Uniqueness of DRS as the 2 Operator Resolvent-Splitting and Impossibility of 3 Operator Resolvent-Splitting

Ernest K. Ryu

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Abstract

Given the success of Douglas-Rachford splitting (DRS), it is natural to ask whether DRS can be generalized. Are there are other 2 operator splittings? Can DRS be generalized to 3 operators? This work presents the answers: no and no. In a certain sense, DRS is the unique 2 operator resolvent-splitting, and generalizing DRS to 3 operators is impossible without lifting, where lifting roughly corresponds to enlarging the problem size. The impossibility result further raises a question. How much lifting is necessary to generalize DRS to 3 operators? This work presents the answer by providing a novel 3 operator resolvent-splitting with provably minimal lifting that directly generalizes DRS.

1 Introduction

In 1979, Lions and Mercier presented Douglas-Rachford splitting (DRS) which solves the monotone inclusion problem

$$\underset{x\in\mathbb{R}^d}{\text{find}} \quad 0 \in (A+B)x$$

with

$$z^{k+1} = (1/2)z^k + (1/2)(2J_{\alpha A} - I)(2J_{\alpha B} - I)z^k$$

for any $\alpha > 0$ [22, 13, 17]. Since its inception, DRS has enjoyed great popularity and has provided great value to the field of optimization.

Given the success of DRS, one may ask the following two questions:

1. Are there other 2 operator splittings?
2. Can we generalize DRS to 3 operators?

In fact, the second question has been a long-standing open problem posed by Lions and Mercier themselves: “[T]he convergence seems difficult to prove ... in the case of a sum of 3 operators.” After all, identifying why a tool works and generalizing it is a common and often fruitful exercise in mathematics.

This work presents the answers to these questions: no and no. In a certain sense, DRS is the unique 2 operator resolvent-splitting. In a certain sense, there is no 3 operator resolvent-splitting without lifting, where lifting roughly corresponds to enlarging the problem size.

This impossibility result further raises the following question:

3. To generalize DRS to 3 operators, how much lifting is necessary?

This work presents the answer by providing a novel 3 operator resolvent-splitting with provably minimal lifting.

Background. To discuss what constitutes a generalization of DRS, we first point out a few key properties of DRS. Perhaps a generalization of DRS should satisfy these as well.

1. DRS is a *resolvent-splitting* in that it is constructed with scalar multiplication, addition, and resolvents.
2. DRS is *frugal* in that it uses $J_{\alpha A}$ and $J_{\alpha B}$ only once per iteration.

3. DRS *converges unconditionally* in that it works for any maximal monotone A and B.
4. DRS uses *no lifting* in that the fixed-point operator maps from $\mathbb{R}^d$ to $\mathbb{R}^d$, where $x \in \mathbb{R}^d$. In other words, DRS does not enlarge the problem size.

Consider the the proximal point method (PPM) [19, 20, 25, 2], which finds an $x \in \mathbb{R}^d$ such that $0 \in Ax$ with

$$x^{k+1} = J_{\alpha A} x^k$$

for any $\alpha > 0$. DRS generalizes PPM, and both methods are frugal, converge unconditionally, use no lifting, and rely on resolvents. Therefore, to require the 4 properties in a generalization of DRS seems reasonable.

Many other splittings have been presented since DRS, and they have certainly provided great value to the field of optimization. Many of them include DRS as a special case and thererfore are generalizations of DRS, in that sense. However, they do not satisfy the 4 stated properties and therefore are not generalizations of DRS, in this sense.

Forward-backward splitting (FBS) [21],

$$x^{k+1} = J_{\alpha B}(I - \alpha A)x^k$$

is frugal, uses no lifting, but is not a resolvent-slpitting. Primal-dual hybrid gradient method (PDHG) [32, 23, 15, 6], also known as Chambolle-Pock method,

$$\begin{aligned} x^{k+1} &= J_A(x^k - \alpha u^k) \\ u^{k+1} &= (I - J_B)(u^k + \alpha(2x^{k+1} - x^k)) \end{aligned}$$

is frugal but uses lifting. David-Yin splitting (DYS) [12], which finds an $x \in \mathbb{R}^d$ such that $0 \in (A + B + C)x$

$$z^{k+1} = (I - J_{\alpha B} + J_{\alpha A} \circ (2J_{\alpha B} - I - \alpha C \circ J_{\alpha B}))z^k$$

for any $\alpha > 0$, is frugal, uses no lifting, but is not a resolvent-splitting. Other methods, such as FBFS [29], PDFP^{2}O/PAPC [18, 7, 14], Condat-Vũ [11, 30], PD3O [31], PDFP [8], AFBA [16], FBHFS [4], and the methods of [3, 10], all fail to satisfy the 4 properties.

Organization of the paper. In Section 2, we show that DRS is the only frugal, unconditionally convergent resolvent-splitting without lifting for the 2 operator problem. We do so by characterizing all frugal resolvent-splittings without lifting and showing that DRS is the only one among them that unconditionally converges.

In Section 3, we show that there is no resolvent-splitting without lifting for the 3 operator problem, even if the splitting is not frugal and not convergent. In particular, we show such a scheme without lifting cannot be a fixed-point encoding.

In Section 4, we define and quantify the notion of lifting for the 3 operator problem. We then provide a novel frugal, unconditionally convergent resolvent-splitting with provably minimal lifting for the 3 operator problem that directly generalizes DRS.

Definitions. We briefly review some standard notation and results of operator theory. Interested readers can find in-depth discussion of these concepts in standard references such as [26, 1].

Write $\langle \cdot, \cdot \rangle$ for the standard Euclidean inner product in $\mathbb{R}^d$. We say A is an operator on $\mathbb{R}^d$ if A maps points of $\mathbb{R}^d$ to subsets of $\mathbb{R}^d$. Given a matrix $M \in \mathbb{R}^{d \times d}$ also write $M : \mathbb{R}^d \to \mathbb{R}^d$ to denote the linear function defined by the matrix M. In particular, write I for both the identity operator and the identity matrix. For any maximal monotone operator A and $\alpha > 0$, write

$$J_{\alpha A} = (I + \alpha A)^{-1}$$

for the resolvent of A. A mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ is nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2$$

for all $x, y \in \mathbb{R}^d$. A mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ is firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in \mathbb{R}^d$. Resolvents are firmly nonexpansive. Given a mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ and a starting point $z^0 \in \mathbb{R}^d$, we call

$$z^{k+1} = Tz^k$$

the fixed-point iteration with respect to T. A fixed-point iteration with respect to a nonexpansive mapping need not converge. A mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ is averaged if it can be expressed as $T = (1-\theta)I + \theta R$, where $R : \mathbb{R}^d \to \mathbb{R}^d$ is nonexpansive and $\theta \in (0,1)$. Note that R and T share the same fixed points. The fixed-point iteration with respect to an averaged mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ converges in that $z^k \to z^\star$ where $Tz^\star = z^\star$, if a fixed point exists.

