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MIQP-Based Algorithm for the Global Solution of Economic Dispatch Problems with Valve-Point Effects

P.-A. Absil, Benoît Sluysmans, and Nicolas Stevens
ICTEAM Institute
University of Louvain
1348 Louvain-la-Neuve, Belgium

***Abstract*—Even in a static setting, the economic load dispatch problem (ELDP)—namely the cost-optimal distribution of power among generating units to meet a specific demand subject to system constraints—turns out to be a challenge owing to the consideration of valve-point effects (VPE), which make the cost function nonsmooth and nonconvex. We present a new method, termed Adaptive Piecewise-Quadratic Under-Approximation (APQUA), for the global solution of the ELDP with VPE. Unlike the many existing methods for this problem, APQUA produces at each iteration an upper and a lower bound on the globally optimal cost, and the gap between the two bounds is guaranteed to converge to zero as the iteration number grows. Consequently, APQUA is guaranteed to compute the global optimum of the ELDP within any user-prescribed accuracy. Even though APQUA has to call an MIQP solver on increasingly large surrogate problems in order to achieve this unprecedented optimality guarantee, our experiments indicate that the total computation time remains reasonable even when the prescribed accuracy is very high.**

***Index Terms*—economic dispatch, global optimization, mixed-integer programming, piecewise-linear approximation, valve-point effect.**

I. Introduction

The economic load dispatch problem (ELDP) consists of minimizing the cost associated with the power generation by optimally scheduling the load across generating units to meet a certain demand subject to system constraints [1]. If the cost function takes highly nonlinear input-output characteristics into account due to the valve-point effect (VPE), then even the static ELDP (see Section II for a precise statement of the problem) that ignores the ramp-rate constraints remains of widespread interest—it appears as a crucial subproblem in full-fledged dispatch problems and several static ELDP instances remain widely used as benchmarks—and turns out to be difficult to solve—standard mathematical-programming methods, which are widely used to address simpler cost functions, are not guaranteed to find the global optimum of the ELDP when the VPE is considered due to the nonlinear, nonsmooth, and multimodal nature of the problem.

These difficulties prompted the development of numerous metaheuristics, mostly population-based stochastic search algorithms, that may provide a sufficiently good solution within reasonable time but without optimality guarantee. Several references can be found, e.g., in [2].

An alternative was proposed in [3], [4], [5], [6], [7] where the VPE cost function is piecewise linearized using pieces of equal length, and the resulting *surrogate* ELDP is solved by a mixed-integer quadratic programming (MIQP) solver. In principle, by adequately choosing the number of pieces, it would be possible to control the optimality gap and compute the global optimum of the *original* ELDP within any user-prescribed accuracy; however the large number of pieces required to achieve even a moderate accuracy would make this approach impractical.

In this paper, we propose instead to choose the piecewise linearization *adaptively*. The outcome is a publicly available [8] deterministic iterative algorithm, termed Adaptive Piecewise-Quadratic Under-Approximation (APQUA), that, while remaining time efficient (the computation time obtained for the 40-unit case of Section V-C actually turns out to be excellent), provides optimality guarantees that are unprecedented in the literature on the ELDP with VPE. Specifically, the main features of the proposed APQUA algorithm (Algorithm 1) are the following. (i) In view of Proposition 1, APQUA produces at each iteration a power distribution $\tilde{p}$ that satisfies the ELDP constraints, as well as a *lower bound* $f(\tilde{p}) - \delta - \gamma$ on the globally optimal value of the ELDP, where $f(\tilde{p})$ is the cost of the power distribution $\tilde{p}$, δ is guaranteed to converge to zero (Theorem 2), and γ is the absolute optimality gap tolerance assigned to the MIQP solver called by APQUA. (ii) As a consequence (see Corollary 4), APQUA is *guaranteed* to return with a power distribution whose cost is *within any user-prescribed accuracy of the global optimum* of the ELDP. (To this end, set δ_{tol} and γ in Algorithm 1 such that their sum is smaller than the prescribed accuracy.) (iii) The VPE cost term is not restricted to be a rectified sine. It can be any Lipschitz-continuous piecewise-concave function, where the endpoints of the pieces (termed "kink points") are specified.

The proposed APQUA algorithm proceeds as follows (see Algorithm 1 for a precise statement). For each power generation unit, the algorithm maintains a set of knots over the

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allowable range. The set of knots includes the kink points of the VPE cost term. The algorithmic loop starts by considering the linear spline interpolation of the VPE term for the current set of knots. This yields a surrogate ELDP problem where the production cost is piecewise quadratic. The surrogate problem can be solved by means of an MIQP solver. The point returned by the MIQP solver is added to the set of knots and the loop starts again. The MIQP solver is thus called repeatedly, each time to solve a new surrogate problem. The analysis of the algorithm crucially relies on the fact that, since the VPE term is piecewise-concave, each surrogate function is an underapproximation of the true function.

