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Bounds in multi-horizon stochastic programs

Francesca Maggioni · Elisabetta Allevi · Asgeir Tomasgard

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Abstract In this paper, we present bounds for *multi-horizon stochastic optimization problems*, a class of problems introduced in [16] relevant in many industry-life applications tipically involving strategic and operational decisions on two different time scales.

After providing three general mathematical formulations of a multi-horizon stochastic program, we extend the definition of the traditional *Expected Value* problem and *Wait-and-See* problem from stochastic programming in a multi-horizon framework. New measures are introduced allowing to quantify the importance of the uncertainty at both strategic and operational levels. Relations among the solution approaches are then determined and chain of inequalities provided. Numerical experiments on a real-life application from energy planning are finally presented.

Keywords bounds · multi-horizon · stochastic programs · energy · strategic decisions · operational decisions

F. Maggioni
Department of Management, Economics and Quantitative Methods, University of Bergamo, Via dei Caniana 2, 24127 Bergamo ITALY E-mail: francesca.maggioni@unibg.it

E. Allevi
Department of Economics and Management, Brescia University, Contrada S. Chiara 50, Brescia 25122, ITALY
E-mail: allevi@eco.unibs.it

A. Tomasgard
Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, Høgskoleringen 1, 7491 Trondheim, NORWAY
E-mail: asgeir.tomasgard@iot.ntnu.no

1 Introduction

Many real-life problems of industries with large capital investments and infrastructure planning, need to combine decisions at long and short-term time scales, which are both typically affected by uncertainty. Long-term uncertainty includes for example costs for infrastructure elements with time horizons of many years. Short-term uncertainty comprises for instance daily variations in demand or prices. In optimization models, this uncertainty can be handled by applying stochastic programming methodology, where the uncertain parameters are represented by discrete values in scenario trees for possible future realizations of the parameters. Computational tractability is usually of great concern in such models, because variables and constraints are duplicated for each scenario in the scenario tree. If all short-term variations are to be included, the scenario tree and, the optimization model will become intractable due to the exponential growth in problem size.

In order to cope with this issue, an alternative formulation of the problem that combines the two time scales has been introduced in [10, 16] using the so-called *multi-horizon* approach. Partitioning the corresponding decision variables in *strategic* and *operational*, the authors showed that the new approach drastically reduces the model size compared to the traditional formulation. This is based on the observation that strategic decisions typically do not depend directly on a particular operational scenario, bur rather on the overall operational performance between two subsequent strategic stages. The multi-horizon approach has been successfully adopted in the literature in order to address real life problems involving different time scales. In particular in [32], it has been adopted for the European power system to handle the challenges related to intermittent energy production and stochastic energy demand in a long-term investment model, avoiding the curse of dimensionality of the traditional stochastic approaches. In [34] a multi-stage multi-horizon stochastic equilibrium model of multi-fuel energy markets has been developed for analyzing energy markets, with particular attention to infrastructure development and renewable energy policies in perfect and imperfect market structures. By decoupling short-term operational decisions feedback from long-term strategic investment decisions, the authors show that the multi-horizon approach allows to consider long-term and short-term uncertainties while maintaining favorable computational complexity. In [35] properties of the risk measures such as time consistency in multi-horizon scenario trees has been investigated and illustrated with a stylized example. Other papers in the literature considering aspects related to handling both time scales in one model are given by [31] who include short-term variations in a strategic model for the Norvegian meat industry, [33] analyze strategic investment decisions in liquefied natural gas transport and discuss the impact of using a stochastic model at the operational level showing that also operational flexibility is important in order to cope with short-term variations and has a significant impact on profitability. In [1, 2] the multi-horizon modeling approach has been applied to a complex pumped storage hydro power plant in a liberalized market environment in order to give decision support for its scheduling. In [28] the authors analyze the short-term uncertainty in long-term energy system models considering a wind power case-study in Denmark and in [29] they consider the impact of policy actions and future energy prices on the cost-optimal development of the energy system in Norway and Sweden.

Clearly, due to the large number of decision variables involved in both the decision scales, approximation techniques which provide lower and upper bounds to the optimal value for multi-horizon problems can be very useful in practice. In this situation easy-to-compute bounds and approximations, by solving much smaller problems instead of the big one associated to the large discrete multi-horizon scenario tree are desirable.

Several bounding techniques are proposed for the *traditional* two-stage and multistage stochastic programs with expectation. In the two-stage case, some of them (see for instance [8, 9, 11, 12, 21]) generalize Jensen's inequality [15] for lower bounding and the Edmundson-Madansky [7, 19, 20] inequality for upper bounding. An alternative method is to aggregate constraints and variables in the extensive-form and solve the resulting problem [5, 26]. Other bounds were introduced in [4] by solving pairs of sub-problems which are much less complex than the general recourse problem and then extended in [27] by considering an alternative way of forming the group sub-problems and merging their results. Notice that scenario grouping approach have been also applied in [6] for chance constrained programs.

Bounds for multistage linear stochastic programs were for the first time proposed in [22], by solving pair sub-problems, by measuring the quality of the deterministic solution and by introducing rolling horizon measures. In [23] the authors extends the bounding approach of [4, 22, 27], for stochastic multistage mixed integer linear programs, solving a sequence of group sub-problems made by a subset of reference scenarios, and a subset of scenarios from the finite support. Besides, in [24] bounds for multistage convex problems with concave risk functionals as objective are provided. In [18] the author elaborates an approximation scheme which integrates stage-aggregation and discretization through coarsening of sigma-algebras to ensure computational tractability, while providing deterministic error bounds.

An alternative approach is to construct two approximating trees, a lower tree and an upper tree, the solution of which lead to upper and lower bounds for the optimal value of the original *continuous* problem. The advantage of this approach is that it generates intervals in which the optimal value lies under guarantee. Results in this direction were for the first time obtained by Frauendorfer [13], followed by [14, 17]. In [17] barycentric discretizations are adopted in a more general setting investigating convex multistage stochastic programs with a generalized non-convex dependence on the random variables. In [25], the authors generalize the bounding ideas of [13, 14, 17] to not necessarily Markovian scenario processes and derive valid lower and upper bounds for the convex case. They construct new discrete probability measures directly from the simulated data of the whole scenario process.

In this paper we provide three general mathematical formulations for the *Multi-Horizon Stochastic Program*, $MHSP$, and new lower and upper bounds from traditional stochastic programs to multi-horizon stochastic formulation. Clearly the strategic and operational decision variables distinction, implies the need of new methods and measures which allow to quantify the importance of the uncertainty at both the decision levels. In particular we extend the definition of the traditional *Expected Value* Problem EV, by introducing the *Multi-Horizon Expected Value* problem, $MHEV$, and the new concept *Multi-Horizon Operative Expected Value* problem, $MHOEV$, obtained by replacing operational uncertain parameters with their expected values. Moreover, the *Multi-Horizon Wait-and-See*, $MHWS$, and the *Multi-Horizon Strategic Wait-and-See*, $MHSWS$, which is obtained by relaxing the nonanticipativity constraints of the strategic decision variables, are proposed. $MHEV$ and $MHOEV$ are compared to $MHSP$, through the *Value of Strategic Decision*, VSD and *Value of Strategic and Operational Decision*, $VSOD$ allowing to quantify the importance of the uncertainty at both strategic and operational levels. Relations among the solution approaches are then determined and analytically proved. Finally, numerical experiments on a real-life application from energy planning inspired from [16] are presented.

The paper is organized as follows: Section 2 introduces the notation and basic definitions. Section 3 introduces the new bounds for multi-horizon stochastic programs and chain of inequalities among them. Section 4 reports numerical results on an energy planning problem. Conclusions follows.

2 Multi-horizons stochastic programs

In this section we propose a general model of multi-horizon stochastic programs introduced in [16]. To the best of our knowledge, this is the first general mathematical formulation for this class of problems.

Let $\mathcal{H} = \{1, \dots, H\}$ be the set of strategic stages and $\mathcal{T}_t = \{1, \dots, O_t\}$, be the set of operational stages at strategic time $t \in \mathcal{H}$. The following definition provides the nested formulation of a *multi-horizon linear mixed-integer stochastic program* in which a decision maker has to take a sequence of strategic decisions $x_1, \dots, x_H$ for long-term planning, and operational decisions $y_t^1, \dots, y_t^{O_t}$, $t \in \mathcal{H}$, for short-term planning, to minimize expected costs.