2 Uniqueness of DRS as the unique frugal, unconditional 2 operator resolvent-splitting without lifting

A pair of functions (T, S) is a *fixed-point encoding* for problem

$$\underset{x\in\mathbb{R}^d}{\text{find}} \quad 0 \in (A+B)x \tag{2op}$$

if

$$\exists z^\star \text{ such that } \begin{pmatrix} T(A,B,z^\star) & = z^\star \\ S(A,B,z^\star) & = x^\star \end{pmatrix} \quad \Leftrightarrow \quad 0 \in (A+B)(x^\star).$$

For notational simplicity, we often drop the dependency on A and B and write Tz and Sz for $T(A,B,z)$ and $S(A,B,z)$. We call T the *fixed-point mapping* and S the *solution mapping*. (T,S) is a *resolvent-splitting* for (2op) if it is a fixed-point encoding constructed with resolvents of A and B, addition, and scalar multiplication. (T,S) is *frugal* if it uses $J_{\alpha A}$ and $J_{\beta B}$ once, in that a single evaluation of $J_{\alpha A}$ and a single evaluation of $J_{\beta B}$ is used to evaluate *both* Tz and Sz for some z. (T,S) converges unconditionally if

$$ST^k z^0 \to x^\star, \quad 0 \in (A+B)x^\star$$

for any $z^0 \in \mathbb{R}^d$ and maximal monotone A and B as $k \to \infty$. (T,S) is *without lifting* if $T : \mathbb{R}^d \to \mathbb{R}^d$ and $S : \mathbb{R}^d \to \mathbb{R}^d$.

Given a fixed-point iteration, we can scale it by a nonzero scalar to get another one that is essentially the same, i.e.,

$$z^{k+1} = T(z^k) \quad \Leftrightarrow \quad az^{k+1} = aT(a^{-1}az^k)$$

for any $a \in \mathbb{R}$ such that $a \neq 0$. Given resolvent-splitting, we can swap the role of A and B to get another one that is conceptually no different, i.e.,

$$(T(A,B,\cdot), S(A,B,\cdot)) \quad \Leftrightarrow \quad (T(B,A,\cdot), S(B,A,\cdot)).$$

Two resolvent-splittings without lifting are *equivalent* if one can be obtained from the other through scaling by a nonzero scalar and/or swapping the role of A and B.

Theorem 1. *Up to equivalence, (T,S) is a frugal resolvent-splittings without lifting for* (2op) *if and only if it is of the form*

$$\begin{aligned} x_1 &= J_{\alpha A} z \\ x_2 &= J_{\beta B}((1+\beta/\alpha)x_1 - (\beta/\alpha)z) \\ T(z) &= z + \theta(x_2 - x_1) \\ S(z) &= \eta x_1 + (1-\eta)x_2 \end{aligned}$$

for some $\alpha, \beta > 0$, $\theta \neq 0$, and $\eta \in \mathbb{R}$.

Theorem 1 characterizes all splittings that encode a fixed-point. So a fixed-point iteration with respect to (T,S) of Theorem 1 stays at a solution if it starts at a solution. However, we of course want a fixed-point iteration to converge to a solution with an arbitrary starting point. Theorem 2 characterizes the ones that do converge.

Theorem 2. *(T,S) of Theorem 1 converges unconditionally if and only if $\alpha = \beta$ and $\theta \in (0,2)$.*

When $\alpha = \beta$, the splitting (T,S) of Theorem 1 is DRS or relaxed Peaceman-Rachford splitting. Theorem 1 and Theorem 2 together states that this is the only frugal, unconditionally convergent resolvent-splitting without lifting for (2op), up to equivalence. In this sense, we conclude DRS is the unique 2 operator resolvent-splitting.

2.1 Proof of Theroem 1

Showing that (T,S) of Theroem 1 is indeed a fixed-point encoding is straightforward. Let $x^\star \in \mathbb{R}^d$ satisfy $0 \in (A+B)x^\star$. Let $\tilde{A}x^\star \in Ax^\star$ and $\tilde{B}x^\star \in Bx^\star$ such that $\tilde{A}x^\star + \tilde{B}x^\star = 0$, and let $z_0 = x^\star + \alpha\tilde{A}x^\star$. Then it is straightforward to verify that $x_1 = x_2 = x^\star$, $Tz_0 = z_0$, and $Sz_0 = x^\star$.

We now need to show that any frugal resolvent-splitting without lifting for (2op) is of the form of Theroem 1, up to equivalence. Before we do so, we will discuss the following lemma, which proof follows from standard linear algebra.

Lemma 3. *Let $M \in \mathbb{R}^{m\times n}$ and $m \in \mathbb{R}^{1\times n}$ be fixed coefficients, and let $v \in \mathbb{R}^n$ be a variable. Then the linear equalities $Mv = 0$ implies the inear equalty $mv = 0$ if and only if there is a $w \in \mathbb{R}^{1\times m}$ such that $wM = m$.*

To clarify, $Mv = 0$ implies $mv = 0$ if all instances of the variable $v \in \mathbb{R}^n$ satisfying $Mv = 0$ satisfies $mv = 0$. If $Mv = 0$ does not imply $mv = 0$, there is an instance of $v \in \mathbb{R}^n$ such that $Mv = 0$ but $mv \neq 0$.

We now proceed onto the main proof. Let (T,S) be any frugal resolvent-splitting with no lifting. Without loss of generality, assume $J_{\alpha A}$ is evaluated before or possibly simultaneously with $J_{\beta B}$.

Consider the evaluation of Tz_0 and Sz_0 for $z_0 \in \mathbb{R}^d$. Write z_1 and z_2 for the points at which $J_{\alpha A}$ and $J_{\beta B}$ are evaluated. Write $x_1 = J_{\alpha A}z_1$ and $x_2 = J_{\beta B}z_2$. Define $\tilde{A}x_1$ and $\tilde{B}x_2$ with $x_1 + \alpha\tilde{A}x_1 = z_1$ and $x_2 + \beta\tilde{B}x_2 = z_2$. By definition of resolvents, we have $\tilde{A}x_1 \in Ax_1$ and $\tilde{B}x_2 \in Bx_2$.

