Hadamard Directional Differentiability of the Optimal Value of a Linear Second-order Conic Programming Problem*

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Abstract

In this paper, we consider perturbation properties of a linear second-order conic optimization problem and its Lagrange dual in which all parameters in the problem are perturbed. We prove the upper semi-continuity of solution mappings for the primal problem and the Lagrange dual problem. We demonstrate that the optimal value function can be expressed as a min-max optimization problem over two compact convex sets, and it is a Lipschitz continuous function and Hadamard directionally differentiable.

Key words: second order conic optimization, optimal value function, solution mapping, Hadamard directional differentiability.

1 Introduction

It is well known that stability theory plays an important role in studying the following linear two-stage stochastic optimization problem

$$\begin{array}{ll} \min\limits_{x\in\Re^n} & d^Tx + \mathbb{E}(\theta(x,\xi)) \\ \text{s.t.} & Ax = b,\ x\geq 0, \\ & \theta(x,\xi) = \begin{array}{ll} \min\limits_{y\in\Re^m} & c^Ty \\ \text{s.t.} & Wy + Tx = h,\ y \geq 0, \end{array} \end{array} \tag{1.1}$$

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where $x \in \Re^n$ is the first stage decision variable and $d \in \Re^n$, $y \in \Re^m$ is the second stage decision variable, $c \in \Re^m$, $W \in \Re^{l\times m}$, $T \in \Re^{l\times n}$,$h \in \Re^l$ and ξ is a random variable which is composed of some elements in $\{c, W, T, h\}$. The continuity and differential properties of $\theta(x, \xi)$ are particularly important in the stability analysis for the linear two-stage problem when the probability distribution is perturbed. There are many publications about the stability of two-stage optimization but among them only a few papers consider the case $\xi = (c, W, T, h)$, namely all parameters in second stage linear program are random. For examples, in Section 3 of [12], Römisch and Wets obtained the Lipschitz continuity of the optimal value $\theta(x, \xi)$ and Han and Chen [3] investigated continuity properties of parametric linear programs.

The literature on perturbation analysis of optimization problems is enormous, and even a short summary about the most important results achieved would be far beyond our reach. For the perturbation analysis of general optimization problem one may refer to [1] and [4]. For structured optimization problems, one may refer to [10] and for stability results about linear complementarity and affine variational inequality problems, see for [7] and [8].

In this paper, instead of linear programming in the second stage, we consider a linear second-order conic optimization problem. The problems can be stated as follows. Given a closed convex set $X \subset \Re^n$ and a point $x \in X$, the second-order conic optimization problem is defined by

$$(\mathrm{P}(x,\xi)) \qquad \begin{array}{ll} \min\limits_{y\in\Re^m} & c^T y \\ \text{s.t.} & a_i^T y + q_i^T x - b_i \geq \|B^i y\|_2, \quad i = 1, \ldots, l, \end{array} \tag{1.2}$$

where $\xi = (c; A; Q; B; b)$ is a given parameter. Here $c \in \Re^m, A = (a_1, \cdots, a_l)^T \in \Re^{l\times m}, Q = (q_1, \cdots, q_l)^T \in \Re^{l\times n}, b \in \Re^l$, $B = (B^1; \ldots; B^l)$ with $B^i \in \Re^{J_i\times m}$, $i = 1, \ldots, l$.

Let $g^i(y, x; \xi) = (B^i y, a_i^T y + q_i^T x - b_i)$, $i = 1, \ldots, l$ and $\mathcal{Q}_{J_i+1} \subset \Re^{J_i+1}$ be the second-order cone in $\Re^{J_i+1}$ defined by

$$\mathcal{Q}_{J_i+1} = \{(s,t) \in \Re^{J_i} \times \Re : t \geq \|s\|_2\}, i = 1, \ldots, l.$$

Then Problem (1.2) is expressed as

$$\begin{array}{ll} \min_y & c^T y \\ \text{s.t.} & g^i(y, x; \xi) \in \mathcal{Q}_{J_i+1}, \quad i = 1, \ldots, l. \end{array} \tag{1.3}$$

We use $\theta(x, \xi)$ to denote the optimal value of Problem $(\mathrm{P}(x, \xi))$. In this paper we will discuss the stability properties of Problem (1.2) when $\xi = (c; A; Q; B; b)$ is perturbed to $\tilde{\xi} = (\tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b})$, especially the differentiability property of $\theta(\cdot, \cdot)$.

The remaining parts of this paper are organized as follows. In Section 2, we demonstrate upper continuity of the solution mapping for Problem $\mathrm{P}(\tilde{x}, \tilde{\xi})$ at some point (x, ξ). In Section

3, we study the upper continuity of the solution mapping for the Lagrange dual of Problem $\mathrm{P}(\tilde{x}, \tilde{\xi})$ at some point (x, ξ). The local Lipschitz continuity of θ and its Hadamard directional differentiability at a point (x, ξ) are established Section 4. We conclude our paper in Section 5.

2 Upper continuity of primal solution mapping

Let $u = (x; \xi) = (x; c; A; Q; B; b)$ where $x \in X$. We consider Problem $(\mathrm{P}(\tilde{x}, \tilde{\xi}))$, which can be expressed as in the compact form:

$$\begin{array}{ll} \min_y & \tilde{c}^T y \\ \text{s.t.} & g(y, \tilde{x}; \tilde{\xi}) \in \mathcal{Q}, \end{array} \tag{2.1}$$

where $g(y, \tilde{x}; \tilde{\xi}) = (g^1(y, \tilde{x}; \tilde{\xi}), \ldots, g^l(y, \tilde{x}; \tilde{\xi}))$ and $\mathcal{Q} = \mathcal{Q}_{J_1+1} \times \cdots \times \mathcal{Q}_{J_l+1}$. Let $f(y, \tilde{u}) = \tilde{c}^T y$. We denote by $\Phi(\tilde{u})$ the feasible set of problem (2.1), namely

$$\Phi(\tilde{u}) = \{y \in \Re^m : g^i(y, \tilde{x}; \tilde{\xi}) \in \mathcal{Q}_{J_i+1}, i = 1, \ldots, l\}, \tag{2.2}$$

and by $Y^*(\tilde{u})$ the set of optimal solutions for Problem (2.1).

For a given parameter (c, A, Q, B, b, x), we analyze properties of the optimal value function $\theta(\cot, \cdot)$ when $u = (c, A, Q, B, b, x)$ is perturbed to $\tilde{u} = (\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$. For this purpose we make the following assumptions about Problem (2.1) and (1.2).

Assumption 2.1. *The set $X \subset \Re^n$ is a non-empty compact convex set.*

Assumption 2.2. *For each $x \in X$, the optimal value of Problem (1.2) is finite and the solution set for Problem (1.2) is compact.*

Assumption 2.3. *The slater condition of Problem (1.2) holds for each $x \in X$, namely for each $x \in \Re^n$, there exists y_x such that*

$$g^i(y_x, x; \xi) \in \operatorname{int} \mathcal{Q}_{J_i+1}, i = 1, \ldots, l, \tag{2.3}$$

which can be written as $a_i^T y_x + q_i^T x - b_i > \|B^i y_x\|_2, i = 1, \ldots, l$.

