Dynamic risked equilibrium *

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Abstract

We revisit the correspondence of competitive partial equilibrium with a social optimum in markets where risk-averse agents solve multistage stochastic optimization problems formulated in scenario trees. The agents trade a commodity that is produced from an uncertain supply of resources which can be stored. The agents can also trade risk using Arrow-Debreu securities. In this setting we define a risk-trading competitive market equilibrium and prove a welfare theorem: competitive equilibrium will yield a social optimum (with a suitably defined social risk measure) when agents have nested coherent risk measures with intersecting polyhedral risk sets, and there are enough Arrow-Debreu securities to hedge the uncertainty in resource supply. We also give a proof of the converse result: a social optimum with an appropriately chosen risk measure will yield a risk-trading competitive market equilibrium when all agents have nested strictly monotone coherent risk measures with intersecting polyhedral risk sets, and there are enough Arrow-Debreu securities to hedge the uncertainty in resource supply.

1 Introduction

In many competitive situations, manufacturers of a product that is sold over several periods use storage to improve their profits. Storage enables the manufacturers to transfer production from periods when prices are low to periods when it is high. In practice, prices are uncertain and so the optimal storage policy becomes the solution to a stochastic control problem in which manufacturers seek to maximize expected profits if risk neutral, or some risk-adjusted profit if they are risk averse.

In this paper we are interested in a setting in which the product to be sold cannot be stored, but the raw materials that are used in its production can be. An example arises in renewable electricity production in which intermittent generation (wind or photovoltaic energy) can be stored in a battery for later sale. Similarly hydro-electric reservoirs can store energy for later conversion to electricity, or farmers can store pasture (or silage, its harvested form) for later conversion into milk by dairy cows. The process by which the storage is replenished has a random element (e.g. wind, sunlight, catchment inflows, and beneficial weather, in the respective examples we cite). Storage of the raw materials enables the producer to maximize their utilization (subject to its production capacity) when sale prices are high, while possibly holding back production during low-priced periods.

Our current interest focuses on a situation in which prices are determined by an equilibrium of several competing producers, where the total sales of product from the manufacturers equals the demand from consumers in each period. Demand is defined in terms of price by a known decreasing demand function.

The simplest case occurs when the future is known with certainty and producers have convex costs. Then an equilibrium time-varying price can be derived from a Lagrangean decomposition of a social planning model that seeks to maximize the consumer and producer surplus summed over all periods. The Second Welfare Theorem (see e.g. [3]) in this setting is a straightforward consequence of Lagrangean duality theory, and states that the optimal social plan can be interpreted as a perfectly competitive equilibrium at the prices that solve the dual problem. The First Welfare Theorem, stating that any perfectly competitive equilibrium maximizes the consumer and producer surplus in the social plan is also immediate from this duality.

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When the supply of the raw material is uncertain, but governed by a known stochastic process, the social planning problem becomes a stochastic optimal control problem, or more generally a multistage stochastic programming problem. Multistage stochastic optimization models have been well studied (see e.g. [2],[10]). If all agents act as price takers, and seek to maximize expected operating profits, then the first and second welfare theorems translate naturally into the stochastic setting. When the stochastic supply process has a finite sample space, it can be represented as a scenario tree [2], and the Lagrangean theory applied to the deterministic equivalent social planning problem and its dual.

Multistage stochastic optimization becomes more complicated when agents are risk-averse. Decisions in this environment affect outcomes at different times and different states of the world, for which consistent preference relations must be defined. A major contribution to the understanding of this field was the theory of coherent risk measures [1] and their extension to a dynamic setting [8]. This defines a conditional risk mapping in each state of the world in terms of current costs added to risk-adjusted uncertain future costs expressed as a certainty equivalent value. The risk-adjustment is defined in terms of a single-step coherent risk measure. By applying the duality theory of coherent risk measures (see e.g. [10]), one can express the certainty equivalent value of future costs in a state of the world as the conditional expectation of future costs with respect to a probability measure that is chosen to be the worst in a convex set of conditional probability distributions that we call the *risk set*.

When the agents are risk averse, the welfare theorems require further assumptions that ensure that the market for risk is complete. The recent paper [5] (building on the models of [4] and [6]) studies this problem for multistage electricity markets when some producers operate hydroelectricity reservoirs with uncertain inflows. Under an assumption that agents can trade risk using Arrow-Debreu securities, [5] show that a risk-averse social planning solution with an appropriately chosen risk measure can be interpreted as a competitive equilibrium in which the agents trade risk. This result corresponds to the Second Welfare Theorem. It is easy to see that this result extends from electricity markets to the more general setting that we study here.

The proof of the result in [5] involves an implicit assumption that all events in the scenario tree occur with positive risk-adjusted probability. This assumption is made in [5] with a claim of no loss of generality (by removing zero probability events), but this is problematic when risk-adjusted probabilities are endogeneous. For example zero risk-adjusted probabilities will occur when risk is measured using Average Value at Risk, which is not a strictly monotone risk measure as discussed in [9]. The welfare result in [5] is therefore only guaranteed for strictly monotone risk measures. In our setting this corresponds to every agent's risk set lying strictly within the positive orthant. This is not surprising. It is well known (see [9]) that strict monotonicity of one-step coherent risk measures will guarantee the time consistency of optimal policies that use nested dynamic versions of these measures, and the time-consistency of agent's policies appears to be a necessary property if we are to consider them to be in equilibrium (since each agent is anticipating future optimal actions by its competitors.)

In this paper we attempt to clarify and resolve this oversight. In doing so we provide a significantly simpler proof of Theorem 11 in [5] that yields greater insights into the relationship between risk-averse optimization and equilibrium when one-step risk measures are coherent and strictly monotone. We also establish the converse to Theorem 11. In other words we demonstrate a form of the First Welfare Theorem: that a perfectly competitive market equilibrium in which risk-averse agents produce and sell a single product, as well as trading Arrow-Debreu securities, will maximize a risk-adjusted social welfare function. As in Theorem 11, the result requires that the agents' risk sets in each state of the world have non-empty intersection.

The current paper also extends the results in [5] to settings in which agents control storage facilities that are linked. The motivating example is a cascade of hydroelectric stations operated by different agents. Water releases from upstream reservoirs affect the actions of downstream agents. Our model shows how a competitive equilibrium will emerge if we introduce a water price at each point in the river chain. This construction is well-understood in a deterministic setting; we show it can be introduced in a stochastic dynamic setting with risk-averse agents.

The establishment of both welfare theorems provides a deeper understanding of markets. Our version of the Second Welfare Theorem (Theorem 11 in [5] under strict monotonicity) shows that a social planner could argue that their actions in solving a risk-averse social planning problem replicates what one might expect to see in a perfectly competitive market with a complete market for trading risk. A number of electricity markets (e.g. Brazil and Chile) operate on this principle. The corresponding analogue of the First Welfare Theorem (newly established in this paper) shows that if electricity markets are perfectly competitive and endowed with a complete market for trading risk, then one might expect them to arrive in equilibrium at policies that maximize risk-adjusted consumer and producer surplus. More importantly perhaps, the theorems indicate how social welfare might be compromised in these markets if risk markets are incomplete or illiquid, even if agents behave as price takers.

The paper is laid out as follows. In the next section we define one-step coherent polyhedral risk measures, and give a lemma that enables us to evaluate these using linear programming. Section 3 defines a scenario tree representation of resource uncertainty, and extends the definition of risk measure to a multistage setting. In Section 4 we introduce two versions of a system optimization problem. The first is a risk-averse optimization problem and the second is a related complementarity problem, which imposes a time-consistent constraint on the solution. They are equivalent when all agents have strictly monotone risk measures. Section 5 defines a social risk measure and recalls the concept of Arrow-Debreu securities. We then prove that any solution to the complementarity version of the social planning problem corresponds to a risk-trading equilibrium when all agent's risk sets have nonempty intersection and the set of Arrow-Debreu securities spans the set of random outcomes in each stage. As a corollary it follows that the social planning problem and risk-trading equilibrium coincide when all agents have strictly monotone risk measures. In the final section we draw some conclusions and discuss extensions to our results.

2 Coherent risk measures

We consider a setting in which decision makers are risk averse when contemplating a decision that has random events defined by a sample space $\mathcal{M}$. For simplicity, we assume throughout this paper that $\mathcal{M}$ is finite. Each decision maker faced with a random cost or disbenefit $Z(m)$, $m \in \mathcal{M}$, measures its risk using a *coherent risk measure* ρ as defined axiomatically by [1]. Thus $\rho(Z)$ is a real number representing the risk-adjusted disbenefit of Z.