Then we can express the evaluation of Tz_0 and Sz_0 as

$$0 = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ * & * & * & * & * & 1 & * & * & 0 \\ * & * & * & * & * & * & * & * & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix},$$

where the $*$ denote unspecified coefficients. With Gaussian elimination and reordering of the rows, we obtain the simpler equivalent system

$$0 = \begin{bmatrix} -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ * & 0 & * & 0 & * & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix}.$$

Assume $a \neq 0$. We will later argue why this must be true. We absorb the top-left a into z_0 and left-multiply by an invertible matrix to get the equivalent system

$$0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ * & 0 & * & 0 & * & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix}.$$

The **boldface symbols** denote where to pay attention in the linear systems. This further simlpifies to the equivalent system

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ * & 0 & * & 0 & * & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ \boldsymbol{aT(a^{-1}az_0)} \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ \boldsymbol{S(a^{-1}az_0)} \end{bmatrix}.$$

By redefining $(T(z_0), S(z_0))$ to be the scaled splitting $(aT(a^{-1}az_0), S(az_0))$, we get the equivalent system

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{\theta_1} & 0 & \boldsymbol{\theta_2} & 1 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{\theta_3} & 0 & \boldsymbol{\theta_4} & 0 & \boldsymbol{\theta_5} & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ \boldsymbol{\theta_6} & 0 & \boldsymbol{\theta_7} & 0 & \boldsymbol{\theta_8} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{z_0} \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ \boldsymbol{Tz_0} \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ \boldsymbol{Sz_0} \end{bmatrix}.$$

Now consider the system of linear equalities with the fixed-point condition $T(z_0) = z_0$ added

$$0 = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & \theta_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \theta_3 & 0 & \theta_4 & 0 & \theta_5 & 1 & 0 & 0 & 0 \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ \theta_6 & 0 & \theta_7 & 0 & \theta_8 & 0 & 0 & 0 & 1 \end{bmatrix}}_{=M} \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix}}_{=v}. \tag{1}$$

The linear equalities (1) must imply that $x_1 = x_2$, $Sz_0 = x_1$, and $\tilde{A}x_1 + \tilde{B}x_2$. We prove these assertions by assuming otherwise and constructing counter examples.

Remember that (T, S) is assumed to be a fixed-point encoding for any maximal monotone operators A and B and any problem size d. Assume for contradiction that (1) does not imply linear equality $x_1 = x_2$. Assume $d = 1$. This means there is a specific instance

$$v' = (z_0', z_1', x_1', z_2', x_2', T(z_0'), \tilde{A}x_1', \tilde{B}x_2', S(z_0')) \in \mathbb{R}^9$$

such that $Mv' = 0$ but $x_1' \neq x_2'$. Define

$$f_1(x) = (\tilde{A}x_1')x + (x - x_1')^2 \qquad f_2(x) = (\tilde{B}x_2')x + (x - x_2')^2,$$

so that $\nabla f_1(x_1') = \tilde{A}x_1'$ and $\nabla f_2(x_2') = \tilde{B}x_2'$. Then $T(\nabla f_1, \nabla f_2, z_0') = z_0'$. Write $x^\star = S(\nabla f_1, \nabla f_2, z_0')$. Since (T, S) is a fixed-point encoding, we have

$$x^\star = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \{f_1(x) + f_2(x)\}.$$

However, $x_1' \neq x_2'$, so either $x_1' \neq x^\star$ or $x_2' \neq x^\star$ or both. Without loss of generality assume $x_1' \neq x^\star$. Define

$$f_3(x) = (\tilde{A}x_1')x + 2(x - x_1')^2.$$

Since, $\nabla f_3(x_1') = \tilde{A}x_1'$ we still have $T(\nabla f_3, \nabla f_2, z_0') = z_0'$ and $S(\nabla f_3, \nabla f_2, z_0') = x^\star$. However,

$$x^\star \neq \operatorname*{argmin}_{x\in\mathbb{R}}\{f_3(x) + f_2(x)\},$$

because $\nabla f_3(x^\star) \neq \nabla f_1(x^\star)$. Therefore (T, S) fails to be a fixed-point encoding for the maximal monotone operators $\nabla f_3, \nabla f_2$, and we have a contradiction.

Next, assume for contradiction that (1) does not imply linear equality $Sz_0 = x_1$. Assume $d = 1$. This means there is a specific instance

$$v' = (z_0', z_1', x_1', z_2', x_2', T(z_0'), \tilde{A}x_1', \tilde{B}x_2', S(z_0')) \in \mathbb{R}^9$$

such that $Mv' = 0$ but $S(z_0') \neq x_1' = x_2'$. Using the same definition of f_1, f_2, f_3, and $x^\star$, the same argument carries over to establish $x^\star = S(\nabla f_1, \nabla f_2, z_0') = S(\nabla f_3, \nabla f_2, z_0')$. However,

$$x^\star = \operatorname*{argmin}_{x\in\mathbb{R}}\{f_1(x) + f_2(x)\} \neq \operatorname*{argmin}_{x\in\mathbb{R}}\{f_3(x) + f_2(x)\},$$

since $x^\star = x_1$. Therefore (T, S) fails to be a fixed-point encoding for the maximal monotone operators $\nabla f_3, \nabla f_2$, and we have a contradiction.

Finally, assume for contradiction that (1) does not imply linear equality $\tilde{A}x_1 + \tilde{B}x_2 = 0$. Assume $d = 1$. This means there is a specific instance

$$v' = (z_0', z_1', x_1', z_2', x_2', T(z_0'), \tilde{A}x_1', \tilde{B}x_2', S(z_0')) \in \mathbb{R}^9$$

such that $Mv' = 0$ but $\tilde{A}x_1' + \tilde{B}x_2' \neq 0$. Note that $x_1' = x_2' = S(z_0')$. Define

$$f_1(x) = (\tilde{A}x_1')x + (x - x_1')^2 \qquad f_2(x) = (\tilde{B}x_2')x + (x - x_2')^2.$$

Then

$$(\nabla f_1 + \nabla f_2)(S(z_0')) = \tilde{A}x_1' + \tilde{B}x_2' \neq 0$$

and

$$S(z_0') \neq \operatorname*{argmin}_{x\in\mathbb{R}}\{f_1(x) + f_2(x)\}$$

which contradicts the assumption that (T, S) is a fixed-point encoding.