If Assumption 2.3 is satisfied, then the dual problem (3.9) has a nonempty compact solution set and the duality gap between (1.2) and (3.9) is zero.

Lemma 2.1. *Let (c, A, Q, B, b) be given if Assumptions 2.1 and 2.3 hold, then there exists $\delta_0 > 0$ such that for any $\tilde{x} \in X$, the Slater condition for Problem (2.1) holds when $\|(\tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b}) - (c; A; Q; B; b)\| \le \delta_0$.*

Proof. From Assumption 2.3, we have that there exist $y_x \in \Re^m$ and $\varepsilon_x > 0$ such that

$$a_i^T y_x + q_i^T x - b_i - \|B^i y_x\|_2 \geq \varepsilon_x, i = 1, \ldots, l.$$

Due to Assumption 2.2, we know that y_x and x are bounded. Let $M_x := \max\{\|y_x\|, \|x\|, 1\}$. When $\max\limits_{1 \leq i \leq l}\{\|\Delta\tilde{a}_i\|, \|\Delta\tilde{q}_i\|, \|\Delta\tilde{b}_i\|, \|\Delta\tilde{B}_i\|\} \leq \dfrac{\varepsilon_x}{6M_x}$ and

$$\|\tilde{x} - x\| \leq \varepsilon_x \left[6\sum_{j=1}^{l} \|q_j\| + \frac{\varepsilon_x}{M_x}\right]^{-1},$$

we have for $i = 1, \cdots, l$,

$$\begin{aligned}
\tilde{a}_i^T y_x + \tilde{q}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}^i y_x\| &\geq a_i^T y_x + q_i^T x - b_i - \|B^i y_x\| - (\|\Delta\tilde{a}_i^T\|\|y_x\| + \|\Delta\tilde{q}_i^T\|\|x\| + \|\tilde{x} - x\|\|\tilde{q}_i\| \\
&+ \|\Delta\tilde{b}_i\| + \|\Delta B^i\|\|y_x\|) \\
&\geq \frac{\varepsilon_x}{6} > 0.
\end{aligned}$$

Thus Slater condition for Problem (2.1) holds when $\tilde{x} \in X$, $\|(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}) - (c, A, Q, B, b)\| \leq \delta_x$ for $\delta_x = \dfrac{\varepsilon_x}{6M_x}$ and $\|\tilde{x} - x\| \leq \omega_x$ with $\omega_x := \varepsilon_x \left[6\sum_{j=1}^{l} \|q_j\| + \dfrac{\varepsilon_x}{M_x}\right]^{-1}$. For such $\omega_x > 0$ at $x \in X$, we have

$$X \subset \cup_{x \in X} \mathbb{B}_{\omega_x}(x),$$

where $\mathbb{B}_r(a)$ stands for an open ball centered at $a \in \Re^n$ with radius $r > 0$. From Assumption 2.1, X is compact, we have from the finite covering theorem that there are a finite number of points $x^1, \ldots, x^{n_0}$ and positive numbers $\omega_{x^1}, \ldots, \omega_{x^{n_0}}$ such that

$$X \subset \cup_{j=1}^{n_0} \mathbb{B}_{\omega_{x^j}}(x^j).$$

Let $\omega_0 = \min\{\omega_{x_j} : j = 1, \ldots, n_0\}$ and $\delta_0 = \min\{\delta_{x_j} : j = 1, \ldots, n_0\}$. Then for any $\tilde{x} \in X$, Slater condition for problem (2.1) holds when $\|(\tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b}) - (c; A; Q; B; b)\| \leq \delta_0$. $\square$

Given $(c; A; Q; B; b)$, let $\delta_0 > 0$ be the positive number in Lemma 2.1 satisfying that the Slater condition holds for (2.1) when $x \in X$ and $\|(\tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b}) - (c; A; Q; B; b)\| \leq \delta_0$. Let us denote by for $r > 0$,

$$\mathcal{U}_r(c; A; Q; B; b) = \{(\tilde{x}; \tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b}) : \tilde{x} \in X, \|(\tilde{c}; \tilde{A}; \tilde{Q}; \tilde{B}; \tilde{b}) - (c; A; Q; B; b)\| \leq r\}.$$

Lemma 2.2. *Let $(c; A; Q; B; b)$ be given with Assumptions 2.1 and 2.3 being satisfied. Then, for any $\widehat{u} \in \mathcal{U}_{\delta_0}(c; A; Q; B; b)$,*

$$\lim_{\tilde{u} \to \widehat{u}} \Phi(\tilde{u}) = \Phi(\widehat{u}).$$

Proof. As the following inclusion

$$\limsup_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u}) \subset \Phi(\widehat{u})$$

is obvious, we only need to verify that

$$\liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u}) \supset \Phi(\widehat{u}).$$

For arbitrary $\hat{y} \in \Phi(\hat{u})$, we now prove $\hat{y} \in \liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u})$. By Lemma 2.1, we have that

$$\exists \ \overline{y} \ \ such \ \ that \ \ \hat{a}_i^T\overline{y} + \hat{q}_i^T x - \hat{b}_i - \|\hat{B}^i\overline{y}\|_2 \geq \hat{\epsilon}, i = 1, \ldots, l.$$

Let $\tilde{u}(t) = (\hat{c}+t\triangle c,\ \hat{A}+t\triangle A,\ \hat{Q}+t\triangle Q,\ \hat{B}+t\triangle B,\ \hat{b}+t\triangle b,\ \hat{x}+t(\tilde{x}-\hat{x}))$, $y(t) = \hat{y}+t(\overline{y}-\hat{y})$, and we obviously have $\tilde{u}(t) \to \hat{u}$ and $y(t) \to \hat{y},\ t \downarrow 0$. Then for $\triangle u_i = (\triangle c_i, \triangle a_i, \triangle q_i, \triangle B^i, \triangle b_i, \triangle x), i = 1, \ldots, l$, we have that