It is well-known that any coherent risk measure $\rho(Z)$ has a dual representation expressing it as

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_\mu[Z]$$

where $\mathcal{D}$ is a convex subset of probability measures on $\mathcal{M}$ (see e.g. [1, 4]). $\mathcal{D}$ is called the *risk set* of the coherent risk measure. We use the notation $[p]_\mathcal{M}$ to denote any vector $\{p(m), m \in \mathcal{M}\}$. So any probability measure $\mu \in \mathcal{D}$ can be written $[\mu]_\mathcal{M}$, where $\mu(m)$ defines the probability of event m. The dual representation using a risk set plays an important role in the analysis we carry out in this paper. We refer to the case where the risk set is a singleton as *risk neutral*.

In the rest of this paper we assume that risk sets are polyhedrons with known extreme points $\left\{[p^k]_\mathcal{M}, k \in \mathcal{K}\right\}$, where $\mathcal{K}$ is a finite index set. Then

$$\sup_{\mu \in \mathcal{D}} \mathbb{E}_\mu[Z] = \sup_{\mu \in \mathcal{D}} \sum_{m \in \mathcal{M}} \mu(m) Z(m) = \max_{k \in \mathcal{K}} \sum_{m \in \mathcal{M}} p^k(m) Z(m)$$

since the maximum of a linear function over $\mathcal{D}$ is attained at an extreme point. By a standard dualization, this gives

$$\sup_{\mu \in \mathcal{D}} \sum_{m \in \mathcal{M}} \mu(m) Z(m) = \left\{ \begin{array}{ll} \min & \theta \\ \text{s.t.} & \theta \geq \sum_{m \in \mathcal{M}} p^k(m) Z(m), \quad k \in \mathcal{K}. \end{array} \right.$$

Lemma 1 *Suppose $\mathcal{D}$ is a polyhedral risk set with extreme points $\{[p^k]_\mathcal{M},\ k \in \mathcal{K}\}$ and $Z(m)$, $m \in \mathcal{M}$ is given. Then*

$$\theta = \sup_{\mu \in \mathcal{D}} \sum_{m \in \mathcal{M}} \mu(m) Z(m)$$

if and only if there is some γ with

$$\begin{aligned} &\sum_{k \in \mathcal{K}} \gamma^k = 1 \\ &0 \leq \gamma^k \perp \theta - \sum_{m \in \mathcal{M}} p^k(m) Z(m) \geq 0, \quad k \in \mathcal{K}. \end{aligned}$$

Furthermore, $\bar{\mu}$, defined by $\bar{\mu}(m) = \sum_{k \in \mathcal{K})} \gamma^k p^k(m)$, is in $\mathcal{D}$ and attains the supremum.

Proof. For the forward implication, just choose $\gamma^k = 1$ for the term involving extreme point k that achieves the supremum. For the reverse implication, since $\theta \geq \sum_{m\in\mathcal{M}} p^k(m)Z(m)$ for each extreme point, it follows that $\theta \geq \sum_{m\in\mathcal{M}} \mu(m)Z(m)$ for each $\mu \in \mathcal{D}$ and hence $\theta \geq \sup_{\mu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \mu(m)Z(m)$. But complementary slackness shows that

$$\theta = \sum_{m\in\mathcal{M}} \bar{\mu}(m)Z(m),$$

where $\bar{\mu}$ is defined in the statement of the theorem and is clearly in $\mathcal{D}$ so $\theta \leq \sup_{\mu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \mu(m)Z(m)$ and thus equality holds. ■

By definition, a coherent risk measure is *monotone*. This means that

$$Z_a \geq Z_b \Rightarrow \rho(Z_a) \geq \rho(Z_b)$$

A stronger condition is *strict monotonicity*. This requires that

$$Z_a \geq Z_b \text{ and } Z_a \neq Z_b \Rightarrow \rho(Z_a) > \rho(Z_b).$$

If strictly monotone coherent risk measures have polyhedral risk sets then these lie strictly inside the positive orthant.

Lemma 2 *Suppose ρ is a coherent risk measure with a polyhedral risk set $\mathcal{D}$. Then $\mathcal{D} \subset int(\mathbb{R}_+^{|\mathcal{M}|})$ if and only if ρ is strictly monotone.*

Proof. Suppose $\mathcal{D}$ lies in $\text{int}(\mathbb{R}_+^{|\mathcal{M}|})$. To show strict monotonicity, we suppose $Z_a \geq Z_b$ and $Z_a(\bar{m}) > Z_b(\bar{m})$ for some $\bar{m} \in \mathcal{M}$. Let $\rho(Z_a) = \sum_{m\in\mathcal{M}} \mu_a^*(m)Z_a(m)$, and $\rho(Z_b) = \sum_{m\in\mathcal{M}} \mu_b^*(m)Z_b(m)$. Then strict monotonicity follows from $\mu_b^*(\bar{m}) > 0$ since

$$\begin{aligned}
\rho(Z_a) &= \sum_{m\in\mathcal{M}} \mu_a^*(m)Z_a(m) \\
&\geq \sum_{m\in\mathcal{M}} \mu_b^*(m)Z_a(m) \\
&> \sum_{m\in\mathcal{M}} \mu_b^*(m)Z_b(m) \\
&= \rho(Z_b).
\end{aligned}$$

Conversely, suppose $\mathcal{D}$ does not lie in $\text{int}(\mathbb{R}_+^{|\mathcal{M}|})$, thus containing some point $\bar{\mu}$ with a zero component, say $\bar{\mu}(m_1) = 0$. Choose $Z(m) > 0$, $m = m_2, m_3, \ldots, m_{|\mathcal{M}|}$, and $Z(m_1) < 0$, so if $\mu(m_1) > 0$ then

$$\begin{aligned}
\sum_{m\in\mathcal{M}} \mu(m)Z(m) &= \mu(m_1)Z(m_1) + \sum_{m\neq m_1} \mu(m)Z(m) \\
&< \sum_{m\neq m_1} \mu(m)Z(m).
\end{aligned}$$

It follows that any $\bar{\mu} \in \arg\max_{\mu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \mu(m)Z(m)$ must have $\bar{\mu}(m_1) = 0$. Let

$$Z'(m) = \begin{cases} Z(m_1) - 1, & m = m_1 \\ Z(m), & \text{otherwise} \end{cases}$$

so $Z' \leq Z$ with $Z' \neq Z$. Any $\bar{\mu}' \in \arg\max_{\mu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \mu(m)Z'(m)$ must have $\bar{\mu}'(m_1) = 0$, and by the optimality of $\bar{\mu}$ satisfy

$$\sum_{m\neq m_1} \bar{\mu}'(m)Z(m) = \sum_{m\neq m_1} \bar{\mu}(m)Z(m)$$

so $\rho(Z') = \rho(Z)$ violating the strict monotonicity of ρ. ■

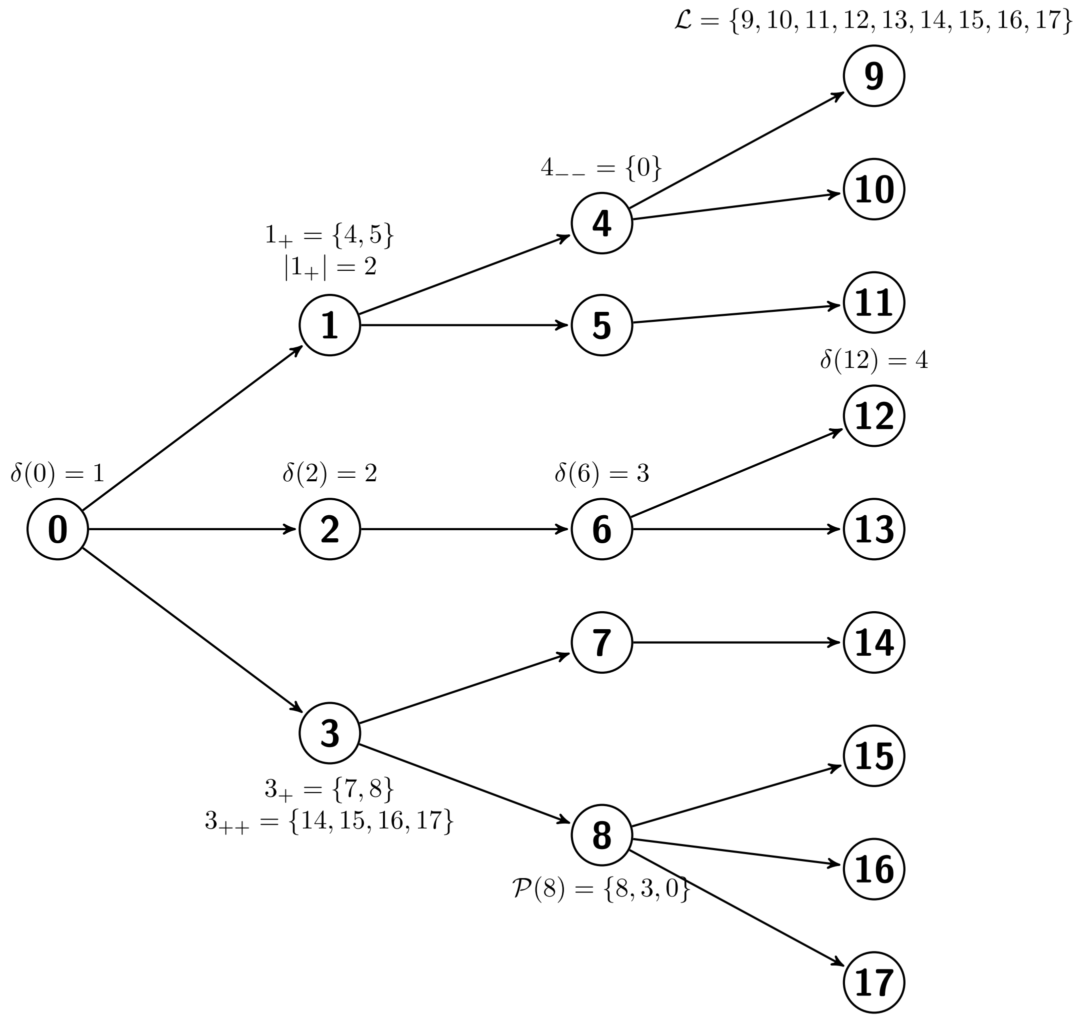

Figure 1: A scenario tree with nodes $\mathcal{N} = \{1, 2, \ldots, 17\}$, and $T = 4$

