{ijkl} (\bar{X}_{ij}\bar{X}_{kl} + \bar{X}_{ij}\bar{X}_{lk} + \bar{X}_{ji}\bar{X}_{kl} + \bar{X}_{ji}\bar{X}_{lk}) \\ &= \langle \bar{X}, \mathcal{A}_1 \bar{X}\rangle = \bar{x}^T A_1 \bar{x}.\end{aligned}$$

The above equation can be obtained from the symmetric structure of $\mathcal{A}_1$, so we have

$$\langle X, \mathcal{A}_1 X\rangle = \langle \mathcal{A}_1, \mathcal{X}^*\rangle < 0 \text{ and } \langle X, \mathcal{A}_2 X\rangle = \langle \mathcal{A}_2, \mathcal{X}^*\rangle < 0.$$

hence (i) holds. This completes the proof. □

Based on Lemma 3.3, we have the following results.

Theorem 3.1. *Let $\mathcal{A}_1, \mathcal{A}_2$ be partial-symmetric 4th-order n-dimensional tensors. Then one and exactly one of the following statements holds:*
(i) $\exists X \in S^n$ such that $\langle X, \mathcal{A}_1 X\rangle < 0, \langle X, \mathcal{A}_2 X\rangle < 0$;
(ii) $\exists \lambda \in [0,1]$ such that $\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2 \in PSD_{4,n}$.

Proof. Suppose that statement (ii) holds; then there exists a $\lambda \in [0,1]$ such that $\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2 \in PSD_{4,n}$. If (i) holds, then there exists an $X \in S^n$ such that $\langle X, \mathcal{A}_1 X\rangle < 0, \langle X, \mathcal{A}_2 X\rangle < 0$. It then follows that

$$0 \le \langle X, (\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2)X\rangle = \lambda\langle X, \mathcal{A}_1 X\rangle + (1-\lambda)\langle X, \mathcal{A}_2 X\rangle < 0,$$

which is impossible. So (i) fails to hold.

Conversely. if (i) fails, it means the following system

$$\langle X, \mathcal{A}_1 X\rangle < 0, \langle X, \mathcal{A}_2 X\rangle < 0$$

has no solution. Then Lemma 3.3 implies that the system

$$\langle \mathcal{X}, \mathcal{A}_1\rangle < 0, \langle \mathcal{X}, \mathcal{A}_2\rangle < 0$$

has no solution whenever $\mathcal{X} \in C_0$. By the generalized Farkas lemma [23], there exists a $\lambda \in [0,1]$ such that

$$\langle \lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2, \mathcal{X}\rangle \ge 0, \qquad \forall \mathcal{X} \in C_0,$$

from which one has

$$\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2 = PSD_{4,n},$$

where $C_0^{\oplus} = PSD_{4,n}$ could be got analogy as the proof of Lemma 2.3 in [14]. Then we complete this proof. □

The following proposition extends the above result to the inhomogeneous case.

Lemma 3.4. *Let $\mathcal{A}_1, \mathcal{A}_2$ be partial-symmetric 4th-order n-dimensional tensors and $B_1, B_2 \in S^n, c_1, c_2 \in R^n$. Then the following statements are equivalent:*

$$\begin{aligned} (i)\ \exists X \in S^n, \qquad & \langle X, \mathcal{A}_1 X\rangle + 2\langle B_1, X\rangle + c_1 < 0, \\ & \langle X, \mathcal{A}_2 X\rangle + 2\langle B_2, X\rangle + c_2 < 0; \end{aligned} \tag{3.1}$$

$$\begin{aligned} (ii)\ & \exists \mathcal{X} \in C_0, X \in S^n, \mathcal{X} - X \otimes X \in PSD_{4,n}, \\ & \langle \mathcal{X}, \mathcal{A}_1\rangle + 2\langle B_1, X\rangle + c_1 < 0, \\ & \langle \mathcal{X}, \mathcal{A}_2\rangle + 2\langle B_2, X\rangle + c_2 < 0. \end{aligned} \tag{3.2}$$

Proof. (i) $\Rightarrow$ (ii) is obvious. We then prove (ii) $\Rightarrow$ (i). Suppose that (ii) holds; then there exists $\mathcal{X}^*, X^*$ satisfying (3.2). Denote $A_1, A_2, \bar{X}^*$ as the matricization of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{X}^*$ and $b_1, b_2, \bar{x}^*$ as the vectorization of B_1, B_2, X^*. Then (3.2) can be written as

$$\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \circ \begin{bmatrix} \bar{X}^* & \bar{x}^* \\ \bar{x}^{*T} & 1 \end{bmatrix} < 0,$$
$$\begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \circ \begin{bmatrix} \bar{X}^* & \bar{x}^* \\ \bar{x}^{*T} & 1 \end{bmatrix} < 0.$$

