A Merit Function Approach for Evolution Strategies

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Abstract

In this paper, we extend a class of globally convergent evolution strategies to handle general constrained optimization problems. The proposed framework handles relaxable constraints using a merit function approach combined with a specific restoration procedure. The unrelaxable constraints in our framework, when present, are treated either by using the extreme barrier function or through a projection approach.

The introduced extension guaranties to the regarded class of evolution strategies global convergence properties for first order stationary constraints. Preliminary numerical experiments are carried out on a set of known test problems as well as on a multidisciplinary design optimization problem.

Keywords: Derivative-free optimization, evolution strategies, merit function, global convergence.

1 Introduction

In this paper, we are interested by the following constrained optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \Omega = \Omega_{\rm r} \cap \Omega_{\rm nr}, \end{array} \tag{1}$$

where the objective function f is assumed to be locally Lipschitz continuous. The feasible region $\Omega \subset \mathbb{R}^n$ of this problem includes two categories of constraints. The first one, denoted by $\Omega_{\rm r}$ and known as relaxable constraints (or soft constraints), is allowed to be violated during the optimization process and may need to be satisfied only approximately or asymptotically. Such set of constraints will be assumed, in the context of this paper, to be of the form:

$$\Omega_{\rm r} = \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, r\}, c_i(x) \le 0\},$$

where the functions c_i are locally Lipschitz continuous. The second category of constraints, denoted by $\Omega_{\rm nr} \subset \mathbb{R}^n$, gathers all unrelaxable constraints (also known as hard constraints), for such constraints no violation is allowed and they require satisfaction during all the optimization process. The set of constraints $\Omega_{\rm nr}$ can be seen as bounds or linear constraints. Many practical optimization problems present both relaxable and unrelaxable constraints. For instance, in

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multidisciplinary design optimization problems [33], one has to deal with different coupled disciplines (e.g., structure, aerodynamics, propulsion) that represent the aircraft model. In this case, the appropriate design can be chosen by maximizing the aircraft range under bounds constraints and subject to many relaxable constraints composed of the involved disciplines.

Evolution strategies (ES's) [34] are one of the most successful stochastic optimization algorithms, seen as a class of evolutionary algorithms that are naturally parallelizable, appropriate for continuous optimization, and that lead to promising results on practical optimization problems [7, 35, 8]. In [15, 16], the authors dealt with a large class of ES's, where a certain number λ of points (called offspring) are randomly generated in each iteration, among which $\mu \leq \lambda$ of them (called parents) are selected. ES's have been growing rapidly in popularity and start to be used for solving challenging optimization problems [21, 6].

In [16], the authors proposed a general globally convergent framework for unrelaxable constraints using two different approaches. The first one relies on techniques inspired from directional direct-search methods [13, 26], where one uses an extreme barrier function to prevent unfeasible displacements together with the possible use of directions that conform to the local geometry of the feasible region. The second approach was based on enforcing all the generated sample points to be feasible, all by using a projection mapping approach. Both proposed strategies were compared to some of the best available solvers for minimizing a function without derivatives. The obtained numerical results confirmed the competitiveness of the two approaches in terms of efficiency as well as robustness. Motivated by the recent availability of massively parallel computing platforms, the authors in [14] proposed a highly parallel globally convergent ES (inspired by [16]) adapted to the full-waveform inversion setting. By combining model reduction and ES's in a parallel environment, the authors helped solving realistic instances of the full-waveform inversion problem.

For ES's enormous state of the art of constraints handling algorithms have been proposed [10]. Coello [11] and Kramer [28] outlined a comprehensive survey of the most popular handling constraints methods currently used with ES's. To the best of our knowledge, all the proposed ES's suffer from the lack of global convergence guarantees when applied to general constrained optimization problems.

In the framework of deterministic Derivative free optimization (DFO), only few works were interested to handle both kinds (relaxable and unrelaxable) of constraints separately. For instance, Audet and Dennis [5] outlines a globally convergent direct-search approach based on a progressive barrier, it combines an extreme barrier approach for unrelaxable constraints and non-dominance filters [17] to handle relaxable constraints. More recently, the authors in [2] extended the progressive barrier approach, developed in [5], to cover the setting of a derivative-free trust-region method. In the frame of directional direct-search methods, Vicente and Gratton [19] proposed an alternative where one handles relaxable constraints by means of a merit function. The latter approach ensures global convergence by imposing a sufficient decrease condition on a merit function combining information from both objective function and constraints violation. Another two-phases derivative-free approach is proposed in [30] to handle specifically the case where finding a feasible point is easier than minimizing the objective function.

In this paper, inspired by the merit function approach for direct search methods [19], we propose to adapt the class of ES algorithms proposed in [16] to handle both relaxable and unrelaxable constraints. The obtained class of ES algorithms relies essentially on a merit function (eventually with a restoration procedure) to decide and control the distribution of the offspring points. The merit function is a standard penalty-based function that has been already proposed in the context of ES [11]. The main advantage of the proposed approach is guaranteeing a form of global convergence. To the best of our knowledge, this paper presents the first globally convergent framework that handles relaxable and unrelaxable constraints in the context of ES's.

The proposed convergence theory generalizes the ES framework in [16] by including relaxable constraints, all in the spirit of the proposed merit function for directional direct search methods [19]. The contributions of this paper are the following. We propose an adaptation of the merit function approach algorithm to the ES setting, a detailed convergence theory of the proposed approach is given. We provide also a practical implementation on some known global optimization problems as well as some tests on a multidisciplinary design optimization (MDO) problem. The performance of our proposed solver are compared to the progressive barrier approach implemented in mesh adaptive direct search (MADS) solver [5].

The paper is organized as follows. Section 2 reminds the class of ES algorithms proposed in [16] to handle unrelaxable constraints. The proposed merit function approach is then given in Section 3 with a detailed description of the changes introduced in a class of ES algorithms in order to handle general constraints. The convergence results of the adapted approach are then detailed in Section 4. In Section 5, we test the proposed algorithm on well-known constrained optimization test problems and an MDO problem. Finally, we conclude the paper in Section 6 with some conclusions and prospects of future work.

2 A globally convergent ES for unrelaxable constraints

This paper focuses on a class of ES's, denoted by $(\mu/\mu_W, \lambda)$–ES, which evolves a single candidate solution. In fact, at the k-th iteration, a new population $y^1_{k+1}, \ldots, y^\lambda_{k+1}$ (called offspring) is generated around a weighted mean x_k of the previous parents (candidate solution). The symbol "$/\mu_W$" in $(\mu/\mu_W, \lambda)$–ES specifies that μ parents are "recombined" into a weighted mean. The parents are selected as the μ best offspring of the previous iteration in terms of the objective function value. The mutation operator of the new offspring points is done by $y^i_{k+1} = x_k + \sigma^{\mathrm{ES}}_k d^i_k$, $i = 1, \ldots, \lambda$, where d^i_k is drawn from a certain distribution $\mathcal{C}_k$ and σ^{ES}_k is a chosen step size. The weights used to compute the means belong to the simplex set $S = \{(\omega^1, \ldots, \omega^\mu) \in \mathbb{R}^\mu : \sum_{i=1}^{\mu} w^i = 1, w^i \geq 0, i = 1, \ldots, \mu\}$. The $(\mu/\mu_W, \lambda)$–ES adapts the sampling distribution to the landscape of the objective function. An adaptation mechanism for the step size parameter is also possible. The latter one increases or decreases depending on the landscape of the objective function. One relevant instance of such an ES is covariance matrix adaptation ES (CMA-ES) [22].

In [15, 16], the authors proposed a framework for making a class of ES's enjoying some global convergence properties while solving optimization problems possibly with unrelaxable constraints. In fact, in [15], by imposing a sufficient decreasing condition on the objective function value, the proposed algorithm was monitoring the step size σ_k to ensure its convergence to zero (which leads then to the existence of a stationary point). The imposed sufficient decreasing condition applied directly to the weighted mean x^{trial}_{k+1} of the new parents. By sufficient decreasing condition we mean $f(x^{\mathrm{trial}}_{k+1}) \leq f(x_k) - \rho(\sigma_k)$, where $\rho(\cdot)$ is a forcing function [26], i.e., a positive, nondecreasing function satisfying $\rho(\sigma)/\sigma \to 0$ when $\sigma \to 0$. To handle unrelaxable constraints [16], one starts with a feasible iterate x_0 and then prevents stepping outside the feasible region by means of a barrier approach. In this context, the sufficient decrease condition is applied not to f but to the extreme barrier function $f_{\Omega_{\mathrm{nr}}}$ associated to f with respect to the constraints set Ω_{nr} [4] (also known as the death penalty function in the terminology of evolutionary algorithms), which is defined by:

$$f_{\Omega_{\mathrm{nr}}}(x) \;=\; \begin{cases} f(x) & \text{if } x \in \Omega_{\mathrm{nr}}, \\ +\infty & \text{otherwise.} \end{cases} \qquad (2)$$

We consider that ties of $+\infty$ are broken arbitrarily in the ordering of the offspring samples. The obtained globally convergent ES is given by Algorithm 1.