3 Dynamic risk measures

We incorporate the risk measures discussed in the previous section into a multistage setting in which agents make production and consumption decisions as well as exchanges of welfare over several time stages. Random events are now defined by a discrete-time stochastic process. Given a finite set of events in each stage, such a process can be modeled using a *scenario tree* with nodes $n \in \mathcal{N}$ and leaves in $\mathcal{L}$. By convention we number the root node $n = 0$. The unique predecessor of node $n \neq 0$ is denoted by n_-. We denote the set of children of node $n \in \mathcal{N} \setminus \mathcal{L}$ by n_+, and denote its cardinality by $|n_+|$. The set of predecessors of node n on the path from n to node 0 is denoted $\mathcal{P}(n)$ (so $\mathcal{P}(n) = \{n, n_-, n_{--}, \ldots, 0\}$), where we use the natural definitions for n_{--} and n_{++}. The depth $\delta(n)$ of node n is the number of nodes on the path to node 0, so $\delta(0) = 1$ and we assume that every leaf node has the same depth, say $\delta_{\mathcal{L}}$. The depth of a node can be interpreted as a time index $t = 1, 2, \ldots, T = \delta_{\mathcal{L}}$. A pictorial representation of a scenario tree with four time stages is given in Figure 1.

For a multistage decision problem, we require a dynamic version of risk. The concept of coherent dynamic risk measures was introduced in [7] and is described for general Markov decision problems in [8]. Formally one defines a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\emptyset, \Omega\} = \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \subset \mathcal{F}_T \subset \mathcal{F}$ of σ-fields where all data in node 0 is assumed to be deterministic, and decisions at time t are $\mathcal{F}_t$-measurable random variables (see [8]). Working with finite probability spaces defined by a scenario tree simplifies this description.

Given a tree defined by $\mathcal{N}$, suppose the random sequence of actions $\{u(n),\ n \in \mathcal{N}\}$ results in a random sequence of disbenefits $\{Z(n), n \in \mathcal{N}\}$. We seek to measure the risk of this disbenefit sequence when viewed by a decision maker at node 0. At node n the decision maker is endowed with a one-step risk set $\mathcal{D}(n)$ that measures the risk of random risk-adjusted costs accounted for in $m \in n_+$. Thus elements of $\mathcal{D}(n)$ are finite probability distributions of the form $[p]_{n_+}$.

The risk-adjusted disbenefit $\theta(n)$ of all random future outcomes at node $n \in \mathcal{N} \setminus \mathcal{L}$ can be defined recursively. We denote the future risk-adjusted disbenefit in each leaf node $n \in \mathcal{L}$ by $\bar{\theta}(n)$. Then $\theta(n)$ is defined recursively to be

$$\theta(n) = \begin{cases} \bar{\theta}(n), & n \in \mathcal{L}, \\ \sup\limits_{\mu \in \mathcal{D}(n)} \sum\limits_{m \in n_+} \mu(m)(Z(m) + \theta(m)), & n \in \mathcal{N} \setminus \mathcal{L}. \end{cases} \tag{1}$$

When viewed in node n, $\theta(n)$ can be interpreted to be the fair one-time charge we would be willing to incur instead of the sequence of random future costs $Z(m)$ incurred in all successor nodes of n. In other words the measure $\theta(n)$ is a *certainty equivalent cost* or *risk-adjusted expected cost* of all the future costs in the subtree rooted at node n. As demonstrated in [8, Theorem 1], any time-consistent dynamic risk measure has this recursive form.

Since we assume for $n \in \mathcal{N} \setminus \mathcal{L}$ that $\mathcal{D}(n)$ is a polyhedron with extreme points $\{[p^k], k \in \mathcal{K}(n)\}$, the recursive structure defined by (1) can then be simplified to

$$\begin{aligned} & \sup_{\mu \in \mathcal{D}(n)} \sum_{m \in n_+} \mu(m)(Z(m) + \theta(m)) \\ = & \begin{cases} \min & \theta \\ \text{s.t.} & \theta \geq \sum\limits_{m \in n_+} p^k(m)\left(Z(m) + \theta(m)\right), \quad k \in \mathcal{K}(n). \end{cases} \end{aligned} \tag{2}$$

For some of our results we will need to make the following assumption.

Assumption 1 *For every $n \in \mathcal{N} \setminus \mathcal{L}$, $\mathcal{D}(n) \subset int(\mathbb{R}_+^{|n_+|})$.*

Under this assumption, Lemma 2 ensures that one-step risk measures are strictly monotone, so the nested risk measure with risk sets $\mathcal{D}(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ gives *time-consistent* policies. To see this observe that when Assumption 1 does not hold it is possible for

$$\bar{p}^k \in \arg\max_{k \in \mathcal{K}(n)} \sum_{m \in n_+} p^k(m)(Z(m) + \theta(m))$$

to have $p^k(\bar{m}) = 0$ for some $\bar{m}$. If so, then evaluating the risk at node 0 will ignore all disbenefits in the subtree of nodes in $\mathcal{N}$ rooted at $\bar{m}$. Decisions in these nodes will not affect the overall risk-adjusted benefit in node 0 unless they change $\bar{p}^k$. If these decisions are suboptimal given that the decision maker is in the state of the world defined by $\bar{m}$ then the policy defined by all the decisions is not time-consistent.

Assumption 1 also avoids some technical difficulties in proving a correspondence between equilibrium and optimization. In particular if no element of any probability vector is ever zero then the risk-adjusted disbenefit at node 0 can be computed by weighting the disbenefit at each node n by a strictly positive unconditional probability, say $\sigma(n)$, to give

$$\theta(0) = \sum_{n \in \mathcal{N} \setminus \{0\}} \sigma(n) Z(n).$$

Given $\sigma(0) = 1$ and $\sigma(n) > 0, n \in \mathcal{N} \setminus \{0\}$, conditional probabilities $\mu(n)$, $n \in \mathcal{N} \setminus \{0\}$ can be computed as $\sigma(n)/\sigma(n_-)$. We extend this concept to risk sets not satisfying Assumption 1 using the following definitions.

Consider a scenario tree with polyhedral risk sets $\mathcal{D}(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$, each having a finite set of extreme points $\{[p^k]_{n_+}, k \in \mathcal{K}(n)\}$. Any set of nonnegative numbers of the form $\{\gamma^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is called a set of *tree multipliers*. A set of tree multipliers is *conditional* if for every $n \in \mathcal{N} \setminus \mathcal{L}$, $\sum_{k \in \mathcal{K}(n)} \gamma^k(n) = 1$. A set of tree multipliers $\{\lambda^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is *unconditional* if

$$0 = 1 - \sum_{k \in \mathcal{K}(0)} \lambda^k(0), \tag{3}$$

$$0 = -\sum_{k \in \mathcal{K}(n)} \lambda^k(n) + \sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0. \tag{4}$$

Observe that any set of conditional tree multipliers γ corresponds to a unique set of unconditional tree multipliers λ defined recursively by setting $\lambda(0) = \gamma(0)$, and defining

$$\lambda(n) = \gamma(n) \sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n), \quad n \in \mathcal{N} \setminus \{0\}. \tag{5}$$

Since $\lambda(0) \geq 0$, repeated application of (5) implies $\lambda(n) \geq 0$ for every $n \in \mathcal{N} \setminus \{0\}$, so $\lambda^k(n)$ are well-defined tree multipliers. These are easily verified to be unconditional since for $n \in \mathcal{N} \setminus \mathcal{L}$

$$\sum_{k \in \mathcal{K}(n)} \lambda^k(n) = \sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n),$$

giving (4) and

$$\sum_{k \in \mathcal{K}(0)} \lambda^k(0) = \sum_{k \in \mathcal{K}(0)} \gamma^k(0) = 1,$$

giving (3). Conversely any unconditional set of tree multipliers corresponds to a unique set of conditional tree multipliers as long as Assumption 1 holds. To see this define $\gamma(0) = \lambda(0)$, and

$$\gamma(n) = \lambda(n) / (\sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n)) \quad n \in \mathcal{N} \setminus \{0\}. \tag{6}$$

By Assumption 1 every component of $p^k(m)$, $m \in 0_+$ is strictly positive, and the vector $(\lambda(0))$ is nonnegative and nonzero by (3), so $\gamma^k(m)$ is well defined by (6) for $m \in 0_+$. However, (4) implies that the vector $(\lambda^k(m))$ is nonnegative and nonzero, and hence recursively that

$$\sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n) > 0, \quad n \in \mathcal{N} \setminus \{0\}. \tag{7}$$

Finally (4) and (6) imply $\sum_{k \in \mathcal{K}(n)} \gamma^k(n) = 1$, showing that $\{\gamma^k(n),\, k \in \mathcal{K}(n),\, n \in \mathcal{N} \setminus \mathcal{L}\}$ is conditional.