The paper is organized as follows. The investigated ELDP with VPE is reviewed in Section II. The proposed method is introduced in Section III, analyzed in Section IV, and illustrated on well known ELDP instances in Section V. Conclusions are drawn in Section VI.

II. Problem Statement

This section gives an overview of the ELDP context and ends with the formulation of the problem considered in the rest of the paper. In order to keep the exposition streamlined, we restrict to the static ELDP with VPE; it already presents the challenging nonlinear nonsmooth multimodal aspects mentioned in Section I, and very widely investigated instances are available, notably those considered in Section V.

In the ELDP, the major component of the cost arises from the generator input, i.e., the fuel. Thus, the objective function associated with the problem is related to how we represent the input-output characteristics of each generator. The total cost is then naturally the sum of each generating unit's contribution. The objective function is therefore expressed in the separable form

$$f(p) = \sum_{i=1}^{n} f_i(p_i),$$

where f is the total cost function in \$/h, f_i is the cost due to the i-th generator, and p_i denotes the power assigned to the i-th unit in MW.

A classical, simple, and straightforward approach to model the cost function is to use a quadratic function for each generator, i.e., $f_i(p_i) = a_i p_i^2 + b_i p_i + c_i$, where a_i, b_i, and c_i are scalar coefficients. However, in reality, performance curves do not behave so smoothly; instead, in generating units with multi-valve steam turbines, ripples will be introduced into these curves. Large steam turbine generators usually have a number of steam admission valves that are opened in sequence to meet an increasing demand from a unit. As each steam admission valve starts to open, a sharp increase in losses will occur due to the wire drawing effect [9]. This is the so-called valve-point effect (VPE). When the VPE is considered, the production cost function becomes

$$f_i(p_i) = a_i p_i^2 + b_i p_i + c_i + f_i^{\text{VPE}}(p_i).$$

A popular model for the VPE is the rectified sine curve

$$f_i^{\text{VPE}}(p_i) = d_i |\sin(e_i(p_i - p_i^{min}))|, \tag{1}$$

where d_i and e_i are positive coefficients [10]. However, the proposed APQUA algorithm and its analysis are not limited to the rectified sine VPE model (1). *Throughout the paper, we only assume for all i that $f_i^{VPE}(p_i)$ is a given Lipschitz-continuous piecewise-concave function.* Specifically, $f_i^{\text{VPE}}(p_i)$ is concave on each interval $[p_{i,j}^{\text{kink}}, p_{i,j+1}^{\text{kink}}]$, $j = 1, \ldots, n_i^{\text{kink}} - 1$, where $p_{i,1}^{\text{kink}} \leq p_i^{\text{min}}$ and $p_{i,n_i^{\text{kink}}}^{\text{kink}} \geq p_i^{\text{max}}$, with p_i^{min} and p_i^{max} as defined below. The proposed method needs to know the kink points (see line 3 of Algorithm 1) but it does not need to know a Lipschitz constant. This is a realistic requirement since, for the popular rectified sine model (1), the kink points are readily computed as $p_{i,j}^{\text{kink}} = (j-1)\pi/e_i + p_i^{\text{min}}$ with $n_i^{\text{kink}} = \lceil (p_i^{\text{max}} - p_i^{\text{min}}) e_i / \pi \rceil + 1$.

Furthermore, the ELDP must also satisfy some constraints, thereby restricting the search space. In keeping with the streamlined exposition announced above, we consider the *power balance constraint* and the *generator capacity constraints*. The power balance constraint imposes that the generated power meets out the demand, despite some wastage of power in the network:

$$\sum_{i=1}^{n} p_i = p_D + p_L(p)$$

with scalar p_D and function p_L being the power demand and the transmission loss in the network in MW, respectively. As often in the literature, we will ignore the transmission losses along the network, i.e., we set $p_L = 0$. The generator capacity constraint imposes that the power generated by each unit lies within an allowable range $[p_i^{\text{min}}, p_i^{\text{max}}]$:

$$p_i^{\text{min}} \leq p_i \leq p_i^{\text{max}},$$

where p_i^{min} and p_i^{max} are the minimum and maximum power output of the i-th generator in MW, respectively.

In summary, the problem of interest in this paper is the static ELDP without losses, stated as the minimization of $f(p) = \sum_{i=1}^{n} f_i(p_i)$ under the lossless power balance constaint and the generator capacity constraints:

$$\min_{p \in \mathbb{R}^n} \; f(p) := \sum_{i=1}^{n} \underbrace{a_i p_i^2 + b_i p_i + c_i + f_i^{\text{VPE}}(p_i)}_{f_i(p)} \tag{2a}$$

$$\text{subject to} \quad \sum_{i=1}^{n} p_i = p_D \tag{2b}$$

$$p_i^{\text{min}} \leq p_i \leq p_i^{\text{max}} \quad \forall i \in \{1, ..., n\}. \tag{2c}$$

The set of points that satisfy the constraints in (2b) and (2c) is termed *feasible set*.