$$\begin{aligned}
&\tilde{a}_i(t)^T y(t) + \tilde{q}_i(t)^T \tilde{x}(t) - \tilde{b}_i(t) - \|\tilde{B}^i(t)y(t)\|_2 \\
&= (\hat{a}_i + t\triangle a_i)^T(t\overline{y} + (1-t)\hat{y}) + (\hat{q}_i + t\triangle q_i)^T(\hat{x} + t\triangle x) - (\hat{b}_i + t\triangle b_i) \\
&\qquad\qquad - \|(\hat{B}^i + t\triangle B^i)(t\overline{y} + (1-t)\hat{y})\|_2 \\
&\geq t(\hat{a}_i^T(\overline{y} - \hat{y}) + \hat{q}_i^T\triangle x + \triangle a_i^T\hat{y} + \triangle q_i^T\hat{x} - \triangle b_i) + t^2(\triangle a_i^T(\overline{y} - \hat{y}) + \triangle q_i^T\triangle x) \\
&\quad + \hat{a}_i^T\hat{y} + \hat{q}_i^T\hat{x} - \hat{b}_i - (1-t)\|\hat{B}^i\hat{y}\|_2 - t\|\hat{B}^i\overline{y}\|_2 - t\|\triangle B^i\hat{y}\|_2 - t^2\|\triangle B^i(\overline{y} - \hat{y})\|_2 \qquad (2.4) \\
&= t(\hat{a}_i^T\overline{y} + \hat{q}_i^T\hat{x} - \hat{b}_i - \|\hat{B}^i\overline{y}\|_2 + \hat{q}_i^T\triangle x + \triangle a_i^T\hat{y} + \triangle q_i^T\hat{x} - \triangle b_i - \|\triangle B^i\hat{y}\|_2) \\
&\quad + t^2(\triangle a_i^T(\overline{y} - \hat{y}) + \triangle q_i^T\triangle x - \|\triangle B^i(\overline{y} - \hat{y})\|_2) + (1-t)(\hat{a}_i^T\hat{y} + \hat{q}_i^T\hat{x} - \hat{b}_i - \|\hat{B}^i\hat{y}\|_2) \\
&\geq t(\hat{\epsilon} + \hat{q}_i^T\triangle x + \triangle a_i^T\hat{y} + \triangle q_i^T\hat{x} - \triangle b_i - \|\triangle B^i\hat{y}\|_2) \\
&\qquad\qquad + t^2(\triangle a_i^T(\overline{y} - \hat{y}) + \triangle q_i^T\triangle x - \|\triangle B^i(\overline{y} - \hat{y})\|_2).
\end{aligned}$$

Therefore we have that, for $\|\triangle u\|$ small enough, there exits $\hat{t} > 0$ such that

$$\tilde{a}_i(t)^T y(t) + \tilde{q}_i(t)^T \tilde{x}(t) - \tilde{b}_i(t) - \|\tilde{B}^i(t)y(t)\|_2 \geq 0, i = 1, \ldots, l, \forall t \in [0, \hat{t}),$$

which implies $\hat{y} \in \liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u})$. The proof is completed. □

Define

$$\Psi(\tilde{u}, \alpha) = \Phi(\tilde{u}) \cap \mathrm{lev}_{\leq\alpha} f(\cdot, \tilde{u})$$

with

$$\mathrm{lev}_{\leq\alpha} f(\cdot, \tilde{u}) = \{y \in \Re^m : f(y, \tilde{u}) \leq \alpha\}, \alpha \in \Re.$$

Lemma 2.3. *For given $(c; A; Q; B; b)$, let Assumptions 2.1 and 2.2 hold. Then for any $\alpha \in \Re^n$, there exists $\delta_1 > 0$ and a bounded set $\mathcal{B} \subset \Re^m$ such that*

$$\Psi(\tilde{u}, \alpha') \subset \mathcal{B}, \forall \alpha' \leq \alpha, \forall \tilde{u} \in \mathcal{U}_{\delta_1}(c; A; Q; B; b).$$

Proof. Without loss of generality, we assume that $\Psi(\tilde{u},\alpha) \neq \emptyset$. Because $\Psi(\tilde{u},\alpha') \subset \Psi(\tilde{u},\alpha), \forall \alpha' \leq \alpha$, we only need to prove $\Psi(\tilde{u},\alpha) \subset \mathcal{B}$. We prove the result by contradiction. Suppose that there exist a sequence $\tilde{u}^k = (x^k, \tilde{\xi}^k)$ such that $x^k \in X$ and $\tilde{\xi}^k \to (c; A; Q; B; b)$ and $y^k \in \Psi(\tilde{u}^k, \alpha)$ with $\|y^k\| \to \infty$. Let $d_y^k = y^k/\|y^k\|$, and notice X is compact, we can find a subsequence k_j such that $x^{k_j} \to x$ and $d_y^{k_j} \to d_y$ for some $x \in X$ and $d_y \in \text{bdry}\mathbf{B}$. In view of $y^{k_j} \in \Psi(\tilde{u}^{k_j}, \alpha)$, one has

$$\begin{aligned}
&\tilde{c}^{k_j T} y^{k_j} \leq \alpha \\
&\tilde{a}_i^{k_j T} y^{k_j} + \tilde{q}_i^{k_j T} x^{k_j} - \tilde{b}_i^{k_j} \geq \|[\tilde{B}^{k_j}]^i y^{k_j}\|_2,\ i = 1, \cdots, l.
\end{aligned}$$

Dividing both sides of the above inequalities, we obtain

$$\begin{aligned}
&\tilde{c}^{k_j T} d_y^{k_j} \leq \alpha/\|y^{k_j}\| \\
&\tilde{a}_i^{k_j T} d_y^{k_j} + \tilde{q}_i^{k_j T} x^{k_j}/\|y^{k_j}\| - \tilde{b}_i^{k_j}/\|y^{k_j}\| \geq \|[\tilde{B}^{k_j}]^i d_y^{k_j}\|_2,\ i = 1, \cdots, l.
\end{aligned}$$

Taking the limits by $j \to \infty$, we have

$$c^T d_y \leq 0, a_i^T d_y \geq \|B^i d_y\|_2 > 0, i = 1, \ldots, l,$$

which contradicts with Assumption 2.2. Since the set of solutions to Problem (1.2) is compact, we have that such d_y must be zero. $\square$

In the following discussions, we need to adopt Proposition 4.4 of Bonnans and Shapiro(2000) [6]. For this, we consider the parameterized optimization problem of the form

$$(\mathrm{P}_u) \qquad \min_{x \in X} f(x,u) \quad \text{s.t.} \quad G(x,u) \in K, \tag{2.5}$$

where $u \in U$, X, Y and U are Banach spaces, K is a closed convex subset of Y. $f : X \times Y \to \Re$ and $G : X \times U \to Y$ are continuous. We denote by

$$\Phi(u) := \{x \in X : G(x,u) \in K\}$$

the feasible set of problem (P_u) and the optimal value function is

$$\nu(u) := \inf_{x \in \Phi(u)} f(x,u),$$

and the associated solution set

$$S(u) := \operatorname*{argmin}_{x \in \Phi(u)} f(x,u).$$

Proposition 2.1. *[6, Proposition 4.4] Let u_0 be a given point in the parameter space U. Suppose that*

(i) the function $f(x,u)$ is continuous on $X \times U$,

(ii) the multifunction $\Phi(\cdot)$ is closed,

(iii) there exist $\alpha \in \Re$ and a compact set $C \subset X$ such that every u in a neighborhood of u_0, the level set

$$lev_{\leq\alpha} f(\cdot, u) := \{x \in \Phi(u) : f(x, u) \leq \alpha\}$$

is nonempty and contained in C,

(iv) for any neighborhood $\mathcal{V}_X$ of the set $S(u_0)$ there exists a neighborhood $\mathcal{V}_U$ of u_0 such that $\mathcal{V}_X \cap \Phi(u)$ is nonempty for all $u \in \mathcal{V}_U$.

Then:

(a) the optimal value function $\nu(u)$ is continuous at $u = u_0$,

(b) the multifunction $S(u)$ is upper semicontinuous at u_0.