To ease notation in the rest of the paper, given any unconditional tree multipliers λ we define tree multipliers σ by

$$\sigma(n) = \begin{cases} 1, & n = 0, \\ \sum\limits_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n), & n \in \mathcal{N} \setminus \{0\}. \end{cases} \tag{8}$$

Observe by (5) that (8) implies

$$\lambda^k(n) = \gamma^k(n) \sigma(n), \quad k \in \mathcal{K}(n), n \in \mathcal{N}, \tag{9}$$

whence multiplying by $p^k(m)$ and summing gives

$$\sum_{k \in \mathcal{K}(n)} \lambda^k(n) p^k(m) = \sigma(m) = \mu(m) \sigma(n), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}, \tag{10}$$

where we define

$$\mu(m) = \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{11}$$

Conditional and unconditional multipliers satisfy the following lemma.

Lemma 3 *If $\theta(n), n \in \mathcal{N}$ and a conditional set of tree multipliers $\{\gamma^k(n),\, k \in \mathcal{K}(n),\, n \in \mathcal{N} \setminus \mathcal{L}\}$ satisfies*

$$0 \leq \gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m) \left(C(m) + \theta(m) \right) \geq 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}, \tag{12}$$

then there exist unconditional multipliers λ satisfying

$$0 \leq \lambda^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m) \left(C(m) + \theta(m) \right) \geq 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}. \tag{13}$$

Conversely, if (λ, θ) satisfies (3),(4),(13), and Assumption 1 holds, then $\sigma(n) > 0$, $n \in \mathcal{N}$ and there exists conditional tree multipliers $\gamma^k(n) = \frac{\lambda^k(n)}{\sigma(n)}, \quad n \in \mathcal{N} \setminus \mathcal{L}$ satisfying (12).

Proof. Given a conditional set of tree multipliers γ construct unconditional multipliers λ from (5) and $\lambda(0) = 1$. Given these values, $\sigma \geq 0$ is defined by (8), so

$$0 \leq \sigma(n)\gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m)\,(C(m) + \theta(m)) \geq 0$$

yielding (13) via (9). Conversely, Assumption 1 implies (7), so we have $\sigma(n) > 0$. The relationship (12) then follows from (13) by dividing through by $\sigma(n) > 0$. ■

4 System optimization

We now turn our attention to a system optimization problem that minimizes risk-adjusted disbenefit using a dynamic risk measure. This is defined by (1) using polyhedral risk sets

$$\mathcal{D}(n) = \operatorname{conv}\{\{[p^k]_{n_+}, k \in \mathcal{K}(n)\}\}, \quad n \in \mathcal{N} \setminus \mathcal{L}.$$

Given actions $u_a(n)$, $n \in \mathcal{N}$ the system disbenefit in node n is measured by a function $\sum_{a \in \mathcal{A}} C_a(u_a(n))$. Here for producer a, C_a measures production cost, and for consumer a, C_a measures consumption disbenefit that increases as u_a increases towards 0. We assume that each C_a is convex.

Each producing agent a consumes resources that come from a vector x_a of storages that are released at rates defined by the vector u_a yielding total production $g_a(u_a)$. The storage is replenished with a vector of random supplies ω_a. This gives a stochastic process defined by

$$x_a(n) \leq x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n).$$

Note that the matrix T_{ab} in the dynamics allows for a network of connections between storage devices controlled by different agents, and the inequality allows for free disposal (or spilling) at the storage device. The dynamics could be expressed a little more generally using a diagonal matrix S_a for gains or losses and making S and T dependent on node as

$$x_a(n) \leq S_a(n) x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab}(n) u_b(n) + \omega_a(n),$$

but since this does not change the subsequent analysis in any substantive way, we assume $S_a(n) \equiv I$ and $T_{ab}(n) \equiv T_{ab}$ in what follows.

The releases u_a and storages x_a are constrained to lie in respective sets $\mathcal{U}_a$ and $\mathcal{X}_a$. For any set $\mathcal{S}$ we define the *normal cone* at $\bar{s}$ to be

$$N_{\mathcal{S}}(\bar{s}) = \{d : d^\top(s - \bar{s}) \leq 0 \text{ for all } s \in S\},$$

and recall that $\bar{s}$ minimizes a convex function $f(s)$ over convex set S if and only if

$$0 \in \nabla_s f(s) + N_{\mathcal{S}}(\bar{s}).$$

Given production u_a, $a \in \mathcal{A}$, and resulting storage x_a, $a \in \mathcal{A}$, the risk-adjusted system disbenefit at node 0 is $\sum_{a \in \mathcal{A}} C_a(u_a(0)) + \theta(0)$ where $\theta(0)$ is defined recursively by

$$\theta(n) = \begin{cases} -\sum_{a \in \mathcal{A}} V_a(x_a(n)), & n \in \mathcal{L}, \\ \max_{\mu \in \mathcal{D}(n)} \sum_{m \in n_+} \mu(m) \left(\sum_{a \in \mathcal{A}} C_a(u_a(m)) + \theta(m) \right), & n \in \mathcal{N} \setminus \mathcal{L}. \end{cases} \tag{14}$$

The risk-averse system optimization problem is then formulated as follows.

$$
\begin{aligned}
&\text{SO}(\mathcal{D}): \\
&\min_{u,x,\theta} \sum_{a\in\mathcal{A}} C_a(u_a(0)) + \theta(0) \\
&\text{s.t. } \theta(n) \geq \sum_{m\in n_+} p^k(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right), \\
&\qquad k\in\mathcal{K}(n),\ n\in\mathcal{N}\setminus\mathcal{L}, && (15)\\
&\quad x_a(n) \leq x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n), \quad a\in\mathcal{A}, n\in\mathcal{N}, && (16)\\
&\quad \sum_{a\in\mathcal{A}} g_a(u_a(n)) \geq 0 \quad n\in\mathcal{N}, && (17)\\
&\quad \theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n)), \quad n\in\mathcal{L}, \\
&\quad u_a(n)\in\mathcal{U}_a, \quad x_a(n)\in\mathcal{X}_a, \quad n\in\mathcal{N}, \quad a\in\mathcal{A}.
\end{aligned}
$$

We proceed to use the Karush-Kuhn-Tucker conditions for SO($\mathcal{D}$) to derive a complementarity version of this problem. To make this possible, the following condition will be assumed throughout the paper.

Assumption 2 *The nonlinear constraints in SO($\mathcal{D}$) satisfy a constraint qualification.*

The Karush-Kuhn-Tucker conditions for SO($\mathcal{D}$) are

$$
\begin{aligned}
&\text{KKT}(\mathcal{D}): \\
&0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0), \\
&0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{k\in\mathcal{K}(n_-)} \lambda^k(n_-)p^k(n), \quad n\in\mathcal{N}\setminus\mathcal{L}, n\neq 0 \\
&0 \leq \lambda^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right) \geq 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L} \\
&\theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n)), \quad n\in\mathcal{L} \\
&0 \in \nabla_{u_a(0)}\left[C_a(u_a(0)) - \tilde{\pi}(0)g_a(u_a(0)) - \sum_{b\in\mathcal{A}} \tilde{\alpha}_b(0)T_{ba}u_a(0)\right] + N_{\mathcal{U}_a}(u_a(0)), \quad a\in\mathcal{A} \\
&0 \in \nabla_{u_a(n)}\left[\sum_{k\in\mathcal{K}(n_-)} \lambda^k(n_-)p^k(n)C_a(u_a(n)) - \tilde{\pi}(n)g_a(u_a(n)) - \sum_{b\in\mathcal{A}} \tilde{\alpha}_b(n)T_{ba}u_a(n)\right] \\
&\qquad + N_{\mathcal{U}_a}(u_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\{0\} \\
&0 \in \tilde{\alpha}_a(n) - \sum_{m\in n_+} \tilde{\alpha}_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\mathcal{L} \\
&0 \in \tilde{\alpha}_a(n) - \sum_{k\in\mathcal{K}(n_-)} \lambda^k(n_-)p^k(n)\nabla_{x_a(n)}V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{L} \\
&0 \leq \tilde{\alpha}_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \geq 0, \quad a\in\mathcal{A}, n\in\mathcal{N} \\
&0 \leq \tilde{\pi}(n) \perp \sum_{a\in\mathcal{A}} g_a(u_a(n)) \geq 0, \quad n\in\mathcal{N}.
\end{aligned}
$$

Since SO($\mathcal{D}$) is a convex optimization problem and the constraint qualification Assumption 2 holds, these KKT conditions are necessary and sufficient for optimality in SO($\mathcal{D}$). Using Lemma 3 we now show under

Assumption 1 that these conditions are equivalent to the following set of conditions.