Definition 2.1 A *Multi-Horizon Stochastic Program, MHSP,* is defined as follows:

$$MHSP := \min_{\mathbf{x},\mathbf{y}} \mathbb{E}_{\boldsymbol{\xi}_{H-1},\boldsymbol{\eta}_1^{O_1},\ldots,\boldsymbol{\eta}_H^{O_H}} z(\mathbf{x},\mathbf{y},\boldsymbol{\xi}_{H-1},\boldsymbol{\eta}_1^{O_1},\ldots,\boldsymbol{\eta}_H^{O_H}) = \quad (1)$$

$$\min_{x_1} c_1 x_1 + \mathbb{E}_{\eta_1^1}\Big[\min_{y_1^1} q_1^1(\boldsymbol{\eta}_1^1)y_1^1(\boldsymbol{\eta}_1^1) + \cdots + \mathbb{E}_{\eta_1^{O_1}}\big[\min_{y_1^{O_1}} q_1^{O_1}(\boldsymbol{\eta}_1^{O_1})y_1^{O_1}(\boldsymbol{\eta}_1^{O_1})\big]\Big] +$$

$$\mathbb{E}_{\xi_1}\bigg[\min_{x_2} c_2(\boldsymbol{\xi}_1)x_2(\boldsymbol{\xi}_1) + \mathbb{E}_{\eta_2^1}\Big[\min_{y_2^1} q_2^1(\boldsymbol{\eta}_2^1)y_2^1(\boldsymbol{\eta}_2^1) + \ldots + \mathbb{E}_{\eta_2^{O_2}}\big[\min_{y_2^{O_2}} q_2^{O_2}(\boldsymbol{\eta}_2^{O_2})y_2^{O_2}(\boldsymbol{\eta}_2^{O_2})\big]\Big]\bigg]$$

$$+ \cdots + \mathbb{E}_{\xi_{H-1}}\bigg[\min_{x_H} c_H(\boldsymbol{\xi}_{H-1})x_H\big(\boldsymbol{\xi}_{H-1}\big) +$$

$$\mathbb{E}_{\eta_H^1}\Big[\min_{y_H^1} q_H^1(\boldsymbol{\eta}_H^1)y_H^1(\boldsymbol{\eta}_H^1) + \cdots + \mathbb{E}_{\eta_H^{O_H}}\big[\min_{y_H^{O_H}} q_H^{O_H}(\boldsymbol{\eta}_H^{O_H})y_H^{O_H}(\boldsymbol{\eta}_H^{O_H})\big]\Big]\bigg]$$

$$\begin{aligned}
&\text{s.t. } Ax_1 = h_1,\\
&T_2(\boldsymbol{\xi}_1)x_1 + W_2(\boldsymbol{\xi}_1)x_2(\boldsymbol{\xi}_1) = h_2(\boldsymbol{\xi}_1),\\
&T_t(\boldsymbol{\xi}_{t-1})x_{t-1}(\boldsymbol{\xi}_{t-2}) + W_t(\boldsymbol{\xi}_{t-1})x_t(\boldsymbol{\xi}_{t-1}) = h_t(\boldsymbol{\xi}_{t-1}),\ t \in \mathcal{H}\setminus\{1,2\},\\
&T_1^1(\boldsymbol{\eta}_1^1)x_1 + W_1^1(\boldsymbol{\eta}_1^1)y_1^1(\boldsymbol{\eta}_1^1) = h_1^1(\boldsymbol{\eta}_1^1),\\
&T_t^1(\boldsymbol{\eta}_t^1)x_t(\boldsymbol{\xi}_{t-1}) + W_t^1(\boldsymbol{\eta}_t^1)y_t^1(\boldsymbol{\eta}_t^1) = h_t^1(\boldsymbol{\eta}_t^1), \qquad t \in \mathcal{H}\setminus\{1\},\\
&T_t^\tau(\boldsymbol{\eta}_t^\tau)y_t^{\tau-1}(\boldsymbol{\eta}_t^{\tau-1}) + W_t^\tau(\boldsymbol{\eta}_t^\tau)y_t^\tau(\boldsymbol{\eta}_t^\tau) = h_t^\tau(\boldsymbol{\eta}_t^\tau), \qquad \tau \in \mathcal{T}_t\setminus\{1\},\ t \in \mathcal{H},\\
&x_t \in \mathbb{R}_+^{n_t-d_t} \times \mathbb{N}^{d_t}, \qquad t \in \mathcal{H},\\
&y_t^\tau \in \mathbb{R}_+^{n_t^\tau-d_t^\tau} \times \mathbb{N}^{d_t^\tau}, \qquad t \in \mathcal{H}, \tau \in \mathcal{T}_t,
\end{aligned}$$

where strategic parameters $c_1 \in \mathbb{R}^{n_1}$, $h_1 \in \mathbb{R}^{m_1}$ and the matrix $A \in \mathbb{R}^{m_1 \times n_1}$ are known. Additionally, let us denote the uncertain strategic parameters vectors and matrices at strategic time $t \in \mathcal{H}\setminus\{1\}$: $c_t \in \mathbb{R}^{n_t}, h_t \in \mathbb{R}^{m_t}$, $T_t \in \mathbb{R}^{m_t \times n_{t-1}}$, $W_t \in \mathbb{R}^{m_t \times n_t}$. We assume that $T_1 = A$ and $W_1 = 0$ (i.e. the null matrix). The strategic uncertainty is described by the random process ξ^t, $t = 1,\ldots,H-1$, which is revealed gradually over time in $H-1$ periods and with $\boldsymbol{\xi}_t := (\xi_1,\ldots,\xi_t)$, $t = 1,\ldots,H-1$ we denote the history of the process up to time t. ξ_t is defined on a probability space $(\Xi_t, \mathscr{A}_t, p)$ with support $\Xi_t \in \mathbb{R}^{n_t}$ and given probability distribution p on the $\sigma-$algebra $\mathscr{A}_t$ (with $\mathscr{A}_t \subseteq \mathscr{A}_{t+1}$) and $\mathbb{E}_{\xi_t}$ denotes the expectation with respect to ξ_t. In general we have $c_t = c_t(\boldsymbol{\xi}_{t-1}),\ h_t = h_t(\boldsymbol{\xi}_{t-1})$, $T_t = T_t(\boldsymbol{\xi}_{t-1})$, $W_t = W_t(\boldsymbol{\xi}_{t-1})$, for $t \in \mathcal{H}\setminus\{1\}$.

The operational uncertainty after the strategic decision at time $t \in \mathcal{H}$ is taken, is described by the random process η_t^τ, $\tau = 1,\ldots,O_t$, revealed gradually in O_t operational periods. With $\boldsymbol{\eta}_t^\tau := (\eta_t^1,\ldots,\eta_t^\tau)$, $\tau = 1,\ldots,O_t$ we denote the history of the process up to time τ at strategic time $t \in \mathcal{H}$. Notice that $\boldsymbol{\eta}_{t+1}^\tau$ is independent by $\boldsymbol{\eta}_t^\tau$, $\tau \in \mathcal{T}_t$ $t \in \mathcal{H}$. Let us denote the uncertain operational parameters vectors and matrices at operational stage $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$, with $q_t^\tau \in \mathbb{R}^{n_t^\tau}$, $h_t^\tau \in \mathbb{R}^{m_t^\tau}$, $T_t^1 \in \mathbb{R}^{m_t^1 \times n_t}$, $T_t^\tau \in \mathbb{R}^{m_t^\tau \times n_t^{\tau-1}}$, and $W_t^\tau \in \mathbb{R}^{m_t^\tau \times n_t^\tau}$. In general we have $q_t^\tau = q_t^\tau(\boldsymbol{\eta}_t^\tau)$, $h_t^\tau = h_t^\tau(\boldsymbol{\eta}_t^\tau)$, $T_t^\tau = T_t^\tau(\boldsymbol{\eta}_t^\tau)$, $W_t^\tau(\boldsymbol{\eta}_t^\tau)$, $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$.

The decisions variables are partitioned as follows:

1. $\mathbf{x} := \{x_t | t \in \mathcal{H}\}$: strategic decision variables, with $x_t \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}$, $t \in \mathcal{H}$.
2. $\mathbf{y} := \{y_t^\tau | \tau \in \mathcal{T}_t, t \in \mathcal{H}\}$: operational decision variables with $y_t^\tau \in \mathbb{R}_+^{n_t^\tau - d_t^\tau} \times \mathbb{N}^{d_t^\tau}, \tau \in \mathcal{T}_t,\ t \in \mathcal{H}$.

The decision processes x_t, y_t^τ, $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$, are both *nonanticipative*, which means they depend on the information respectively up to time t and up to times t and τ.

The problem expressed in (1) can be formulated through the corresponding dynamic programming equations (see [30]). This formulation takes into account that at the last operational and strategic stage, the values of all the problem data $\boldsymbol{\xi}_{H-1}$ and $\boldsymbol{\eta}_H^{O_H}$ are already known and the values of the earlier strategic decisions $x_1, x_2, \ldots, x_H$ and operational ones $y_t^1, x_t^2, \ldots, y_t^{H-1}$, $t \in \mathcal{H}$, have been already chosen. The problem therefore becomes

$$\begin{aligned} &Q_H^{O_H}(y_H^{O_H-1}, \eta_H^{O_H}) := \qquad (2)\\ &\min_{y_H^{O_H}} q_H^{O_H}(\eta_H^{O_H}) y_H^{O_H}(\eta_H^{O_H})\\ &\quad \text{s.t. } T_H^{O_H}(\eta_H^{O_H}) y_H^{O_H-1}(\eta_H^{O_H-1}) + W_H^{O_H}(\eta_H^{O_H}) y_H^{O_H}(\eta_H^{O_H}) = h_H^{O_H}(\eta_H^{O_H}) \ ,\\ &\qquad y_H^{O_H}(\eta_H^{O_H}) \in \mathbb{R}_+^{n_H^{O_H} - d_H^{O_H}} \times \mathbb{N}^{d_H^{O_H}} \ . \end{aligned}$$