Theorem 2.4. *For given $(c; A; Q; B; b)$, let Assumptions 2.1,2.2 and 2.3 hold. For any $\widehat{u} \in \mathcal{U}_{\delta_1}(c; A; Q; B; b)$ with δ_1 defined in Lemma 2.3, one has that θ is continuous at $\widehat{u}$ and the solution set mapping Y^* is upper semi-continuous at $\widehat{u}$, namely for $\epsilon > 0$ there exists a number $\delta > 0$ such that*

$$Y^*(\tilde{u}) \subset Y^*(\widehat{u}) + \epsilon \boldsymbol{B}, \forall \tilde{u} \in \mathbb{B}_\delta(\widehat{u}).$$

Proof. Let

$$f(y, \tilde{u}) = \tilde{c}^T y, G_i(y, \tilde{u}) = \tilde{a}_i^T y + \tilde{q}_i^T \tilde{x} - \tilde{b}_i - \|\tilde{B}^i y\|_2, i = 1, \cdots, l \text{ and } K = \Re^l_+.$$

Then the constraint set $\Phi(\tilde{u})$ is expressed as

$$\Phi(\tilde{u}) = \{y \in \Re^m : G_i(y, \tilde{u}) \in K_i, i = 1, \cdots, l\}$$

and the problem is expressed in the setting of Proposition 2.1. Obviously we have that $f(y, \tilde{u})$ is continuous in $\Re^m \times \mathcal{U}_{\delta_1}(c; A; Q; B; b)$, namely condition (i) of Proposition 2.1 holds. From Lemma 2.2 and noticing the equivalence between the outer semi-continuity and the closedness for set-value mappings, we have that Φ is a closed set-value mapping so that (ii) of Proposition 2.1 holds. Condition (iii) of Proposition 2.1 comes from Lemma 2.3. Since Assumption 2.3 implies Robison constraint qualification for $\Phi(\widehat{u})$ at any point $\widehat{y} \in Y^*(\widehat{u})$. Then it follows form Theorem 2.87 in [6] that

$$\text{dist}(\widehat{y}, \Phi(\tilde{u})) \leq \kappa(\text{dist}\,(G(\widehat{y}, \tilde{u}), K)) \leq \kappa \|G(\widehat{y}, \tilde{u}) - G(\widehat{y}, \widehat{u})\| \tag{2.6}$$

for $\tilde{u} \in \mathcal{V}_U$, where $\mathcal{V}_U$ is some neighborhood of $\widehat{u}$ and $\kappa > 0$. Since G is Lipschitz continuous, we have that condition (iv) of Proposition 2.1 holds.

Therefore, we have from Proposition 2.1 that the optimal value function θ is continuous at at $\widehat{u}$ and the solution set $Y^*(\tilde{u})$ is upper semicontinuous at $\widehat{u}$, namely for $\epsilon > 0$ there exists a number $\delta_2 > 0$ such that

$$Y^*(\tilde{u}) \subset Y^*(\widehat{u}) + \epsilon \mathbf{B}, \forall \tilde{u} \in \mathbb{B}_{\delta_2}(\widehat{u}).$$

The proof is completed. □

3 Upper continuity of dual solution mapping

First of all, we derive the Lagrange dual of the second-order conic optimization problem (2.1). The Lagrangian function of problem (2.1) is defined by

$$L(y, \tilde{\lambda}; \tilde{u}) = \tilde{c}^T y - \sum_{i=1}^{l} \langle \tilde{\lambda}^i, g^i(y, \tilde{x}; \tilde{\xi}) \rangle, \; \tilde{\lambda} = (\tilde{\lambda}^1; \ldots; \tilde{\lambda}^l). \tag{3.7}$$

The Lagrangian function can expressed as

$$L(y, \tilde{\lambda}; \tilde{u})) = \tilde{c}^T y - \langle \tilde{\lambda}, \tilde{\mathcal{A}} y \rangle - \sum_{i=1}^{l} \tilde{\lambda}^i_{J_i+1}(\tilde{q}_i^T x - \tilde{b}_i),$$

where $\tilde{\mathcal{A}} : \Re^m \to \Re^{J_1+1} \times \cdots \times \Re^{J_l+1}$ is a linear operator defined by

$$\tilde{\mathcal{A}} y = \big((\tilde{B}^1 y, \tilde{a}_1^T y); \cdots; (\tilde{B}^l y, \tilde{a}_l^T y) \big). \tag{3.8}$$

Then the Lagrange dual of Problem (2.1) becomes

$$\begin{aligned} \max \quad & \sum_{i=1}^{l} \tilde{\lambda}^i_{J_i+1} \left(\tilde{b}_i - \tilde{q}_i^T x \right) \\ \text{s.t.} \quad & \tilde{c} - \tilde{\mathcal{A}}^* \tilde{\lambda} = 0, \\ & \tilde{\lambda} \in \mathcal{Q}, \end{aligned} \tag{3.9}$$

where $\tilde{\mathcal{A}}^*$ is the adjoint of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^* \tilde{\lambda}$ is calculated by

$$\tilde{\mathcal{A}}^* \tilde{\lambda} = \sum_{i=1}^{l} [\tilde{B}^{iT} \; \tilde{a}_i] \tilde{\lambda}^i. \tag{3.10}$$

We denote the feasible set for Problem (3.9) by

$$\mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) = \{ \tilde{\lambda} = (\tilde{\lambda}^1; \cdots; \tilde{\lambda}^l) \in \mathcal{Q} : \tilde{c} - \tilde{\mathcal{A}}^* \tilde{\lambda} = 0 \}, \tag{3.11}$$

where $\tilde{\mathcal{A}}$ is defined by (3.8).

We denote $\phi(\tilde{\lambda}, \tilde{u}) = \sum_{i=1}^{l} \tilde{\lambda}^i_{J_i+1} \left(\tilde{b}_i - \tilde{q}_i^T x \right)$ the objective function for problem (3.9) and by $\Lambda^*(\tilde{u})$ the set of optimal solutions of problem (3.9).

Lemma 3.1. *Let (c, A, B) be given. If Assumption 2.2 holds, then there exists $\delta_2 > 0$ such that the Slater condition for problem (3.9) holds when $\|(\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B)\| \leq \delta_2$, namely there exists $\tilde{\lambda}_{(\tilde{c},\tilde{A},\tilde{B})}$ such that*

$$\tilde{c} - \tilde{\mathcal{A}}^* \tilde{\lambda}_{(\tilde{c},\tilde{A},\tilde{B})} = 0,\ \tilde{\lambda}_{(\tilde{c},\tilde{A},\tilde{B})} \in \operatorname{int} \mathcal{Q}$$

when $\|(\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B)\| \leq \delta_2$.

Proof. From Assumption 2.2, we know that Slater condition for Problem (3.9) holds, namely there exists a λ such that

$$c - \mathcal{A}^*\lambda = 0,\ \lambda \in \operatorname{int}\mathcal{Q}. \tag{3.12}$$

where $\mathcal{A}y = ((B^1 y, a_1^T y); \ldots; (B^l y, a_l^T y))$. The operator $\mathcal{A}^*$ is onto when Assumption 2.2 holds. In fact, suppose that there exist $d_y \in \Re^m$ such that $\mathcal{A}d_y = 0$, which implies that $B^i d_y = 0, a_i^T d_y = 0, i = 1, \ldots, l$. We obtain from (3.12) that $c^T d_y = 0$. Therefore we obtain $d_y \in Y^*(u)^\infty$ and this implies $d_y = 0$ because otherwise $Y^*(u)$ is unbounded, a contradiction with Assumption 2.2. Thus we have that $\ker \mathcal{A} = \{0\}$ and operator $\mathcal{A}^*$ is onto.