Since $\mathcal{X}^* - X^* \otimes X^* \in PSD_{4,n}$ and is partially symmetric, $\forall u \in R^{n^2}$, let U be the folding matricization of u; we have

$$\begin{aligned} u^T(\bar{X}^* - \bar{x}^*\bar{x}^{*T})u &= \langle U, (\mathcal{X}^* - X^* \otimes X^*)U\rangle \\ &= \langle \frac{U+U^T}{2}, (\mathcal{X}^* - X^* \otimes X^*)\frac{U+U^T}{2}\rangle \geq 0. \end{aligned}$$

So $\bar{X}^* - \bar{x}^*\bar{x}^{*T} \succeq 0$. Then the system

$$\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \circ Z < 0,$$
$$\begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \circ Z < 0,$$
$$Z \succeq 0, \quad Z_{n^2+1,n^2+1} = 1.$$

has a solution. From [23], there exists a $p \in R^{n^2}$ such that

$$\begin{bmatrix} p \\ 1 \end{bmatrix}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} < 0,$$
$$\begin{bmatrix} p \\ 1 \end{bmatrix}^T \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} < 0.$$

Let P be the folding matricization of p; then

$$\langle \frac{P+P^T}{2}, \mathcal{A}_1 \frac{P+P^T}{2}\rangle = \langle P, \mathcal{A}_1 P\rangle = p^T A_1 p \text{ and}$$
$$\langle \frac{P+P^T}{2}, B_1\rangle = \langle P, B_1\rangle = p^T b_1.$$

So we have

$$\langle \frac{P+P^T}{2}, \mathcal{A}_1 \frac{P+P^T}{2}\rangle + 2\langle \frac{P+P^T}{2}, B_1\rangle + c_1 < 0,$$
$$\langle \frac{P+P^T}{2}, \mathcal{A}_2 \frac{P+P^T}{2}\rangle + 2\langle \frac{P+P^T}{2}, B_2\rangle + c_2 < 0.$$

which implies that (i) holds. □

Using Lemma 3.4, we have the following results.

Theorem 3.2. *Let $\mathcal{A}_1, \mathcal{A}_2$ be partial-symmetric 4th-order n-dimensional tensors, and $B_1, B_2 \in S^n, c_1, c_2 \in R^n$. Then one and exactly one of the following statements holds:*

$$\begin{aligned} (i)\ \exists X \in S^n, \quad & \langle X, \mathcal{A}_1 X\rangle + 2\langle B_1, X\rangle + c_1 < 0, \\ & \langle X, \mathcal{A}_2 X\rangle + 2\langle B_2, X\rangle + c_2 < 0; \\ (ii)\ \exists \lambda \in [0,1] \quad & \forall \mathcal{X} \in C_0, X \in S^n, \mathcal{X} - X \otimes X \in PSD_{4,n}, \\ & \langle \mathcal{X}, \lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2\rangle + 2\langle \lambda B_1 + (1-\lambda)B_2, X\rangle + \lambda c_1 + (1-\lambda)c_2 \geq 0. \end{aligned}$$

Proof. (ii)$\Rightarrow$ not (i). Suppose that statement (ii) holds, then there exists $\lambda \in [0,1]$ such that $\forall \mathcal{X} \in C_0, X \in S^n, \mathcal{X} - X \otimes X \in PSD_{4,n}$. We have

$$\langle \mathcal{X}, \lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2\rangle + 2\langle \lambda B_1 + (1-\lambda)B_2, X\rangle + \lambda c_1 + (1-\lambda)c_2 \geq 0.$$

Assume that (i) holds also, namely, there exists an $X^* \in S^n$ such that

$$\begin{aligned} &\langle X^*, \mathcal{A}_1 X^*\rangle + 2\langle B_1, X^*\rangle + c_1 < 0 \text{ and} \\ &\langle X^*, \mathcal{A}_2 X^*\rangle + 2\langle B_2, X^*\rangle + c_2 < 0. \end{aligned}$$

Let $\mathcal{X}^* = X^* \otimes X^*$; it is obvious that $\mathcal{X}^* \in C_0$ and $\mathcal{X}^* - X^* \otimes X^* \in PSD_{4,n}$. Then we have

$$\begin{aligned} 0 &\leq \langle \mathcal{X}^*, \lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2\rangle + 2\langle \lambda B_1 + (1-\lambda)B_2, X^*\rangle + \lambda c_1 + (1-\lambda)c_2 \\ &= \lambda(\langle X^*, \mathcal{A}_1 X^*\rangle + 2\langle B_1, X^*\rangle + c_1) + (1-\lambda)(\langle X^*, \mathcal{A}_1 X^*\rangle + 2\langle B_1, X^*\rangle + c_1) < 0, \end{aligned}$$

which is impossible, so (i) must fail.