Algorithm 1: A globally convergent ES for unrelaxable constraints ($\Omega = \Omega_{\mathrm{nr}}$)

Data: choose positive integers λ and μ such that $\lambda \geq \mu$. Select an initial $x_0 \in \Omega_{\mathrm{nr}}$ and evaluate $f(x_0)$. Choose initial step lengths $\sigma_0, \sigma_0^{\mathrm{ES}} > 0$ and initial weights $(\omega_0^1, \ldots, \omega_0^\mu) \in S$. Choose constants $\beta_1, \beta_2, d_{\min}, d_{\max}$ such that $0 < \beta_1 \leq \beta_2 < 1$ and $0 < d_{\min} < d_{\max}$. Select a forcing function $\rho(\cdot)$.

for $k = 0, 1, \ldots$ **do**

Step 1: compute new sample points $Y_{k+1} = \{y_{k+1}^1, \ldots, y_{k+1}^\lambda\}$ such that

$$y_{k+1}^i \;=\; x_k + \sigma_k \tilde{d}_k^i, \; i = 1, \ldots, \lambda,$$

where the directions $\tilde{d}_k^i$'s are computed from the original ES directions d_k^i's (which in turn are drawn from a chosen ES distribution $\mathcal{C}_k$ and scaled if necessary to satisfy $d_{\min} \leq \|d_k^i\| \leq d_{\max}$);

Step 2: evaluate $f_{\Omega_{\mathrm{nr}}}(y_{k+1}^i)$, $i = 1, \ldots, \lambda$, and reorder the offspring points in $Y_{k+1} = \{\tilde{y}_{k+1}^1, \ldots, \tilde{y}_{k+1}^\lambda\}$ by increasing order: $f_{\Omega_{\mathrm{nr}}}(\tilde{y}_{k+1}^1) \leq \cdots \leq f_{\Omega_{\mathrm{nr}}}(\tilde{y}_{k+1}^\lambda)$.
Select the new parents as the best μ offspring sample points $\{\tilde{y}_{k+1}^1, \ldots, \tilde{y}_{k+1}^\mu\}$, and compute their weighted mean

$$x_{k+1}^{\mathrm{trial}} \;=\; \sum_{i=1}^{\mu} \omega_k^i \tilde{y}_{k+1}^i.$$

Evaluate $f_{\Omega_{\mathrm{nr}}}(x_{k+1}^{\mathrm{trial}})$;

Step 3: if $f_{\Omega_{\mathrm{nr}}}(x_{k+1}^{\mathrm{trial}}) \leq f(x_k) - \rho(\sigma_k)$ **then**

consider the iteration successful, set $x_{k+1} = x_{k+1}^{\mathrm{trial}}$, and $\sigma_{k+1} \geq \sigma_k$ (for example $\sigma_{k+1} = \max\{\sigma_k, \sigma_k^{\mathrm{ES}}\}$).;

else

consider the iteration unsuccessful, set $x_{k+1} = x_k$ and $\sigma_{k+1} = \bar{\beta}_k \sigma_k$, with $\bar{\beta}_k \in (\beta_1, \beta_2)$;

end

Step 4: update the ES step length $\sigma_{k+1}^{\mathrm{ES}}$, the distribution $\mathcal{C}_{k+1}$, and the weights $(\omega_{k+1}^1, \ldots, \omega_{k+1}^\mu) \in S$;

end

We note that, due to the local geometry of the unrelaxable constraints around the current point x_k, the directions $\tilde{d}_k^i$ used to compute the offspring are not necessarily randomly generated following a pure ES paradigm. In fact, two approaches were proposed in [16], the first one was based on the use of the extreme barrier function and also the inclusion of positive generators of the tangent cone of the constraints. In this case, whenever the current iterate x_k is getting close to the boundary of the feasible region, the set of directions $\{\tilde{d}_k^i\}$ will include (in addition to the ES randomly generated directions d_k^i) positive generators of the tangent cone. The second proposed approach for generating the set of directions $\{\tilde{d}_k^i\}$ is based on projecting onto the feasible domain all the generated sampled points $x_k+\sigma_k d_k^i$, and then taking instead $\Phi_{\Omega_{\rm nr}}(x_k+\sigma_k d_k^i)$ where $\Phi_{\Omega_{\rm nr}}$ is a given projection operator on the constraints set $\Omega_{\rm nr}$. This procedure is the same as considering $\tilde{d}_k^i = \frac{\Phi_{\Omega_{\rm nr}}(x_k+\sigma_k d_k^i)-x_k}{\sigma_k}$ in the framework of Algorithm 1. For typical choices of the projection $\Phi_{\Omega_{\rm nr}}$, one can use the ℓ_2-projection in the case of bound constraints (as it is trivial to evaluate), in the case of linear constraints one may use the ℓ_1-projection (as it reduces to the solution of an LP problem). For both approaches, we note that Steps 2 and 3 of Algorithm 1 make use of the extreme barrier function (2).

Due to the sufficient decrease condition, one can guarantee that a subsequence of step sizes will converge to zero. From this property and the fact that the step size is significantly reduced (at least by β_2) in unsuccessful iterations, one proves that there exists a subsequence K of unsuccessful iterates driving the step size to zero [16, Lemma 2.1]. The global convergence is then achieved by establishing that some type of directional derivatives are nonnegative at limit points of refining subsequences along certain limit directions (see [16, Theorem 2.1]).

3 A globally convergent ES for general constraints

The challenge of this paper consists in changing Algorithm 1, in a minimal way, to a globally convergent framework that takes into account both relaxable constraints and unrelaxable constraints. For this, we define a merit function as follows:

$$M(x) \;=\; \begin{cases} f(x) + \bar{\delta} g(x) & \text{if } x \in \Omega_{\rm nr}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

where $\bar{\delta} > 0$ is a given positive constant and g defines a constraint violation function with respect to relaxable constraints. The ℓ_1-norm is commonly used to define the constraint violation function, i.e., $g(x) = \sum_{i=1}^r \max(c_i(x), 0)$. Other choices for g exist, for instance, using the ℓ_2-norm i.e., $g(x) = \sum_{i=1}^r \max(c_i(x), 0)^2$. The merit function will be used to evaluate a trial step and hence decide whether such step will be accepted or not. The extension of the globally convergent ES (given by Algorithm 1) to a general constrained setting can be seen as a combination of two approaches, a feasible one where either the extreme barrier or a projection operator will be used to handle the unrelaxable constraints, and a merit function approach (possibly with a restoration procedure) to handle relaxable constraints.

The description of the proposed framework is as follows. For a given iteration k, a trial mean parent $x_{k+1}^{\rm trial}$ is computed as the weighted mean of the μ best points in terms of the merit function value. The current trial mean parent will be considered as a "***Successful point***" if one of the two following situations occur. The first scenario happens when one is sufficiently away from the feasible region (i.e., $g(x_k) > C\rho(\sigma_k)$ for some constant $C > 1$) and $x_{k+1}^{\rm trial}$ sufficiently decreases the constraint violation function g is observed (i.e., $g_{\Omega_{\rm nr}}(x_{k+1}^{\rm trial}) < g(x_k) - \rho(\sigma_k)$, where $g_{\Omega_{\rm nr}}$ denotes the extreme barrier function associated to g with respect to $\Omega_{\rm nr}$). The second situation occurs when the merit function is sufficiently decreased (i.e., $M(x_{k+1}^{\rm trial}) < M(x_k) - \rho(\sigma_k)$).