SE($\mathcal{D}$):

$$
\begin{aligned}
&0 = 1 - \sum_{k\in\mathcal{K}(n)} \gamma^k(n), \quad n \in \mathcal{N}\setminus\mathcal{L} \\
&0 \le \gamma^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N}\setminus\mathcal{L} \\
&\theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n)), \quad n \in \mathcal{L} \\
&0 \in \nabla_{u_a(n)}\left[C_a(u_a(n)) - \pi(n)g_a(u_a(n)) - \sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N} \\
&0 \in \alpha_a(n) - \sum_{m\in n_+}\sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}\setminus\mathcal{L} \\
&0 \in \alpha_a(n) - \nabla_{x_a(n)}V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L} \\
&0 \le \alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N} \\
&0 \le \pi(n) \perp \sum_{a\in\mathcal{A}} g_a(u_a(n)) \ge 0, \quad n \in \mathcal{N}.
\end{aligned}
$$

Theorem 1 *(i) Any solution to SE($\mathcal{D}$) solves SO($\mathcal{D}$) and satisfies*

$$
\begin{aligned}
\theta(n) &= \max_{\mu\in\mathcal{D}(n)} \sum_{m\in n_+} \mu(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right) \\
&= \sum_{m\in n_+} \bar{\mu}(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right),
\end{aligned}
$$

where $\bar{\mu}(m) = \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)$.

(ii) Under Assumption 1 any solution to SO($\mathcal{D}$) satisfies SE($\mathcal{D}$) for some π, α, γ.

Proof. (i) Suppose $(u, \theta, x, \gamma, \alpha, \pi)$ is a solution of SE($\mathcal{D}$). Since γ are conditional multipliers, and $\theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n))$, $n \in \mathcal{L}$, and (θ, γ) satisfies (12), Lemma 3 provides unconditional multipliers λ (and therefore σ from (8)) such that (13) holds for $C(m) = \sum_{a\in\mathcal{A}} C_a(u_a(m))$. Using these observations, the problem SE($\mathcal{D}$)

leads to the conditions:

$$0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0),$$

$$0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{k\in\mathcal{K}(n_-)} \lambda^k(n_-)p^k(n), \quad n \in \mathcal{N}\setminus\mathcal{L}, n \neq 0$$

$$0 \le \lambda^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_a(u_a(m)) + \theta(m)) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N}\setminus\mathcal{L}$$

$$\theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n)), \quad n \in \mathcal{L}$$

$$0 \in \nabla_{u_a(n)}\left[\sigma(n)C_a(u_a(n)) - \sigma(n)\pi(n)g_a(u_a(n)) - \sigma(n)\sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+}\sum_{k\in\mathcal{K}(n)} \lambda^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}\setminus\mathcal{L}$$

$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)}V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L}$$

$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_a(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

The relationships involving normal cones follow from multiplication by $\sigma(n)$ and (9), while the complementarity conditions follow from Lemma 3 and multiplication by $\sigma(n)$. If we let $\tilde{\alpha}_a(n) = \sigma(n)\alpha_a(n)$ and $\tilde{\pi}(n) = \sigma(n)\pi(n)$ then recalling (8) these conditions yield KKT($\mathcal{D}$), the KKT conditions for SO($\mathcal{D}$). Since any solution of SE($\mathcal{D}$) satisfies (12) in Lemma 3, (9) and Lemma 1 imply that

$$\theta(n) = \sup_{\mu\in\mathcal{D}(n)} \sum_{m\in n_+} \mu(m)\left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m)\right)$$

is attained by $\bar{\mu}(m) = \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)$, which gives the last statement of (i).
(ii) For the converse result, suppose that we have a solution $(u, x, \theta, \lambda, \tilde{\pi}, \tilde{\alpha})$ of the KKT conditions of SO($\mathcal{D}$) as shown above. Then Assumption 1 and Lemma 3 provide $\sigma(n) > 0$ and a conditional set of multipliers $\gamma^k(n) = \lambda^k(n)/\sigma(n)$ satisfying (12) for $C(m) = \sum_{a\in\mathcal{A}} C_a(u_a(m))$. Substituting $\alpha_a(n) = \tilde{\alpha}_a(n)/\sigma(n)$ and $\pi(n) = \tilde{\pi}(n)/\sigma(n)$ into the KKT conditions of SO($\mathcal{D}$) and using (12) and (10) leads to

$$1 = \sum_{k\in\mathcal{K}(n)} \gamma^k(n), \quad n \in \mathcal{N}\setminus\mathcal{L}$$

$$0 \le \gamma^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_a(u_a(m)) + \theta(m)) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N}\setminus\mathcal{L}$$

$$\theta(n) = -\sum_{a\in\mathcal{A}} V_a(x_a(n)), \quad n \in \mathcal{L}$$

$$0 \in \nabla_{u_a(n)}\left[\sigma(n)C_a(u_a(n)) - \sigma(n)\pi(n)g_a(u_a(n)) - \sigma(n)\sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+} \sigma(n) \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}\setminus\mathcal{L}$$

$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)}V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L}$$

$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_a(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

Dividing through by $\sigma(n)$ appropriately leads to a solution of SE($\mathcal{D}$) as required. ■

5 Risk trading

We now turn our attention to the situation where agents with polyhedral risk sets can trade financial contracts to reduce their risk. We will show that the system optimal solution to a social planning problem corresponds to a perfectly competitive equilibrium with risk trading. We use the notation $Z_a(n)$, $n \in \mathcal{N}$ to denote the disbenefit of agent a, and $\mathcal{D}_a(n)$ to denote the risk set of agent a, which is a polyhedral set with extreme points $\{[p_a^k]_{n_+}, k \in \mathcal{K}_a(n)\}$. In order to get some alignment between the objectives of agents and a social planner, we establish a connection between their risk sets using the following assumption and definitions.

Assumption 3 *For $n \in \mathcal{N} \setminus \mathcal{L}$*

$$\bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n) \neq \emptyset.$$

Definition 1 *For $n \in \mathcal{N} \setminus \mathcal{L}$ the* social planning risk set *is*

$$\mathcal{D}_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n), .$$

The financial instruments that are traded are assumed to take a specific form.

Definition 2 *Given any node $n \in \mathcal{N} \setminus \mathcal{L}$, an* Arrow-Debreu security *for node $m \in n_+$ is a contract that charges a price $\mu(m)$ in node n to receive a payment of* 1 *in node $m \in n_+$.*

We shall assume throughout this section that the market for risk is *complete*. Formally this means that the set of Arrow-Debreu securities traded at each node n spans the set of possible outcomes in n_+.

Assumption 4 *At every node $n \in \mathcal{N} \setminus \mathcal{L}$, there is an Arrow-Debreu security for each child node $m \in n_+$.*

To reduce its risk, suppose that each agent a in node n purchases $W_a(m)$ Arrow-Debreu securities for node $m \in n_+$. Each agent a's optimization problem with risk trading is then formulated as

$$\begin{aligned}
&\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a): \\
&\qquad \min_{u_a, x_a, W_a, \theta_a} \quad Z_a(0; u, x, W) + \theta_a(0) \\
&\qquad\qquad \text{s.t.} \quad \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)), \\
&\qquad\qquad\qquad\qquad k \in K_a(n),\ n \in \mathcal{N} \setminus \mathcal{L}, \\
&\qquad\qquad\qquad \theta_a(n) = -V_a(x_a(n)), \quad n \in \mathcal{L}, \\
&\qquad\qquad\qquad u_a(n) \in \mathcal{U}_a,\ x_a(n) \in \mathcal{X}_a, \quad n \in \mathcal{N},
\end{aligned}$$

where we use the shorthand notation

$$\begin{aligned}
Z_a(n; u, x, W) = {} & C_a(u_a(n)) - \pi(n) g_a(u_a(n)) + \alpha_a(n) \left(x_a(n) - x_a(n_-) - \omega_a(n) \right) \\
& - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n) + \sum_{m \in n_+} \mu(m) W_a(m), \quad n \in \mathcal{N}.
\end{aligned} \tag{18}$$

Essentially the agent minimizes immediate cost plus the (insurance) cost of the security along with future costs, in the understanding that the security will pay back in the next period according to the situation realized.