With the assertions proved, we proceed with the proof. Gaussian elimination on (1) gives us the equivalent system

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & \theta_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{\theta_3 + 1} & 0 & \theta_4 & 0 & \theta_5 & \mathbf{0} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ \theta_6 & 0 & \theta_7 & 0 & \theta_8 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix}.$$

Because the linear equality must imply $x_1 = x_2$ and because of where the zeros and nonzeros are placed, we have $\theta_3 = -1$ and $\theta_4 = -\theta_5 = \theta$ for some $\theta \neq 0$. Because the linear equality must imply $x_1 = Sz_0$ and because of where the zeros and nonzeros are placed, $\theta_6 = 0$, $\theta_7 = -1 + \eta$, and $\theta_8 = -\eta$ for some $\eta \in \mathbb{R}$. Plugging these in, we get

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & \theta_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & \boldsymbol{\theta} & 0 & \boldsymbol{-\theta} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ \mathbf{0} & 0 & \boldsymbol{-1 + \eta} & 0 & \boldsymbol{-\eta} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}x_2 \\ Sz_0 \end{bmatrix}.$$

We perform Gaussian elimination again to get the equivalent system

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & \theta_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1+\theta_1+\theta_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{-(1+\theta_2)\alpha} & \boldsymbol{\beta} & \mathbf{0} \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \beta & 0 \\ 0 & 0 & -1+\eta & 0 & -\eta & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ Tz_0 \\ \tilde{A}x_1 \\ \tilde{B}z_2 \\ Sz_0 \end{bmatrix}.$$

Because the linear equality must imply $\tilde{A}x_1 + \tilde{B}z_2 = 0$ and because of where the zeros and nonzeros are placed, $\theta_1 = \beta/\alpha$ and $\theta_2 = -1 - \beta/\alpha$. Finally, plugging in the parameters and expressing the splitting in functional form, we get the splitting of Theroem 1.

All that remains is to prove the assertion that $a \neq 0$. Assume for contradiction that $a = 0$. Assume $d = 1$. Let

$$f_1(x) = cx^2 \quad f_2(x) = dx$$

where $c > 0$ is unspecified and $d \neq 0$. Then $J_{\alpha A}0 = 0$. If $A = \nabla f_1$ and $B = \nabla f_2$, the output of Tz_0 and Sz_0 is independent of the value of c. Since (T, S) is assumed to be a fixed-point encoding, there must be a $z^\star$ such that $Tz^\star = z^\star$ and

$$Sz^\star = \underset{x\in\mathbb{R}}{\operatorname{argmin}}\{f_1(x) + f_2(x)\}.$$

As mentioned, the left-hand side is independent of c, but the right-hand side is not, which is a contradiction. □

2.2 Proof of Theorem 2

Consider the problem

$$\underset{x\in\mathbb{R}^2}{\text{find}} \quad 0 = (A + B)x,$$

where

$$A = \begin{bmatrix} 0 & \tan(\omega)/\alpha \\ -\tan(\omega)/\alpha & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -\tan(\omega)/\beta \\ \tan(\omega)/\beta & 0 \end{bmatrix}$$

and $\alpha, \beta > 0$ and $\alpha \neq \beta$. We identify A and B as maximal monotone operators from $\mathbb{R}^2 \to \mathbb{R}^2$. Note that $x^\star = 0$ is the unique solution.

With basic algebra, we can show that

$$Tz = \begin{bmatrix} 1 & \theta(\alpha-\beta)\cos(\omega)\sin(\omega) \\ -\theta(\alpha-\beta)\cos(\omega)\sin(\omega) & 1 \end{bmatrix} z$$

With basic eigenvalue computation, we get

$$|\lambda_1|^2 = |\lambda_2|^2 = 1 + (\theta(1-\beta/\alpha)\cos(\omega)\sin(\omega))^2 > 1,$$

where λ_1, λ_2 are the eivengalues of the matrix that defines T. So if $z^0 \neq 0$, the iteration $z^{k+1} = Tz^k$ diverges in that $\|z^k\| \to \infty$ and $\|Sz^k\| \to \infty$.

When $\alpha = \beta$, the splitting (T, S) of Theorem 1 reduces to DRS or relaxed Peaceman-Rachford splitting. It is well known that the iteration converges for all maximal monotone A and B if and only if $\theta \in (0, 2)$ in this case. □

3 Impossibility of 3 operator resolvent-splitting without lifting

A pair of functions (T, S) is a fixed-point encoding for problem

$$\underset{x\in\mathbb{R}^d}{\text{find}} \quad 0 \in (A + B + C)x \tag{3op}$$

if

$$\exists z^\star \text{ such that } \begin{pmatrix} T(A,B,C,z^\star) & = z^\star \\ S(A,B,C,z^\star) & = x^\star \end{pmatrix} \quad \Leftrightarrow \quad 0 \in (A+B+C)(x^\star).$$

The terms resolvent-splitting, frugal, unconditional convergence, and no lifting are defined similarly as before.

If one could find a frugal, unconditionally convergent resolvent-splitting without lifting for (3op), it would be a satisfying generalization of DRS to 3 operators. However, this is impossible. Even if we drop frugality and convergence as requirements, this is impossible.

Theorem 4. *There is no resolvent-splitting **without lifting** for* (3op).

Clarification. Assume $T : \mathbb{R}^d \to \mathbb{R}^d$ and $S : \mathbb{R}^d \to \mathbb{R}^d$ are constructed with finitely many resolvents,

$$\begin{array}{c} J_{\alpha(1)A}, J_{\alpha(2)A}, \dots, J_{\alpha(n_A)A} \\ J_{\beta(1)B}, J_{\beta(2)B}, \dots, J_{\beta(n_B)B} \\ J_{\gamma(1)C}, J_{\gamma(2)C}, \dots, J_{\gamma(n_C)C} \end{array}$$

where the parameters $\alpha(i)$, $\beta(j)$, $\gamma(k)$ may be different. Theorem 4 states that (T,S) will fail to be a fixed-point encoding.

Remark. The set of mappings constructed with resolvents, identity operator, and scalar multiplication forms a "near-ring". Theorem 4 states that no element of this near-ring is a fixed-point encoding for (3op). The set is not a ring because $T \circ (U+V) \neq T \circ U + T \circ V$ for non-linear functions.

3.1 Proof of Theorem 4

Assume for contradiction that (T,S) is a resolvent-splitting without lifting. Assume $d = 1$, and let n be the total number of resolvent evaluations required to compute T and S. The specific value of n depends on how you count, i.e., whether you simplify things and whether some resolvent evaluations are counted redundantly. All that matters is that n is finite.