Before checking whether the trial point is successful or not, the algorithm will try first to restore the feasibility or at least decrease the constraints violation if needed. The restoration process will be activated if the current mean parent x_k is far away from the feasible region and the trial point x_{k+1}^{trial} sufficiently decreases the constraint violation function g but not the merit function. More specifically, a "***Restoration identifier***" will be activated if one has

$$g_{\Omega_{\text{nr}}}(x_{k+1}^{\text{trial}}) \ < \ g(x_k) - \rho(\sigma_k) \quad \text{and} \quad g(x_k) > C\rho(\sigma_k)$$

and

$$M(x_{k+1}^{\text{trial}}) \ \geq \ M(x_k).$$

The restoration algorithm will be left as far as progress on the reduction of the constraint violation can not be achieved all without any considerable increase in f. The complete description of the restoration procedure is given in Algorithm 3.

As a result, the main iteration of the proposed merit function approach can be devided to two steps: restoration and minimization. In the restoration step the aim is to decrease infeasibility (by minimizing essentially the function $g_{\Omega_{\text{nr}}}$) while in the minimization step the objective function f is improved over a relaxed set of constraints by using the merit function M. The final obtained approach is described is given in Algorithm 2.

For both algorithms (main and restoration), our global convergence analysis will be done independently of the choice of the distribution $\mathcal{C}_k$, the weights $(\omega_k^1, \ldots, \omega_k^\mu) \in S$, and the step size σ_k^{ES} . Therefore and similarly to Algorithm 1, the update of the ES parameters is left unspecified. Note that one also imposes bounds on all directions d_k^i used by the algorithm. This modification is, however, very mild since the lower bound $d_{\min}$ can be chosen very close to zero and the upper bound $d_{\max}$ set to a very large number. The construction of the set of directions $\{\tilde{d}_k^i\}$ can be done with respect to the local geometry of the unrelaxable constraints as proposed in [16].

4 Global convergence

The convergence results presented in this section are in the vein of those first established for the merit function approach for direct search methods [19]. For the convergence analysis, we will consider a sequence of iterations generated by Algorithm 2 without any stopping criterion. The analysis is organized depending on the number of times where restoration is entered. To keep the presentation of the paper simpler, only the case where restoration is entered for finitely times will be treated in our convergence analysis of this section. For completeness, the analysis of the two other cases (namely when (a) an infinite run of consecutive steps inside Restoration or (b) one enters the restoration an infinite number of times) is given in Appendix A. The analysis of both cases shows that such behaviors would lead to feasibility and optimality results similar to the case where the restoration is entered finitely times.

When the restoration is entered for finitely times, one can guarantee that a subsequence of the step sizes $\{\sigma_k\}$ will converge to zero. In fact, due to the sufficient decrease condition imposed on the merit function along the iterates (or in the constraints violation function if the iterates are sufficiently away from the feasible region) and the control on the step size (reduced at least by β_2 for unsuccessful iterations), one can ensure the existence of a subsequence K of unsuccessful iterates driving the step size to zero.

Algorithm 2: A globally convergent ES for general constraints (Main)

Data: choose positive integers λ and μ such that $\lambda \geq \mu$. Select an initial $x_0 \in \Omega_{\text{nr}}$ and evaluate $f(x_0)$. Choose initial step lengths $\sigma_0, \sigma_0^{\text{ES}} > 0$ and initial weights $(\omega_0^1, \ldots, \omega_0^\mu) \in S$. Choose constants $\beta_1, \beta_2, d_{\min}, d_{\max}$ such that $0 < \beta_1 \leq \beta_2 < 1$ and $0 < d_{\min} < d_{\max}$. Select a forcing function $\rho(\cdot)$.

for $k = 0, 1, \ldots$ **do**

Step 1: compute new sample points $Y_{k+1} = \{y_{k+1}^1, \ldots, y_{k+1}^\lambda\}$ such that

$$y_{k+1}^i \;=\; x_k + \sigma_k \tilde{d}_k^i, \; i = 1, \ldots, \lambda,$$

where the directions $\tilde{d}_k^i$'s are computed from the original ES directions d_k^i's (which in turn are drawn from a chosen ES distribution $\mathcal{C}_k$ and scaled if necessary to satisfy $d_{\min} \leq \|d_k^i\| \leq d_{\max}$).;

Step 2: evaluate $M(y_{k+1}^i)$, $i = 1, \ldots, \lambda$, and reorder the offspring points in $Y_{k+1} = \{\tilde{y}_{k+1}^1, \ldots, \tilde{y}_{k+1}^\lambda\}$ by increasing order: $M(\tilde{y}_{k+1}^1) \leq \cdots \leq M(\tilde{y}_{k+1}^\lambda)$.

Select the new parents as the best μ offspring sample points $\{\tilde{y}_{k+1}^1, \ldots, \tilde{y}_{k+1}^\mu\}$, and compute their weighted mean

$$x_{k+1}^{\text{trial}} \;=\; \sum_{i=1}^{\mu} \omega_k^i \tilde{y}_{k+1}^i;$$

Step 3: if $x_{k+1}^{\text{trial}} \notin \Omega_{\text{nr}}$ **then**

  the iteration is declared unsuccessful;

else

  if x_{k+1}^{trial} *is a **"Restoration identifier"*** **then**

    enter Restoration (with $k_r = k$);

  else

    if x_{k+1}^{trial} *is a **"Successful point"*** **then**

      declare the iteration successful, set $x_{k+1} = x_{k+1}^{\text{trial}}$, and $\sigma_{k+1} \geq \sigma_k$ (for example $\sigma_{k+1} = \max\{\sigma_k, \sigma_k^{\text{ES}}\}$);

    else

      the iteration is declared unsuccessful;

    end

  end

end

if *the iteration is declared unsuccessful* **then**

  set $x_{k+1} = x_k$ and $\sigma_{k+1} = \beta_k \sigma_k$, with $\beta_k \in (\beta_1, \beta_2)$;

end

Step 4: update the ES step length σ_{k+1}^{ES}, the distribution $\mathcal{C}_{k+1}$, and the weights $(\omega_{k+1}^1, \ldots, \omega_{k+1}^\mu) \in S$;

end

Lemma 4.1 *Let f be bounded below and assuming that the restoration is not entered after a*

Algorithm 3: A globally convergent ES for general constraints (Restoration)

Data: Start from $x_{k_r} \in \Omega_{\rm nr}$ given from the Main algorithm and consider the same parameter as in there.

for $k = k_r, k_r+1, k_r+2, \ldots$ **do**

Step 1: compute new sample points $Y_{k+1} = \{y^1_{k+1}, \ldots, y^\lambda_{k+1}\}$ such that

$$y^i_{k+1} \;=\; x_k + \sigma_k \tilde{d}^i_k, \;\, i = 1, \ldots, \lambda,$$

where the directions $\tilde{d}^i_k$'s are computed from the original ES directions d^i_k's (which in turn are drawn from a chosen ES distribution $\mathcal{C}_k$ and scaled if necessary to satisfy $d_{\min} \le \|d^i_k\| \le d_{\max}$);

Step 2: evaluate $g_{\Omega_{\rm nr}}(y^i_{k+1})$, $i = 1, \ldots, \lambda$, and reorder the offspring points in $Y_{k+1} = \{\tilde{y}^1_{k+1}, \ldots, \tilde{y}^\lambda_{k+1}\}$ by increasing order: $g_{\Omega_{\rm nr}}(\tilde{y}^1_{k+1}) \le \cdots \le g_{\Omega_{\rm nr}}(\tilde{y}^\lambda_{k+1})$.
Select the new parents as the best μ offspring sample points $\{\tilde{y}^1_{k+1}, \ldots, \tilde{y}^\mu_{k+1}\}$, and compute their weighted mean

$$x^{\rm trial}_{k+1} \;=\; \sum_{i=1}^{\mu} \omega^i_k \tilde{y}^i_{k+1};$$

Step 3: if $x^{\rm trial}_{k+1} \notin \Omega_{\rm nr}$ **then**
the iteration is declared unsuccessful;
else
if $g(x^{\rm trial}_{k+1}) \;<\; g(x_k) - \rho(\sigma_k)$ *and* $g(x_k) > C\rho(\sigma_k)$ **then**
the iteration is declared successful, set $x_{k+1} = x^{\rm trial}_{k+1}$, and $\sigma_{k+1} \ge \sigma_k$ (for example $\sigma_{k+1} = \max\{\sigma_k, \sigma^{\rm ES}_k\}$);
else
the iteration is declared unsuccessful;
end
end
if *the iteration is declared unsuccessful* **then**
if $M(x^{\rm trial}_{k+1}) \;<\; M(x_k)$ **then**
leave Restoration and return to the Main algorithm (starting at a new $(k+1)$-th iteration using x_{k+1} and σ_{k+1});
else
set $x_{k+1} = x_k$ and $\sigma_{k+1} = \beta_k \sigma_k$, with $\beta_k \in (\beta_1, \beta_2)$;
end
end
Step 4: update the ES step length $\sigma^{\rm ES}_{k+1}$, the distribution $\mathcal{C}_{k+1}$, and the weights $(\omega^1_{k+1}, \ldots, \omega^\mu_{k+1}) \in S$;

end

certain order. Then,

$$\liminf_{k \to +\infty} \sigma_k = 0.$$

Proof. Suppose that there exists a $\bar{k} > 0$ and $\sigma > 0$ such that $\sigma_k > \sigma$ and $k \geq \bar{k}$ is a given iteration of Algorithm 2. If there is an infinite sequence J_1 of successful iterations after $\bar{k}$, this leads to a contradiction with the fact that g and f are bounded below.