We then show that not (i) $\Rightarrow$ (ii). If (i) fails, it means that the following system has no solution,

$$\exists X \in S^n, \quad \langle X, \mathcal{A}_1 X\rangle + 2\langle B_1, X\rangle + c_1 < 0, \langle X, \mathcal{A}_2 X\rangle + 2\langle B_2, X\rangle + c_2 < 0.$$

Lemma 3.4 implies that system

$$\begin{aligned} &\exists \mathcal{X} \in C_0, X \in S^n, \mathcal{X} - X \otimes X \in PSD_{4,n}, \\ &\langle \mathcal{X}, \mathcal{A}_1\rangle + 2\langle B_1, X\rangle + c_1 < 0, \\ &\langle \mathcal{X}, \mathcal{A}_2\rangle + 2\langle B_2, X\rangle + c_2 < 0 \end{aligned}$$

has no solution. From the generalized Farkas lemma [23], there exists a $\lambda \in [0,1]$ such that

$$\begin{aligned} &\forall \mathcal{X} \in C_0, X \in S^n, \mathcal{X} - X \otimes X \in PSD_{4,n}, \\ &\langle \mathcal{X}, \lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2\rangle + 2\langle \lambda B_1 + (1-\lambda)B_2, X\rangle + \lambda c_1 + (1-\lambda)c_2 \geq 0. \end{aligned}$$

This completes the proof of theorem. □

Hu et al. [14] extended Yuan's lemma to even-order symmetric tensor system based on the usual definition of positive semidefiniteness (c.f. Definition 2.3) and under the assumption of essential nonpositivity (c.f. Definition 2.2). We recall there results.

Theorem 3.3. *Let $n, p \in \mathcal{N}$ and let m be an even number. Let $\mathcal{F}_l, l = 0, 1, \ldots, p$ be mth-order n-dimensional symmetric tensors. Suppose that there exists a nonsingular matrix P such that $P^m \mathcal{F}_l, l = 0, 1, \ldots, p$ are all essentially nonpositive tensors. Then, one and exactly on of the following statements holds:*
(i) $(\exists x \in R^n)(\langle \mathcal{F}_l, x^{\otimes m}\rangle < 0, l = 0, 1, \ldots, p)$;
(ii) $(\exists \lambda_l \geq 0, l = 0, 1, \ldots, p, \sum_{l=0}^{p} \lambda_l = 1)(\sum_{l=0}^{p} \lambda_l \mathcal{F}_l \in SOS_{m,n})$.

In what follows, we present another extension of Yuan's lemma to 4th-order tensor system with the definition of positive semidefiniteness in Definition A2, and also under the assumption of essential nonpositivity.

Lemma 3.5. *Let $\mathcal{F}^l$ be partial-symmetric essentially nonpositive 4th-order n-dimensional tensors, $l = 0, 1, \ldots, p$. Then, the following statements are equivalent:*
(i) $\exists X \in S^n_+$ such that $\langle X, \mathcal{F}^l X\rangle < 0, l = 0, 1, \ldots, p$;
(ii) $\exists \mathcal{X} \in conv\{X \otimes X, X \in S^n_+\} \triangleq C$ such that $\langle \mathcal{X}, \mathcal{F}^l\rangle < 0, l = 0, 1, \ldots, p$.

Proof. (i)$\Rightarrow$ (ii) is obvious. To get (ii) $\Rightarrow$ (i), suppose that there exists an $\mathcal{X} \in C$ such that

$$\langle \mathcal{X}, \mathcal{F}^l\rangle < 0, l = 0, 1, \ldots, p.$$

Since C is a closed and convex cone, there exists $X^j \in S^n_+$ such that

$$\mathcal{X} = \sum_{j=1}^{J} X^j \otimes X^j.$$

It is easy to see that $\mathcal{X}_{iiii} \geq 0$. Denote $\bar{X} \in S^n$ by

$$\bar{X}_{ij} = \sqrt[4]{\mathcal{X}_{iiii}\mathcal{X}_{jjjj}}, i, j, k, l = 1, ..., n.$$

Let $\bar{x} = (\sqrt[4]{\mathcal{X}_{1111}}, \ldots, \sqrt[4]{\mathcal{X}_{nnnn}})^T$; then there holds that

$$\bar{X} = \bar{x}\bar{x}^T,$$

and so $\bar{X} \in S^n_+$. Then, using the same step of proof for Proposition 3.1 in [14], we have, for each $l = 0, 1, \ldots, p$,

$$\langle \bar{X}, \mathcal{F}^l \bar{X}\rangle = \langle \mathcal{F}^l, \bar{x}^{\otimes 4}\rangle = \langle \mathcal{F}^l, \mathcal{X}\rangle < 0.$$