The solution value of problem (2), depends on the operational decision $y_H^{O_H-1}$ at the previous stage and on the realization of the operational data process $\eta_H^{O_H}$. Problem (1) is then solved recursively computing the operational *cost-to-go* functions $Q_H^\tau(y_H^{\tau-1}, \boldsymbol{\eta}_H^\tau)$, going backward in the operational stages. At stage $t = H$ and $\tau = 2, \ldots, O_H - 1$ the operational problem is formulated as follows:

$$\begin{aligned} &Q_H^\tau(y_H^{\tau-1}, \boldsymbol{\eta}_H^\tau) := \qquad (3)\\ &\min_{y_H^\tau} q_H^\tau(\boldsymbol{\eta}_H^\tau) y_H^\tau(\boldsymbol{\eta}_H^\tau) + \mathbb{E}_{\eta_H^{\tau+1}}[Q_H^{\tau+1}(y_H^\tau, \boldsymbol{\eta}_H^{\tau+1})]\\ &\text{s.t. } T_H^\tau(\boldsymbol{\eta}_H^\tau) y_H^{\tau-1}(\boldsymbol{\eta}_H^{\tau-1}) + W_H^\tau(\boldsymbol{\eta}_H^\tau) y_H^\tau(\boldsymbol{\eta}_H^\tau) = h_H^\tau(\boldsymbol{\eta}_H^\tau), \quad \tau = 2, \ldots, O_H - 1,\\ &\quad y_H^\tau(\boldsymbol{\eta}_H^\tau) \in \mathbb{R}_+^{n_H^\tau - d_H^\tau} \times \mathbb{N}^{d_H^\tau} \qquad \tau = 2, \ldots, O_H - 1. \end{aligned}$$

When $t = H$ and $\tau = 1$ we have:

$$\begin{aligned} Q_H^1(x_H, \boldsymbol{\eta}_H^1) := \min_{y_H^1} \ & q_H^1(\boldsymbol{\eta}_H^1) y_H^1(\boldsymbol{\eta}_H^1) + \mathbb{E}_{\eta_H^2}[Q_H^2(y_H^1, \boldsymbol{\eta}_H^2)] \qquad (4)\\ &\text{s.t. } T_H^1(\boldsymbol{\eta}_H^1) x_H(\boldsymbol{\xi}^{H-1}) + W_H^1(\boldsymbol{\eta}_H^1) y_H^1(\boldsymbol{\eta}_H^1) = h_H^1(\boldsymbol{\eta}_H^1),\\ & x_H \in \mathbb{R}_+^{n_H - d_H} \times \mathbb{N}^{d_H},\\ & y_H^1 \in \mathbb{R}_+^{n_H^1 - d_H^1} \times \mathbb{N}^{d_H^1}, \end{aligned}$$

and then the strategic cost-to-go function at $t = H$ becomes:

$$\begin{aligned} Q_H(x_{H-1}, \boldsymbol{\xi}_{H-1}) := \min_{x_H} \ & c_H(\boldsymbol{\xi}_{H-1}) x_H(\boldsymbol{\xi}_{H-1}) + \mathbb{E}_{\eta_H^1}[Q_H^1(x_H, \boldsymbol{\eta}_H^1)] \qquad (5)\\ &\text{s.t. } T_H(\boldsymbol{\xi}_{H-1}) x_{H-1}(\boldsymbol{\xi}_{H-2}) + W_H(\boldsymbol{\xi}_{H-1}) x_H(\boldsymbol{\xi}_{H-1}) = h_H(\boldsymbol{\xi}_{H-1}),\\ & x_H \in \mathbb{R}_+^{n_H - d_H} \times \mathbb{N}^{d_H}. \end{aligned}$$

Similarly when $t = 2, \dots, H-1$ and $\tau = 1$, problem (4) becomes

$$\begin{aligned} Q_t^1(x_t, \boldsymbol{\eta}_t^1) := \min_{y_t^1} \; & q_t^1(\boldsymbol{\eta}_t^1) y_t^1(\boldsymbol{\eta}_t^1) + \mathbb{E}_{\eta_t^2}[Q_t^2(y_t^1, \boldsymbol{\eta}_t^2)] \\ \text{s.t. } & T_t^1(\boldsymbol{\eta}_t^1) x_t(\boldsymbol{\xi}_{t-1}) + W_t^1(\boldsymbol{\eta}_t^1) y_t^1(\boldsymbol{\eta}_t^1) = h_t^1(\boldsymbol{\eta}_t^1), \\ & x_t \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}, \\ & y_t^1 \in \mathbb{R}_+^{n_t^1 - d_t^1} \times \mathbb{N}^{d_t^1}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} Q_t^2(y_t^1, \boldsymbol{\eta}_t^2) := \min_{y_t^2} \; & q_t^2(\boldsymbol{\eta}_t^2) y_t^2(\boldsymbol{\eta}_t^2) + \mathbb{E}_{\eta_t^3}[Q_t^3(y_t^2, \boldsymbol{\eta}_t^3)] \\ \text{s.t. } & T_t^2(\boldsymbol{\eta}_t^2) y_t^1(\boldsymbol{\eta}_t^1) + W_t^2(\boldsymbol{\eta}_t^2) y_t^2(\boldsymbol{\eta}_t^2) = h_t^2(\boldsymbol{\eta}_t^2), \\ & y_t^2(\boldsymbol{\eta}_t^2) \in \mathbb{R}_+^{n_t^2 - d_t^2} \times \mathbb{N}^{d_t^2}, \end{aligned} \tag{7}$$

and then the strategic cost-to-go function at stage $t = 2, \dots, H-1$ becomes:

$$\begin{aligned} Q_t(x_{t-1}, \boldsymbol{\xi}_{t-1}) := \min_{x_t} \; & c_t(\boldsymbol{\xi}_{t-1}) x_t(\boldsymbol{\xi}_{t-1}) + \mathbb{E}_{\eta_t^1}[Q_t^1(x_t, \boldsymbol{\eta}_t^1)] \\ \text{s.t. } & T_t(\boldsymbol{\xi}_{t-1}) x_{t-1}(\boldsymbol{\xi}_{t-2}) + W_t(\boldsymbol{\xi}_{t-1}) x_t(\boldsymbol{\xi}_{t-1}) = h_t(\boldsymbol{\xi}_{t-1}), \\ & x_t \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}. \end{aligned} \tag{8}$$

On top of all these problems we have to find the first decision variable x_1, as the solution of the model

$$\begin{aligned} \min_{x_1} \; & c_1 x_1 + \mathbb{E}_{\eta_1^1}\left[Q_1^1(x_1, \eta_1^1)\right] + \mathbb{E}_{\xi_1}\left[Q_2(x_1, \xi_1)\right] \\ \text{s.t. } & A x_1 = h_1 \;, \\ & x_1 \in \mathbb{R}_+^{n_1 - d_1} \times \mathbb{N}^{d_1} \;. \end{aligned} \tag{9}$$

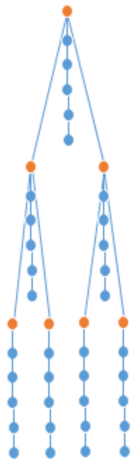
where $Q_1^1(x_1, \eta_1^1)$ is the same of (7) with $t = 1$ and $\tau = 1$.

2.1 Scenario tree approximations for multi-horizon stochastic programs

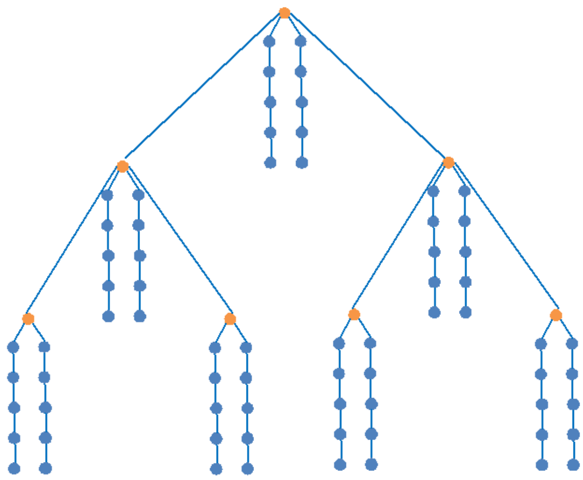
In order to proceed with numerical computations, it is useful to have a discretization of the underlying random process. This is obtained by considering a finite number of realizations of the random processes $\xi_1, \dots, \xi_{H-1}$ and $\eta_t^1, \dots, \eta_t^{O_t}$, $t \in \mathcal{H}$.

The information structure at both the strategic and operational levels can be described in the form of a *multi-horizon scenario tree* $\mathscr{T}$ where at each strategic stage $t \in \mathcal{H}$ there is a discrete number of strategic nodes where a specific realization of the uncertain parameters at strategic level takes place. Let $\mathcal{N}_t$ be the set of ordered strategic nodes at stage $t = 1, \dots, H$ and $\mathcal{N} = \bigcup_{t=1}^{H} \mathcal{N}_t$. In order to identify the operational uncertainty we now consider proper operational sub-trees in each strategic node $\ell \in \mathcal{N}_t$: let $\mathcal{S}$ be the set of possible

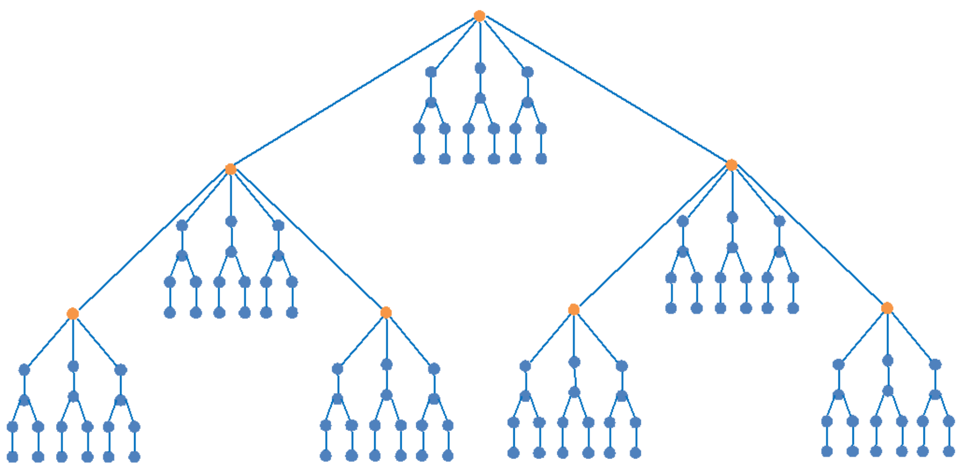
(a) Multi-Horizon Expected Value ($MHEV$) scenario tree structure.