In fact, since ρ is a nondecreasing positive function, one has $\rho(\sigma_k) \geq \rho(\sigma) > 0$. Hence, if $g(x_{k+1}) < g(x_k) - \rho(\sigma_k)$ and $g(x_k) > C\rho(\sigma_k)$ for all $k \in J_1$, then

$$g(x_{k+1}) < g(x_k) - \rho(\sigma),$$

which obviously contradicts the boundness below of g by 0. Thus there must exists an infinite subsequence $J_2 \subseteq J_1$ of iterates for which $M(x_{k+1}) \; < \; M(x_k) - \rho(\sigma_k)$. Hence,

$$M(x_{k+1}) \; < \; M(x_k) - \rho(\sigma) \qquad \text{for all } k \in J_2.$$

Thus $M(x_k)$ tends to $-\infty$ which is a contradiction, since both f and g are bounded below.

The proof is thus completed if there is an infinite number of successful iterations. However, if no more successful iterations occur after a certain order, then this also leads to a contradiction. The conclusion is that one must have a subsequence of iterations driving σ_k to zero. ∎

Theorem 4.1 *Let f be bounded below and assuming that the restoration is not entered after a certain order.*

There exists a subsequence K of unsuccessful iterates for which $\lim_{k\in K} \sigma_k = 0$. Moreover, if the sequence $\{x_k\}$ is bounded, there exists an x_* and a refining subsequence K' such that $\lim_{k\in K} x_k = x_*$.

Proof. From Lemma 4.1, there must exist an infinite subsequence K of unsuccessful iterates for which σ_{k+1} goes to zero. In a such case we have $\sigma_k = (1/\beta_k)\sigma_{k+1}$, $\beta_k \in (\beta_1, \beta_2)$, and $\beta_1 > 0$, and thus $\sigma_k \to 0$, for $k \in K$, too.

The second part of the theorem is proved by extracting a convergent subsequence $K' \subset K$ for which x_k converges to x_*. ∎

The global convergence will be achieved by establishing that some type of directional derivatives are nonnegative at limit points of refining subsequences along certain limit directions (known as refining directions). By refining subsequence [4], we mean a subsequence of unsuccessful iterates in the Main algorithm (see Algorithm 2) for which the step-size parameter converges to zero.

When h is Lipschitz continuous near $x_* \in \Omega_{\text{nr}}$, one can make use of the Clarke-Jahn generalized derivative along a direction d

$$h^\circ(x_*; d) \; = \; \limsup_{\substack{x \to x_*, x \in \Omega_{\text{nr}} \\ t \downarrow 0, x+td \in \Omega_{\text{nr}}}} \frac{h(x+td) - h(x)}{t}.$$

(Such a derivative is essentially the Clarke generalized directional derivative [9], adapted by Jahn [25] to the presence of constraints). However, for the proper definition of $h^\circ(x_*; d)$, one needs to guarantee that $x + td \in \Omega_{\text{nr}}$ for $x \in \Omega_{\text{nr}}$ arbitrarily close to x_* which is assured if d is hypertangent to Ω_{nr} at x_*. In the following, $B(x; \epsilon)$ is the closed ball formed by all points which dist no more than ϵ to x.

Definition 4.1 *A vector $d \in \mathbb{R}^n$ is said to be a hypertangent vector to the set $\Omega_{\rm nr} \subseteq \mathbb{R}^n$ at the point x in $\Omega_{\rm nr}$ if there exists a scalar $\epsilon > 0$ such that*

$$y + tw \in \Omega_{\rm nr}, \quad \forall y \in \Omega_{\rm nr} \cap B(x;\epsilon), \quad w \in B(d;\epsilon), \quad \textit{and} \quad 0 < t < \epsilon.$$

The hypertangent cone to $\Omega_{\rm nr}$ at x, denoted by $T^{\rm H}_{\Omega_{\rm nr}}(x)$, is the set of all hypertangent vectors to $\Omega_{\rm nr}$ at x. Then, the Clarke tangent cone to $\Omega_{\rm nr}$ at x (denoted by $T^{\rm CL}_{\Omega_{\rm nr}}(x)$) can be defined as the closure of the hypertangent cone $T^{\rm H}_{\Omega_{\rm nr}}(x)$. The Clarke tangent cone generalizes the notion of tangent cone in Nonlinear Programming [32], and the original definition $d \in T^{\rm CL}_{\Omega_{\rm nr}}(x)$ is given below.

Definition 4.2 *A vector $d \in \mathbb{R}^n$ is said to be a Clarke tangent vector to the set $\Omega_{\rm nr} \subseteq \mathbb{R}^n$ at the point x in the closure of $\Omega_{\rm nr}$ if for every sequence $\{y_k\}$ of elements of $\Omega_{\rm nr}$ that converges to x and for every sequence of positive real numbers $\{t_k\}$ converging to zero, there exists a sequence of vectors $\{w_k\}$ converging to d such that $y_k + t_k w_k \in \Omega_{\rm nr}$.*

Given a direction v in the tangent cone, possibly not in the hypertangent one, one can consider the Clarke-Jahn generalized derivative to $\Omega_{\rm nr}$ at x_* as the limit

$$h^\circ(x_*; v) \;=\; \lim_{d \in T^{\rm H}_{\Omega_{\rm nr}}(x_*), d \to v} h^\circ(x_*; d)$$

(see [4]). A point $x_* \in \Omega_{\rm nr}$ is considered Clarke stationary if $h^\circ(x_*; d) \geq 0$, $\forall d \in T^{\rm CL}_{\Omega_{\rm nr}}(x_*)$.

It remains now to define the notion of refining direction [4], associated with a convergent refining subsequence K, as a limit point of $\{a_k/\|a_k\|\}$ for all $k \in K$ sufficiently large such that $x_k + \sigma_k a_k \in \Omega_{\rm nr}$, where, in the particular case of taking the weighted mean as the object of evaluation, one has $a_k = \sum_{i=1}^{\mu} \omega_k^i \tilde{d}_k^i$. The following convergence result is concerning the determination of feasibility.

Theorem 4.2 *Let $a_k = \sum_{i=1}^{\mu} \omega_k^i d_k^i$ and assume that f is bounded below. Suppose that the restoration is not entered after a certain order. Let $x_* \in \Omega_{\rm nr}$ be the limit point of a convergent subsequence of unsuccessful iterates $\{x_k\}_K$ for which $\lim_{k \in K} \sigma_k = 0$. Assume that g is Lipschitz continuous near x_* with constant $\nu_g > 0$.*

If $d \in T^{\rm H}_{\Omega_{\rm nr}}(x_*)$ is a refining direction associated with $\{a_k/\|a_k\|\}_K$, then either $g(x_*) = 0$ or $g^\circ(x_*; d) \geq 0$.

Proof. Let d be a limit point of $\{a_k/\|a_k\|\}_K$. Then it must exist a subsequence K' of K such that $a_k/\|a_k\| \to d$ on K'. On the other hand, we have for all k that

$$x_{k+1}^{\rm trial} \;=\; \sum_{i=1}^{\mu} \omega_k^i \tilde{y}_{k+1}^i \;=\; x_k + \sigma_k \sum_{i=1}^{\mu} \omega_k^i d_k^i \;=\; x_k + \sigma_k a_k,$$

Since the iteration $k \in K'$ is unsuccessful, $g(x_{k+1}^{\rm trial}) \geq g(x_k) - \rho(\sigma_k)$ or $g(x_k) \leq C\rho(\sigma_k)$, and then either there exists an infinite number of the first inequality or the second one as follows:

1. For the case where there exists a subsequence $K_1 \subseteq K'$ such that $g(x_k) \leq C\rho(\sigma_k)$, it is trivial to obtain $g(x_*) = 0$ using both the continuity of g and the fact that σ_k tends to zero in K_1.