Define $M = [(B^{1T}, a_1), \ldots, (B^{lT}, a_l)]$, in view of (3.10) for $\mathcal{A}^*$, we have that matrix M is of row full rank.

The validity of Slater condition for Problem (3.9) is equivalent to the solvability of the following system in variable $\tilde{\lambda}$:

$$\tilde{c} - \tilde{\mathcal{A}}^*\tilde{\lambda} = 0,\ \tilde{\lambda} \in \operatorname{int} \mathcal{Q}. \tag{3.13}$$

For $(\tilde{c}, \tilde{\mathcal{A}}) = (c, \mathcal{A}) + (\Delta c, \Delta\mathcal{A})$ with $(\tilde{c}, \tilde{A}, \tilde{B}) = (c, A, B) + (\Delta c, \Delta A, \Delta B)$ and $\tilde{\lambda} = \lambda + \Delta\lambda$, the first equality in (3.13) is equivalent to

$$\begin{aligned} 0 &= \tilde{c} - (\mathcal{A}^* + \Delta\mathcal{A}^*)(\lambda + \Delta\lambda) \\ &= c + \Delta c - (\mathcal{A}^* + \Delta\mathcal{A}^*)(\lambda + \Delta\lambda) \\ &= \Delta c - \mathcal{A}^*\Delta\lambda - \Delta\mathcal{A}^*\lambda - \Delta\mathcal{A}^*\Delta\lambda \end{aligned}$$

or

$$-\tilde{M}\Delta\lambda = -\Delta c + \Delta M\lambda. \tag{3.14}$$

Since $\ker\mathcal{A} = \{0\}$ implies that matrix M is full rank in row, we have that MM^T is positive definite. Let $\Delta\mathcal{N} = \Delta MM^T + M\Delta M^T + \Delta M\Delta M^T$, then when ΔM is small enough, $\tilde{M}\tilde{M}^T = MM^T + \Delta\mathcal{N}$ is nonsingular. We assume that $\delta_3 > 0$ satisfies that $\tilde{M}\tilde{M}^T$ is nonsingular $\|\Delta M\| \leq \delta_3$. Then we obtain from Sherman-Morrison-Woodbury formula that

$$\begin{aligned} &\tilde{M}^\dagger = \tilde{M}^T(\tilde{M}\tilde{M}^T)^{-1} \\ &= (M^T + \Delta M^T)(MM^T + \Delta\mathcal{N})^{-1} \\ &= (M^T + \Delta M^T)[(MM^T)^{-1} - (MM^T)^{-1}\Delta\mathcal{N}[I_m + (MM^T)^{-1}\Delta\mathcal{N}]^{-1}(MM^T)^{-1}] \\ &= M^\dagger + \Delta\Sigma, \end{aligned}$$

where $\Delta\Sigma$ satisfies $\|\Delta\Sigma\| = \mathrm{O}(\|\Delta M\|)$. Since $\tilde{M}\tilde{M}^T$ is nonsingular when $\|\Delta M\| \le \delta_3$, we have that

$$\Delta\lambda^*(\Delta M) := -\tilde{M}^\dagger(-\Delta c + \Delta M\lambda) = -[M^\dagger + \Delta\Sigma](-\Delta c + \Delta M\lambda) \tag{3.15}$$

is a particular solution to (3.14). From the expression for $\Delta\lambda^*$ in (3.15), we may assume that $\delta_3 > 0$ small enough such that $\|\Delta\mu^*(\Delta M)\| < \min\left\{\|\lambda\|/2, (\|M^\dagger\| + \delta_3)(1 + \|\bar{\lambda}\|)\delta_3\right\}$ when $\|(\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B)\| \le \delta_3$. Therefore, for $\Delta\lambda = \Delta\lambda^*$, we have that

$$\tilde{\lambda} := \lambda + \Delta\lambda$$

satisfies (3.13) when $\|(\tilde{c}, \tilde{A}, \tilde{B}) - (c, A, B)\| \le \delta_3$. The proof is completed. □

Lemma 3.2. *Let (c, A, B) be given with Assumption 2.2 being satisfied. Then, for any $(\widehat{c}, \widehat{A}, \widehat{B}) \in \mathbb{B}_{\delta_2}(c, A, B)$,*

$$\lim_{(\tilde{c},\tilde{A},\tilde{B})\to(\widehat{c},\widehat{A},\widehat{B})} \mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) = \mathcal{E}(\widehat{c}, \widehat{A}, \widehat{B}).$$

Proof. As the following inclusion

$$\limsup_{(\tilde{c},\tilde{A},\tilde{B})\to(\widehat{c},\widehat{A},\widehat{B})} \mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) \subset \mathcal{E}(\widehat{c}, \widehat{A}, \widehat{B}).$$

is obvious, we only need to verify that

$$\liminf_{(\tilde{c},\tilde{A},\tilde{B})\to(\widehat{c},\widehat{A},\widehat{B})} \mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B}) \supset \mathcal{E}(\widehat{c}, \widehat{A}, \widehat{B}).$$

For arbitrary $\widehat{\lambda} \in \mathcal{E}(\widehat{c}, \widehat{A}, \widehat{B})$, we now prove $\widehat{\lambda} \in \liminf_{(\tilde{c},\tilde{A},\tilde{B})\to(\widehat{c},\widehat{A},\widehat{B})} \mathcal{E}(\tilde{c}, \tilde{A}, \tilde{B})$. By Lemma 3.1, we have that there exists $\overline{\lambda}$ such that

$$\widehat{c} - \widehat{\mathcal{A}}^*\overline{\lambda} = 0,\ \overline{\lambda} \in \operatorname{int} \mathcal{Q}.$$

For $(\Delta c, \Delta\mathcal{A})$, let $(\tilde{c}(t), \tilde{\mathcal{A}}(t)) = (\widehat{c} + t\Delta c,\ \widehat{\mathcal{A}} + t\Delta\mathcal{A})$ with $(\tilde{c}(t), \tilde{B}(t), \tilde{A}(t)) = (\widehat{c} + t\Delta c, \widehat{B} + t\Delta B, \widehat{A} + t\Delta A)$, we obviously have $(\tilde{c}(t), \tilde{B}(t), \tilde{A}(t)) \to (\widehat{c},\ \widehat{B}, \widehat{A})$ as $t \downarrow 0$. Define $\lambda(t) = (\lambda^1(t); \ldots; \lambda^l(t))$ by

$$\lambda(t) = (1 - t)\widehat{\lambda} + t(\overline{\lambda} + d_\lambda(t)) = \widehat{\lambda} + t(\overline{\lambda} - \widehat{\lambda}) + td_\lambda(t). \tag{3.16}$$

We consider the system

$$\tilde{c}(t) - \tilde{\mathcal{A}}^*(t)\lambda(t) = 0. \tag{3.17}$$

Define

$$\tilde{\xi}(t) = \widehat{\lambda} + t(\overline{\lambda} - \widehat{\lambda}).$$