III. Proposed Approach

The proposed approach consists in building a sequence of surrogate problems, each of which can be tackled by an MIQP solver.

A. Surrogate Problems

For all i, given knots $X_{i,1} < X_{i,2} < \cdots < X_{i,n_i^{\text{knot}}}$ including the kink points $p_{i,1}^{\text{kink}}, \ldots, p_{i,n_i^{\text{kink}}}^{\text{kink}}$, consider the surrogate of (2) that consists of replacing f_i^{VPE} by its linear spline interpolation s_i with respect to the given knots. Explicitly,

$$s_i(p_i) = \alpha_{i,j} p_i + \beta_{i,j},\ p_i \in [X_{i,j}, X_{i,j+1}],\ j = 1, 2, \ldots, n_i^{\text{knot}} - 1 \quad (3)$$

where

$$\alpha_{i,j} = \frac{f_i^{\text{VPE}}(X_{i,j+1}) - f_i^{\text{VPE}}(X_{i,j})}{X_{i,j+1} - X_{i,j}}$$

is the slope of the jth linear piece and

$$\begin{aligned}\beta_{i,j} &= f_i^{\text{VPE}}(X_{i,j}) - \alpha_{i,j} X_{i,j} \\ &= \frac{X_{i,j+1} f_i^{\text{VPE}}(X_{i,j}) - X_{i,j} f_i^{\text{VPE}}(X_{i,j+1})}{X_{i,j+1} - X_{i,j}}\end{aligned}$$

is its vertical intercept. The *surrogate problem* induced by the knots $X_{i,1} < X_{i,2} < \cdots < X_{i,n_i^{\text{knot}}}$ is thus

$$\min_{p \in \mathbb{R}^n} \ g(p) := \sum_{i=1}^{n} \underbrace{a_i p_i^2 + b_i p_i + c_i + s_i(p_i)}_{g_i(p)} \quad (4a)$$

$$\text{subject to} \quad \sum_{i=1}^{n} p_i = p_D \quad (4b)$$

$$p_i^{\min} \le p_i \le p_i^{\max} \quad \forall i \in \{1, ..., n\}. \quad (4c)$$

The linear spline s_i is an underapproximation of f_i^{VPE} in view of the piecewise concavity assumption on f_i^{VPE}. Hence the surrogate objective function (4a) is an underapproximation of the original objective function (2a).

B. MIP Background

Among several possible mixed-integer programming (MIP) formulations, the optimization of a piecewise-linear cost function

$$s(p) = \alpha_j p + \beta_j, \quad p \in [X_{j-1}, X_j],\ j = 1, 2, \ldots, n^{\text{knot}} - 1$$

admits the SOS1 (special ordered set of type 1) formulation, termed *multiple choice model* in [11]:

$$\begin{aligned}
\min_{p,\chi,\eta} \ & \sum_{j=1}^{n^{\text{knot}}-1} \alpha_j \chi_j + \beta_j \eta_j \\
\text{subject to } \ & p = \sum_{j=1}^{n^{\text{knot}}-1} \chi_j \\
& X_j \eta_j \le \chi_j \le X_{j+1} \eta_j \quad \forall j \in \{1, \ldots, n^{\text{knot}} - 1\} \\
& \sum_{j=1}^{n^{\text{knot}}-1} \eta_j = 1 \\
& \eta_j \in \{0, 1\} \quad \forall j \in \{1, \ldots, n^{\text{knot}} - 1\}.
\end{aligned} \quad (5)$$

The constraints $\sum_j \eta_j = 1$ and $\eta_j \in \{0, 1\}$ impose that one and only one of the η_j's—say $\eta_{\hat{\jmath}}$—is equal to one, the others being zero. This forces to select one of the linear pieces, here piece $\hat{\jmath}$. The constraint $X_j \eta_j \le \chi_j \le X_{j+1} \eta_j$ forces $\chi_{\hat{\jmath}}$ to be in $[X_{\hat{\jmath}}, X_{\hat{\jmath}+1}]$, all the others being zero. The objective reduces to its expression on the active segment, $\alpha_{\hat{\jmath}} \chi_{\hat{\jmath}} + \beta_{\hat{\jmath}} \eta_{\hat{\jmath}}$, and the constraint $p = \sum_j \chi_j$ is just $p = \chi_{\hat{\jmath}}$.