2. For the case where there exists a subsequence $K_2 \subseteq K'$ such that the sequence $\{a_k/\|a_k\|\}_{K_2}$ converges to $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$ in K_2 and the sequence $\{\|a_k\|\sigma_k\}_{k\in K_2}$ goes to zero in K_2 (a_k is bounded above for all k, and so $\sigma_k\|a_k\|$ tends to zero when σ_k does). Thus one must have necessarily for k sufficiently large in K_2, $x_k + \sigma_k a_k \in \Omega_{\mathrm{nr}}$ such that

$$g(x_k + \sigma_k a_k) \geq g(x_k) - \rho(\sigma_k).$$

From the definition of the Clarke-Jahn generalized derivative along directions $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$,

$$\begin{aligned} g^\circ(x_*; d) &= \limsup_{x\to x_*, t\downarrow 0, x+td\in\Omega_{\mathrm{nr}}} \frac{g(x+td) - g(x)}{t} \\ &\geq \limsup_{k\in K_2} \frac{g(x_k + \sigma_k\|a_k\|d) - g(x_k)}{\sigma_k\|a_k\|} \\ &\geq \limsup_{k\in K_2} \frac{g(x_k + \sigma_k\|a_k\|(a_k/\|a_k\|)) - g(x_k)}{\sigma_k\|a_k\|} - g_k, \end{aligned}$$

where,

$$g_k = \frac{g(x_k + \sigma_k a_k) - g(x_k + \sigma_k\|a_k\|d)}{\sigma_k\|a_k\|}$$

from the Lipschitz continuity of g near x_*

$$\begin{aligned} g_k &= \frac{g(x_k + \sigma_k a_k) - g(x_k + \sigma_k\|a_k\|d)}{\sigma_k\|a_k\|} \\ &\leq \nu_g \left\| \frac{a_k}{\|a_k\|} - d \right\| \end{aligned}$$

tends to zero on K_2. Finally,

$$\begin{aligned} g^\circ(x_*; d) &\geq \limsup_{k\in K_2} \frac{g(x_k + \sigma_k a_k) - g(x_k) + \rho(\sigma_k)}{\sigma_k\|a_k\|} - \frac{\rho(\sigma_k)}{\sigma_k\|a_k\|} - g_k \\ &= \limsup_{k\in K_2} \frac{g(x_k + \sigma_k a_k) - g(x_k) + \rho(\sigma_k)}{\sigma_k\|a_k\|}. \end{aligned}$$

One then obtains $g^\circ(x_*; d) \geq 0$.

■

Moreover, assuming that the set of the refining directions $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$, associated with $\{a_k/\|a_k\|\}_K$, is dense in the unite sphere. One can show that the limit point x_* is Clarke stationary for the flowing optimization problem, known as the constraint violation problem:

$$\begin{array}{ll} \min & g(x) \\ s.t. & x \in \Omega_{\mathrm{nr}}. \end{array} \tag{4}$$

Theorem 4.3 *Let* $a_k = \sum_{i=1}^{\mu} \omega_k^i d_k^i$ *and assume that* f *is bounded below. Suppose that the restoration is not entered after a certain order. Assume that the directions* $\tilde{d}_k^i$*'s and the weights* ω_k^i*'s are such that (i)* $\sigma_k \|a_k\|$ *tends to zero when* σ_k *does, and (ii)* $\rho(\sigma_k)/(\sigma_k \|a_k\|)$ *also tends to zero.*

Let $x_* \in \Omega_{\rm nr}$ *be the limit point of a convergent subsequence of unsuccessful iterates* $\{x_k\}_K$ *for which* $\lim_{k \in K} \sigma_k = 0$ *and that* $T_\Omega^{\rm CL}(x_*) \neq \emptyset$*. Assume that* g *is Lipschitz continuous near* x_* *with constant* $\nu > 0$

Then either (a) $g(x_*) = 0$ *(implying* $x_* \in \Omega_{\rm r}$ *and thus* $x_* \in \Omega$*) or (b) if the set of refining directions* $d \in T_{\Omega_{\rm nr}}^{\rm CL}(x_*)$ *associated with* $\{a_k/\|a_k\|\}_{K'}$ *(where* K' *is a subsequence of* K *for which* $g(x_k + \sigma_k a_k) \geq g(x_k) - \rho(\sigma_k)$*) is dense in* $T_{\Omega_{\rm nr}}^{\rm CL}(x_*) \cap \{d \in \mathbb{R}^n : \|d\| = 1\}$*, then* $g^\circ(x_*; v) \geq 0$ *for all* $v \in T_{\Omega_{\rm nr}}^{\rm CL}(x_*)$ *and* x_* *is a Clarke stationary point of the constraint violation problem (4).*

Proof. See the proof of [19, Theorem 4.2]. ∎

We now move to an intermediate optimality result. As in [19], we will not use $x_* \in \Omega_{\rm r}$ explicitly in the proof but only $g^\circ(x_*; d) \leq 0$. The latter inequality describes the cone of first order linearized directions under feasibility assumption $x_* \in \Omega_{\rm r}$.

Theorem 4.4 *Let* $a_k = \sum_{i=1}^{\mu} \omega_k^i d_k^i$ *and assume that* f *is bounded below. Suppose that the restoration is not entered after a certain order.*

Let $x_* \in \Omega_{\rm nr}$ *be the limit point of a convergent subsequence of unsuccessful iterates* $\{x_k\}_K$ *for which* $\lim_{k \in K} \sigma_k = 0$*. Assume that* g *and* f *are Lipschitz continuous near* x_**.*

If $d \in T_{\Omega_{\rm nr}}^{\rm H}(x_*)$ *is a refining direction associated with* $\{a_k/\|a_k\|\}_K$ *such that* $g^\circ(x_*; d) \leq 0$*. Then* $f^\circ(x_*; d) \geq 0$*.*

Proof. By assumption there exists a subsequence $K' \subseteq K$ such that the sequence $\{a_k/\|a_k\|\}_{K'}$ converges to $d \in T_{\Omega_{\rm nr}}^{\rm H}(x_*)$ in K' and the sequence $\{\|a_k\|\sigma_k\}_{K'}$ goes to zero in K', Thus one must have necessarily for k sufficiently large in K', $x_{k+1}^{\rm trial} = x_k + \sigma_k a_k \in \Omega_{\rm nr}$.

Since the iteration $k \in K'$ is unsuccessful, one has $M(x_{k+1}^{\rm trial}) \geq M(x_k) - \rho(\sigma_k)$, and thus

$$\frac{f(x_k + \sigma_k a_k) - f(x_k)}{\|a_k\|\sigma_k} \quad \geq \quad -\bar{\delta}\frac{g(x_k + \sigma_k a_k) - g(x_k)}{\|a_k\|\sigma_k} - \frac{\rho(\sigma_k)}{\sigma_k\|a_k\|} \tag{5}$$

On the other hand,

$$\begin{aligned} f^\circ(x_*; d) &= \limsup_{x \to x_*, t \downarrow 0, x + td \in \Omega} \frac{f(x + td) - f(x)}{t} \\ &\geq \limsup_{k \in K'} \frac{f(x_k + \sigma_k \|a_k\| d) - f(x_k)}{\sigma_k \|a_k\|} \\ &\geq \limsup_{k \in K'} \frac{f(x_k + \sigma_k \|a_k\| (a_k/\|a_k\|)) - f(x_k)}{\sigma_k \|a_k\|} - f_k, \end{aligned}$$

where,

$$f_k = \frac{f(x_k + \sigma_k a_k) - f(x_k + \sigma_k \|a_k\| d)}{\sigma_k \|a_k\|},$$

which then implies from (5)

$$\begin{aligned}
f^\circ(x_*;d) &\geq \limsup_{k\in K'} \frac{f(x_k+\sigma_k\|a_k\|(a_k/\|a_k\|))-f(x_k)}{\sigma_k\|a_k\|} - f_k,\\
&\geq \limsup_{k\in K'} -\bar\delta\frac{g(x_k+\sigma_k a_k)-g(x_k)}{\|a_k\|\sigma_k} - \frac{\rho(\sigma_k)}{\sigma_k\|a_k\|} - f_k\\
&\geq \limsup_{k\in K'} -\bar\delta\frac{g(x_k+\sigma_k\|a_k\|d)-g(x_k)}{\sigma_k\|a_k\|} + \bar\delta g_k - \frac{\rho(\sigma_k)}{\sigma_k\|a_k\|} - f_k,
\end{aligned}$$

where

$$g_k = \frac{g(x_k+\sigma_k a_k)-g(x_k+\sigma_k\|a_k\|d)}{\sigma_k\|a_k\|}.$$

From the assumption $g^\circ(x_*;d) \leq 0$, one has

$$\limsup_{k\in K'} \frac{g(x_k+\sigma_k\|a_k\|d)-g(x_k)}{\sigma_k\|a_k\|} \leq \limsup_{x\to x_*, t\downarrow 0, x+td\in\Omega_{\mathrm{nr}}} \frac{g(x+td)-g(x)}{t} \leq 0,$$

one obtains then

$$f^\circ(x_*;d) \geq \limsup_{k\in K'} \bar\delta g_k - \frac{\rho(\sigma_k)}{\sigma_k\|a_k\|} - f_k. \tag{6}$$