Then the equation (3.17) is equivalent to

$$\tilde{\mathcal{A}}^*(t)d_\lambda(t) = -(\Delta c - [\Delta \mathcal{A}]^*\tilde{\xi}(t)). \tag{3.18}$$

Let $d^*_\lambda(t)$ be the following least square norm solution to (3.18):

$$d^*_\lambda(t) = -[\tilde{\mathcal{A}}(t)^*]^\dagger(\Delta c - [\Delta \mathcal{A}]^*\tilde{\xi}(t)). \tag{3.19}$$

Similar to the analysis in the proof of Lemma 3.1, we obtain $\tilde{\mathcal{A}}(t)^\dagger = \widehat{\mathcal{A}}^\dagger + \mathrm{O}(t\|\Delta\mathcal{A}\|)$. We may assume that $\|\tilde{\mathcal{A}}(t)^\dagger\| \le 2\|\widehat{\mathcal{A}}^\dagger\|$ for $(\tilde{c},\tilde{A},\tilde{B}) \in \mathbb{B}_{\delta_3}(\widehat{c},\widehat{A},\widehat{B})$ when $t > 0$ small enough. Let

$$\kappa = 2\|\widehat{\mathcal{A}}^\dagger\|\max\{1,\|\widehat{\lambda}\|,\|\overline{\lambda}\|\} + 1.$$

Then for $\|(\Delta c,\Delta B,\Delta A)\| \le \delta_3$, one has

$$\|d^*_\lambda(t)\| \le \|\tilde{\mathcal{A}}(t)^\dagger\|\|\Delta c - \Delta\mathcal{A}^*\tilde{\xi}(t)\| \le 2\|\widehat{\mathcal{A}}^\dagger\|\max\{1,\|\widehat{\lambda}\|,\|\overline{\lambda}\|\}\|(\Delta c,\Delta B,\Delta A)\| < \kappa\delta_3.$$

Since $\overline{\lambda} \in \operatorname{int}\mathcal{Q}$ one has that $\overline{\lambda} + d^*_\lambda(t) \in \operatorname{int}\mathcal{Q}$ when $\delta_3 \le \delta_2$ is small enough. Then we obtain for $t \in [0,1]$ that

$$\lambda(t) = (1-t)\widehat{\lambda} + t(\overline{\lambda} - d^*_\lambda(t)) \in \mathcal{Q}$$

and satisfies (3.17). Therefore, when $(\tilde{c},\tilde{A},\tilde{B}) \in \mathbb{B}_{\delta_3}(\widehat{c},\widehat{A},\widehat{B})$ and for small $t > 0$, one has

$$\lambda(t) \in \mathcal{E}(\widehat{c} + t\Delta c,\ \widehat{\mathcal{A}} + t\Delta\mathcal{A})$$

and $\lambda(t) \to \widehat{\lambda}$. This implies $\widehat{\lambda} \in \liminf\limits_{(\tilde{c},\tilde{A},\tilde{B})\to(\widehat{c},\widehat{A},\widehat{B})} \mathcal{E}(\tilde{c},\tilde{A},\tilde{B})$. This proof is completed. □

Define

$$\Gamma(\tilde{u},\alpha) = \mathcal{E}(\tilde{c},\tilde{A},\tilde{B}) \cap \mathrm{lev}_{\ge\alpha}\phi(\cdot,\tilde{u}) \tag{3.20}$$

with

$$\mathrm{lev}_{\ge\alpha}\phi(\cdot,\tilde{u}) = \{\tilde{\lambda} \in \Re^{J_1+1} \times \cdots \times \Re^{J_l+1} : \phi(\tilde{\lambda},\tilde{u}) \ge \alpha\}, \alpha \in \Re.$$

Lemma 3.3. *For given $(c;A;Q;B;b)$, let Assumptions 2.1,2.2 and 2.3. Then for any $\alpha \in \Re^n$, there exists $\delta_3 > 0$ and a bounded set $\mathcal{D} \subset \Re^{J_1+1} \times \cdots \times \Re^{J_l+1}$ such that*

$$\Gamma(\tilde{u},\alpha') \subset \mathcal{D}, \forall\alpha' \ge \alpha, \forall\tilde{u} \in \mathcal{U}_{\delta_3}(c;A;Q;B;b).$$

Proof. Without loss of generality, we assume that $\Gamma(\tilde{u},\alpha) \neq \emptyset$. Because $\Gamma(\tilde{u},\alpha') \subset \Gamma(\tilde{u},\alpha), \forall\alpha' \le \alpha$, we only need to prove $\Gamma(\tilde{u},\alpha) \subset \mathcal{D}$.

We first prove that, for any $\tilde{\lambda} \in \Gamma(\tilde{u},\alpha)$, $\tilde{\lambda}$ is bounded by contradiction. Suppose that there exist a sequence $\tilde{u}^k = (x^k,\tilde{\xi}^k)$ such that $x^k \in X$ and $\tilde{\xi}^k \to (c;A;Q;B;b)$ and $\tilde{\lambda}^k \in \Gamma(\tilde{u}^k,\alpha)$ with

$\|\tilde{\lambda}^k\| \to \infty$. Let $\tilde{d}^k_\lambda = \tilde{\lambda}^k/\|\tilde{\lambda}^k\|$ and notice X is compact, we can find a subsequence k_j such that $x^{k_j} \to x$ and $\tilde{d}^{k_j}_\lambda \to \tilde{d}_\lambda$ for some $x \in X$ with, $\tilde{d}_\lambda \in \text{bdry}\mathbf{B}$. In view of $\tilde{\lambda}^{k_j} \in \Gamma(\tilde{u}^{k_j}, \alpha)$, one has

$$\begin{aligned}
&\sum_{i=1}^{l} [\tilde{\lambda}^i_{J_i+1}]^{k_j T}(\tilde{b}^{k_j}_i - \tilde{q}^{k_j T}_i x^{k_j}) \geq \alpha \\
&\tilde{c}^{k_j} - [\tilde{\mathcal{A}}^{k_j}]^* \tilde{\lambda}^{k_j} = 0, \\
&\tilde{\lambda}^{k_j} \in \mathcal{Q}.
\end{aligned} \tag{3.21}$$

Dividing the above inequalities by $\|\tilde{\lambda}^{k_j}\|$, we get

$$\begin{aligned}
&\sum_{i=1}^{l} [\tilde{d}^i_\lambda]^{k_j T}(\tilde{b}^{k_j}_i - \tilde{q}^{k_j T}_i x^{k_j}) \geq \alpha/\|\tilde{\lambda}^{k_j}\|, \\
&\tilde{c}^{k_j}/\|\tilde{\lambda}^{k_j}\| - [\tilde{\mathcal{A}}^{k_j}]^* \tilde{d}^{k_j}_\lambda = 0, \\
&\tilde{d}^{k_j}_\lambda \in \mathcal{Q}.
\end{aligned}$$