(b) Multi-Horizon scenario tree structure with one operational scenario for each strategic node. This is the tree structure associated to the Multi-Horizon Operative Expected Value ($MHOEV$).



(c) Multi-Horizon scenario tree structure with two operational scenarios for each strategic node.



(d) Multi-Horizon scenario tree structure with proper multi-stage stochastic programming operational model for each strategic node. This is the typical tree structure associated to the Multi-Horizon Stochastic Program ($MHSP$).

Fig. 1 Examples of multi-horizons scenario trees (strategic nodes in red and operational nodes in blue).

operational scenarios. Examples of multi-horizons scenario trees with different operational sub-models for each strategic node are showed in Figure 1.

Another formulation of the *Multi-Horizon Stochastic Program* 1, based on a multi-horizon scenario tree (see Figure 1(d)) and node notation, is given as follows:

$$
\begin{aligned}
MHSP := \min_{\mathbf{x},\mathbf{y}} & \sum_{t=1}^{H} \sum_{\ell \in \mathcal{N}_t} \pi_\ell (c_\ell x_\ell + \sum_{s \in \mathcal{S}_t} w_\ell^s \sum_{\tau \in \mathcal{T}_t} q_\ell^{s,\tau} y_\ell^{s,\tau}) \\
\text{s.t.} \;\; & A x_\ell = h_\ell, \qquad \ell \in \mathcal{N}_1, \\
& T_\ell x_{a(\ell)} + W_\ell x_\ell = h_\ell, \qquad \ell \in \mathcal{N}_t,\ t \in \mathcal{H} \backslash \{1\}, \\
& T_\ell^{s,1} x_\ell + W_\ell^{s,1} y_\ell^{s,1} = h_\ell^{s,1}, \qquad \ell \in \mathcal{N}_t,\ s \in \mathcal{S}_t,\ t \in \mathcal{H}, \\
& T_\ell^{s,\tau} y_\ell^{s,\tau-1} + W_\ell^{s,\tau} y_\ell^{s,\tau} = h_\ell^{s,\tau}, \qquad \ell \in \mathcal{N}_t,\ \tau \in \mathcal{T}_t \backslash \{1\},\ s \in \mathcal{S}_t,\ t \in \mathcal{H}, \\
& x_\ell \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}, \qquad \ell \in \mathcal{N}_t, t \in \mathcal{H}, \\
& y_\ell^{s,\tau} \in \mathbb{R}_+^{n_t^\tau - d_t^\tau} \times \mathbb{N}^{d_t^\tau}, \qquad \ell \in \mathcal{N}_t,\ \tau \in \mathcal{T}_t,\ s \in \mathcal{S}_t,\ t \in \mathcal{H},
\end{aligned}
\tag{10}
$$

where $c_\ell \in \mathbb{R}^{n_t}, h_\ell \in \mathbb{R}^{m_t}$, $T_\ell \in \mathbb{R}^{m_t \times n_{t-1}}$, $W_\ell \in \mathbb{R}^{m_t \times n_t}$ be vectors and matrices at strategic node $\ell \in \mathcal{N}_t$, $t \in \mathcal{H} \backslash \{1\}$. If $\ell \in \mathcal{N}_1$ we assume $T_\ell = A$, $W_\ell = 0$ (i.e., the null matrix), $c_\ell \in \mathbb{R}^{n_1}$ and $h_\ell \in \mathbb{R}^{m_1}$ be known vectors.

Each strategic node at stage t, except the root, is connected to a unique node at stage $t-1$ called ancestor and to nodes at stage $t+1$ called successors. For each strategic node ℓ, we denote its ancestor with $a(\ell)$, with π_ℓ the probability of node ℓ at strategic stage t and with $\pi_{a(\ell),\ell}$ the conditional probability of the random process in node ℓ given its history up to the ancestor node ℓ. We have $\sum\limits_{\ell \in \mathcal{N}_t} \pi_\ell = 1$, $t \in \mathcal{H}$.

Operational vectors and matrices at operational stage τ, in operational scenario s derived by node $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$, are then given by $q_\ell^{s,\tau} \in \mathbb{R}^{n_t^\tau}$, $h_\ell^{s,\tau} \in \mathbb{R}^{m_t^\tau}$, and $T_\ell^{s,\tau} \in \mathbb{R}^{m_t^\tau \times n_t^{\tau-1}}$, $W_\ell^{s,\tau} \in \mathbb{R}^{m_t^\tau \times n_t^\tau}$. When $\tau = 1$ we have $T_\ell^{s,1} \in \mathbb{R}^{m_t^1 \times n_t}$. We indicate with w_ℓ^s the probability of operational scenario s derived by node ℓ. Moreover $\sum\limits_{s \in \mathcal{S}_t} w_\ell^s = 1$, $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$. The strategic decision variable is given by $\mathbf{x} := \{x_\ell \mid \ell \in \mathcal{N}_t, t \in \mathcal{H}\}$, with $x_\ell \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}$. The operational decision variable is $\mathbf{y} := \{y_\ell^{s,\tau} \mid \ell \in \mathcal{N}_t, \tau \in \mathcal{T}_t, s \in \mathcal{S}_t,\ t \in \mathcal{H}\}$, with $y_\ell^{s,\tau} \in \mathbb{R}_+^{n_t^\tau - d_t^\tau} \times \mathbb{N}^{d_t^\tau}$. In the following, for a simpler presentation, the feasibility condition on $\mathbf{x}$ and $\mathbf{y}$ will be omitted even if assumed to be satisfied.

We now introduce a second multi-horizon scenario tree notation of problem (1) based on strategic and operational scenarios: let $\mathscr{S}$ the set of strategic scenarios, denoting with $T_t^{\mathbf{s}}$ and $W_t^{\mathbf{s}}$ respectively the matrices T_ℓ and W_ℓ, operational parameters $(q_\ell^{s,\tau}, h_\ell^{s,\tau}, T_\ell^{s,\tau}, W_\ell^{s,\tau})$ with $\left(q_{\mathbf{s},t}^{s,\tau}, h_{\mathbf{s},t}^{s,\tau}, T_{\mathbf{s},t}^{s,\tau}, W_{\mathbf{s},t}^{s,\tau}\right)$, the strategic decision variables x_ℓ with $x_t^{\mathbf{s}}$, the operational decision variables $y_\ell^{s,\tau}$ with $y_{\mathbf{s},t}^{s,\tau}$, the probability w_ℓ^s with $w_{\mathbf{s},t}^s$ in strategic scenario $\mathbf{s} \in \mathscr{S}$, and the probability of the strategic scenario $\mathbf{s}$ with $\pi^{\mathbf{s}}$, the multi-horizon linear stochastic program (10) can be expressed as follows:

$$MHSP := \min_{\mathbf{x},\mathbf{y}} \sum_{t=1}^{H} \sum_{\mathbf{s}\in\mathscr{S}} \pi^{\mathbf{s}} (c_t^{\mathbf{s}} x_t^{\mathbf{s}} + \sum_{s\in\mathcal{S}_t} w_{\mathbf{s},t}^{s} \sum_{\tau\in\mathcal{T}_t} q_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau}) \tag{11}$$

$$\begin{aligned}
&\text{s.t. } \ Ax_1 = h_1, \\
&T_t^{\mathbf{s}} x_{t-1}^{\mathbf{s}} + W_t^{\mathbf{s}} x_t^{\mathbf{s}} = h_t^{\mathbf{s}}, \qquad \mathbf{s}\in\mathscr{S},\ t\in\mathcal{H}\backslash\{1\}, \\
&T_{\mathbf{s},t}^{s,1} x_t^{\mathbf{s}} + W_{\mathbf{s},t}^{s,1} y_{\mathbf{s},t}^{s,1} = h_{\mathbf{s},t}^{s,1}, \qquad \mathbf{s}\in\mathscr{S}, s\in\mathcal{S}_t,\ t\in\mathcal{H}, \\
&T_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau-1} + W_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau} = h_{\mathbf{s},t}^{s,\tau}, \qquad \mathbf{s}\in\mathscr{S},\ s\in\mathcal{S}_t,\ \tau\in\mathcal{T}_t\backslash\{1\}, t\in\mathcal{H}, \\
&x_t^{\mathbf{s}'} = x_t^{\mathbf{s}''},\ \forall \mathbf{s}',\mathbf{s}''\in\mathscr{S} \text{ for which } \mathbf{s}' = \mathbf{s}'' \text{up to strategic stage } t \\
&y_{\mathbf{s},t}^{s',\tau} = y_{\mathbf{s},t}^{s'',\tau},\ \forall s', s''\in\mathcal{S}_t \text{ for which } s' = s'' \text{up to operational stage } \tau,
\end{aligned}$$

where the nonanticipativity of strategic and operational decision variables is enforced by the last two constraints.

We note that the model (1), and its scenario-tree approximations (10) and (11) are exact if the following assumptions are satisfied:

- strategic uncertainty is independent of the operational uncertainty of previous time periods;
- strategic decisions are independent of earlier operational decisions;
- there is no connection between operational scenarios of two consecutive strategic nodes.

3 Bounds for multi-horizon stochastic programs

In this section we introduce new bounds for the multi-horizon stochastic program (1). For simplicity we adopt the scenario tree notation introduced before.