The Lipschitz continuity of both g and f near x_* guaranties that the quantities f_k and g_k tend to zero in K'. Thus, the proof is completed since the right-hand-side of (6) tends to zero in K'. ■

Theorem 4.5 *Assuming that f is bounded below and that Restoration is not entered after a certain order.*

Let $x_* \in \Omega_{\mathrm{nr}}$ be the limit point of a convergent subsequence of unsuccessful iterates $\{x_k\}_{k\in K}$ for which $\lim_{k\in K}\sigma_k = 0$. Assume that g and f are Lipschitz continuous near x_*.

Assume that the set

$$T(x_*) = T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*) \cap \{d\in\mathbb{R}^n : \|d\|=1, g^\circ(x_*,d)\leq 0\} \tag{7}$$

has a non-empty interior.

Let the set of refining directions be dense in $T(x_*)$. Then $f^\circ(x_*,v)\geq 0$ for all $v\in T^{\mathrm{CL}}_{\Omega_{\mathrm{nr}}}(x_*)$ such that $g^\circ(x_*,v)\leq 0$, and x_* is a Clarke stationary point of the problem (1).

Proof. See the proof of [19, Theorem 4.4]. ■

5 Numerical experiments

To quantify the efficiency of the proposed merit approach, we compare our solver with the direct search method MADS where the progressive barrier approach has been implemented [5] to handle both relaxable and unrelaxable constraints. The progressive barrier approach, proposed in MADS, enjoys similar convergence properties as for our algorithm, hence, a comparison between the two solvers is very meaningful. For MADS solver, we used the implementation given in the NOMAD package [1, 3, 29], version 3.6.1 (C++ version linked to Matlab via a mex interface), where we enabled the option `DISABLE MODELS`, meaning that no modeling is used in MADS. The models are disabled since our solvers are not using any modeling to speed up the convergence.

The parameter choices of Algorithm 2 and Algorithm 3 followed those in [16]. The values of λ and μ and of the initial weights are those of CMA-ES for unconstrained optimization (see [20]): $\lambda = 4 + \text{floor}(3\log(n))$, $\mu = \text{floor}(\lambda/2)$, where floor$(\cdot)$ rounds to the nearest integer, and $\omega_0^i = a_i/(a_1 + \cdots + a_\mu)$, $a_i = \log(\lambda/2 + 1/2) - \log(i)$, $i = 1, \ldots, \mu$. The choices of the distribution $\mathcal{C}_k$ and of the update of σ_k^{ES} also followed CMA-ES for unconstrained optimization (see [20]). The forcing function selected was $\rho(\sigma) = 10^{-4}\sigma^2$. To reduce the step length in unsuccessful iterations we used $\sigma_{k+1} = 0.9\sigma_k$ which corresponds to setting $\beta_1 = \beta_2 = 0.9$. In successful iterations we set $\sigma_{k+1} = \max\{\sigma_k, \sigma_k^{\text{CMA-ES}}\}$ (with $\sigma_k^{\text{CMA-ES}}$ the CMA step size used in ES). The directions d_k^i, $i = 1, \ldots, \lambda$, were scaled if necessary to obey the safeguards $d_{\min} \leq \|d_k^i\| \leq d_{\max}$, with $d_{\min} = 10^{-10}$ and $d_{\max} = 10^{10}$. For the update of the penalty parameter we picked $\bar{\delta} = \max\{10, g(x_0)\}$ and $C = 100$. The measure for constraint violation was using the ℓ_1-norm penalty.

The initial step size is estimated using only the bound constraints: If there is a pair of finite lower and upper bounds for a variable, then σ_0 is set to the half of the minimum of such distances, otherwise $\sigma_0 = 1$.

In our test results, we consider that all the constraints are relaxable except the bounds. In this case, the merit function (MF) and the progressive approaches (PB) are respectively enabled, the related solvers will be called ES-MF and MADS-PB respectively.

5.1 Results on known test problems

Our test set is the one used in [23, 24, 27, 31] and comprises 13 well-known test problems `G1`–`G13` (see Table 1). These test problems, coded in Matlab, exhibit a diversity of features and the kind of difficulties that appear in constrained global optimization. In addition to such problems, we added three other engineering optimization problems [23, 12]: `PVD` the pressure vessel design problem, `TCS` the tension-compression string problem, and `WBD` the welded beam design problem. Problems `G2`, `G3`, and `G8` are maximization problems and were converted to minimization. Problems `G3`, `G5`, `G11`, and `WBD` contain equality constraints. When a constraint is of the form $c_i^e(x) = 0$, we use the following relaxed inequality constraint instead $c_i(x) = |c_i^e(x)| - 10^{-4} \leq 0$.

The starting point x_0 is chosen to be the same for all solvers and set to $(LB + UB)/2$ when the bounds LB and UB are given, otherwise it is generated randomly in the search space. Two different maximal budgets are considered for our experiments; firstly, we use a small one (i.e., 1000 objective function evaluations) to analyse the performance of the algorithms during the early stages of the optimization; secondly, a large maximal budget (i.e., 20000 objective function evaluations) is used to allow the analysis of the asymptotic behavior of the tested solvers. We note that MADS-PB is a deterministic solver while ES-MF is stochastic, thus different runs of ES-MF will be used. We describe our finding based upon average results over 10 runs (as different runs of ES-MF yielded close results).

Tables 2 and 3, report results for both ES-MF and MADS-PB using a maximal budget of 1000 and 20000, respectively. For each problem, we display the optimal objective value found

Name	G1	G2	G3	G4	G5	G6	G7	G8	G9	G10	G11	G12	G13	PVD	TCS	WBD
n	13	20	20	5	4	2	10	2	7	8	2	3	5	4	3	4
m	9	2	1	6	5	2	8	2	4	6	1	1	3	3	4	6

Table 1: Some of the features of the non-linear constrained optimization problems: the dimension n and the number of the constraints m (in addition to the bounds).

by the solver $f(x_*)$, the associated constrained violation $g(x_*)$, and the number of objective function evaluations $\#f$ needed to reach x_*. When a solver returns a flag error or encounters an internal problem, we display '−' instead of the values of $f(x_*)$ and $g(x_*)$. At the solution x_*, one requires at least tolerance of 10^{-5} on the constraints violation (i.e. $g(x_*) < 10^{-5}$) to consider x_* feasible with respect to relaxable constraints.

Table 2 gives the obtained results for a maximal budget of 1000 function evaluations. For both starting strategies (feasible or not) and except few problems, the tested solvers were not able to converge with the regarded budget. ES-MF is shown to have comparable performance with MADS-PB on the tested problems. In fact, with a feasible starting point, ES-MF is performing well on the problems G1, G2, G3, G8, G11, G12, and G13. While MADS-PB is being the best on the problems G4, G5, G6, G7, G9, G10, PVD , TCS and WBD. Using an infeasible starting point, ES-MF is performing better on the problems G1, G2, G3, G7, G8, G11, G12, TCS, and WBD.

For a large maximal number of function evaluation of 20000 (Table 3), ES-MF and MADS-PB achieve convergence to a stationary point on more problems. We note that MADS-PB requires more function evaluations for four problems G2, G3, G10 and G13. When the starting point is feasible, enlarging the budget allows having exact feasibility at the solution point. In this case, regarding the value of the objective function, we note that increasing the number of function evaluation does not affect the results compare to the ones obtained using 1000 function evaluations. When using an infeasible starting point, the advantage of ES-MF over MADS-PB is more evident compared to the small budget case. In fact, one can observe that ES-MF is better than MADS-PB on nine of the sixteen tested problems (i.e. G1, G2, G5, G6, G7, G8, G9, G11, G12, G13 and WBD). MADS-PB is shown to be better on the following four problems: G3, G4, G6 and PVD. Both solvers did not succeed to find a feasible solution for the problem G10 , for the TCS problem MADS returns a flag error while ES-MF converge to an unfeasible solution.