C. MIQP Formulation of the Surrogate Problem

Mimicking the above strategy, we can reformulate the surrogate problem (4) as the following MIQP problem, which can be viewed as a simplified version of [6, §3.1]:

$$\begin{aligned}
\min_{p,\chi,\eta} \ & \sum_{i=1}^{n} \left[a_i p_i^2 + b_i p_i + c_i + \sum_{j=1}^{n_i^{\text{knot}}-1} (\alpha_{ij} \chi_{ij} + \beta_{ij} \eta_{ij}) \right] \\
\text{subject to } \ & \sum_{i=1}^{n} p_i = p_D \\
& p_i^{\min} \le p_i \le p_i^{\max} \\
& p_i = \sum_{j=1}^{n_i^{\text{knot}}-1} \chi_{i,j} \\
& X_{i,j} \eta_{i,j} \le \chi_{i,j} \le X_{i,j+1} \eta_{i,j} \\
& \sum_{j=1}^{n_i^{\text{knot}}-1} \eta_{i,j} = 1 \\
& \eta_{i,j} \in \{0, 1\}
\end{aligned} \quad (6)$$

D. Knot Update Mechanism and Algorithm Statement

The proposed APQUA algorithm (Algorithm 1) alternates between (i) obtaining with an MIQP solver a feasible (a reasonable requirement since the constraints are very simple) approximate (within the tolerance of the MIQP solver) solution $\tilde{p}$ of the surrogate problem (4) induced by the current set of knots, and (ii) updating the set of knots by inserting $\tilde{p}$ therein.

In our implementation of APQUA, the initial knots (line 3 of Algorithm 1) are chosen to be the kink points and their mid-points, as we have observed that including the mid-points speeds up the algorithm.

The MIQP solver is called in line 12. The "if" statement in line 15 prevents repeated knots. The "sort" operation in line 16 is straightforward since it merely consists in inserting $\tilde{p}_i$ in the already sorted list. In line 13, g refers to the surrogate function (4a) induced by the current knots $X_{i,1}, \ldots, X_{i,n_i^{\text{knot}}}$ (this function will be called g^m in the analysis below).

IV. Optimality Guarantees

We first introduce notation and assumptions that will be in force throughout this section. Recall that f is the cost function of the original ELDP (2a). Let $\check{f}$ denote the optimal value of the original ELDP (2); m the counter of the `repeat` loop of Algorithm 1; g^m the surrogate function (4a) used at iteration m (i.e., the mth execution of the `repeat` loop); $\check{g}^m$ the optimal value of the surrogate problem (4) at iteration m; $\tilde{p}^m$ the feasible approximate solution of (4) returned in line 12 of Algorithm 1;

$$\delta^m = f(\tilde{p}^m) - g^m(\tilde{p}^m);$$

Algorithm 1 APQUA: Adaptive piecewise-quadratic under-approximation

1: Set tolerance parameter $\delta_{\text{tol}} \geq 0$;
2: **for** $i = 1, \ldots, n$ **do**
3: Choose ordered knots $(X_{i,1}, \ldots, X_{i,n_i^{\text{knot}}})$ including the kink points $p_{i,1}^{\text{kink}}, \ldots, p_{i,n_i^{\text{kink}}}^{\text{kink}}$.
4: **end for**
5: **repeat**
6: **for** $i = 1, \ldots, n$ **do**
7: **for** $j = 1, \ldots, n_i^{\text{knot}} - 1$ **do**
8: $\alpha_{i,j} \leftarrow (f^{\text{VPE}}(X_{i,j+1}) - f^{\text{VPE}}(X_{i,j}))/(X_{i,j+1} - X_{i,j})$;
9: $\beta_{i,j} \leftarrow f^{\text{VPE}}(X_{i,j}) - \alpha_{i,j} X_{i,j}$;
10: **end for**
11: **end for**
12: $\tilde{p} \leftarrow$ feasible (approximate) solution of the MIQP surrogate problem (4), obtained with an MIQP solver with absolute optimality gap tolerance γ;
13: $\delta \leftarrow f(\tilde{p}) - g(\tilde{p})$;
14: **for** $i = 1, \ldots, n$ **do**
15: **if** $\min_{j \in \{1,\ldots,n_i^{\text{knot}}\}} |\tilde{p}_i - X_{i,j}| > 0$ **then**
16: $(X_{i,1}, \ldots, X_{i,n_i^{\text{knot}}+1}) \leftarrow \text{sort}(X_{i,1}, \ldots, X_{i,n_i^{\text{knot}}}, \tilde{p}_i)$
17: $n_i^{\text{knot}} \leftarrow n_i^{\text{knot}} + 1$;
18: **end if**
19: **end for**
20: **until** $\delta < \delta_{\text{tol}}$

γ^m the optimality gap assigned to the MIQP solver at iteration m, guaranteeing that

$$\check{g}^m \leq g^m(\tilde{p}^m) \leq \check{g}^m + \gamma^m. \tag{7}$$

At each iteration, APQUA provides a lower bound and an upper bound on the optimal value $\check{f}$ of the ELDP (2):