A sequence of upper bounds can be obtained by inserting feasible solutions from other problems. This is the case for the solutions obtained by the following problem:

Definition 3.1 The *Multi-Horizon Expected Value problem, MHEV,* is obtained by replacing both strategic uncertain parameters $(c_\ell, h_\ell, T_\ell, W_\ell)$, and operational uncertain parameters $(q_\ell^{s,\tau}, h_\ell^{s,\tau}, T_\ell^{s,\tau}, W_\ell^{s,\tau})$ in problem (10) with their expected values

$$\left(\sum_{\ell\in\mathcal{N}_t} \pi_\ell c_\ell, \sum_{\ell\in\mathcal{N}_t} \pi_\ell h_\ell, \sum_{\ell\in\mathcal{N}_t} \pi_\ell T_\ell, \sum_{\ell\in\mathcal{N}-t} \pi_\ell W_\ell\right) := \left(\bar{c}_t, \bar{h}_t, \bar{T}_t, \bar{W}_t\right),\ t\in\mathcal{H}\backslash\{1\},$$

and

$$\left(\sum_{s\in\mathcal{S}_t} w_\ell^s q_\ell^{s,\tau}, \sum_{s\in\mathcal{S}_t} w_\ell^s h_\ell^{s,\tau}, \sum_{s\in\mathcal{S}_t} w_\ell^s T_\ell^{s,\tau}, \sum_{s\in\mathcal{S}_t} w_\ell^s W_\ell^{s,\tau}\right) := \left(\bar{q}_t^\tau, \bar{h}_t^\tau, \bar{T}_t^\tau, \bar{W}_t^\tau\right),\ \tau\in\mathcal{T}_t,\ t\in\mathcal{H},$$

and solving the deterministic program:

$$MHEV := \min_{\mathbf{x},\mathbf{y}} \sum_{t=1}^{H} (\bar{c}_t x_t + \sum_{\tau \in \mathcal{T}_t} \bar{q}_t^\tau y_t^\tau) \tag{12}$$
$$\begin{aligned} &\text{s.t.} \;\; Ax_1 = \bar{h}_1, \\ &\bar{T}_t x_{t-1} + \bar{W}_t x_t = \bar{h}_t, \qquad t \in \mathcal{H} \setminus \{1\}, \\ &\bar{T}_t^1 x_t + \bar{W}_t^1 y_t^1 = \bar{h}_t^1, \qquad t \in \mathcal{H}, \\ &\bar{T}_t^\tau y_t^{\tau-1} + \bar{W}_t^\tau y_t^\tau = \bar{h}_t^\tau, \qquad \tau \in \mathcal{T}_t \setminus \{1\}, \; t \in \mathcal{H}. \end{aligned}$$

See Figure 1(a) for an example of scenario tree structure for problem $MHEV$. The optimal strategic solution of problem $MHEV$ is denoted with $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_H)$ and the optimal operational solution is denoted with $\bar{\mathbf{y}} = (\bar{y}_1^\tau, \ldots, \bar{y}_H^\tau)$, where $\bar{y}_t^\tau, \tau \in \mathcal{T}_t$ is the set of all deterministic operational solution derived at strategic stage $t \in \mathcal{H}$.

Definition 3.2 The *Multi-Horizon Operative Expected Value problem, MHOEV,* is obtained by replacing operational uncertain parameters $(q_\ell^{s,\tau}, h_\ell^{s,\tau}, T_\ell^{s,\tau}, W_\ell^{s,\tau})$ in problem (10) with their expected values $\left(\bar{q}_\ell^\tau, \bar{h}_\ell^\tau, \bar{T}_\ell^\tau, \bar{W}_\ell^\tau\right), \; \tau \in \mathcal{T}_t, \; \ell \in \mathcal{N}_t$, $t \in \mathcal{H}$, and solving the stochastic program:

$$MHOEV := \min_{\mathbf{x},\mathbf{y}} \sum_{t=1}^{H} \sum_{\ell \in \mathcal{N}_t} \pi_\ell (c_\ell x_\ell + \sum_{\tau \in \mathcal{T}_t} \bar{q}_\ell^\tau y_\ell^\tau) \tag{13}$$
$$\begin{aligned} &\text{s.t.} \;\; Ax_\ell = h_\ell, \qquad \ell \in \mathcal{N}_1, \\ &T_\ell x_{a(\ell)} + W_\ell x_\ell = h_\ell, \qquad \ell \in \mathcal{N}_t, \; t \in \mathcal{H} \setminus \{1\}, \\ &\bar{T}_\ell^1 x_\ell + \bar{W}_\ell^1 y_\ell^1 = \bar{h}_\ell^1, \qquad \ell \in \mathcal{N}_t, \; t \in \mathcal{H} \\ &\bar{T}_\ell^\tau y_\ell^{\tau-1} + \bar{W}_\ell^\tau y_\ell^\tau = \bar{h}_\ell^\tau, \qquad \ell \in \mathcal{N}_t, \; t \in \mathcal{H}, \; \tau \in \mathcal{T}_t \setminus \{1\}. \end{aligned}$$

which equivalently, using scenario notation, can be expressed as follows:

$$MHOEV := \sum_{t=1}^{H} \sum_{\mathbf{s} \in \mathscr{S}} \pi^{\mathbf{s}} \min_{\mathbf{x},\mathbf{y}} (c_t^{\mathbf{s}} x_t^{\mathbf{s}} + \sum_{\tau \in \mathcal{T}_t} \bar{q}_{\mathbf{s},t}^\tau y_{\mathbf{s},t}^\tau) \tag{14}$$
$$\begin{aligned} &\text{s.t.} \;\; Ax_{_1} = h_{_1}, \\ &T_t^{\mathbf{s}} x_{t-1}^{\mathbf{s}} + W_t^{\mathbf{s}} x_t^{\mathbf{s}} = h_t^{\mathbf{s}}, \qquad \mathbf{s} \in \mathscr{S}, \; t \in \mathcal{H} \setminus \{1\}, \\ &\bar{T}_{\mathbf{s},t}^1 x_t^{\mathbf{s}} + \bar{W}_{\mathbf{s},t}^1 y_{\mathbf{s},t}^1 = \bar{h}_{\mathbf{s},t}^1, \qquad \mathbf{s} \in \mathscr{S}, \; t \in \mathcal{H}, \\ &\bar{T}_{\mathbf{s},t}^\tau y_{\mathbf{s},t}^{\tau-1} + \bar{W}_{\mathbf{s},t}^\tau y_{\mathbf{s},t}^\tau = \bar{h}_{\mathbf{s},t}^\tau, \qquad \mathbf{s} \in \mathscr{S}, \; \tau \in \mathcal{T}_t \setminus \{1\}, \; t \in \mathcal{H}, \end{aligned}$$

where

$$(\sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^s q_{\mathbf{s},t}^{s,\tau}, \sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^s h_{\mathbf{s},t}^{s,\tau}, \sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^s T_{\mathbf{s},t}^{s,\tau}, \sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^s W_{\mathbf{s},t}^{s,\tau}) := \left(\bar{q}_{\mathbf{s},t}^\tau, \bar{h}_{\mathbf{s},t}^\tau, \bar{T}_{\mathbf{s},t}^\tau, \bar{W}_{\mathbf{s},t}^\tau\right),$$

for $\mathbf{s} \in \mathscr{S}, \; \tau \in \mathcal{T}_t, \; t \in \mathcal{H}$.

See Figure 1(b) for an example of scenario tree structure for problem $MHOEV$. Relaxing the nonanticipativity constraints on the strategic and operational decision variables in formulation (11), we obtain the Multi-Horizon Wait-and-See Program defined as follows:

Definition 3.3 A *Multi-Horizon Wait-and-See Program, MHWS,* is as follows:

$$MHWS := \sum_{t=1}^{H} \sum_{\mathbf{s} \in \mathscr{S}} \pi^{\mathbf{s}} \min_{\mathbf{x},\mathbf{y}} (c_t^{\mathbf{s}} x_t^{\mathbf{s}} + \sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^{s} \sum_{\tau \in \mathcal{T}_t} q_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau}) \tag{15}$$
$$\text{s.t. } Ax_1 = h_1,$$
$$T_t^{\mathbf{s}} x_{t-1}^{\mathbf{s}} + W_t^{\mathbf{s}} x_t^{\mathbf{s}} = h_t^{\mathbf{s}}, \quad \mathbf{s} \in \mathscr{S},\ t \in \mathcal{H} \backslash \{1\},$$
$$T_{\mathbf{s},t}^{s,1} x_t^{\mathbf{s}} + W_{\mathbf{s},t}^{s,1} y_{\mathbf{s},t}^{s,1} = h_{\mathbf{s},t}^{s,1}, \quad \mathbf{s} \in \mathscr{S}, s \in \mathcal{S}_t,\ t \in \mathcal{H},$$
$$T_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau-1} + W_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau} = h_{\mathbf{s},t}^{s,\tau}, \quad \mathbf{s} \in \mathscr{S},\ s \in \mathcal{S}_t,\ \tau \in \mathcal{T}_t \backslash \{1\},\ t \in \mathcal{H}.$$

Relaxing the nonanticipativity constraints only on the strategic decision variables in formulation (11) we obtain the Multi-Horizon Strategic Wait-and-See Program defined as follows:

Definition 3.4 A *Multi-Horizon Strategic Wait-and-See Program, MHSWS,* is defined as follows:

$$MHSWS := \sum_{t=1}^{H} \sum_{\mathbf{s} \in \mathscr{S}} \pi^{\mathbf{s}} \min_{\mathbf{x},\mathbf{y}} (c_t^{\mathbf{s}} x_t^{\mathbf{s}} + \sum_{s \in \mathcal{S}_t} w_{\mathbf{s},t}^{s} \sum_{\tau \in \mathcal{T}_t} q_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau}) \tag{16}$$
$$\text{s.t. } Ax_1 = h_1,$$
$$T_t^{\mathbf{s}} x_{t-1}^{\mathbf{s}} + W_t^{\mathbf{s}} x_t^{\mathbf{s}} = h_t^{\mathbf{s}}, \quad \mathbf{s} \in \mathscr{S},\ t \in \mathcal{H} \backslash \{1\},$$
$$T_{\mathbf{s},t}^{s,1} x_t^{\mathbf{s}} + W_{\mathbf{s},t}^{s,1} y_{\mathbf{s},t}^{s,1} = h_{\mathbf{s},t}^{s,1}, \quad \mathbf{s} \in \mathscr{S}, s \in \mathcal{S}_t,\ t \in \mathcal{H},$$
$$T_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau-1} + W_{\mathbf{s},t}^{s,\tau} y_{\mathbf{s},t}^{s,\tau} = h_{\mathbf{s},t}^{s,\tau}, \quad \mathbf{s} \in \mathscr{S},\ s \in \mathcal{S}_t,\ \tau \in \mathcal{T}_t \backslash \{1\},\ t \in \mathcal{H},$$
$$y_{\mathbf{s},t}^{s',\tau} = y_{\mathbf{s},t}^{s'',\tau},\ \forall s', s'' \in \mathcal{S}_t \text{ for which } s' = s'' \text{up to operational stage } \tau.$$

Definition 3.5 The *Multi-Horizon Expected result of Strategic Decision, $MHEES$,* is defined as the solution value of problem (1), having the strategic decision variables $\bar{\mathbf{x}}$ fixed at the optimal values obtained by problem (12).

Proposition 3.1 *Given a Multi-Horizon Stochastic program (1), the following inequality is satisfied:*

$$MHSP \leq MHEES.$$

Proof Any feasible solution of problem $MHEES$ is also a solution of problem $MHSP$, since the former is more restricted than the latter, and the relation holds true. If $MHEES = \infty$, the inequality is automatically satisfied. □

Definition 3.6 The *Value of Strategic Decision, VSD,* is defined as

$$VSD := MHEES - MHSP \geq 0.$$

Notice that $MHEES$ could be infeasible since too many variables are fixed from the expected value solutions. For this reason we define the following measures:

Definition 3.7 The *Multi-Horizon Expected result of Strategic Decision at strategic stage t, $MHEES_t$ $t \in \mathcal{H}$,* is defined as the solution value of problem (1), having the strategic decision variables $\bar{\mathbf{x}}$ fixed at the optimal values obtained by problem (12) up to stage t.

Definition 3.8 The *Value of Strategic Decision at strategic stage t, VSD_t,* is then defined as follows

$$VSD_t := MHEES_t - MHSP \geq 0, \quad t \in \mathcal{H}.$$

Similarly other upper bounds can be obtained by fixing both the strategic and operational decisions:

Definition 3.9 The *Multi-Horizon Expected result of Strategic and Operational Decision at strategic stage t and operational stage τ, $MHEESO_t^\tau$,* $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}\backslash\{H\}$, is defined as the solution value of problem (1), having both the strategic and operational decision variables fixed at the optimal values $(\bar{x}_1, \ldots, \bar{x}_t)$ and $(\bar{y}_1^\tau, \ldots, \bar{y}_t^\tau)$ obtained by problem (12) up to strategic stage t and operational stage τ.

Definition 3.10 The *Value of Strategic and Operational Decision at strategic stage t and operational stage τ, $VSOD_t^\tau$,* is defined as follows:

$$VSOD_t^\tau := MHEESO_t^\tau - MHSP \geq 0, \ \tau \in \mathcal{T}_t, \ t \in \mathcal{H}\backslash\{H\}.$$

Proposition 3.2 *The following chains of inequalities hold true:*

- $MHSP \leq MHEES_1 \leq MHEES_2 \leq \ldots \leq MHEES_{H-1}$;
- $MHSP \leq MHEESO_1^\tau \leq MHEESO_2^\tau \leq \ldots \leq MHEESO_{H-1}^\tau$, $\tau \in \mathcal{T}_t$;
- $MHEES_t \leq MHEESO_t^\tau$, $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}\backslash\{H\}$.

Proof See the proof of property 3.1. □

Proposition 3.3 *Given the Multi-Horizon Stochastic program $MHSP$, the Multi-Horizon Strategic Wait-and-See Program, $MHSWS$, and the Multi-Horizon Wait-and-See Program, $MHWS$ the following relation holds true*

$$MHWS \leq MHSWS \leq MHSP. \tag{17}$$

Proof For every strategic scenario $\mathbf{s} \in \mathscr{S}$ and operational scenario $s \in \mathcal{S}_t$, $t \in \mathcal{H}$ in problem (11), denoting with $\bar{x}^s_{\mathbf{s}}$ an optimal solution under these specific scenarios $\mathbf{s}$ and s, with $\bar{x}_{\mathbf{s}}$ an optimal solution under the strategic scenario $\mathbf{s}$, with x^* an optimal solution to the multi-horizon stochastic problem (11), and with z the total cost, the following relation holds true:

$$z(\bar{x}^s_{\mathbf{s}}) \leq z(\bar{x}_{\mathbf{s}}) \leq z(x^*). \tag{18}$$

The thesis is obtained taking the expectation of all the sides of (18). □

Proposition 3.4 *Given the Multi-Horizon Wait-and-See Program, MHWS, and the Multi-Horizon Expected Value problem, MHEV with fixed objective function coefficients and recourse matrix, the following relation holds true*

$$MHEV \leq MHWS. \tag{19}$$

Proof From (9) we have:

$$\begin{aligned} &\min_{\mathbf{x},\mathbf{y}} \mathbb{E}_{\boldsymbol{\xi}_{H-1},\boldsymbol{\eta}_1^{O_1},\dots,\boldsymbol{\eta}_H^{O_H}} z(\mathbf{x},\mathbf{y},\boldsymbol{\xi}_{H-1},\boldsymbol{\eta}_1^{O_1},\dots,\boldsymbol{\eta}_H^{O_H}) = \\ &\min_{x_1} c_1x_1 + \mathbb{E}_{\eta_1^1}\left[Q_1^1(x_1,\eta_1^1)\right] + \mathbb{E}_{\xi^1}\left[Q_2(x_1,\xi^1)\right] \\ &\ \text{s.t. } Ax_1 = h_1\ , \\ &x_1 \in \mathbb{R}_+^{n_1-d_1} \times \mathbb{N}^{d_1}\ , \end{aligned} \tag{20}$$

with $Q_1^1(x_1,\eta_1^1)$ and $Q_2(x_1,\xi_1)$ defined in section 2. We note that, due to the assumptions of fixed objective function coefficients and recourse matrices, we can write
$\boldsymbol{\xi}_{H-1} = (h_2,T_2,\dots,h_H,T_H)$ and
$(\boldsymbol{\eta}_1^{O_1};\dots;\boldsymbol{\eta}_H^{O_H}) = (h_1^1,T_1^1,\dots,h_1^{O_1},T_1^{O_1};\dots;h_H^1,T_H^1,\dots,h_H^{O_H},T_H^{O_H})$ and we denote $\boldsymbol{\psi} := (\boldsymbol{\xi}_{H-1},\boldsymbol{\eta}_1^{O_1},\dots,\boldsymbol{\eta}_H^{O_H})$.
We note that

$$\begin{aligned} z(\mathbf{x},\mathbf{y},\boldsymbol{\psi}) = c_1x_1 + Q_1^1(x_1,\eta_1^1) + Q_2(x_1,\xi^1) + \\ \delta(x_1|Ax_1 = h_1,\ x_1 \in \mathbb{R}_+^{n_1-d_1} \times \mathbb{N}^{d_1}), \end{aligned}$$

where $\delta(x|X)$ is the indicator function of the point x for the set X, is jointly convex in x_1, η_1^1 and ξ_1. Notice also that Q_t, $t = 2\dots,H$ and Q_t^τ, $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$ are convex in $\mathbf{x}$ and $\mathbf{y}$: by induction because $Q_H^{O_H}$ is convex for all $\eta_H^{O_H}$, so is $\mathbb{E}_{\eta_H^{O_H}}\left[Q_H^{O_H}(y_H^{O_H-1},\eta_H^{O_H})\right]$. We can then carry this back to each $\tau < O_H$ and $t < H$ and the convexity in the decision variables $\mathbf{x},\mathbf{y}$ follows. To show that $f(\boldsymbol{\psi}) := \min_{\mathbf{x},\mathbf{y}} z(\mathbf{x},\mathbf{y},\boldsymbol{\psi})$ is convex in $\boldsymbol{\psi}$ we proceed as follows: consider $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ where $z(\mathbf{x},\mathbf{y},\boldsymbol{\psi}') = f(\boldsymbol{\psi}')$ and $z(\mathbf{x},\mathbf{y},\boldsymbol{\psi}'') = f(\boldsymbol{\psi}'')$, then:

$$\begin{aligned} \lambda f(\boldsymbol{\psi}') + (1-\lambda)f(\boldsymbol{\psi}'') &= \lambda z(\mathbf{x},\mathbf{y},\boldsymbol{\psi}') + (1-\lambda)z(\mathbf{x},\mathbf{y},\boldsymbol{\psi}'') \\ &\geq z(\mathbf{x},\mathbf{y},\lambda\boldsymbol{\psi}' + (1-\lambda)\boldsymbol{\psi}'') \\ &\geq \min_{\mathbf{x},\mathbf{y}} z(\mathbf{x},\mathbf{y},\lambda\boldsymbol{\psi}' + (1-\lambda)\boldsymbol{\psi}'') \\ &= f(\lambda\boldsymbol{\psi}' + (1-\lambda)\boldsymbol{\psi}''), \end{aligned}$$

which establishes convexity of $f(\boldsymbol{\psi})$. The thesis follows from Jensen's inequality [15], which states that for any convex function $f(\boldsymbol{\psi})$ of $\boldsymbol{\psi}$, $\mathbb{E}_{\boldsymbol{\psi}} f(\boldsymbol{\psi}) \geq f(\mathbb{E}\boldsymbol{\psi})$. □

From the traditional Expected Value of Information $EVPI$ [20], the following two definitions can be given:

Definition 3.11 The *Multi-Horizon Expected Value of Perfect Strategic Information, $MHEVPSI$,* is defined as

$$MHEVPSI := MHSP - MHSWS \geq 0.$$

Definition 3.12 The *Multi-Horizon Expected Value of Perfect Information, $MHEVPI$,* is defined as

$$MHEVPI := MHSP - MHWS \geq 0.$$

Notice that from Proposition 3.3 we trivially have $EVPSOI \geq EVPSI$.