5.2 Application to a multidisciplinary design optimization problem

MDO problems are typical real applications where one has to minimize a given objectif function subject to a set of relaxable and unrelaxable constraints. In this section, we test our proposed algorithm in an MDO problem taken from [18, 36] where a simplified wing design (built around a tube) is regarded. In this test case, one tries to find the best wing design considering interdisciplinary trade-off, which is between aerodynamic (a minimum drag) and structural (a minimum weight) performances. Typically, the two disciplines are evaluated sequentially by means of a fixed point iterative method until the coupling is solved with the appropriate accuracy. More details on the problem are given in [18].

The optimization problem has 7 design variables, see Table 5.2. In addition to the bounds, the test case has three nonlinear constraints which are treated as relaxable. The bound contraints x_{lb} and x_{ub} are regarded as unrelaxable and will be treated using l_2 projection approach. We run our code using the proposed starting guess $x_0 = (37.5,\ 9.0,\ 0.39,\ 1.1,\ 1.0,\ 3.3,\ 0.545)$ as

Name	Best known	ES-MF			MADS-PB		
	f_{opt}	$f(x_*)$	$\#f$	$g(x_*)$	$f(x_*)$	$\#f$	$g(x_*)$
Starting feasible							
G1	-15	-12.4895	1000	$4.3e-07$	-8.99982	1000	0
G2	-0.803619	-0.271234	1000	0	-0.173894	1000	0
G3	-1	-0.254056	1000	$2.7e-06$	-0.0436105	1000	0
G4	-30665.5	-30723.2	1000	0.016	-30498.1	1000	0
G5	5126.5	5999.48	1000	$7.6e-05$	5976.11	973	0
G6	-6961.81	-7588.41	1000	0.22	-6961.26	1000	0
G7	24.3062	147.259	320	0	30.0327	1000	0
G8	-0.095825	-0.095825	330	0	-0.095825	453	0
G9	680.63	691.948	1000	0	683.871	1000	0
G10	7049.33	16607.4	1000	0	7843.26	1000	0
G11	0.75	0.749403	1000	$2.5e-07$	0.9998	331	0
G12	-1	-1	161	0	-1	309	0
G13	0.0539498	1.45074	1000	$5.9e-07$	2.78621	1000	0
PVD	5868.76	$3.21995e+06$	1000	2.2	8115.01	978	0
TCS	0.0126653	0.0135886	817	0	0.0126658	836	0
WBD	1.725	3.13076	559	0	3.01286	1000	0
Starting infeasible							
G1	-15	-11.0679	1000	0	-8.93833	1000	0
G2	-0.803619	-0.271234	1000	0	-0.173894	1000	0
G3	-1	-0.000743875	1000	0	$-1.40301e-06$	1000	0
G4	-30665.5	-31003.2	1000	0.22	-30643.8	1000	0
G5	5126.5	5603.69	1000	$4.4e+04$	5236.08	1000	0.63
G6	-6961.81	-3351.17	1000	0	-6961.81	1000	0
G7	24.3062	49.1948	1000	0	83.6455	1000	0
G8	-0.095825	-0.095825	204	0	-0.095825	525	0
G9	680.63	691.948	1000	0	683.871	1000	0
G10	7049.33	9626.33	1000	4.8	6013.14	1000	0.031
G11	0.75	0.749403	1000	$2.5e-07$	0.9998	331	0
G12	-1	-1	161	0	-1	309	0
G13	0.0539498	1	1000	1	0.998918	1000	0
PVD	5868.76	2284.14	1000	2.2	6344.92	997	0
TCS	0.0126653	0.000149129	617	0.97	–	–	–
WBD	1.725	3.78146	1000	0	3.89919	1000	0

Table 2: Obtained results using a maximal budget of 1000 (average of 10 runs).

in [36]. The provided starting point is infeasible towards the nonlinear constraints. A large maximal number of function evaluation of 20000 is used to quantify the asymptotic efficiency and the robustness of the tested methods.

From the obtained results, one can see that ES-MF converges to an asymptotically feasible solution $x_* = (43.043,\ 6.738,\ 0.28,\ 3.000,\ 0.749,\ 3.942,\ 0.300)$ with $f(x_*) = -16.61198$ and $g(x_*) = 2 \times 10^{-14}$ using 12781 function evaluations. For MADS-PB, using 3848 function eval-

Name	Best known	ES-MF			MADS-PB		
	f_{opt}	$f(x_*)$	$\#f$	$g(x_*)$	$f(x_*)$	$\#f$	$g(x_*)$
			Starting feasible				
G1	-15	-12.9999	7645	0	-9.00207	4817	0
G2	-0.803619	-0.27127	2107	0	-0.226599	20000	0
G3	-1	-1.05473	5227	0	-0.652199	20000	0
G4	-30665.5	-31009.8	3664	0.17	-30503.1	3064	0
G5	5126.5	5976.79	244	0	5976.79	1132	0
G6	-6961.81	-6942.57	1261	0	-6961.81	1381	0
G7	24.3062	147.259	320	0	25.5112	5160	0
G8	-0.095825	-0.095825	330	0	-0.095825	453	0
G9	680.63	680.63	5071	0	680.799	3568	0
G10	7049.33	15116.7	5094	0.02	7687.35	5067	0
G11	0.75	0.75	1177	0	0.9998	331	0
G12	-1	-1	161	0	-1	309	0
G13	0.0539498	1	2287	0	2.66335	20000	0
PVD	5868.76	396143	3179	0.0037	7890.36	1385	0
TCS	0.0126653	0.0135886	817	0	0.0126658	836	0
WBD	1.725	3.13076	559	0	3.01285	1292	0
			Starting infeasible				
G1	-15	-14.9951	3901	0	-8.99999	4222	0
G2	-0.803619	-0.27127	2107	0	-0.226599	20000	0
G3	-1	-0.000743875	1015	0	-0.00413072	20000	0
G4	-30665.5	-30990.3	2746	0.19	-30665.4	2846	0
G5	5126.5	5334.29	2782	0	5240.95	5291	0.008
G6	-6961.81	-6961.81	2500	0	-6961.81	1078	0
G7	24.3062	24.3062	11562	0	27.1991	12426	0
G8	-0.095825	-0.095825	204	0	-0.095825	525	0
G9	680.63	680.63	5071	0	680.799	3568	0
G10	7049.33	9681.53	7195	0.082	6192.82	20000	0.021
G11	0.75	0.75	1177	0	0.9998	331	0
G12	-1	-1	161	0	-1	309	0
G13	0.0539498	0.438745	12367	0	0.996284	20000	0
PVD	5868.76	$2.57711e+12$	5575	$1.1e+04$	6342.85	1515	0
TCS	0.0126653	0.000149129	617	0.97	–	–	–
WBD	1.725	2.70832	2971	0	3.7413	2801	0

Table 3: Obtained results using a maximal budget of 20000 (average of 10 runs).

uations, converges to the feasible point $x_* = (44.170,\ 6.746,\ 0.28,\ 3.000,\ 0.721,\ 4.028,\ 0.300)$ with $f(x_*) = -16.60627$. We note that while MADS-PB seems to converge to a local minimum but with a reasonable budget, the obtained solution using ES-MF solver seems to be better than even the best know optimum but with a very small constraints violation. To confirm the obtained performance of ES-MF on this MDO problem, we test also 10 random starting points

Design variable	Best known	x_{lb}	x_{ub}	Starting guess
Wing span x_1	44.19	30.0	45.0	37.5
Root cord x_2	6.75	6.0	12.0	9.0
Taper ratio x_3	0.28	0.28	0.50	0.39
Angle of attack at root x_4	3.0	-1.0	3.0	1.1
Angle of attack at tip and at rest x_5	0.72	-1.0	3.0	1.0
Tube external diameter x_6	4.03	1.6	5.0	3.3
Tube thickness x_7	0.3	0.3	0.79	0.545
Objective function value	-16.61011	10^{20}	-8.0157	-10.93552
Constraint violation	0	3×10^{40}	0	2.01×10^{7}

Table 4: Description of the MDO problem variables. The coordinates and the value of the best known solution have been rounded.

generated inside the hyper-cube $x_{\text{lb}} \times x_{\text{ub}}$ as follows

$$x_0 = \alpha x_{\text{lb}} + (1 - \alpha) x_{\text{ub}},$$

for 10 values of α uniformly generated in $(0, 1)$.