As we outlined above for the system optimization problem, $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ is equivalent to its KKT conditions, which are derived by applying nonnegative Lagrange multipliers $\lambda^k(n)$ to the inequality constraints. Since θ is unconstrained, λ satisfies (3) and (4), so they are unconditional tree multipliers. This enables us to substitute $\sigma(n)$ for $\sum_{k \in \mathcal{K}_a(n_-)} \lambda^k(n_-) p_a^k(n)$ in the following.

KKT$_a$:

$$0 = 1 - \sum_{k \in \mathcal{K}_a(0)} \lambda^k(0), \tag{19a}$$

$$0 = - \sum_{k \in \mathcal{K}_a(n)} \lambda^k(n) + \sigma(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0 \tag{19b}$$

$$0 \leq \lambda^k(n) \perp \theta_a(n) - \sum_{m \in n_+} p_a^k(m) \left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \right) \geq 0,$$

$$\quad k \in \mathcal{K}_a(n), n \in \mathcal{N} \setminus \mathcal{L} \tag{19c}$$

$$\theta_a(n) = -V_a(x_a(n)), \quad n \in \mathcal{L}. \tag{19d}$$

$$0 \in \nabla_{u_a(0)} Z_a(0; u, x, W) + N_{\mathcal{U}_a}(u_a(0)), \tag{19e}$$

$$0 \in \sigma(n) \nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \setminus \{0\} \tag{19f}$$

$$0 \in \sigma(n)\alpha_a(n) - \sum_{m \in n_+} \sigma(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L} \tag{19g}$$

$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)} V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \tag{19h}$$

$$0 = \mu(m) - \sum_{k \in \mathcal{K}_a(0)} \lambda^k(0) p_a^k(m), \quad m \in 0_+ \tag{19i}$$

$$0 = \sigma(q_-)\mu(q) - \sigma(q), \quad q \in n_{++} \cap \mathcal{N}, \quad n \in \mathcal{N}. \tag{19j}$$

with $Z_a(n; u, x, W)$ defined by (18).

We also define a complementarity form of $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$

$\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$:

$$0 = 1 - \sum_{k \in \mathcal{K}_a(n)} \gamma^k(n), \quad n \in \mathcal{N} \setminus \mathcal{L} \tag{20a}$$

$$0 \leq \gamma^k(n) \perp \theta_a(n) - \sum_{m \in n_+} p_a^k(m) \Big(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \Big) \geq 0,$$

$$\quad k \in \mathcal{K}_a(n), n \in \mathcal{N} \setminus \mathcal{L} \tag{20b}$$

$$\theta_a(n) = -V_a(x_a(n)), \quad n \in \mathcal{L} \tag{20c}$$

$$0 \in \nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \tag{20d}$$

$$0 \in \alpha_a(n) - \sum_{m \in n_+} \mu(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L} \tag{20e}$$

$$0 \in \alpha_a(n) - \nabla_{x_a(n)} V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \tag{20f}$$

$$0 = \mu(m) - \sum_{k \in \mathcal{K}_a(n)} \gamma^k(n) p_a^k(m), \quad m \in n_+, n \in \mathcal{N} \setminus \mathcal{L} \tag{20g}$$

where $Z_a(n; u, x, W)$ is defined by (18).

Theorem 2 *(i) Any solution to $AE_a(\pi, \alpha, \mu, \mathcal{D}_a)$ provides a solution to the optimization problem $AO_a(\pi, \alpha, \mu, \mathcal{D}_a)$, and satisfies*

$$\begin{aligned} \theta_a(n) &= \max_{\mu \in \mathcal{D}_a(n)} \sum_{m \in n_+} \mu(m) \left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \right) \\ &= \sum_{m \in n_+} \bar{\mu}(m) \left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \right), \end{aligned}$$

where $\bar{\mu}(m) = \sum_{k \in \mathcal{K}_a(n)} \gamma^k(n) p_a^k(m)$.

(ii) If Assumption 1 holds, then any solution of $AO_a(\pi, \alpha, \mu, \mathcal{D}_a)$ provides a solution to $AE_a(\pi, \alpha, \mu, \mathcal{D}_a)$ for some γ.

Proof. (i) First suppose that $(u_a, x_a, W_a, \theta_a, \gamma)$ is a solution of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. It follows from Lemma 3 that there exists λ which satisfies (13) with $C(m) = Z_a(m; u, x, W) - W_a(m)$ and $\theta_a(m)$ replacing $\theta(m)$. Given λ we can define σ using (8).

Putting these relationships together and substituting into the $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ conditions (observing that $\sigma(n) \geq 0$) gives

$$
\begin{aligned}
&0 = 1 - \sum_{k \in \mathcal{K}_a(0)} \lambda^k(0), \\
&0 = -\sum_{k \in \mathcal{K}_a(n)} \lambda^k(n) + \sum_{k \in \mathcal{K}_a(n_-)} \lambda^k(n_-) p_a^k(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0 \\
&0 \leq \lambda^k(n) \perp \theta_a(n) - \sum_{m \in n_+} p_a^k(m) \Big(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \Big) \geq 0, \quad k \in \mathcal{K}_a(n), n \in \mathcal{N} \setminus \mathcal{L} \\
&\ \theta_a(n) = -V_a(x_a(n)), \quad n \in \mathcal{L} \\
&0 \in \sigma(n) \nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \\
&0 \in \sigma(n) \alpha_a(n) - \sum_{m \in n_+} \sigma(m) \alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L} \\
&0 \in \sigma(n) \alpha_a(n) - \sigma(n) \nabla_{x_a(n)} V_a(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \\
&0 = \sigma(n) \mu(m) - \sum_{k \in \mathcal{K}_a(n)} \lambda^k(n) p_a^k(m), \quad m \in n_+, n \in \mathcal{N} \setminus \mathcal{L}
\end{aligned}
$$

with $Z_a(n, u, x, W)$ defined by (18).

Clearly we recover (19a)–(19h). It simply remains to show that λ satisfies (19i) and (19j). Since $\lambda^k(0) = \gamma^k(0)$, (19i) is immediate from (20g). Since $\sigma(n) = \sum_{k \in \mathcal{K}_a(n_-)} \lambda^k(n_-) p_a^k(n)$, (20g) is equivalent to $\sigma(n)\mu(m) = \sigma(m)$ for $m \in n_+$, $n \in \mathcal{N} \setminus \mathcal{L}$, which gives (19j) if we identify q with m.
(ii) For the converse, suppose that we have a solution of (19), then Lemma 3 coupled with Assumption 1 provides $\sigma(n) > 0$ and conditional multipliers $\gamma^k(n) = \lambda^k(n)/\sigma(n)$ that satisfy (12) for $C(m) = Z_a(m; u, x, W) - W_a(m)$ and $\theta(n) = \theta_a(n)$. Thus (20a), (20b) and (20c) are satisfied in the definition of the $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ problem. Now (20g) follows by dividing (19j) by $\sigma(q_-)$ and using (8) and (9). Noting (10) and then dividing (19g) and (19h) by $\sigma(n)$ then gives (20e) and (20f) respectively. The relationship (20d) follows from the definition of σ and (19e) and (19f). ■

Definition 3 *A multistage risk-trading equilibrium $RTE(\mathcal{D}_\mathcal{A})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$, and a corresponding collection of actions, $\{u_a(n), n \in \mathcal{N}\}$, $\{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ with the property that for some γ, $(u_a, x_a, W_a, \theta_a, \gamma)$ solves the problem $AE_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and*

$$
\begin{aligned}
&0 \leq \pi(n) \quad \perp \quad \sum_{a \in \mathcal{A}} g_a(u_a(n)) \geq 0, \quad n \in \mathcal{N}, &(21) \\
&0 \leq \alpha_a(n) \quad \perp \quad -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \geq 0, \\
&\qquad\qquad\qquad\qquad a \in \mathcal{A}, n \in \mathcal{N}, &(22) \\
&0 \leq \mu(n) \quad \perp \quad -\sum_{a \in \mathcal{A}} W_a(n) \geq 0, \quad n \in \mathcal{N} \setminus \{0\}. &(23)
\end{aligned}
$$

Theorem 3 *Consider a set of agents $a \in \mathcal{A}$, each endowed with polyhedral node-dependent risk sets $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Suppose $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading equilibrium $RTE(\mathcal{D}_\mathcal{A})$ in which agent a solves $AE_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ with a policy defined by $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ together with a policy of trading Arrow-Debreu securities defined by $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$. Then* *(i)* $\bar{\mu} \in \mathcal{D}_a$ *for all* $a \in \mathcal{A}$*, and hence* $\bar{\mu} \in \mathcal{D}_s$*,*

(ii)

$$\bar{\theta}(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} C_a(\bar{u}_a(m)) + \bar{\theta}(m) \right), \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{24}$$