Since T and S are explicitly constructed with resolvents, we can find a sequential ordering to them under which the resolvents are evaluated one after another. More specifically, we label the resolvents $J_1, J_2, \dots, J_n$, where J_i is one of $J_{\alpha A}$, $J_{\beta B}$, or $J_{\gamma C}$ for some $\alpha > 0$, $\beta > 0$, or $\gamma > 0$ for each $i = 1, \dots, n$. We call z_i the point at which J_i is evaluated and $x_i = J_i(z_i)$ for $i = 1, \dots, n$. In the process of evaluating Tz_0 and Sz_0, we get $z_0, z_1, x_1, z_2, x_2, \dots, z_n, x_n$. By nature of the ordering, z_i is defined as a linear combination of $z_0, z_1, x_1, z_2, x_2, \dots, z_{i-1}, x_{i-1}$ for each $i = 1, \dots, n$. Likewise, Tz_0 can be expressed as a linear combination of $z_0, z_1, x_1, z_2, x_2, \dots, z_n, x_n$. Assume $J_{\alpha A}$, $J_{\beta B}$, and $J_{\gamma C}$ are all used least once in T or S with some $\alpha > 0$, $\beta > 0$, and $\gamma > 0$. Otherwise, if, for example, $J_{\alpha A}$ is never used, we let $z_{n+1} = 0$ and $J_{n+1} = J_A$ to fix the issue.

Say $J_{\alpha A}$, $J_{\beta B}$, and $J_{\gamma C}$ are evaluated n_A, n_B, and n_C times, respectively. So $n_1 + n_2 + n_3 = n$. Define the indices $a(1), a(2), \dots, a(n_A) \in \{1, 2, \dots, n\}$ and the parameters $\alpha(1), \alpha(2), \dots, \alpha(n_A) >$ so that

$$x_{a(\ell)} = J_{a(\ell)}(z_{a(\ell)}) = J_{\alpha(\ell)A}(z_{a(\ell)}).$$

In other words, $x_{a(1)}, x_{a(2)}, \dots, x_{a(n_A)}$ are the outputs of the resolvents of A. (Define $a(1), a(2), \dots, a(n_A)$ so that they are distinct, i.e., $x_{a(1)}, x_{a(2)}, \dots, x_{a(n_A)}$ must cover all n_A outputs of the resolvents of A.) Define the indices $b(1), b(2), \dots, b(n_B) \in \{1, 2, \dots, n\}$ and $c(1), c(2), \dots, c(n_C) \in \{1, 2, \dots, n\}$ and the parameters $\beta(1), \beta(2), \dots, \beta(n_B) > 0$ and $\gamma(1), \gamma(2), \dots, \gamma(n_C) > 0$ likewise.

We express the evaluation of Tz_0 with the following system of linear and nonlinear equalities:

$$0 = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & * & * & 1 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & & & \\ & & & \vdots & & & & & & \\ * & * & * & * & * & * & \cdots & 1 & 0 & 0 \\ * & * & * & * & * & * & \cdots & * & * & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ z_3 \\ \vdots \\ z_n \\ x_n \\ Tz_0 \end{bmatrix}$$

$$x_1 = J_1(z_1),\ x_2 = J_2(z_2),\ \ldots,\ x_n = J_n(z_n),$$

where the $*$ denote unspecified coefficients. Each linear equality except the last one defines z_i for $i = 1, \ldots, n$. The last linear equality defines $T(z_0)$. With Gaussian elimination, we obtain the simpler equivalent system

$$0 = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & & & \\ & & & \vdots & & & & & & \\ * & 0 & * & 0 & * & 0 & \cdots & 1 & 0 & 0 \\ * & 0 & * & 0 & * & 0 & \cdots & 0 & * & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ z_3 \\ \vdots \\ z_n \\ x_n \\ Tz_0 \end{bmatrix}$$

$$x_1 = J_1(z_1),\ x_2 = J_2(z_2),\ \ldots,\ x_n = J_n(z_n).$$

Now we add the fixed-point condition $z_0 = Tz_0$ and perform Gaussian elimination:

$$0 = \underbrace{\begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & 0 & * & 0 & * & 1 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & & & \\ & & & \vdots & & & & & & \\ * & 0 & * & 0 & * & 0 & \cdots & 1 & 0 & 0 \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \boldsymbol{*} & \mathbf{0} & \boldsymbol{*} & \mathbf{0} & \boldsymbol{*} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{*} & \mathbf{0} \end{bmatrix}}_{=M} \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ x_1 \\ z_2 \\ x_2 \\ z_3 \\ \vdots \\ z_n \\ x_n \\ Tz_0 \end{bmatrix}}_{=v} \tag{2}$$

$$x_1 = J_1(z_1),\ x_2 = J_2(z_2),\ \ldots,\ x_n = J_n(z_n),$$

The **boldface symbols** denote where to pay attention in the linear system. Consider the system of linear equalities $Mv = 0$ of (2) combined with the linear equalities

$$\begin{aligned} x_{a(1)} &= x_{a(2)} = \cdots = x_{a(n_A)} \\ x_{b(1)} &= x_{b(2)} = \cdots = x_{b(n_B)} \\ x_{c(1)} &= x_{c(2)} = \cdots = x_{c(n_C)}. \end{aligned} \tag{3}$$

It is impossible to perform Gaussian elimination on these linear equalities to establish $x_{a(1)} = x_{b(1)} = x_{c(1)}$. Every row of M, except the last one, cannot be used in Gaussian elimination to prove a linear equality only involving the

x's, and the linear equalities of (3) do not help either. The last row of M can establish $x_{a(1)} = x_{b(1)}$ or $x_{b(1)} = x_{c(1)}$ but not both.

However, the fact that we cannot establish $x_{a(1)} = x_{b(1)} = x_{c(1)}$ with the linear equalities does not immediately imply that we cannot do so if we also use the nonlinear equalities. Therefore we construct a counter example: maximal monotone operators A, B, and C on $\mathbb{R}$ for which (T, S) fails to be a fixed-point encoding. By Lemma 3, there is a specific instance

$$v' = (z'_0, z'_1, x'_1, z'_2, x'_2, \dots, z'_n, x'_n, T(z'_0))$$

that satisfies $Mv' = 0$ and the linear equalities of (3), but $x'_{a(1)} \neq x'_{b(1)}$ or $x'_{b(1)} \neq x'_{c(1)}$. Without loss of generality, say $x'_{b(1)} \neq x'_{c(1)}$.