Hence (i) holds. □

Theorem 3.4 (Extension of Yuan's Lemma)**.** *Let $\mathcal{F}^l, l = 0, 1, \ldots, p$ be 4th-order n-dimensional essentially nonpositive partial-symmetric tensors. If $\langle X, \mathcal{F}^l X\rangle \geq 0,\ X \in E^l, E^l \subseteq S^n_+, l = 0, 1, \ldots, p$, and $\cup_{l=0}^{p} E^l = S^n_+$, then there exist $\lambda_l \geq 0, l = 0, 1 \ldots, p, \sum_{l=0}^{p} \lambda_l = 1$ such that*

$$\sum_{l=0}^{p} \lambda_l \mathcal{F}^l \in PSD^+_{4,n}.$$

Proof. First, the assumptions means that the following system

$$\langle \mathcal{X}, \mathcal{F}^l \rangle < 0, \quad l = 0, \dots, p, \quad \forall \mathcal{X} \in C \tag{3.3}$$

has no solution. If not, then there exists a solution $\mathcal{X}^*$ to (3.3). From Lemma 3.5, we know that there exists an $X^* \in S^n_+$ such that

$$\langle X^*, \mathcal{F}^l X^* \rangle < 0, \quad l = 0, \dots, p. \tag{3.4}$$

Since $\cup_{l=0}^p E^l = S^n_+$, there must be an i such that $X^* \in E^i$, and $\langle X^*, \mathcal{F}^i X^* \rangle \geq 0$, which contradicts (3.4). So system (3.3) is not solvable. Hence for any $\epsilon > 0$, the system

$$\langle \mathcal{X}, \mathcal{F}^0 \rangle < 0, \quad \langle \mathcal{X}, \mathcal{F}^l \rangle + \epsilon \leq 0, \quad l = 1, \dots, p, \quad \mathcal{X} \in C$$

is not solvable. From the generalized Farkas lemma, there exists $\lambda_\epsilon^l \geq 0, l = 1, \dots, p$ such that

$$\langle \mathcal{X}, \mathcal{F}^0 \rangle + \sum_{l=1}^p \lambda_\epsilon^l (\langle \mathcal{X}, \mathcal{F}^l \rangle + \epsilon) \geq 0 \qquad \forall \mathcal{X} \in C,$$

which equals

$$\frac{1}{1 + \sum_{l=1}^p \lambda_\epsilon^l} \langle \mathcal{X}, \mathcal{F}^0 \rangle + \sum_{l=1}^p \frac{\lambda_\epsilon^l}{1 + \sum_{l=1}^p \lambda_\epsilon} (\langle \mathcal{X}, \mathcal{F}^l \rangle + \epsilon) \geq 0 \qquad \forall \mathcal{X} \in C. \tag{3.5}$$

Without loss of generality, we may assume that $\frac{1}{1+\sum_{l=1}^p \lambda_\epsilon^l} \to \lambda_0 (\epsilon \to 0^+)$ and $\frac{\lambda_\epsilon^l}{1+\sum_{l=1}^p \lambda_\epsilon^l} \to \lambda_l (\epsilon \to 0^+)(l = 1, \dots, p)$; then $\lambda_l \in [0, 1], l = 0, \dots, p$ and $\sum_{l=0}^p \lambda_l = 1$. In (3.5), let $\epsilon \to 0^+$, one has

$$\langle \mathcal{X}, \sum_{l=0}^p \lambda_l \mathcal{F}^l \rangle \geq 0, \quad \forall \mathcal{X} \in C.$$

This completes the proof. □

Lemma 3.6. *Let $\mathcal{F}^l, l = 0, 1, \dots, p$, be 4th-order n-dimensional essentially nonpositive partial symmetric tensors, B_l be nonpositive matrices, $c_l \in R$, $l = 0, 1, \dots, p$. Then, the following statements are equivalent:*
(i) $\exists X \in S^n_+$ such that $\langle X, \mathcal{F}^l X \rangle + 2\langle B_l, X \rangle + c_l < 0, l = 0, 1, \dots, p$;
(ii) $\exists \mathcal{X} \in C, X \in S^n_+$ and $\mathcal{X} - X \otimes X \in PSD^+_{4,n}$ such that

$$\langle \mathcal{X}, \mathcal{F}^l \rangle + 2\langle B_l, X \rangle + c_l < 0, l = 0, 1, \dots, p.$$

Proof. (i)⇒ (ii) is obvious. To get (ii) ⇒ (i), suppose that there exists an $\mathcal{X} \in C, X \in S^n_+, \mathcal{X} - X \otimes X \in PSD^+_{4,n}$ such that

$$\langle \mathcal{X}, \mathcal{F}^l \rangle + 2\langle B_l, X \rangle + c_l < 0, l = 0, 1, \dots, p.$$

Define $\bar{X} \in S^n$ given by

$$\bar{X}_{ij} = \sqrt[4]{\mathcal{X}_{iiii} \mathcal{X}_{jjjj}}.$$

Then from $\mathcal{X} - X \otimes X \in PSD^+_{4,n}$ and $X \in S^n_+$, we get

$$\bar{X}_{ij} = \sqrt[4]{\mathcal{X}_{iiii}\mathcal{X}_{jjjj}} \geq \sqrt{X_{ii}X_{jj}} \geq X_{ij}.$$

It follows from the nonpositivity of B_l that

$$\langle B_l, \bar{X} \rangle \leq \langle B_l, X \rangle.$$

From the proof of Lemma 3.5, we know that $\bar{X} \in S^n_+$ and

$$\langle \bar{X}, \mathcal{F}^l \bar{X} \rangle \leq \langle \mathcal{X}, \mathcal{F}^l \rangle.$$

Combine the above pieces together, we have

$$\langle \bar{X}, \mathcal{F}^l \bar{X} \rangle + \langle B_l, \bar{X} \rangle + c_l \leq \langle \mathcal{X}, \mathcal{F}^l \rangle + \langle B_l, X \rangle + c_l, \quad l = 0, 1, \ldots, p.$$

and the proof is completed. □

The following result provides the inhomogeneous version of generalized Yuan's lemma.