(iii) there exist multipliers γ *such that* $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \bar{\pi}, \bar{\alpha})$ *is a solution to* $SE(\mathcal{D}_0)$ *with* $\mathcal{D}_0 = \{\bar{\mu}\}$*,*

(iv) there exist multipliers γ *such that* $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \bar{\pi}, \bar{\alpha})$ *is a solution to* $SE(\mathcal{D}_s)$

where $\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$ *and* $\bar{\mu}(n) = \sum_{k \in K_s(n)} \gamma^k(n) p_a^k(m)$*.*

Proof. (i) If we have a solution of $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ for each $a \in \mathcal{A}$, it follows from (20g) that $[\bar{\mu}]_{n_+} \in \mathcal{D}_a(n)$ for each n and thus $\bar{\mu} \in \mathcal{D}_a$ for all a, and hence $\bar{\mu} \in \mathcal{D}_s$ by Definition 1.
(ii) For each $a \in \mathcal{A}$ it follows from Theorem 2 that for $n \in \mathcal{N} \setminus \mathcal{L}$,

$$\bar{\theta}_a(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right). \tag{25}$$

Summing over $a \in \mathcal{A}$ and invoking (23) gives

$$\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} \bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) + \bar{\theta}(m) \right).$$

Recalling the definition of $\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W})$ from (18), summing over $a \in \mathcal{A}$, and invoking (21), (22) and (23) gives (24).
(iii) Suppose $\mathcal{D}_0 = \{\bar{\mu}\}$. It follows that $\mathcal{K}_0(n) = \{1\}$ for $n \in \mathcal{N} \setminus \mathcal{L}$ where $p_0^1(m) = \bar{\mu}(m)$, for $m \in n_+$. Define $\gamma^1(n) = 1$, $n \in \mathcal{N} \setminus \mathcal{L}$. It then follows that the first, second and fifth conditions of $SE(\mathcal{D}_0)$ simplify to

$$\begin{aligned}
\gamma^1(n) &= 1, \quad n \in \mathcal{N} \setminus \mathcal{L}, \\
\theta(n) &= \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} C_a(u_a(m)) + \theta(m) \right), \quad n \in \mathcal{N} \setminus \mathcal{L}, \\
0 &\in \alpha_a(n) - \sum_{m \in n_+} \bar{\mu}(m) \alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L}.
\end{aligned}$$

Combining these with the other conditions in (20) shows that $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \pi, \alpha)$ solves $SE(\mathcal{D}_0)$.
(iv) Suppose $(\bar{u_a}, \bar{x_a}, \bar{\theta_a}, \gamma_a)$ solves $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$. Let $\bar{Z}_a(n; \bar{u}, \bar{x}, \bar{W})$ be defined using (18) so that it follows from (21), (22) and (23) that

$$\begin{aligned}
&\sum_{a \in \mathcal{A}} \sum_{m \in n_+} \bar{\mu}(m) \left(C_a(\bar{u}_a(m)) + \bar{\theta}_a(m) \right) \\
&\qquad = \sum_{a \in \mathcal{A}} \sum_{m \in n_+} \bar{\mu}(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right)
\end{aligned}$$

which by (24)

$$= \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$$

which by (20a) and (20b) and Lemma 1

$$= \sum_{a \in \mathcal{A}} \sup_{\mu \in \mathcal{D}_a(n)} \sum_{m \in n_+} \mu(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right)$$

which by Assumption 3 and Definition 1,

$$\geq \sum_{a\in\mathcal{A}} \sup_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m)\right)$$

and interchanging supremum and summation

$$\geq \sup_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \sum_{a\in\mathcal{A}} \left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m)\right)$$

since feasibility implies $-\mu(m)\sum_{a\in\mathcal{A}} \bar{W}_a(m) \geq 0$

$$\geq \sup_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \sum_{a\in\mathcal{A}} \left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W}) + \bar{\theta}_a(m)\right)$$

by (21), (22) and (23)

$$= \sup_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \sum_{a\in\mathcal{A}} \left(C_a(\bar{u}_a(m)) + \bar{\theta}_a(m)\right)$$

by (i)

$$\geq \sum_{m\in n_+} \bar{\mu}(m) \sum_{a\in\mathcal{A}} \left(C_a(\bar{u}_a(m)) + \bar{\theta}_a(m)\right).$$

Hence equality holds throughout and thus $[\bar{\mu}]_{n_+}$ solves

$$\sup_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \left(\sum_{a\in\mathcal{A}} C_a(\bar{u}_a(m)) + \bar{\theta}(m)\right).$$

Lemma 1 then shows that these conditions are equivalent to the first two conditions of SE($\mathcal{D}_s$), which combined with the other conditions in $\mathrm{AE}_a(\bar{\pi},\bar{\alpha},\bar{\mu},\mathcal{D}_a)$ gives the remaining conditions of SE($\mathcal{D}_s$). ■

We now proceed to prove the converse of Theorem 3. We will require a preliminary lemma that uses the following formulations. For each $n \in \mathcal{N}\setminus\mathcal{L}$, suppose $Z_a^s(m)$, $\theta^s(m)$ and $\theta_a^s(m)$ are given for each $m \in n_+$ and satisfy $\theta^s(m) = \sum_{a\in\mathcal{A}} \theta_a^s(m)$. Consider the problems:

$$\mathrm{R}(n,\mathcal{D}_s): \quad \max_{\mu\in\mathcal{D}_s(n)} \sum_{m\in n_+} \mu(m) \left(\sum_{a\in\mathcal{A}} Z_a^s(m) + \theta^s(m)\right)$$

$\mathrm{T}(n,\mathcal{D}_{\mathcal{A}})$:

$$\begin{aligned}
\min_{[[W_a]_{n_+}]_{a\in\mathcal{A}},\theta_a(n)} \quad & \sum_{a\in\mathcal{A}} \theta_a(n) \\
\text{s.t.} \quad & \theta_a(n) \geq \sum_{m\in n_+} p_a^k(m)\left(Z_a^s(m) - W_a(m) + \theta_a^s(m)\right), \quad k\in\mathcal{K}_a(n), a\in\mathcal{A} \\
& -\sum_{a\in\mathcal{A}} W_a(m) \geq 0, \quad m\in n_+
\end{aligned}$$

TD($n, \mathcal{D}_\mathcal{A}$):

$$\max_{\mu,\phi} \sum_{m\in n_+} \sum_{a\in\mathcal{A}} \left(\sum_{k\in\mathcal{K}_a(n)} p_a^k(m)\phi_a^k(n) \right) (Z_a^s(m) + \theta_a^s(m))$$

$$\sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n) = 1, \quad a \in \mathcal{A},$$

$$\mu(m) = \sum_{k\in\mathcal{K}_a(n)} p_a^k(m)\phi_a^k(n), \quad m \in n_+, a \in \mathcal{A}$$

$$\mu(m) \geq 0, \quad m \in n_+, \quad \phi_a^k(n) \geq 0, \quad k \in \mathcal{K}_a(n), a \in \mathcal{A}$$

and

TOC($n, \mathcal{D}_\mathcal{A}$):

$$0 = 1 - \sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n), \quad a \in \mathcal{A}$$

$$0 = \mu(m) - \sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n)p_a^k(m), \quad m \in n_+, a \in \mathcal{A}$$

$$0 \leq \phi_a^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m) \left(Z_a^s(m) - W_a(m) + \theta_a^s(m)\right), \quad k \in \mathcal{K}_a(n), a \in \mathcal{A}$$

$$0 \leq \mu(m) \perp -\sum_{a\in\mathcal{A}} W_a(m) \geq 0, \quad m \in n_+$$

The formulation R evaluates the one stage risk of the random disbenefit $\sum_{a\in\mathcal{A}} Z_a$ using the coherent risk measure with risk set $D_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$. The problem T on the other hand accumulates the risk measure of each agent a in a setting where they can exchange welfare W (constrained so that it cannot be created out of nothing). If the model has a variable $W_a(m)$ defined for each outcome $m \in n_+$, then the following analysis demonstrates that an exchange exists in node n that will yield the risk-adjusted value of the total social disbenefit faced by all agents if evaluated with risk set $\mathcal{D}_s(n)$.

Lemma 4 *Let $n \in \mathcal{N}$ and suppose $\mathcal{D}_\mathcal{A}$ satisfies Assumption 3. The problems T, TD, TOC and R all have optimal solutions with the same optimal value. Any solution to one of these problems yields a solution to all of the others.*

Proof. Observe that T and TD are dual linear programs, and TOC gives the optimality conditions for T. The constraints of TD entail that $\mu(m), m \in n_+$ is a finite probability distribution that is constrained to lie in each $\mathcal{D}_a(n)$. Definition 1 means that TD is equivalent to R. So any optimal solution of one of these four formulations yields solutions to all the others. Observe that the feasible region of TD is compact and and nonempty by Assumption 3, so T, TD, TOC and R all have optimal solutions with the same optimal value. ■

Theorem 4 *Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Now let $(u, x, \theta^s, \gamma, \pi, \alpha)$ be a solution to SE($\mathcal{D}_s$) with risk sets $D_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$. Let μ be defined by*

$$\mu(m) = \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}.$$

The prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$ and actions $\{u_a(n) \mid n \in \mathcal{N}\}$, $\{W_a(n) \mid n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading equilibrium RTE($\mathcal{D}_\mathcal{A}$).