Define A such that

$$J_{\alpha(i)A}(z'_{a(i)}) = x'_{a(1)}$$

for all $i = 1, \dots, n_A$. In particular, we achieve this by defining

$$A(x'_{a(1)}) = \left[\min_{i=1,\dots,n_A} \left(z'_{a(i)} - x'_{a(i)}\right)/\alpha(i), \max_{i=1,\dots,n_A} \left(z'_{a(i)} - x'_{a(i)}\right)/\alpha(i)\right].$$

For the moment, leave $A(x)$ for $x \neq x'_{a(1)}$ unspecified. Define $B(x'_{b(1)})$ and $C(x'_{c(1)})$ likewise. By construction, $z'_0 = T(A, B, C, z'_0)$, even though A, B, and C are not yet fully specified. Write $x' = S(z'_0)$. We have $x' \neq x'_{b(1)}$ or $x' \neq x'_{c(1)}$ since $x'_{b(1)} \neq x'_{c(1)}$. Without loss of generality, let $x' \neq x'_{c(1)}$.

Now we define

$$A(x) = \begin{cases} (x - x'_{a(1)}) + \min\{A(x'_{a(1)})\} & \text{for } x < x'_{a(1)} \\ (x - x'_{a(1)}) + \max\{A(x'_{a(1)})\} & \text{for } x > x'_{a(1)} \end{cases}$$

and

$$B(x) = \begin{cases} (x - x'_{b(1)}) + \min\{A(x'_{b(1)})\} & \text{for } x < x'_{b(1)} \\ (x - x'_{b(1)}) + \max\{A(x'_{b(1)})\} & \text{for } x > x'_{b(1)}. \end{cases}$$

(This makes A and B maximal monotone.) By construction, $(A + B)(x')$ is a bounded subset of $\mathbb{R}$, and $C(x')$ is unspecified. Depending on whether $x' < x'_{c(1)}$ or $x' > x'_{c(1)}$, we can make $C(x')$ an arbitrarily small or large value, respectively (and still have C be monotone). In either case, we make $C(x')$ single-valued and so small or so large that $0 \notin (A + B + C)(x')$. We extend the definition of C to all of $\mathbb{R}$ to make it maximal monotone.

So we have maximal monotone operators A, B, and C, such that $z'_0 = T(A, B, Cz'_0)$ but the $x' = S(z'_0)$ does not satisfy $0 \in (A + B + C)x'$. This contradicts the assumption that (T, S) is a fixed-point encoding. □

4 Attainment of 3 operator resolvent-splitting with minimal lifting

We say a resolvent-splitting $(\boldsymbol{T}, S)$ for (3op) uses *ℓ-fold lifting* if

$$\boldsymbol{T} : \mathbb{R}^{\ell d} \to \mathbb{R}^{\ell d} \qquad S : \mathbb{R}^{\ell d} \to \mathbb{R}^d.$$

Note that 1-fold lifting corresponds to no lifting. Theorem 4 states a resolvent-splitting for (3op) requires lifting. Then how much? The answer is 2-fold lifting.

A standard trick to solve (3op) is to "copy" variables to form an enlarged problem

$$\underset{x_1,x_2,x_3\in\mathbb{R}^d}{\text{find}} \quad 0 \in \begin{bmatrix} Ax_1 \\ Bx_2 \\ Cx_3 \end{bmatrix} + N_{\{(x_1,x_2,x_3)\,|\,x_1=x_2=x_3\}}(x_1, x_2, x_3),$$

where N_K is the normal cone operator with respect to the set K, and apply DRS to get

$$\begin{aligned} \bar{z} &= (1/3)(z_A + z_B + z_C) \\ T_A(\boldsymbol{z}) &= z_A + J_{\alpha A}(2\bar{z} - z_A) - \bar{z} \\ T_B(\boldsymbol{z}) &= z_B + J_{\alpha B}(2\bar{z} - z_B) - \bar{z} \\ T_C(\boldsymbol{z}) &= z_C + J_{\alpha C}(2\bar{z} - z_C) - \bar{z} \\ S(\boldsymbol{z}) &= J_{\alpha A}(2\bar{z} - z_A), \end{aligned} \tag{4}$$

where $\alpha > 0$ and $\boldsymbol{z} = (z_A, z_B, z_C)$. This frugal, unconditionally convergent resolvent-splitting uses 3-fold lifting, since $\boldsymbol{T} = (T_A, T_B, T_C) : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$.

So constructing a resolvent-splitting for (3op) is impossible with 1-fold lifting but is possible with 3-fold lifting. It turns out that 2-fold lifting is sufficient, and we therefore call 2-fold lifting the *minimal lifting* for (3op).

Theorem 5. *The pair $(\boldsymbol{T}, S)$, where $\boldsymbol{T} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $S : \mathbb{R}^{2d} \to \mathbb{R}^d$ are defined as*

$$\begin{aligned}
x_1 &= J_{\alpha A}(z_1) \\
x_2 &= J_{\alpha B}(x_1 + z_2) \\
x_3 &= J_{\alpha C}\left(x_1 - z_1 + x_2 - z_2\right) \\
T_1(\boldsymbol{z}) &= z_1 + \theta(x_3 - x_1) \\
T_2(\boldsymbol{z}) &= z_2 + \theta(x_3 - x_2) \\
S(\boldsymbol{z}) &= \eta_1 x_1 + \eta_2 x_2 + (1 - \eta_1 - \eta_2)x_3,
\end{aligned}$$

is a fixed-point encoding, and $(\boldsymbol{T}, S)$ converges unconditionally for $\theta \in (0, 1)$, $\alpha > 0$, and $\eta_1, \eta_2 \in \mathbb{R}$.

Therefore, $(\boldsymbol{T}, S)$ for any $\theta \in (0, 1)$ is a frugal, unconditionally convergent, resolvent-splitting with minimal lifting for (3op). When $B = 0$, the splitting of Theorem 5 reduces to DRS. In this sense, this splitting is a direct generalization of DRS with minimal lifting.

Remark. To the best of the author's knowledge, the splitting of Theorem 5 is new and cannot be reduced to an instance of a known splitting method. This is why Theorem 5 is proved from first principles.

4.1 Proof of Theorem 5

Without loss of generality, assume $\alpha = 1$. We first show that $(\boldsymbol{T}, S)$ is a fixed-point encoding.