Theorem 3.5. *Let $\mathcal{F}^l, l = 0, 1, \ldots, p$, be 4th-order n-dimensional essentially nonpositive partial symmetric tensors, B_l be nonpositive matrix, $c_l \in R$, $l = 0, 1, \ldots, p$. If $\langle X, \mathcal{F}^l X \rangle + 2\langle B_l, X \rangle + c_l \geq 0$, $X \in E^l, E^l \subseteq S^n_+, l = 0, 1, \ldots, p$, and $\cup_{l=0}^p E^l = S^n_+$, then there exist $\lambda_l \geq 0, l = 0, 1 \ldots, p, \sum_{l=0}^p \lambda_l = 1$ such that $\forall \mathcal{X} \in C, X \in S^n_+, \mathcal{X} - X \otimes X \in PSD^+_{4,n}$,*

$$\langle (\sum_{l=0}^p \lambda_l \mathcal{F}^l), \mathcal{X} \rangle + 2\langle (\sum_{l=0}^p \lambda_l B_l), X \rangle + \sum_{l=0}^p \lambda_l c_l \geq 0.$$

Proof. The condition in the theorem means that the following system

$$\begin{aligned} &\langle \mathcal{F}^l, \mathcal{X} \rangle + 2\langle B_l, X \rangle + c_l < 0, \quad l = 0, 1, \ldots, p, \\ &\forall \mathcal{X} \in C, X \in S^n_+, \mathcal{X} - X \otimes X \in PSD^+_{4,n} \end{aligned} \tag{3.6}$$

has no solution. If there exists a pair of solutions $\mathcal{X}^*, X^*$, from Lemma 3.6, we know there exists an $\bar{X} \in S^n_+$ such that

$$\langle \bar{X}, \mathcal{F}^l \bar{X} \rangle + 2\langle B_l, \bar{X} \rangle + c_l < 0, \quad l = 0, \ldots, p. \tag{3.7}$$

Since $\cup_{l=0}^p E^l = S^n_+$, there exists an i such that $\bar{X} \in E^i$, and $\langle \bar{X}, \mathcal{F}^i \bar{X} \rangle + 2\langle B_i, \bar{X} \rangle + c_i \geq 0$, which contradicts (3.7). So system (3.6) is not solvable. Using similar arguments in the proof of Theorem 3.4, the required result follows. □

One difference between our extension of Yuan's lemma and [14] is that when applying to 4th-order polynomial optimization without odd terms, we first relax the 4th-order polynomial optimization to quadratic Semidefinite programming. If this quadratic matrix programming is convex, we could solve it by SDP. Otherwise, we may further relax quadratic SDP programming to linear 4th-order tensor optimization, and investigate the relationship between them with the extended Yuan's lemma. In Addition, a kind of quadratic matrix programming, with form $\langle X, \mathcal{F}X\rangle + 2\langle B, X\rangle + c$ as objective and constrained functions while $\mathcal{F}$ is partial-symmetric 4th-order tensor, B is nonpositive symmetric matrix, we may use Theorem 3.5 to get the strong duality of this kind of matrix programming.

4 The discretization of the BEC Problem and corresponding SDP relaxation

Consider the following tensor optimization of the form [32]:

$$\begin{aligned} (P) \qquad & min \quad f(x) = \langle \mathcal{A}, x^{\otimes 4}\rangle + \langle B, x^{\otimes 2}\rangle \\ & s.t. \quad x^T x \le 1, \end{aligned}$$

where $\mathcal{A}$ is a 4th-order partial-symmetric tensor and B is a symmetric matrix. All off-diagonal entries of $\mathcal{A}$ and B are nonpositive.

We then consider the semidefinite relaxation of (P), which is a quadratic matrix programming:

$$\begin{aligned} (RP) \qquad & min \quad g(X) = \langle X, \mathcal{A}X\rangle + \langle B, X\rangle \\ & s.t. \quad \langle I, X\rangle \le 1 \\ & \qquad\quad X \succeq 0 \end{aligned}$$

We have the following proposition, where $v(\cdot)$ denotes the optimal value of the concerned problem.

Proposition 4.1. *The semidefinite relaxation* (RP) *is tight, i.e.,* $v(P) = v(RP)$.