Proposition 1. *For each m, by calling in line 12 the MIQP solver with gap tolerance γ^m, APQUA produces $f(\tilde{p}^m)$ and $g^m(\tilde{p}^m)$ such that*

$$g^m(\tilde{p}^m) - \gamma^m \leq \check{f} \leq f(\tilde{p}^m),$$

or equivalently, since $\delta^m := f(\tilde{p}^m) - g^m(\tilde{p}^m)$,

$$g^m(\tilde{p}^m) - \gamma^m \leq \check{f} \leq g^m(\tilde{p}^m) + \delta^m,$$
$$f(\tilde{p}^m) - \delta^m - \gamma^m \leq \check{f} \leq f(\tilde{p}^m).$$

Proof. In view of the MIQP solver gap (7), we have $g^m(\tilde{p}^m) - \gamma^m \leq \check{g}^m$. Since function g^m is an underapproximation of f, we also have $\check{g}^m \leq \check{f}$. This yields $g^m(\tilde{p}^m) - \gamma^m \leq \check{f}$. Finally $\check{f} \leq f(\tilde{p}^m)$ since $\tilde{p}^m$ is feasible (line (12) of Algorithm 1). ☐

We now consider $\delta_{\text{tol}} = 0$ in APQUA, which then generates an infinite sequence of iterates. The next result holds regardless of the choice of the MIQP gaps γ^m.

Theorem 2. $\lim_{m\to\infty} \delta^m = 0$.

Proof. We first point out that f and g^m, $m = 0, 1, \ldots$, are Lipschitz continuous on the ELDP feasible set. Indeed, for every i, the cost function f_i in (2a) satisfies the Lipschitz property $|f_i(p_i + \Delta) - f_i(p_i)| \leq (2a_i p_i^{\max} + b_i + K_i^{\text{VPE}})\Delta$—where K_i^{VPE} is a Lipschitz constant for f_i^{VPE}, with $K_i^{\text{VPE}} = d_i e_i$ in the case of the rectified sine (1)—for all p_i and $p_i + \Delta$ that satisfy the generator capacity constraints in (2c). The ELDP cost function f in (2a), being the sum of Lipschitz continuous functions, is thus Lipschitz continuous on the ELDP feasible set, with a constant $K = \sum_{i=1}^{n} 2a_i p_i^{\max} + b_i + K_i^{\text{VPE}}$. Since g_i^m is obtained by replacing f_i^{VPE} by chords, it follows that $2a_i p_i^{\max} + b_i + K_i^{\text{VPE}}$ is still a Lipschitz constant for g_i^m, and hence K is also a Lipschitz constant for g^m.

By contradiction, suppose that the claim, $\lim_{m\to\infty} \delta^m = 0$, does not hold. Since $\delta^m = f(\tilde{p}^m) - g^m(\tilde{p}^m) \geq 0$ for all m, this means that there exists $\epsilon > 0$ and a subsequence $(\tilde{p}^{m_k})_{k=1,2,\ldots}$ such that $\delta^{m_k} \geq \epsilon$ for all k. Restrict $(\tilde{p}^{m_k})_{k=1,2,\ldots}$ to a convergent subsequence, which exists by the Bolzano-Weierstrass theorem since the feasible set is bounded. We then have

$$\begin{aligned}
\epsilon &\leq \delta^{m_k} \\
&= f(\tilde{p}^{m_k}) - g^{m_k}(\tilde{p}^{m_k}) \\
&= f(\tilde{p}^{m_k}) - g^{m_k}(\tilde{p}^{m_{k-1}}) + g^{m_k}(\tilde{p}^{m_{k-1}}) - g^{m_k}(\tilde{p}^{m_k}) \\
&= f(\tilde{p}^{m_k}) - f(\tilde{p}^{m_{k-1}}) + g^{m_k}(\tilde{p}^{m_{k-1}}) - g^{m_k}(\tilde{p}^{m_k}) \\
&\leq |f(\tilde{p}^{m_k}) - f(\tilde{p}^{m_{k-1}})| + |g^{m_k}(\tilde{p}^{m_{k-1}}) - g^{m_k}(\tilde{p}^{m_k})| \\
&\leq K\|\tilde{p}^{m_k} - \tilde{p}^{m_{k-1}}\| + K\|\tilde{p}^{m_k} - \tilde{p}^{m_{k-1}}\| \\
&= 2K\|\tilde{p}^{m_k} - \tilde{p}^{m_{k-1}}\|,
\end{aligned}$$

where the 4th line comes from $g^{m_k}(\tilde{p}^{m_{k-1}}) = f(\tilde{p}^{m_{k-1}})$—in line 16 of Algorithm 1, $\tilde{p}$ is added to the set of knots and knots are never removed—and the 6th line comes from the Lipschitz property. Hence $0 < \epsilon \leq 2K\|\tilde{p}^{m_k} - \tilde{p}^{m_{k-1}}\|$, contradicting the convergence of $(\tilde{p}^{m_k})_{k=1,2,\ldots}$. ☐

The next convergence result readily follows.