Assume $\boldsymbol{z} = (z_1, z_2)$ is a fixed point of $\boldsymbol{T}$. Write

$$\begin{aligned}
a &= z_1 - x_1 \\
b &= x_1 + z_2 - x_2 \\
c &= x_1 - z_1 + x_2 - z_2 - x_3.
\end{aligned}$$

Add the three and use $x_1 = x_2 = x_3$ to get

$$a + b + c = 0.$$

Since $a \in Ax_1$, $b \in Bx_2$, and $c \in Cx_3$, by the definitions of x_1, x_2, and x_3, this proves $x_1 = x_2 = x_3$ is a solution to (3op).

Now assume $x^\star$ is a solution to (3op), and let $a \in Ax^\star$, $b \in Bx^\star$, and $c \in Cx^\star$ so that $a + b + c = 0$. We then define

$$\boldsymbol{z}^\star = (a + x^\star, b)$$

It is straightforward to verify that $\boldsymbol{T}(\boldsymbol{z}^\star) = \boldsymbol{z}^\star$ and $S(\boldsymbol{z}) = x^\star$.

Next we show that $(\boldsymbol{T}, S)$ converges unconditionally for $\theta \in (0, 1)$. We show this by showing $\boldsymbol{T}$ is nonexpansive for $\theta = 1$. Let $\theta = 1$ and define $\boldsymbol{U}$ as

$$\boldsymbol{U}(\boldsymbol{z}) = \begin{bmatrix} J_A(z_1) - z_1 \\ J_B(z_2 + x_1) - z_2 \end{bmatrix}.$$

Then we can write

$$\boldsymbol{T} = -\boldsymbol{U} + \begin{bmatrix} J_C \\ J_C \end{bmatrix} \circ M \circ \boldsymbol{U}.$$

Define the matrices

$$M = \begin{bmatrix} I & I \\ I & I \end{bmatrix} \qquad N = \begin{bmatrix} -I & I \\ I & -I \end{bmatrix}.$$

Then

$$\begin{aligned}
&\|\boldsymbol{T}(\boldsymbol{y})-\boldsymbol{T}(\boldsymbol{z})\|^2\\
=&\|\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})\|^2+\left\|\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{y})-\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{z})\right\|^2\\
&-2\left\langle\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z}),\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{y})-\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{z})\right\rangle\\
\le&\|\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})\|^2\\
&+\left\langle M\circ(\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})),\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{y})-\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{z})\right\rangle\\
&-2\left\langle\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z}),\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{y})-\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{z})\right\rangle\\
=&\|\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})\|^2\\
&+(\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z}))^TN\left(\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{y})-\begin{bmatrix}J_C\\J_C\end{bmatrix}\circ M\circ\boldsymbol{U}(\boldsymbol{z})\right)\\
=&\|\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})\|^2
\end{aligned}$$

The inequality follows from firm-nonexpansiveness of J_C. Next we have

$$\begin{aligned}
&\|\boldsymbol{U}(\boldsymbol{y})-\boldsymbol{U}(\boldsymbol{z})\|^2\\
=&\|\boldsymbol{y}-\boldsymbol{z}\|^2+\left\|\begin{bmatrix}J_A(y_1)-J_A(z_1)\\J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\end{bmatrix}\right\|^2\\
&-2\left\langle\boldsymbol{y}-\boldsymbol{z},\begin{bmatrix}J_A(y_1)-J_A(z_1)\\J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\end{bmatrix}\right\rangle\\
\le&\|\boldsymbol{y}-\boldsymbol{z}\|^2\\
&+\left\langle\begin{bmatrix}y_1-z_1\\y_2+J_A(y_1)-z_2+J_A(z_1)\end{bmatrix},\begin{bmatrix}J_A(y_1)-J_A(z_1)\\J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\end{bmatrix}\right\rangle\\
&-2\left\langle\boldsymbol{y}-\boldsymbol{z},\begin{bmatrix}J_A(y_1)-J_A(z_1)\\J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\end{bmatrix}\right\rangle\\
\le&\|\boldsymbol{y}-\boldsymbol{z}\|^2-\|J_A(y_1)-J_A(z_1)\|^2-\|J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\|^2\\
&+2\langle J_A(y_1)-J_A(z_1),J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\rangle\\
=&\|\boldsymbol{y}-\boldsymbol{z}\|^2-\|J_A(y_1)-J_A(z_1)+J_B(y_2+J_A(y_1))-J_B(z_2+J_A(z_1))\|^2\\
\le&\|\boldsymbol{y}-\boldsymbol{z}\|^2
\end{aligned}$$

Both inequalities follows from firm-nonexpansiveness of J_A and J_B. □

4.2 Numerical examples

Whether the splitting of Theorem 5 is fast or efficient is somewhat besides the point, as the purpose of Theorem 5 is to establish attainment of minimal lifting. Nevertheless, we run 2 computational experiments to establish that the splitting of Theorem 5 is, at least in some cases, competitive with or even better than the existing approach of copying variables and applying DRS.

Portfolio optimization. Consider the Markowitz portfolio optimization [5] problem

$$\begin{array}{ll}\underset{x\in\mathbb{R}^d}{\text{minimize}} & (1/n)\sum_{i=1}^n(a_i^Tx-b)^2\\ \text{subject to} & x\in\Delta\\ & \mu^Tx\ge b,\end{array}$$

where d is the number of assets, $a_1,\dots,a_n\in\mathbb{R}^d$ are n realizations of the returns on the assets, $\Delta=\{x\in\mathbb{R}^d\,|\,x_1,\dots,x_d\ge 0,\,x_1+\cdots+x_d=1\}$ is the standard simplex for portolios with no short positions, $\mu\in\mathbb{R}^d$