Proof. Note that $v(P) \ge v(RP)$ is obvious, so we just need to prove $v(P) \le v(RP)$. Assume that X^* is solution of (RP); denote $x^* = \sqrt{\text{diag}(X^*)}$. First we observe that ${x^*}^T x^* = \langle I, X^*\rangle \le 1$. So x^* is a feasible point of (P). Since X^* is positive semidefinite, we get

$$X^*_{ij} \le \sqrt{X^*_{ii} X^*_{jj}}.$$

As all off-diagonal entries of B are nonpositive, we have

$$\begin{aligned} \langle B, {x^*}^{\otimes 2}\rangle &= \sum_{i,j=1} B_{ij} x^*_i x^*_j = \sum_{i,j=1} B_{ij} \sqrt{X^*_{ii} X^*_{jj}} \\ &\le \sum_{i,j=1} B_{ij} X^*_{ij} = \langle B, X^*\rangle. \end{aligned}$$

It follows from the nonpositivity of $\mathcal{A}$ that

$$\begin{aligned}\langle \mathcal{A}, x^{*\otimes 4}\rangle &= \sum_{i,j=1}\sum_{k,l=1}\mathcal{A}_{ijkl}x_i^*x_j^*x_k^*x_l^* \\ &\leq \sum_{i,j=1}\sum_{k,l=1}\mathcal{A}_{ijkl}X_{ij}^*X_{kl}^* = \langle X^*, \mathcal{A}X^*\rangle.\end{aligned}$$

So we have $f(x^*) \leq g(X^*)$, yield $v(P) \leq v(RP)$. This completes the proof. □

We illustrate an application about nonlinear eigenvalue problem. For instance, the calculation of the Gross-Pitaevskii equation describing the ground states of Bose-Einstein condensates [30,31]. In [27], the following problem was considered:

$$\begin{cases} -\triangle u + Wu + \zeta|u|^2u = \lambda u, \quad \text{in} \quad \Omega, \\ \qquad\qquad\qquad u = 0, \quad \text{on} \quad \partial\Omega, \\ \qquad \int_\Omega u^2 d\Omega = 1, \end{cases}$$

where Ω denotes the three dimensional domain $[0,1]^3$, $\zeta = 1$ and $W = x_1^2 + x_2^2 + x_3^2$. By using mesh grid and discretization, one obtains the following problem:

$$\begin{cases} \mathcal{A}u^3 + Bu = \lambda u \\ \qquad \|u\|_2^2 = (1/h)^3. \end{cases} \tag{4.8}$$

In one dimensional case, $\mathcal{A}$ is a 4th-order partial-symmetric tensor whose diagonal entries is 1 and all off-diagonal entries are 0. B is the following matrix,

$$B = 1/h^2 \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 2 \end{bmatrix} + h^2 \begin{bmatrix} 1^2 & & & \\ & 2^2 & & \\ & & \ddots & \\ & & & (n-1)^2, \end{bmatrix}$$

where h is the length of grid and n is number of grid. So the off-diagonal entries of B is nonpositive as well. Equation (4.8) is the nonlinear eigenvalue problem. Here we are concerned with the solution of the minimum eigenvalue. We can see that (4.8) is the optimal condition of the following problem

$$\begin{aligned} (P_1) \qquad min \quad & (1/2)\mathcal{A}x^4 + Bx^2 \\ s.t. \quad & \|x\|_2^2 = (1/h)^3. \end{aligned}$$

The optimal solution is a eigenvector of (4.8), but it might not be eigenvector corresponding to minimum eigenvalue. So (P_1) gives a upper bound of minimum eigenvalue and we denote it as λ_{sup}. The lower bound is given from the following problem which we denote as λ_{inf}.

$$\begin{aligned} (P_2) \qquad min \quad & \mathcal{A}x^4 + Bx^2 \\ s.t. \quad & \|x\|_2^2 = (1/h)^3. \end{aligned}$$

Since (P_1), (P_2) are not convex problem, we consider the semidefinite relaxation problem (RP_1)

$$(RP_1) \qquad min \quad 1/2\langle X, \mathcal{A}X\rangle + \langle B, X\rangle$$
$$s.t. \quad \langle I, X\rangle = (1/h)^3, \quad X \succeq 0,$$

and (RP_2) is analogous to (RP_1). Since $\mathcal{A}$ and B have nonpositive off-diagonal entries, Proposition 4.1 tells us that that $v(P_1) = v(RP_1)$ and $v(P_2) = v(RP_2)$. Moreover, the proof of Proposition 4.1 implies that and if X_i is an optimal solution to (RP_1), then $\sqrt{\text{diag}(X_i)}$ is an optimal solution to (P_i), $i = 1, 2$.

5 Numerical Experiments

In this section, we present numerical results on the semidefinite relaxations on the BEC problem and overdamped condition. The supporting software is cvx1.2.1 in MATLAB 2014.a, The configuration of the computer is: Intel(R) Core(TM) i5-2410M CPU @2.30GHZ, RAM=2GB.