Corollary 3. *For $\delta_{tol} = 0$, denoting $\bar{\gamma} = \limsup_{m\to\infty} \gamma^m$, we have $\limsup_{m\to\infty} f(\tilde{p}^m) \leq \check{f} + \bar{\gamma}$. If moreover the sequence of MIQP gap tolerances $(\gamma^1, \gamma^2, \ldots)$ is chosen to converge to zero, then $\lim_{m\to\infty} f(\tilde{p}^m) = \check{f}$.*

Proof. From Proposition 1, $f(\tilde{p}^m) \leq \check{f} + \gamma^m + \delta^m$. Take the $\limsup$ and invoke Theorem 2 to get the first claim. If moreover $\lim_{n\to\infty} \gamma^n = 0$, then $\bar{\gamma} = 0$ and thus $\limsup_{m\to\infty} f(\tilde{p}^m) \leq \check{f}$. But $\check{f} \leq f(\tilde{p}^m)$ for all m since $\tilde{p}^m$ is feasible. This proves the second claim. ☐

We now formalize the optimality guarantee of APQUA.

Corollary 4. *For $\delta_{tol} > 0$ (arbitrarily small), APQUA terminates in finitely many (say m) iterations with a power distribution $\tilde{p}^m$ such that*

$$f(\tilde{p}^m) < \check{f} + \delta_{tol} + \gamma^m.$$

Proof. The stopping criterion $\delta^m < \delta_{\text{tol}}$ is satisfied for m large enough in view of Theorem 2. Then by Proposition 1 we have $f(\tilde{p}^m) \leq \check{f} + \delta^m + \gamma^m < \check{f} + \delta_{\text{tol}} + \gamma^m$. ☐

V. Numerical Experiments

We have implemented APQUA (Algorithm 1) using the AMPL modeling language and the Gurobi MIQP solver. The code to reproduce the results is publicly available [8].

Since we report for information the wallclock computation times, we mention that the experiments were run on an x86_64 GNU/Linux platform with 3.40GHz CPU (Intel(R) Core(TM) i7-4770) and 16GB RAM. We stress however that the CPU clock rate is not the only factor that influences the computation times. The number of cores and threads certainly plays a role, since a CPU monitor revealed that all the 8 threads of the CPU were exploited in the computations. The choice of the MIQP solver also has an influence, as well as the chosen tolerances. In this respect, we set $\delta_{\text{tol}} = 10^{-5}$ and the MIQP solver gap tolerance $\gamma = 10^{-5}$ throughout.

The numerical experiments presented here are sanity checks that consist of running our implementation of APQUA on some widely investigated ELDP instances, thereby returning an interval of length $\delta^m + \gamma < \delta_{\text{tol}} + \gamma$ which is guaranteed to contain the optimal value of the ELDP instance. If the best value found in the literature is: (i) above the interval, then it means that APQUA has found a better solution, an unexpected situation when the problem instance has been investigated by means of countless heuristics; (ii) in the interval, then APQUA provides an optimality guarantee to the best known value; (iii) under the interval, then either the best value in the literature is incorrect (this is actually a likely situation in view of the inaccuracies in the literature as pointed out recently in [2]; see more details below) or else APQUA fails the sanity check (this has not occurred).

A. Case 1: 3-Unit System with Valve-Point Loading Effects

We consider the 3-unit system with valve-point loading effect addressed, e.g., in [12, §5.2.1] with demand $p_D = 850$ MW. The results are reported in Table I; see also Figure 1 where several quantities produced by the APQUA algorithm are visualized on a plot of the cost function of each generator. In Table I, the three powers are the power distribution produced by APQUA when it terminates. “Total cost” is the ELDP cost $f(\tilde{p})$ associated with the final computed power distribution. “Lower bound” is $f(\tilde{p}) - \delta - \gamma$, which is guaranteed by Proposition 1 to be a lower bound on the globally optimal value of the ELDP. In the experiments that we conducted, APQUA turned out to return with $\delta \ll \delta_{\text{tol}}$, yielding an interval length $\delta + \gamma$ practically equal to 1e-5.

According to [12, Table 10], the best known value is 8234.05, however the solution is not feasible since it achieves $p_D = 849.9990$. The best value reported in [12, Table 10] for a feasible solution is 8234.0717. It is slightly below the lower bound provided by APQUA, but this is to be attributed to the rounding error: from the powers reported in the last row of [12, Table 10], one obtains more precisely 8234.071732.

Interestingly, Figure 1 reveals that, at the global optimum, there are two units (namely units 1 and 3) whose outputs are not located at singular points. The global minimum is thus of Type 2 in the sense of [13, §III.B].

TABLE I
Results obtained by APQUA for case 1.

Unit	Power (MW)
p_1	300.266900
p_2	149.733100
p_3	400.000000
Total cost ($/h)	8234.071732
Lower bound	8234.071722
Run time (s)	0.06

TABLE II
Results obtained by APQUA for case 2.