Proposition 3.5 *Given the Multi-Horizon Expected Value problem, $MHEV$, and the Multi-Horizon Operative Expected Value problem, $MHOEV$ with fixed objective function coefficients and recourse matrix, the following relation holds:*

$$MHEV \leq MHOEV. \tag{21}$$

Proof We observe that $MHEV$ corresponds to the expected value problem of $MHOEV$, which is a traditional stochastic program having only one level of uncertainty (strategic). The relation then follows from $EV \leq WS \leq RP$ [3], where the first inequality holds true for standard stochastic optimization problems with fixed objective function coefficients and recourse matrix. □

4 Numerical Results

4.1 Problem Description

In this section we report a multi-horizon stochastic optimization problem taken from [16] to test the bounds introduced before. The problem is inspired by the EnRiMa model *Energy Efficiency and Risk Management in Public Buildings* for installing photovoltaic panels on a building with the aim to determine the quantity of capacity to install and when. The value of the panels is provided by how they can cover the electricity demand $D_{\ell}^{s,\tau}$ [kWh] in strategic node ℓ of the set of ordered strategic nodes $\mathcal{N}_t$, profile $s \in \mathcal{S}_t$ and operational time $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$, taking into account of a factor $R_{\ell}^{s,\tau}$ specifying what percentage of the nominal capacity the panel actually produce in a given hour τ and scenario s. $\Delta_\ell^{s,\tau}$ is the duration of the operational periods (1 hour) in node ℓ, scenario s and operational time τ. As in [16] we consider strategic periods one year long and three profiles: summer, winter and the rest of the year. Given the installation cost CI_ℓ [€/kW], the cost of energy at time t in scenario s $CE_\ell^{s,\tau}$ [€/kW] and w_ℓ^s the probability of scenario (or profile) s derived from the node ℓ, the problem is to determine the capacity to be installed x_ℓ [kW], the total installed capacity y_ℓ [kW] in each strategic node $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$, and the amount of purchased electricity $z_\ell^{s,\tau}$ [kWh] at time $\tau \in \mathcal{T}_t$ in scenario (profile) $s \in \mathcal{S}_t$, $t \in \mathcal{H}$, to minimize the expected cost.

We assume the following notation based on a multi-horizon scenario tree and node notation as in model (10).
Sets:

$\mathcal{H} = \{t \mid t = 1, \dots, H\}$, set of strategic stages;
$\mathcal{N}_t = \{\ell \mid \ell = 1, \dots, \ell_t\}$, set of ordered nodes of the strategic tree at stage $t \in \mathcal{H}$;
$\mathcal{T}_t = \{\tau \mid \tau = 1, \dots, O_t\}$, set of operational stages at strategic time $t \in \mathcal{H}$
$\mathcal{S}_t = \{s \mid s = 1, \dots, S_t\}$, set of operational scenarios (or profiles) at strategic time $t \in \mathcal{H}$.

Stategic Stochastic Parameters:

CI_ℓ [€/kW] , the PV installation cost at strategic node $\ell \in \mathcal{N}_t,\ t \in \mathcal{H}$;
$a(\ell)$, ancestor of the node $\ell \in \mathcal{N}_t,\ t \in \mathcal{H} \setminus \{1\}$ in the strategic scenario tree;
π_ℓ , probability of strategic node $\ell \in \mathcal{N}_t,\ t \in \mathcal{H}$.

Operational Stochastic Parameters:

$CE_\ell^{s,\tau}$ [€/kW] , the cost of energy at time $\tau \in \mathcal{T}_t$ in scenario $s \in \mathcal{S}_t$ derived from strategic node $\ell \in \mathcal{N}_t,\ t \in \mathcal{H}$;
$\Delta_\ell^{s,\tau}$, the duration of the operational periods in node $\ell \in \mathcal{N}_t,$ scenario $s \in \mathcal{S}_t$ and operational time $\tau \in \mathcal{T}_t,\ t \in \mathcal{H}$;
$R_\ell^{s,\tau}$, percentage of the nominal capacity the panel actually produce in a given hour $\tau \in \mathcal{T}_t,$ and scenario $s \in \mathcal{S}_t,\ t \in \mathcal{H}$;
$D_\ell^{s,\tau}$[kWh] , electricity demand in a given hour $\tau \in \mathcal{T}_t,$ and scenario $s \in \mathcal{S}_t,\ t \in \mathcal{H}$;
w_ℓ^s , the probability of scenario (or profile) $s \in \mathcal{S}_t,\ t \in \mathcal{H}$.

Strategic Variables:

x_ℓ [kW] , the capacity to be installed in each strategic node $\ell \in \mathcal{N}_t, t \in \mathcal{H}$;
u_ℓ [kW] , the total installed capacity in each strategic node $\ell \in \mathcal{N}_t, t \in \mathcal{H}$.

Operational Variables:

$y_\ell^{s,\tau}$ [kWh] , amount of purchased electricity at time $\tau \in \mathcal{T}_t$, in scenario (profile) $s \in \mathcal{S}_t$, $t \in \mathcal{H}$.

The multi-horizon stochastic programming problem derived for this problem is formulated as follows:

$$\min_{\mathbf{x},\mathbf{u},\mathbf{y}} \sum_{t=1}^{H} \sum_{\ell \in \mathcal{N}_t} \pi_\ell (CI_\ell x_\ell + \sum_{s \in \mathcal{S}_t} w_\ell^s \sum_{\tau \in \mathcal{T}_t} CE_\ell^{s,\tau} y_\ell^{s,\tau}) \tag{22}$$
$$\text{t.c. } u_\ell = u_{a(\ell)} + x_\ell,\ \ell \in \mathcal{N}_t,\ t \in \mathcal{H} \setminus \{1\},$$
$$\Delta_\ell^{s,\tau} R_\ell^{s,\tau} u_\ell + y_\ell^{s,\tau} \geq D_\ell^{s,\tau},\ \ell \in \mathcal{N}_t,\ \tau \in \mathcal{T}_t,\ s \in \mathcal{S}_t,\ t \in \mathcal{H}.$$

The first sum in the objective function of problem (22) takes into account the expected installation cost while the second sum represents the expected cost derived by buying electricity in period τ and profile s. The first constraint keep track of the installed capacity u_ℓ, at node $\ell \in \mathcal{N}_t$, in strategic stage $t \in \mathcal{H}$. Finally the last constraint ensures that we have enough power to satisfy the demand $D_\ell^{s,\tau}$, in node $\ell \in \mathcal{N}_t$, period $\tau \in \mathcal{T}_t$, profile $s \in \mathcal{S}_t$ at stategic time $t \in \mathcal{H}$.

4.2 The data

We consider a multi-horizon scenario tree defined by the user with strategic tree given by 3 strategic stages with 2 branches from the root, and 2 from each of the second-stage nodes resulting in 4 strategic scenarios and 7 strategic nodes. One strategic period is one year long. Table 1 reports the strategic scenario tree structure with the ancestor $a(\ell)$ of node ℓ, its probability π_ℓ and the values of the PV installation costs CI_ℓ, $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$. Last column refers to an additional parameter SCE_ℓ modeling the long-term averages of the price. Consequently the electricity cost can be obtained as $CE_\ell^{s,\tau} = SCE_\ell \times ICE^{s,\tau}$, where $ICE^{s,\tau}$ are dimensionless multipliers modeling the daily price profiles for each $s \in \mathcal{S}_t$ $t \in \mathcal{H}$. We assume in summer it is equal to 0.7, in winter 1.5 and in the rest of the year 0.9 and constant over operational periods.
For the demand $D_\ell^{s,\tau}$ and PV-production factors $R_\ell^{s,\tau}$ we adopt the same approach and data as in [16]: $D_\ell^{s,\tau} = SD_\ell \times ID^{s,\tau}$, where $SD_\ell = 10$ kWh in all the strategic nodes ℓ and $ID^{s,\tau}$ denotes the multipliers from the operational profiles. The PV-production factors $R_\ell^{s,\tau}$ are assumed to be constant in the long term, i.e. $R_\ell^{s,\tau} = R^{s,\tau}$. For more details on data adopted we refer to [16]. We assume that the duration of the operational periods $\Delta_\ell^{s,\tau}$ is the same over all periods and profiles and is one hour long. The number of operational periods in the trees derived by each strategic node $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$, is 24 (the number of hours in a day) with weight $w_\ell^s = 0.246575$, with $s = 1$ (winter), $s = 2$ (summer) and $w_\ell^s = 0.506849$ with $s = 3$ (the rest of the year) derived by taking into account the number of days in a year respectively of winter and

summer (90 days) and for the rest (185 days). All the operational sub-trees can be classified as two-stage trees plus 23 extra-periods.