Example 1. The corresponding numerical results of 4 are presented in Table 1(one dimension), Table 2(two dimension) and Table 3(three dimension)

n	λ_{sup}	λ_{inf}	error	$cpu(\lambda_{sup})$	$cpu(\lambda_{inf})$
100	11.6421	11.6391	0.0030	0.21	0.28
200	11.6427	11.6398	0.0029	0.63	0.99
300	11.6428	11.6399	0.0029	1.23	1.01
400	11.6429	11.6399	0.0030	2.05	1.99
500	11.6429	11.6399	0.0030	3.59	3.66
600	11.6429	11.6399	0.0030	5.01	4.97
700	11.6429	11.6400	0.0029	6.98	7.23

Table 1: one dimensional case $\Omega = [0, 1]$

n	λ_{sup}	λ_{inf}	error	$cpu(\lambda_{sup})$	$cpu(\lambda_{inf})$
10×10	22.3486	22.3352	0.0134	0.49	0.97
20×20	22.4724	22.4597	0.0127	1.44	1.21
30×30	22.4954	22.4828	0.0126	7.00	7.17
40×40	22.5196	22.5031	0.0164	20.38	17.52
50×50	22.5233	22.5068	0.0165	86.54	92.41

Table 2: two dimensional case $\Omega = [0, 1]^2$

n	λ_{sup}	λ_{inf}	error	$cpu(\lambda_{sup})$	$cpu(\lambda_{inf})$
10×10×10	33.4450	33.4053	0.0397	5.38	5.65
12×12×12	33.5763	33.5214	0.0549	33.69	34.71
14×14×14	33.6223	33.5681	0.0542	131.93	128.61

Table 3: three dimensional case $\Omega = [0,1]^3$

In the BEC problem, the energy functional is defined as

$$E(\phi) = \int_{R^d} |\nabla\phi(x)|^2 + V(x)|\phi(x)|^2 + \frac{\beta}{2}|\phi(x)|^4 dx.$$

The ground state of a BEC is usually defined as the minimizer of the energy function on a spherical constraint S

$$\phi_g = \arg\min_{\phi\in S} E(\phi),$$

where S is defined as

$$S = \{\phi | E(\phi) < \infty, \int_{R^d} |\phi(x)|^2 dx = 1\}.$$

When energy function and constraint be discretized, it becomes a 4th-order polynomial problem

$$\begin{aligned} min \quad & (1/2)\mathcal{A}x^4 + Bx^2 \\ s.t. \quad & \|x\|_2^2 = (1/h)^3, \end{aligned}$$

which is (P_1). So the solution of (P_1) we got is the eigenvector of (4.8) and minimizer of the energy function. From the table, we could see the difference between two bounds is very little. It means that the minimizer of the energy function is a eigenvector of (4.8) and the corresponding eigenvalue is closed to minimum eigenvalue. But how we estimate the error is a question. However, Xie et al. (citation missing) solved the original nonlinear eigenvalue problem directly by a full multigrid method. So whether the solution they got is the minimizer of the energy function or not is a question. In [16], they also computed the BEC problem and they also relaxed this problem to SDP. But they didn't consider the special structure of $\mathcal{A}$ and B. So the original problem and relaxed problem are not tight.

Example 2. (See [10]) In structural mechanics, the differential system is said to be overdamped when the overdamping condition

$$\min_{\|x\|_2=1} [(x^T Cx)^2 - 4(x^T Mx)(x^T Kx)] > 0$$

is satisfied [26]. Where M, C and K are real symmetric, $M \succ 0, C \succ 0$ and $K \succeq 0$. We can rewrite this problem as

$$\min_{\|x\|_2=1} \langle C\otimes C - 2M\otimes K - 2K\otimes M, x^{\otimes 4}\rangle > 0,$$

and we could solve this problem by the relaxation. If the off-diagonal entries of tensor $C\otimes C - 2M\otimes K - 2K\otimes M$ are nonpositive, then two problem have the same optimal value.

In the connected damped mass-spring system. The mass matrix $M = \mathrm{diag}(m_1, \ldots, m_n)$ is diagonal matrix and damping matrix C and stiffness matrix K are symmetric tridiagonal. If we take all the spring(respectively, dampers) to have the same constant and take $m_i = 1$. Then

$$M = I, \quad C = \tau \mathrm{tridiag}(-1, 3, -1), \quad K = \kappa \mathrm{tridiag}(-1, 3, -1).$$

We take $n = 50$ and first choose $\kappa = 5$ and $\tau = 3$. The problem is not overdamped. Second ,we take $\kappa = 5$ and $\tau = 10$. The system is overdamped. Now we use our method to solve this problem. The corresponding results are represented in Table 4. The Y (resp. N) in last column represent the system is overdamped (resp. not overdamped).

n	τ	κ	optimal value	time	overdamped
50	3	5	-27777.8	0.44	N
50	10	5	201711	0.36	Y
50	3	10	-111111	0.43	N
50	10	10	151521	0.52	Y
200	3	5	-444444	2.50	N
200	10	5	3.20176e+06	2.67	Y
200	3	10	-1.77778e+06	1.97	N
200	10	10	2.40156e+06	2.37	Y

Table 4:

And we find that the results are correct.