Unit	Power (MW)
p_1	628.318531
p_2	299.199300
p_3	299.199300
p_4	159.733100
p_5	159.733100
p_6	159.733100
p_7	159.733100
p_8	159.733100
p_9	159.733100
p_{10}	77.399913
p_{11}	77.399913
p_{12}	87.684530
p_{13}	92.399913
Total cost ($/h)	24169.917726
Lower bound	24169.917716
Run time (s)	1.31

B. Case 2: 13-Unit System with Valve-Point Loading Effects

This case is a 13-generator system with valve-point loading effect. The coefficients of the fuel cost functions are provided in [14, Table 1]. The load is $p_D = 2520$ MW.

The best value reported in [6, Table 2] is 24169.9177 $/h. The APQUA results in Table II indicate that the global optimum is more precisely in the interval $[24169.917716, 24169.917726]$.

The minimal cost values reported in [12, Table 15] for RTO, ICA-PSO, and IPSO are below the interval returned by APQUA. We are thus in case (iii) mentioned in the beginning of this Section V. However, we computed the cost for the power distribution reported in [12, Table 15] for RTO, ICA-PSO and IPSO and obtained instead 24174.1781, 24178.6982, and 24173.7523, respectively. They are thus above the interval $[24169.917716, 24169.917726]$ in which we now know that the global minimum is located. We also point out that the power distribution reported for RTO is not exactly feasible, since the sum of the powers is equal to 25200.0760 MW. Several results found in the literature suffer from such inaccuracies, as pointed out in [2].

C. Case 3: 40-Unit System with Valve-Point Loading Effects

This case is a 40-generator system with valve-point loading effect. The coefficients of the fuel cost functions are provided in [15, Table 7] and the load is $p_D = 10500$ MW. The results returned by APQUA are given in Table III.

Solutions found in the literature are reported and discussed, e.g., in [16, Table 6], [17, §IV.A], [18, Table IX], and [2, §2.2]. For example, in [18, Table IX], the best value 121412.53 $/h

TABLE III
RESULTS OBTAINED BY APQUA FOR CASE 3.

Unit	Power (MW)	Unit	Power (MW)
p_1	110.79982509	p_{21}	523.27937031
p_2	110.79982509	p_{22}	523.27937031
p_3	97.39991254	p_{23}	523.27937031
p_4	179.73310011	p_{24}	523.27937031
p_5	87.79990459	p_{25}	523.27937031
p_6	140.00000000	p_{26}	523.27937031
p_7	259.59965017	p_{27}	10.00000000
p_8	284.59965017	p_{28}	10.00000000
p_9	284.59965017	p_{29}	10.00000000
p_{10}	130.00000000	p_{30}	87.79990459
p_{11}	94.00000000	p_{31}	190.00000000
p_{12}	94.00000000	p_{32}	190.00000000
p_{13}	214.75979010	p_{33}	190.00000000
p_{14}	394.27937031	p_{34}	164.79982509
p_{15}	394.27937031	p_{35}	200.00000000
p_{16}	394.27937031	p_{36}	194.39777798
p_{17}	489.27937031	p_{37}	110.00000000
p_{18}	489.27937031	p_{38}	110.00000000
p_{19}	511.27937031	p_{39}	110.00000000
p_{20}	511.27937031	p_{40}	511.27937031
	Total cost (\$/h)	121412.535519	
	Lower bound	121412.535509	
	Run time (s)	1.81	

is obtained in about 39.33 seconds, to be compared with the dispatch with cost 121412.535519 \$/h obtained by APQUA in less than 2 seconds. We also point out that the many methods reported in [16, Table 6] all have a computation time above 2 seconds (and do not provide an optimality guarantee).

It is mentioned in [2, §2.2] that "the global optimal solution of the 40 units system has a total generation cost near to 121,412.5355 \$/h". APQUA makes it possible to give a precise meaning to "near to": the global optimal cost is now known to be between 121412.535509 \$/h and 121412.535519 \$/h. (We also ran APQUA with γ set to 1e-6 instead of 1e-5. It returned the even more precise lower bound of 121412.535518 \$/h in less than 2 seconds.)

VI. CONCLUSION

We have proposed APQUA (Algorithm 1), a deterministic MIQP-based iterative method for the ELDP with VPE (2). At each iteration m, APQUA calls an MIQP solver on a new surrogate problem, updates the surrogate cost (4a) based on the power dispatch $\tilde{p}^m$ returned by the MIQP solver, and provides an interval $[f(\tilde{p}^m) - \delta^m - \gamma^m, f(\tilde{p}^m)]$ in which the globally optimal value of the ELDP is guaranteed to lie, where γ^m is the gap tolerance assigned to the MIQP solver and δ^m can be made arbitrarily small by taking sufficiently many iterations.