Problem Instance	f at x_0 $f(x_0)$	ES-MF			MADS-PB		
		$f(x_*)$	$g(x_*)$	$\#f$	$f(x_*)$	$\#f$	$g(x_*)$
MDO1	-10.93552	-16.61198	$1.859e-14$	13491	-16.48513285	6231	0
MDO2	-0.803619	-16.61198	$1.862e-14$	13731	-16.60096482	3894	0
MDO3	-1	-16.61198	$1.975e-14$	11811	-16.50912211	3758	0
MDO4	-30665.5	-16.61198	$1.256e-14$	11031	-16.38935905	7061	0
MDO5	5126.5	-16.61198	$1.944e-14$	12681	-16.41189544	6053	0
MDO6	-6961.81	-16.61198	$1.899e-14$	14571	-16.5696096	5138	0
MDO7	24.3062	-16.61198	$1.904e-14$	13731	-16.60579898	5357	0
MDO8	-0.095825	-16.61198	$2.147e-14$	11371	-16.59635846	3662	0
MDO9	680.63	-16.61198	$1.726e-14$	12321	-16.60317325	4979	0
MDO10	7049.33	-16.61198	$1.473e-14$	9671	-16.04842191	3296	0

Table 5: Comparison results obtained on the tested MDO problem using 10 different starting points and with a maximal budget of 20000 (average of 10 runs for each starting point).

The obtained results using MADS-PB and ES-MF are given in Table 5. One can see that for all the chosen starting points ES-MF converges to the global minimum of the MDO problem while MADS-PB gets trapped by local minima. We note that in general ES-MF requires more function evaluations than MADS for all the tested instances.

6 Conclusion

In this paper, we propose a globally convergent class of ES algorithms where a merit function (with eventually a restoration procedure) is used to decide and control the distribution of the generated points. The obtained algorithm generalized the work [16] by including relaxable constraints in the spirit of what is done in [19]. To the best of our knowledge, the proposed approach is the first globally convergent framework that handles relaxable and unrelaxable constraints in the context of ES's. The proposed convergence analysis was organized depending on the number of times Restoration is entered.

We provided preliminary numerical tests on well-known global optimization problems as well as a multidisciplinary design optimization problem. The obtained results showed the potential of the merit approach compared to the progressive barrier approach (proposed in MADS algorithm). Finally, we acknowledge that, we are concurrently working on performing a study of extensive numerical experiments to analyse the performance of the proposed algorithm.

A Appendix

A.1 Case where algorithm is never left

Theorem A.1 *Assume that f is bounded below and that the restoration is entered and never left.*

(i) *Then there exists a refining subsequence.*

(ii) *Let $x_* \in \Omega_{\mathrm{nr}}$ be the limit point of a convergent subsequence of unsuccessful of iterates $\{x_k\}_K$ for which $\lim_{k\in K} \sigma_k = 0$. Assume that g is Lipschitz continuous near x_*, and let $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$ a corresponding refining direction. Then either $g(x_*) = 0$ or $g^\circ(x_*; d) \geq 0$.*

(iii) *Let $x_* \in \Omega_{\mathrm{nr}}$ be the limit point of a convergent subsequence of unsuccessful of iterates $\{x_k\}_K$ for which $\lim_{k\in K} \sigma_k = 0$. Assume that g and f are Lipschitz continuous near x_*, and let $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$ a corresponding refining direction such that $g^\circ(x_*; d) \leq 0$. Then $f^\circ(x_*; d) \geq 0$.*

(iv) *Assume that the interior of the set $T(x_*)$ given in (7) is non-empty. Let the set of refining directions be dense in $T(x_*)$. Then $f^\circ(x_*, v) \geq 0$ for all $v \in T^{\mathrm{CL}}_{\Omega_{\mathrm{nr}}}(x_*)$ such that $g^\circ(x_*, v) \leq 0$, and x_* is a Clarke stationary point of the problem (1).*

Proof. (i) There must exist a refining subsequence K within this call of the restoration, by applying the same argument of the case where one has $g(x_{k+1}) < g(x_k) - \rho(\sigma_k)$ and $g(x_k) > C\rho(\sigma_k)$ for an infinite subsequence of successful iterations (see the proof of Theorem 4.1). By assumption there exists a subsequence $K' \subseteq K$ such that the sequence $\{a_k/\|a_k\|\}_{k\in K'}$ converges to $d \in T^{\mathrm{H}}_{\Omega_{\mathrm{nr}}}(x_*)$ in K' and the sequence $\{\|a_k\|\sigma_k\}_{k\in K'}$ goes to zero in K'. Thus one must have necessarily for k sufficiently large in K', $x^{\mathrm{trial}}_{k+1} = x_k + \sigma_k a_k \in \Omega_{\mathrm{nr}}$.

(ii) Since the iteration $k \in K'$ is unsuccessful in the Restoration, $g(x_k + \sigma_k a_k) \geq g(x_k) - \rho(\sigma_k)$ or $g(x_{k+1}) \leq C\rho(\sigma_k)$, and the proof follows an argument already seen (see the proof of Theorem 4.2).

(iii) Since at the unsuccessful iteration $k \in K'$, Restoration is never left, so one has $M(x_k + \sigma_k a_k) \geq M(x_k)$, and the proof follows an argument already seen (see the proof of Theorem 4.4).

(iv) The same proof as [19, Theorem 4.4]. ∎

A.2 Case where restoration algorithm is entered and left infinite times

Theorem A.2 *Consider Algorithm 2 and assume that f is bounded below. Assume that Restoration is entered and left an infinite number of times.*

(i) *Then there exists a refining subsequence.*

(ii) Let $x_* \in \Omega_{\rm nr}$ be the limit point of a convergent subsequence of unsuccessful of iterates $\{x_k\}_K$ for which $\lim_{k\in K}\sigma_k = 0$. Assume that g is Lipschitz continuous near x_*, and let $d \in T^{\rm H}_{\Omega_{\rm nr}}(x_*)$ a corresponding refining direction. Then either $g(x_*) = 0$ (implying $x_* \in \Omega_r$ and thus $x_* \in \Omega$) or $g^\circ(x_*; d) \geq 0$.

(iii) Let $x_* \in \Omega_{\rm nr}$ be the limit point of a convergent subsequence of unsuccessful of iterates $\{x_k\}_K$ for which $\lim_{k\in K}\sigma_k = 0$. Assume that g and f are Lipschitz continuous near x_*, and let $d \in T^{\rm H}_{\Omega_{\rm nr}}(x_*)$ a corresponding refining direction such that $g^\circ(x_*; d) \leq 0$. Then $f^\circ(x_*; d) \geq 0$.

(iv) Assume that the interior of the set $T(x_*)$ given in (7) is non-empty. Let the set of refining directions be dense in $T(x_*)$. Then $f^\circ(x_*, v) \geq 0$ for all $v \in T^{\rm CL}_{\Omega_{\rm nr}}(x_*)$ such that $g^\circ(x_*, v) \leq 0$, and x_* is a Clarke stationary point.

Proof. *(i)* Let $K_1 \subseteq K$ and $K_2 \subseteq K$ be two subsequences where Restoration is entered and left respectively.

Since the iteration $k \in K_2$ is unsuccessful in the Restoration, one knows that the step size σ_k is reduced and never increased, one then obtains that σ_k tends to zero. By assumption there exists a subsequence $K' \subseteq K_2$ such that the sequence $\{a_k/\|a_k\|\}_{k\in K'}$ converges to $d \in T^{\rm H}_{\Omega_{\rm nr}}(x_*)$ in K_2 and the sequence $\{\|a_k\|\sigma_k\}_{k\in K'}$ goes to zero in K'.

(ii) For all $k \in K'$, one has $g(x_k + \sigma_k a_k) \geq g(x_k) - \rho(\sigma_k)$ or $g(x_k) \leq C\rho(\sigma_k)$, one concludes that either $g(x_*) = 0$ or $g^\circ(x_*; d) \geq 0$.

(iii) For all $k \in K'$, one has $M(x_k + \sigma_k a_k) \geq M(x_k)$, and from this we conclude that $f^\circ(x_*; d) \geq 0$ if $g^\circ(x_*; d) \leq 0$.

(iv) The same proof as [19, Theorem 4.4]. ■

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