Example 3. Let f be a 4th-order polynomial on R^n, and Let $\mathcal{F}$ be the coefficient tensor of f such that

$$f(x) = \langle \mathcal{F}, \tilde{x}^{\otimes 4} \rangle,$$

where $\tilde{x} = (x^T, 1)^T$.

Consider the following 4th-order polynomial optimization where $\mathcal{F}_l, l = 0, 1, \ldots, p$ have the nonpositive off-diagonal elements

$$\begin{aligned} (P) \qquad min \quad & f_0(x) = \langle \mathcal{F}_0, \tilde{x}^{\otimes 4} \rangle \\ s.t. \quad & f_l(x) = \langle \mathcal{F}_l, \tilde{x}^{\otimes 4} \rangle \leq 0, \quad l = 1, \ldots, p \\ & \tilde{x} = (x^T, 1)^T. \end{aligned}$$

The semidefinite relaxation of (P) is (RP)

$$\begin{aligned} (RP) \qquad min \quad & f_0(x) = \langle \tilde{X}, \mathcal{F}_0 \tilde{X} \rangle \\ s.t. \quad & f_l(x) = \langle \tilde{X}, \mathcal{F}_l \tilde{X} \rangle \le 0, \quad l = 1, \dots, p \\ & \tilde{X} \succeq 0, X_{n+1,n+1} = 1. \end{aligned}$$

From [32], we know that $v(P) = v(RP)$. If (RP) is a convex problem, we can solve it by semidefinite programming. However, If it is not convex, we have to further relax it to

$$\begin{aligned} (RRP) \qquad min \quad & f_0(x) = \langle \mathcal{F}_0, \tilde{\mathcal{X}} \rangle \\ s.t. \quad & f_l(x) = \langle \mathcal{F}_l, \tilde{\mathcal{X}} \rangle \le 0, \quad l = 1, \dots, p \\ & \tilde{\mathcal{X}} \in conv\{X \otimes X, X \succeq 0\}, \tilde{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1. \end{aligned}$$

From Theorem 3.4, we have $v(RP) = v(RRP)$. But we note that checking the membership problem $\tilde{\mathcal{X}} \in conv\{X \otimes X, X \succeq 0\}$ is a hard problem.If $\mathcal{F}_l, l = 0, 1, \dots, p$ are completely symmetric tensors, denote $\bar{\mathcal{X}}$ as the complete symmetrization of $\tilde{\mathcal{X}}$. Then

$$\langle \mathcal{F}_l, \bar{\mathcal{X}} \rangle = \langle \mathcal{F}_l, \tilde{\mathcal{X}} \rangle, l = 0, 1, \dots, p,$$

and problem (RRP) becomes

$$\begin{aligned} (RRP_1) \qquad min \quad & f_0(x) = \langle \mathcal{F}_0, \tilde{\mathcal{X}} \rangle \\ s.t. \quad & f_l(x) = \langle \mathcal{F}_l, \tilde{\mathcal{X}} \rangle \le 0, \quad l = 1, \dots, p \\ & \tilde{\mathcal{X}} \in conv\{x^{\otimes 4}, x \in R^{n+1}\}, \tilde{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1. \end{aligned}$$

Wile [18] showed that $conv\{x^{\otimes 4}, x \in R^{n+1}\}$ is the complete symmetrization of $conv\{X \otimes X, X \succeq 0\}$. Then (RRP_1) is exactly the relaxed problem in Hu et.al [14], where (RRP_1) can be computed by the following SOS program

$$\begin{aligned} (SOS) \qquad max \quad & \mu \\ s.t. \quad & f_0(x) + \sum_{l=1}^{p} \lambda_l f_l - \mu = \sigma_0 \\ & \lambda_l \ge 0, l = 1, \dots, p, \sigma_0 \ is \ SOS, deg\sigma_0 \le 4, \end{aligned}$$

and $v(SOS) = v(RRP_1)$, so we can get the optimal value of the original problem (P) by solving the above SOS program.

We consider the following homogeneous polynomial optimization problem

$$\begin{aligned} (HP) \qquad min \quad & f_0(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_3^4 - 20x_1x_2x_3^2 \\ s.t. \quad & f_1(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_3^4 - 1 \le 0 \end{aligned}$$

By computing the corresponding the SOS program of (HP) via YALMIP [19, 20], we obtain min(HP) = -6.0711.

6 Concluding Remarks

In this paper, by generalizing the definition of positive semidefinite tensors, we extend the Yuan's Lemma to a special quadratic system of quadratic forms characterized by 4th-order tensors. We also consider the difference between our extension and Hu [14]. As an application, we consider the BEC problem which is minimizing a polynomial function over a sphere surface. Because of the special structure of this problem, we show that there is no gap between the corresponding SDP relaxation and the original problem. As a result, we can solve the BEC problem by solving its SDP relaxation. The drawback is that the computational cost of the SDP relaxations is usually more expensive. For small-size cases, we obtain good results. And we will continue our research on large-scale problems.

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