Table 1 Strategic scenario tree structure, values of strategic parameters CI_ℓ and SCE_ℓ, $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$.

t	ℓ	$a(\ell)$	π_ℓ	CI_ℓ	SCE_ℓ
1	1	-	1	50	0.05
2	2	1	0.5	25	0.06
2	3	1	0.5	30	0.04
3	4	2	0.25	12	0.03
3	5	2	0.25	10	0.05
3	6	3	0.25	13	0.04
3	7	3	0.25	15	0.06

4.3 Computational tests

This section presents computational tests on the bounds introduced in Section 3 applied to the problem (22). We use Ampl environment along with the callable library of CPLEX 12.5.1.0 to solve the linear problem derived from our case study. All the computations have been done on a 64-bit machine with 12 GB of RAM and a 2.90 GHz processor. CPLEX solves the $MHSP$ problem with the scenario tree structure described before in 198 dual simplex iterations with a CPU time of 0.015625 seconds. After eliminating 238 constraints, the adjusted problem is composed by 518 linear variables, and 273 linear constraints of which 7 equality and 266 inequality constraints. We use it as a benchmark to evaluate the cost of optimal solutions obtained using the other reduced multi-horizon scenario trees.

The results are presented in Table 2 which reports total costs, optimal strategic solution values x_ℓ and u_ℓ, $\ell \in \mathcal{T}$, $t \in \mathcal{H}$, of problems $MHSP$, $MHOEV$ and $MHEV$. Results show that a better description of the operational stochasticity of model $MHSP$ compared to $MHOEV$, leads to splitting the installation capacity among strategic stages, with a lower quantity at the first one and a consequently larger total cost due to the larger amount of purchased electricity at each operational time and profile. Lower costs are obtained by the deterministic model $MHEV$ with deterministic parameters in both strategic and operational stages. However, since the objective function coefficients and recourse matrix are not fixed, Proposition 3.5 relating $MHEV$ and $MHOEV$ is not verified. The case of fixed objective function coefficients and recourse matrix will be addressed in Section 4.4.

Moreover, the total costs from the three models, $MHSP$, $MHOEV$ and $MHEV$ are not directly comparable because they refer to different multi-horizon scenario trees. The optimal solutions are then compared on the multi-horizon scenario tree associated to problem $MHSP$ that is used as a bench-

Table 2 Optimal total cost and optimal strategic solution values of problems $MHSP$, $MHOEV$ and $MHEV$ over nodes and strategic stages.

		$MHSP$		$MHOEV$	
total cost		6373.512		4499.2669	
t	ℓ	x_ℓ	u_ℓ	x_ℓ	u_ℓ
1	1	15.1515	15.1515	45.7374	45.7374
2	2	34.8485	50	6.4232	52.1606
2	3	0.9775	16.129	0	45.7374
3	4	0	50	0	52.1606
3	5	5.5556	55.5556	0.9784	53.1389
3	6	17.2043	33.3333	4.4344	50.1718
3	7	33.871	50	6.4231	52.1606

		$MHEV$	
total cost		4504.7396	
t	ℓ	$\bar{x}$	$\bar{u}$
1	1	45.7374	45.7374
2	2	4.4344	50.1718
3	3	1.9887	52.1606

mark. First of all we evaluate the optimal solutions of the $MHEV$ model by fixing the quantity of installed capacity x_ℓ, $\ell = 1, 2, 3$ until stage t ($t = 1, 2, 3$), in the multi-horizon stochastic framework by means of $MHEES_1$, $MHEES_2$ and $MHEES_2 = MHEES$ (see Table 3). The associated chain

$$VSD_1 = 213.4495 < VSD_2 = 223.854 < VSD = 230.3545 \ ,$$

shows the losses obtained by installing the electricity capacity suggested by the deterministic solution (see Table 2). The chain above verifies Proposition 3.1 and the first inequality of Proposition 3.2. By fixing both the strategic and operational variables from the expected value solution at $t = \tau = 1$, we get $MHEESO_1^1 = \infty$ and consequently $VSOD_1^1 = \infty$ showing that the inappropriateness of the deterministic solution in a multi-horizon stochastic setting. The second and third inequalities of Proposition 3.2 are then verified.

Table 3 Optimal Strategic Solution Values of problem $MHEES_1$, $MHEES_2$ and $MHEES$ over nodes and strategic stages.

		$MHEES_1$		$MHEES_2$		$MHEES$	
total cost		6586.9615		6597.366		6603.8665	
t	ℓ	x_ℓ	u_ℓ	x_ℓ	u_ℓ	x_ℓ	u_ℓ
1	1	45.7374	45.7374	45.7374	45.7374	45.7374	45.7374
2	2	4.2626	50	4.4344	50.1718	4.4344	50.1718
2	3	0	45.7374	4.4344	50.1718	4.4344	50.1718
3	4	0	50	0	50.1718	1.9887	52.1605
3	5	5.5556	55.5556	5.3838	55.5556	1.9887	52.1605
3	6	0	45.7374	0	50.1718	1.9887	52.1605
3	7	4.2626	50	0	50.1718	1.9887	52.1605

We then compute the Multi-Horizon Wait-and-See Program, $MHWS$ obtained by relaxing the nonanticipativity constraints on the strategic and operational decision variables and the Multi-Horizon Strategic Wait-and-See Program, $MHSWS$ by relaxing the nonanticipativity constraints only on the

Table 4 Values of $MHSP$ and other bounds

model	total cost
$MHSP$	6373.512
$MHEV$	4504.7396
$MHOEV$	4499.2669
$MHEES_1$	6586.9615
$MHEES_2$	6597.366
$MHEES_3 = MHEES$	6603.8665
$MHEESO_1^1$	∞
$MHSWS$	6372.8001
$MHWS$	6001.3994

strategic variables. The measure $MHEVPSI = 0.7119$ (see Definition 3.11) allows to quantify the value of knowing in advance the PV installation cost, while $MHEVPI = 372.1126$ (see Definition 3.12) to quantify the value of knowing in advance the PV installation cost, the cost of energy and the demand at each operational time. By comparing the values of the different approaches we have:

$$MHWS = 6001.3994 < MHSWS = 6372.8001 < MHSP = 6373.512,$$

which verifies Proposition 3.3.
Notice that for the problem considered, Proposition 3.5 is not verified since the objective function coefficients and recourse matrix are not fixed; thus we have: $MHEV = 4504.7396 > MHOEV = 4499.2669$. In order to verify Proposition 3.5, in the following section we will assume fixed objective function coefficients and recourse matrix.

4.4 Case of fixed recourse and objective function coefficients

In this section we consider the case in which objective function coefficients and recourse matrix are fixed. Strategic parameters CI_ℓ and SCE_ℓ, are defined as in Table 5, and operational parameters as follows: the dimensionless multipliers modeling the daily price profiles are assumed constant over days and profiles $ICE^{s,\tau} = 0.998631$, $s \in \mathcal{S}_t$, $\tau \in \mathcal{T}_t$, $t \in \mathcal{H}$ and the fraction of PV production $IR^{s,\tau} = \overline{IR}^{\tau} := \sum_{s \in \mathcal{S}_t} w_\ell^s IR^{s,\tau}$ $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$.
Results are reported in Table 6 which reports the total costs of problems $MHSP$, $MHEV$, $MHOEV$, $MHWS$ and $MHSWS$. The values obtained show that:

$$MHEV = 4504.7396 \leq MHWS = 4524.6039,$$
$$MHEV = 4504.7396 = MHOEV = 4504.7396,$$

verifying propositions 3.5 and 3.4. All the other inequalities are also verified.

Table 5 Strategic scenario tree structure, values of strategic parameters CI_ℓ and SCE_ℓ, $\ell \in \mathcal{N}_t$, $t \in \mathcal{H}$ in case of fixed recourse and objective function coefficients.

t	ℓ	$a(\ell)$	π_ℓ	CI_ℓ	SCE_ℓ
1	1	-	1	50	0.05
2	2	1	0.5	27.5	0.05
3	3	1	0.5	27.5	0.05
3	4	2	0.25	12.5	0.045
3	5	2	0.25	12.5	0.045
3	6	3	0.25	12.5	0.045
3	7	3	0.25	12.5	0.045

Table 6 Values of $MHSP$, $MHEV$, $MHOEV$ under fixed objective function coefficients and recourse matrices.

model	total cost
$MHSP$	4782.759
$MHEV$	4504.7396
$MHOEV$	4504.7396
$MHWS$	4524.6039
$MHSWS$	4782.7587

5 Conclusions

In this paper, we have developed lower and upper bounds for multi-horizon stochastic optimization programs [16]. The structure of such a kind of problems allows to model and solve problems that need to combine strategic (long-term) and operational (short-term) uncertainty, without the computational intractability that would follow from using a standard multistage stochastic model.

Three general mathematical formulations, the definitions of the expected value problem and wait-and-see problem from stochastic programming, have been introduced and discussed for a multi-horizon framework.

New measures and bounds have been defined, allowing to quantify the importance of the uncertainty at both strategic and operational levels and the systematic error done by most stochastic programs, including either short-run or long-run uncertainty, replacing the other type with expectation.

Relations among the solution approaches are then determined and chain of inequalities provided and proved. For illustration, numerical results on a real-life application for energy planning inspired by [16] have been presented.

Future works will consider hierarchy of bounds based on solving smaller multi-horizon problems based on partitions of the strategic and operational scenario sets.

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