Proof. Suppose $(u, x, \theta^s, \gamma, \pi, \alpha)$ is a solution of SE($\mathcal{D}_s$). It follows from Theorem 1 that defining $\mu(m) = \sum_{k\in\mathcal{K}_s(n)} \gamma^k(n) p_s^k(m) \in \mathcal{D}_s$ for each $m \in n_+$ we have

$$\begin{aligned}
\theta^s(n) = &\sum_{m\in n_+} \mu(m)\Bigg(\sum_{a\in\mathcal{A}}\Bigg(C_a(u_a(m)) - \pi(m)g_a(u_a(m)) \\
&+ \alpha_a(m)\Big(x_a(m) - x_a(m_-) - \sum_{b\in\mathcal{A}} T_{ab}u_b(m) - \omega_a(m)\Big)\Bigg) + \theta^s(m)\Bigg) \\
= &\sum_{m\in n_+} \mu(m)\Bigg(\sum_{a\in\mathcal{A}}\Big(C_a(u_a(m)) - \pi(m)g_a(u_a(m)) \\
&+ \alpha_a(m)\left(x_a(m) - x_a(m_-) - \omega_a(m)\right) - \sum_{b\in\mathcal{A}} \alpha_b(m)T_{ba}u_a(m)\Big) + \theta^s(m)\Bigg).
\end{aligned}$$

Consider the leaf nodes $m \in \mathcal{L}$. At these nodes $\theta^s(m) = -\sum_{a\in\mathcal{A}} V_a(x_a(m))$ so defining $\theta_a^s(m) = -V_a(x_a(m))$ for each $a \in \mathcal{A}$ we have $\sum_{a\in\mathcal{A}} \theta_a^s(m) = \theta^s(m)$. Letting

$$\begin{aligned}
Z_a^s(m) = &C_a(u_a(m)) - \pi(m)g_a(u_a(m)) \\
&+ \alpha_a(m)\left(x_a(m) - x_a(m_-) - \omega_a(m)\right) - \sum_{b\in\mathcal{A}} \alpha_b(m)T_{ba}u_a(m)
\end{aligned}$$

for the given solution values of SE($\mathcal{D}_s$), Lemma 4 shows that $[\mu]_{n_+}$ and values $[\phi_a^k(n)]_{a\in\mathcal{A},k\in\mathcal{K}_a(n)}$, $[[W_a]_{n_+}]_{a\in\mathcal{A}}, \theta_a(n)$ solves TOC($n, \mathcal{D}_\mathcal{A}$) for each node $n = m_-$, and that the solution value of R($n, \mathcal{D}_s$) (namely $\theta^s(n)$) is equal to $\sum_{a\in\mathcal{A}} \theta_a(n)$.

We now recursively apply this argument. For each node n in the penultimate stage, we let $\theta_a^s(n) = \theta_a(n)$, the above computed solution value, so that $\sum_{a\in\mathcal{A}} \theta_a^s(n) = \theta^s(n)$. Further, we define

$$\begin{aligned}
Z_a^s(n) = &C_a(u_a(n)) - \pi(n)g_a(u_a(n)) + \alpha_a(n)\left(x_a(n) - x_a(n_-) - \omega_a(n)\right) \\
&- \sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n) + \sum_{m\in n_+} \mu(m)W_a(m)
\end{aligned}$$

for the given solution values of SE($\mathcal{D}_s$) and the previous step computed solution values for $W_a(m)$. For each node $q = n_-$, Lemma 4 constructs solution values $[\mu]_{q+}$, $[\phi_a^k(q)]_{a\in\mathcal{A},k\in\mathcal{K}_a(q)}$, $[\theta_a(q), [W_a]_{q+}]_{a\in\mathcal{A}}$ for TOC($q, \mathcal{D}_\mathcal{A}$) such that $\theta^s(q) = \sum_{a\in\mathcal{A}} \theta_a(q)$. This argument can then be repeated until we reach the root node of $\mathcal{N}$.

This process generates μ and values of (u, x, α, π) that satisfy (20c), (20d), (20e) and (20f) for every $a \in \mathcal{A}$ since they are solutions to SE($\mathcal{D}_s$). Furthermore, for each $a \in \mathcal{A}$, extracting $\gamma_a^k(n) = \phi_a^k(n)$ and $Z_a(n; u, x, W) = Z_a^s(n)$ from the solutions of TOC($n, \mathcal{D}_\mathcal{A}$), it follows from the definition of TOC($n, \mathcal{D}_\mathcal{A}$) that (20a), (20b) and (20g) are also satisfied with $\gamma(n) = \gamma_a(n)$. Thus we have constructed solutions for each problem AE$_a(\pi, \alpha, \mu, \mathcal{D}_a)$.

Since for each $n \in \mathcal{N} \setminus \mathcal{L}$, TOC($n, \mathcal{D}_\mathcal{A}$) includes the condition that

$$0 \le \mu(m) \perp -\sum_{a\in\mathcal{A}} W_a(m) \ge 0, \quad m \in n_+$$

it follows that (23) holds. The final conditions (21) and (22) follow as they are part of the original solution of SE($\mathcal{D}_s$). ■

We can derive a version of the welfare theorems in which each agent solves a multistage optimization problem AO$_a(\pi, \alpha, \mu, \mathcal{D}_a)$.

Definition 4 *A multistage risk-trading optimization equilibrium (RTOE($\mathcal{D}_\mathcal{A}$)) is a version of RTE($\mathcal{D}_\mathcal{A}$) in which each agent solves AO$_a(\pi, \alpha, \mu, \mathcal{D}_a)$ rather than AE$_a(\pi, \alpha, \mu, \mathcal{D}_a)$.*

The equivalence of RTOE with the risk-averse social planning problem is expressed by the following corollaries, the proofs of which are immediate from Theorem 3 and Theorem 4 and the fact that Assumption 1 gives the equivalence of AE$_a(\pi, \alpha, \mu, \mathcal{D}_a)$and AO$_a(\pi, \alpha, \mu, \mathcal{D}_a)$, and SE($\mathcal{D}_s$)and SO($\mathcal{D}_s$).

Corollary 5 *Suppose Assumption 1 holds. Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Suppose $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading optimization equilibrium RTOE($\mathcal{D}_{\mathcal{A}}$) in which agent a solves $AO_a(\pi, \alpha, \mu, \mathcal{D}_a)$ with a policy defined by $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ together with a policy of trading Arrow-Debreu securities defined by $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$. Then $(\bar{u}, \bar{x}, \bar{\theta}, \bar{\pi}, \bar{\alpha})$ is a solution to SO($\mathcal{D}_s$) where $D_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$ and $\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$.*

Corollary 6 *Suppose Assumption 1 holds. Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Now let (u, θ^s) be a solution to SO($\mathcal{D}_s$) with risk sets $D_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$. Suppose this gives rise to Lagrange multipliers $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ for constraints (17) and (16) respectively. Then for some γ*

1. *(u, θ^s, γ) satisfies SE($\mathcal{D}_s$),*
2. *If $\mu(m) = \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m), \quad m \in n_+, \ n \in \mathcal{N} \setminus \mathcal{L}$ then the prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ and actions $\{u_a(n) \mid n \in \mathcal{N}\}$, $\{W_a(n) \mid n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading equilibrium RTOE($\mathcal{D}_{\mathcal{A}}$).*

6 Conclusions

This paper has established versions of the first and second welfare theorems in a setting where agents can trade risk. The proofs of these results can be seen to be relatively straightforward consequences of Lagrangean duality. There are two features of these results that are worthy of some discussion.

Our optimization versions of the welfare theorems (Corollaries 5 and 6) rely on Assumption 1. This is equivalent to the assertion that the one-step risk measure is strictly monotone, thus guaranteeing a nested risk measure that yields a time-consistent optimal solution. Competitive equilibrium specifies an optimal action for each agent in every state of the world, even if this is discounted in equilibrium to have zero risk-adjusted disbenefit. It is therefore necessary for a social plan to specify a set of actions for the agents in such states. This can be done either by constraining it to be time consistent using the formulation SE($\mathcal{D}_s$) in the absence of Assumption 1, or by imposing strict monotonicity on each agent's one-step risk measure.

Observe that the welfare results rely also on Assumption 3. The risk sets of the agents must intersect to enable trade to be bounded. In a non-polyhedral setting we would require the stronger condition that the intersection of the relative interiors of the risk sets is nonempty (see e.g. [6]). If one agent believes that the risk-adjusted price of a given Arrow-Debreu contract strictly exceeds that of a prospective purchaser, then an infinite trade will result.

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