Taking the limits by $j \to \infty$, we have

$$\sum_{i=1}^{l} \tilde{d}^{iT}_\lambda (\tilde{b}_i - \tilde{q}^T_i x) \geq 0, \ \tilde{\mathcal{A}}^* \tilde{d}_\lambda = 0, \ \tilde{d}_\lambda \in \mathcal{Q}, \ \|\tilde{d}_\lambda\| = 1,$$

which contradicts with the compactness of the optimal solution set of problem (3.9), this is implied by Slater condition for primal SOCP problem (2.1) proved in Lemma 2.1. □

Theorem 3.1. *For given $(c; A; Q; B; b)$, let Assumptions 2.1,2.2 and 2.3 hold. For any $\widehat{u} \in \mathcal{U}_{\delta_3}(c; A; Q; B; b)$ with δ_3 defined in Lemma 3.3, one has that the solution set mapping Λ^* is upper semi-continuous at $\widehat{u}$, namely for $\epsilon > 0$ there exists a number $\delta > 0$ such that*

$$\Lambda^*(\tilde{u}) \subset \Lambda^*(\widehat{u}) + \epsilon \boldsymbol{B}, \forall \tilde{u} \in \mathbb{B}_\delta(\widehat{u}).$$

Proof. The results in this theorem can be proved by Lemma 3.2 and Lemma 3.3. The proof is similar to that of Theorem 2.4. We omit it here. □

4 Differentiability of optimal value function

From Lemma 2.3 and Lemma 3.3, we assume that for some $\delta_4 > 0$, $\alpha \in \Re$, and bounded sets $\mathcal{B} \in \Re^m$, $\mathcal{D} \subset \Re^{J_1+1} \times \cdots \times \Re^{J_l+1}$,

$$\Psi(\tilde{u}, \alpha) \subset \mathcal{B}, \Gamma(\tilde{u}, \alpha) \subset \mathcal{D}$$

for any $\tilde{x} \in X$ and $\|\tilde{\xi} - (c; A; Q; B; b)\| \leq \delta_4$. Therefore, by the Lagrange duality theory, the optimal value can be written as

$$\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x}) = \max_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \min_{y \in \mathcal{B}} L(y, \lambda; \tilde{u}). \tag{4.22}$$

Proposition 4.1. *For given* $(c; A; Q; B; b)$ *and* $x \in X$*, let Assumptions 2.1,2.2 and 2.3 hold. Then* $\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$ *is locally Lipschitz continuous around* $(c; A; Q; B; b, x)$*, namely there exists some* $\kappa \geq 0$ *depending on* $(c; A; Q; B; b, x)$ *such that*

$$|\theta(\tilde{u}) - \theta(u')| \leq \kappa \|\tilde{u} - u'\|, \tag{4.23}$$

when $\tilde{u}, u' \in \mathbb{B}_{\delta_5}(c; A; Q; B; b, x)$ *for some positive constant* $\delta_5 > 0$ *depending on* $(c; A; Q; B; b, x)$*. Here* $\tilde{u} = (\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$*,* $u' = (c', A', Q', B', b', x')$ *and*

$$\|\tilde{u} - u'\| = \|\tilde{c} - c'\| + \sum_{j=1}^{l} \|\tilde{B}^j - B'^j\| + \|\tilde{A} - A'\| + \|\tilde{Q} - Q'\| + \|\tilde{b} - b'\| + \|\tilde{x} - x'\|.$$

Proof. Since $\tilde{L}(\cdot, \cdot)$ is continuous, the max-min values of $\tilde{L}$ at $\tilde{u}$ and u' can be arrived. Let $(\tilde{y}, \tilde{\lambda}), (y', \lambda') \in \mathcal{B} \times [\mathcal{Q} \cap \mathcal{D}]$ satisfy

$$\theta(\tilde{u}) = L(\tilde{y}, \tilde{\lambda}; \tilde{u}), \theta(u') = L(y', \lambda'; u').$$

Without less of generality, we assume that $\theta(\tilde{u}) \leq \theta(u')$. Then we have

$$\begin{aligned}
&|\theta(\tilde{u}) - \theta(u')| \\
&= |\sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \inf_{y \in \mathcal{B}} L(y, \lambda, \tilde{u}) - \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} \inf_{y \in \mathcal{B}} L(y, \lambda, u')| \\
&= |L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(y', \lambda'; u')| \\
&= |L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(\tilde{y}, \lambda'; \tilde{u}) + L(\tilde{y}, \lambda'; \tilde{u}) - L(y', \lambda'; u')| \\
&\leq |L(\tilde{y}, \lambda'; \tilde{u}) - L(y', \lambda'; u')| \\
&\leq |L(\tilde{y}, \lambda'; \tilde{u}) - L(\tilde{y}, \lambda'; u')| \\
&\leq \sup_{y \in \mathcal{B}} \sup_{\lambda \in \mathcal{Q} \cap \mathcal{D}} |L(y, \lambda; \tilde{u}) - L(y, \lambda; u')|.
\end{aligned} \tag{4.24}$$

Choose $\delta_5 \leq \min\{\delta_4, \|Q\|\}$ and define

$$\kappa = \max\{1, \mathrm{Diam}(X), \mathrm{Diam}(\mathcal{B}) 2\|Q\|\} \times \mathrm{Diam}(\mathcal{D}).$$

Then, when $\|\tilde{u} - u'\| \leq \delta_5$, for $y \in \mathcal{B}$ and $\lambda \in \mathcal{Q} \cap \mathcal{D}$, we have

$$\begin{aligned}
&|L(y, \lambda; \tilde{u}) - L(y, \lambda; u')| \\
&= |[\tilde{c} - c']^T y + \langle \lambda, g(y, \tilde{x}; \tilde{\xi}) - g(y, x'; \tilde{\xi}') | \leq \|c - c'\| \|y\| \\
&+ \|\lambda\| \times \left\{ \left(\sum_{j=1}^{l} \|\tilde{B}^j - B'^j\| + \|\tilde{A} - A'\| \right) \|y\| + \|\tilde{Q}\| \|\tilde{x} - x'\| + \|\tilde{Q} - Q'\| \|x'\| + \|\tilde{b} - b'\| \right\} \\
&\leq \kappa \left\{ \|c - c'\| + \sum_{j=1}^{l} \|\tilde{B}^j - B'^j\| + \|\tilde{A} - A'\| + \|\tilde{Q} - Q'\| + \|\tilde{b} - b'\| + \|\tilde{x} - x'\| \right\} \\
&= \kappa \|\tilde{u} - u'\|.
\end{aligned}$$

Combing the above inequality with (4.24), we obtain the inequality (4.23) when $\tilde{u}, u' \in \mathbb{B}_{\delta_5}(c; A; Q; B; b, x)$. $\square$.

For the discussions in the following, we need use the result in Theorem 7.24 of [11], for this we consider the minimax problem

$$\min_{x \in X} \left\{ \phi(x) := \sup_{y \in Y} f(x, y) \right\} \tag{4.25}$$

where $X \subset \Re^n$ and $Y \subset \Re^m$ are convex and compact and the function $f : X \times Y \to \Re$ is continuous. Consider the perturbation of the minimax problem (4.25) :

$$\min_{x \in X} \sup_{y \in Y} \{ f(x, y) + t\eta_t(x, y) \}, \tag{4.26}$$

where $\eta_t(x, y)$ is continuous in $X \times Y$, $t \geq 0$. Moreover we assume that $f(x, y)$ is convex in $x \in X$ and concave in $y \in Y$. Denoted by $v(t)$ the optimal value of the above problem (4.26). Clearly $v(0)$ is the optimal value of the unperturbed problem (4.25). Then the following lemma holds.