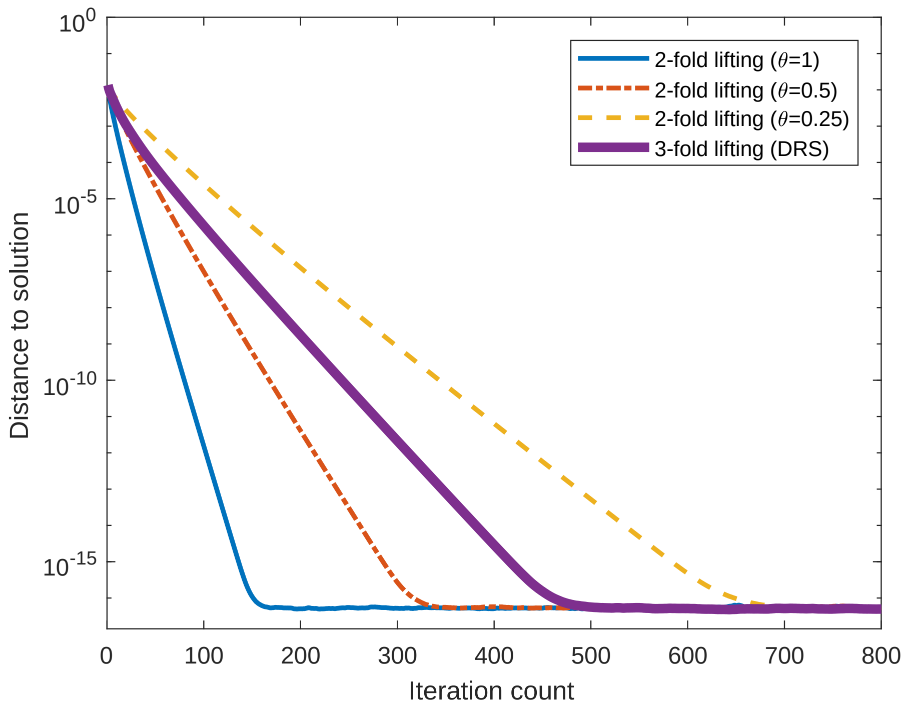

Figure 1: Distance to solution vs. iterations for the portfolio optimization problem. Theorem 5 proves convergence for $\theta \in (0, 1)$.

is the (estimated) average return of the assets, and $b \in \mathbb{R}$ is the desired expected return. We reformulate this problem as

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} \quad \underbrace{\frac{1}{n}\sum_{i=1}^{n}(a_i^T x - b)^2}_{=f(x)} + \underbrace{\delta_\Delta(x)}_{=g(x)} + \underbrace{\delta_{\{x \,|\, \mu^T x \geq b\}}(x)}_{=h(x)}$$

and apply the splittings of (4) and of Theorem 5 with $A = \partial f$, $B = \partial g$, and $C = \partial h$. To compute the projection onto the simplex, we use the algorithm and code of [9]. For the experiments, we used synthetic data with $n = 30000$ and $d = 10000$, which make the data approximately 2GB in size. The code for data generation and optimization is provided in the author's website for scientific reproducibility.

Figure 1 shows the results. We can see that the splitting of Theorem 5, which uses 2-fold lifting, is competitive with and even better than the splitting of (4), which derives from DRS and uses 3-fold lifting. For both splittings, α was roughly tuned for best performance.

Fused lasso. Consider the fused lasso [27, 24, 28] problem

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} \quad (\lambda/(2n))\|Ax - b\|^2 + \textstyle\sum_{i=1}^{d-1} |x_{i+1} - x_i|$$

where $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, and $\lambda > 0$. For simplicity, assume d is odd. We reformulate this problem as

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} \quad \underbrace{\frac{\lambda}{2n}\|Ax - b\|^2}_{=f(x)} + \underbrace{\sum_{i=1,3,\ldots,d-2} |x_{i+1} - x_i|}_{=g(x)} + \underbrace{\sum_{i=2,4,\ldots,d-1} |x_{i+1} - x_i|}_{=h(x)}$$

and apply the splittings of (4) and of Theorem 5 with $A = \partial f$, $B = \partial g$, and $C = \partial h$. For the experiments, we used synthetic data with $n = 3000$, $d = 1001$, and $\lambda = 10$. The code for data generation and optimization is provided in the author's website for scientific reproducibility.

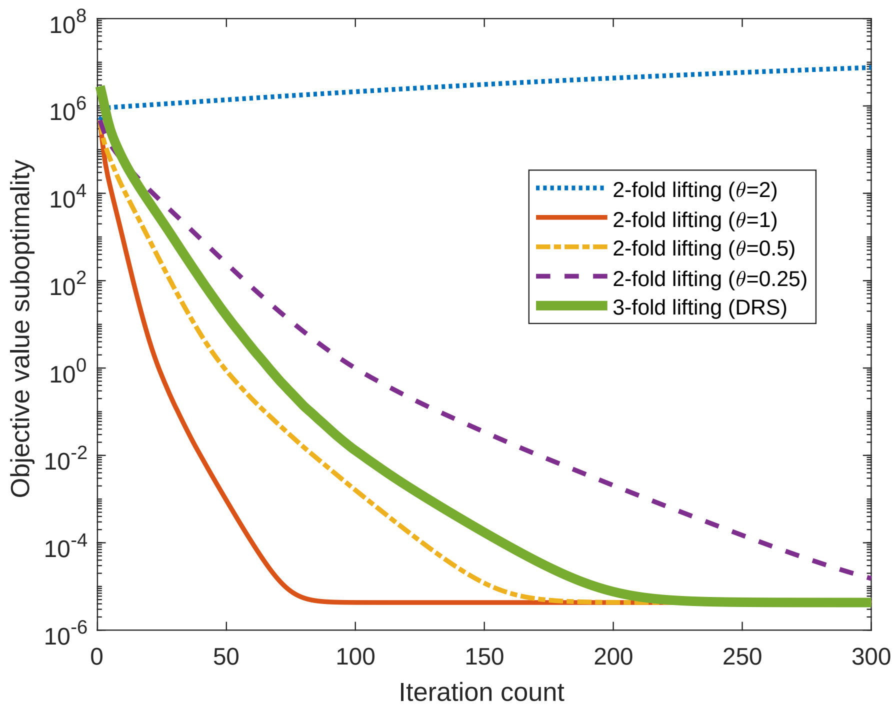

Figure 2: Suboptimality of function value vs. iterations for the fused lasso problem. Theorem 5 proves convergence for $\theta \in (0, 1)$. Indeed, the splitting diverges for $\theta = 2$.

Figure 2 shows the results. We can see that the splitting of Theorem 5, which uses 2-fold lifting, is competitive with and even better than the splitting of (4), which derives from DRS and uses 3-fold lifting. For both splittings, α was roughly tuned for best performance.

5 Conclusion

This work establishes that DRS is the unique frugal, unconditionally convergent resolvent-splitting without lifting for the 2 operator problem and that there is no resolvent-splitting without lifting for the 3 operator problem. Furthermore, this work presents a novel, frugal, unconditionally convergent resolvent-splitting for the 3 operator problem that directly generalizes DRS. This splitting proves that 2-fold lifting is the minimal lifting necessary for the 3 operator problem. In other words, the presented splitting is optimal in terms of frugality and lifting.

The potential for future work based on the ideas presented in this work is large. Analyzing and establishing uniqueness or optimality of other splittings is one direction of future work. Characterizing all splittings of a given setup is another. In particular, there is no reason to believe the splitting of Theorem 5 is unique, so characterizing all frugal, unconditionally convergent resolvent-splittings for the 3 operator problem would be interesting.

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