In spite of its guaranteed convergence and the optimality bounds that it produces, the proposed method does not make heuristics obsolete. First of all, at the expense of weakening the convergence result (Theorem 2), heuristics can be introduced in APQUA to make it even more time-efficient by slowing down the growth of the number of integer variables in the successive surrogate problems, much in the spirit of [5, §III.C] and [6, §3.2]. As for stochastic population-based heuristics, they have the advantage of being readily applicable to a wider class of nonconvex optimization problems, and they may remain an asset on very large size problems in order to obtain a reasonably good solution within the imparted computation time. Assessing this superiority in a systematic way, however, will require making benchmark problems and solvers publicly available in a ready-to-use format, as we have done with [8].

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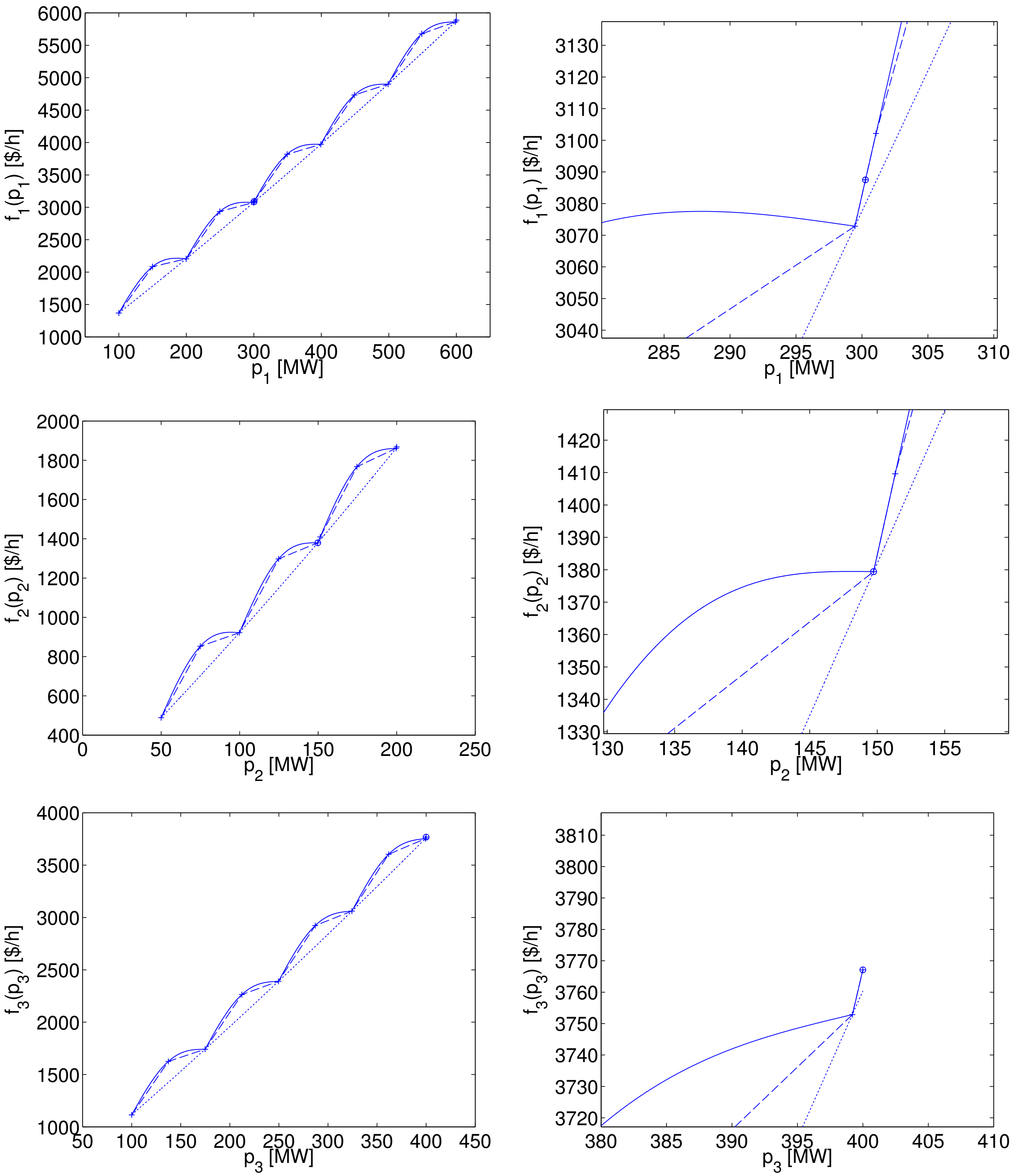

Figure 1. Results for the case of Section V-A. The right-hand plots are zooms on the left-hand plots. Solid line: ELDP cost f_i (2a). Dotted line: quadratic term of the ELDP cost. Dashed line: surrogate cost g_i (4a) when APQUA terminates. "o": power distribution returned by APQUA in the final iteration. "+": knots when APQUA terminates.