Lemma 4.1. *[11, Theorem 7.24] Suppose that the following conditions hold:*

(i) the sets $X \subset \Re^n$ and $Y \subset \Re^m$ are convex and compact,

(ii) for all $t \geq 0$, the function $\zeta_t := f + t\eta_t$ is continuous on $X \times Y$, convex respects to $x \in X$ and concave respects to $y \in Y$,

(iii) η_t converges uniformly as $t \downarrow 0$ to a function $\gamma(x, y) \in C(X, Y)$.

Then we have

$$\lim_{t \downarrow 0} \frac{v(t) - v(0)}{t} = \inf_{x \in X^*} \sup_{y \in Y^*} \gamma(x, y).$$

Theorem 4.1. *For given $(c; A; Q; B; b)$ and $x \in X$, let Assumptions 2.1, 2.2 and 2.3 hold. Then the optimal value function $\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$ is directionally differentiable at (c, A, Q, B, b, x). Moreover, $\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$ is Hadamard directionally differentiable at (c, A, Q, B, b, x). Thus we have the the following Taylor expansion of $\theta(\tilde{u})$ at u*

$$\theta(\tilde{u}) = \theta(u) + \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} (\Delta c - \Delta \mathcal{A}^* \lambda)^T y + \sum_{i=1}^{l} \lambda^i_{J_i+1} (\Delta b_i - \Delta q_i^T x - q_i^T \Delta x) + \mathrm{o}(\|\Delta u\|), \tag{4.27}$$

where $\tilde{u} = (\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$, $u = (c, A, Q, B, b, x)$ and $\Delta u = \tilde{u} - u$ satisfying $\|\Delta u\| \leq \delta_4$.

Proof. In the setting of Lemma 4.1, for the direction $\tilde{u}-u$, we define

$$\zeta_t = L(y,\lambda;u_t), \quad f = L(y,\lambda;u) \tag{4.28}$$

where $u_t = u + t\Delta u$ and $u_t = (c_t, A_t, Q_t, B_t, b_t, x_t)$, then we can write

$$\lim_{t\downarrow 0} \frac{L(y,\lambda;u_t) - L(y,\lambda;u)}{t} = \gamma(y,\lambda).$$

Because of the Lagrange function (3.7), we have that ζ_t is continuous, convex respects to $y \in \mathcal{B}$ and concave respects to $\lambda \in [\mathcal{Q} \cap \mathcal{D}]$. For the convex and compact set of saddle points $Y^*(u) \times \Lambda^*(u)$, the directional derivative of θ at u in direction Δu can be derived by Lemma 4.1 as follows:

$$\theta'(u;\Delta u) = \lim_{t\downarrow 0} \frac{\theta(u_t)-\theta(u)}{t} = \inf_{y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} \lim_{t\downarrow 0} \frac{L(y,\lambda;u_t) - L(y,\lambda;u)}{t}$$

$$= \inf_{y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} \lim_{t\downarrow 0} \frac{1}{t}\left((c_t - c)^T y - \langle \lambda, (\mathcal{A}_t - \mathcal{A})y\rangle + \sum_{i=1}^{l} \lambda^i_{J_i+1}((b_t-b)_i - [(q_i)_t]^T x_t + q_i^T x)\right)$$

$$= \inf_{y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} \lim_{t\downarrow 0} \left((\Delta c - \Delta\mathcal{A}^*\lambda)^T y + \sum_{i=1}^{l} \lambda^i_{J_i+1}(\Delta b_i - \Delta q_i^T x - q_i^T \Delta x - t\Delta q_i^T \Delta x)\right)$$

$$= \inf_{y\in y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} (\Delta c - \Delta\mathcal{A}^*\lambda)^T y + \sum_{i=1}^{l} \lambda^i_{J_i+1}(\Delta b_i - \Delta q_i^T x - q_i^T \Delta x).$$

Combining with the Lipschitz continuity of $\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$ from Proposition 4.1, we have that $\theta(\tilde{c}, \tilde{A}, \tilde{Q}, \tilde{B}, \tilde{b}, \tilde{x})$ is Hadamard directionally differentiable at u. Therefore we obtain the Taylor expansion of $\theta(\tilde{u})$ at u:

$$\theta(\tilde{u}) = \theta(u) + \inf_{y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} (\Delta c - \Delta\mathcal{A}^*\lambda)^T y + \sum_{i=1}^{l} \lambda^i_{J_i+1}(\Delta b_i - \Delta q_i^T x - q_i^T \Delta x) + \mathrm{o}(\|\Delta u\|).$$

The proof is completed. □

5 Conclusions

We consider the stability of a second-order conic optimization problem when all parameters in the problem are perturbed. Under Slater constraint qualification, we prove the upper semi-continuity of the solution sets of both the original problem and the dual problem. Furthermore, we show that the optimal value function is locally Lipschitz continuous and Hadamard directionally differentiable. Interestingly, as we express the optimal value function as a min-max optimization problem over two compact convex sets, we may use Theorem 7.59 of [11] to discuss the asymptotic distribution of the optimal value when ξ is a random variable and the sample average approach is adopted.

References

[1] Bonnans, J. F., Shapiro, A., *Perturbation analysis of optimization problems*, Springer, New York, 2000.

[2] B. Bereanu, *The continuity of the optimum in parametric programming and applications to stochastic programming*, Journal of Optimization Theory and Applications, **18**(1976), 319–332.

[3] Youpan Han and Zhiping Chen, *Quantitative stability of full random two-stage stochastic programs with recourse*,Optim. Lett., **9**(2015),1075-1090.

[4] A.L. Dontchev and R.T. Rockafellar. *Implicit Functions and Solution Mappings*, Springer, New York, 2009.

[5] D. Bertsimas, *Theory and applications of robust optimization*, SIAM Review, **53**(2011), 464–501.

[6] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Series in Operations Research, New York, 2000.

[7] M.S. Gowda and J.-S. Pang, *On solution stability of linear complementarity problem*, Mathematics of Operation Reseach, **17**(1992), 77-83.

[8] M.S. Gowda and J.-S. Pang, *On the boundedness and stability of solutions to the affine variational inequality problem*, Mathematics of Operation Reseach, **32**(1994), 421-441.

[9] D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Mathematics of Operation Reseach, **28**(2003), 1–38.

[10] G. M. Lee, N.N. Tam and N.D. Yen, *Quadratic Programming and Affine Variational Inequalities, A Qualitative Study*,Springer, New York, 2005.

[11] A. Shapiro, D. Dentcheva, A. Ruszczyński, *Lectures on Stochastic Programming Modeling and Theory*, SIAM, Philadelphia, 2009.

[12] W. Römisch and R. J.-B. Wets, *Stability of ε-approximate solutions to convex stochastic programs*, SIAM J. Optim., **18**(3)(2007),961-979.