

# **Lectures on Parametric Optimization: An Introduction**

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# CHAPTER 1

## Introduction and notation

### 1.1. Introduction

The present report aims to provide an overview of results from parametric optimization which could be called classical results on the subject. These are results, mostly obtained in the 80th and 90th.

We deal with programs of the type: For  $t$  in a parameter set  $T \subset \mathbb{R}^p$  we wish to find local minimizers  $x = x(t)$  of

$$(1.1) \quad P(t) : \min_{x \in \mathbb{R}^n} f(x, t) \quad \text{s.t.} \quad x \in F(t) = \{x \mid g_j(x, t) \leq 0, j = 1, \dots, m\} .$$

Note that for any fixed parameter  $\bar{t} \in T$ , the program  $P(\bar{t})$  represents a standard optimization problem.

Suppose for fixed  $\bar{t}$  we have given the feasible set  $F(\bar{t})$ , and a local minimizer  $\bar{x}$  of  $P(\bar{t})$  with corresponding value  $v(\bar{t}) = f(\bar{x}, \bar{t})$ . A main question in parametric optimization is then the following:

**Main question of parametric optimization.** For parameter  $t \in T$  close to  $\bar{t}$ ,

- how does the feasible set  $F(t)$  change compared to  $F(\bar{t})$ ?
- does there exist a local minimizer  $x(t)$  of  $P(t)$  close to  $\bar{x}$  and if yes, how does  $x(t)$  and the value  $v(t) = f(x(t), t)$  change with  $t$ ? Continuously, differentially?

To give a concrete example we consider the parametric program

$$P(t) \quad \min_{x \in \mathbb{R}^2} f(x, t) = (x_1 - t)^2 + (x_2 - 2)^2 \quad \text{s.t.} \quad x \in F(t) ,$$

$$\text{where} \quad F(t) = \{(x_1, x_2) \mid x_2 \leq 2t, x_1 \leq 1\} .$$

It is not difficult to see (cf., Figure 2.2) that for  $\bar{t} = 0$  the (unique) minimizer of  $P(0)$  is  $\bar{x} = (0, 0)$ . More precisely for  $t \in \mathbb{R}$  the (unique) minimizer  $x(t)$  of  $P(t)$  with corresponding value function  $v(t) = f(x(t), t)$  is given by (see Figure 2.2)

$$x(t) = \begin{cases} (t, 2t) & \text{for } t \leq 1 \\ (1, 2) & \text{for } t > 1 \end{cases} \quad v(t) = \begin{cases} 2(t-1)^2 & \text{for } t \leq 1 \\ (t-1)^2 & \text{for } t > 1 \end{cases} .$$

The minimizer function is piecewise linear and the value function is (differentiable and) piecewise quadratic (see Figure 1.1). We refer to Section 6.6 to a general result for this type of parametric quadratic program. This example indicates as a rule of thumb the following behavior in parametric optimization:

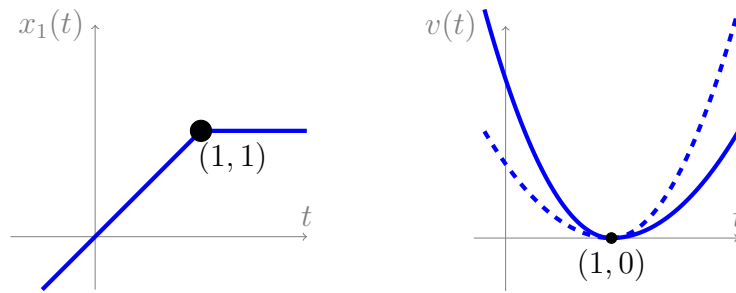


FIGURE 1.1. Minimizer function  $x_1(t)$  (1st component), value function  $v(t)$ .

- In general, we cannot expect that the minimizer function  $x(t)$  of  $P(t)$  is globally differentiable. We only can hope for piecewise differentiable, Lipschitz continuous functions  $x(t)$ .
- The value function however may be smoother than the minimizer function  $x(t)$ .

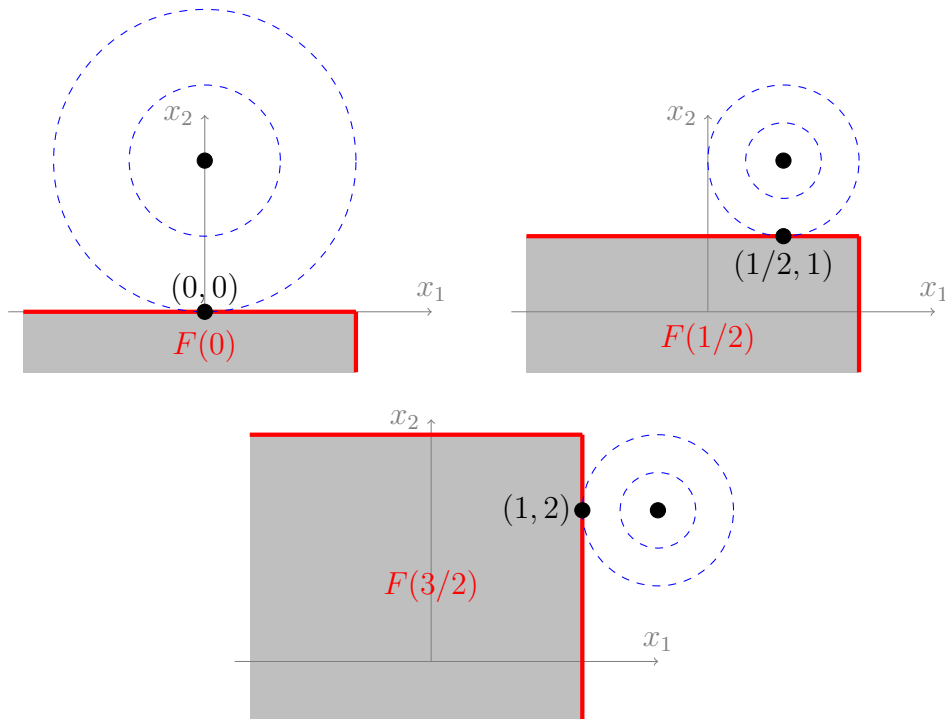


FIGURE 1.2. Sketch for  $P(0)$  (above, left), for  $P(1/2)$  (above, right), for  $P(3/2)$  (below).

Why such a report? There are books on this topic. Fiacco [10] wrote the first introduction on parametric optimization. But this exposition is essentially restricted

to programs where the assumptions of the Implicit Function Theorem are satisfied (see Chapter 4). The early book of Bank, Guddat, Klatte, Kummer, and Tammer [2] does not yet contain many of the standard results on nonlinear problems. The book Bonnans, and Shapiro [6] considers parametric programs under more general cone constraints and reports on results beyond an introduction. In Klatte, and Kummer [18] mainly nonsmooth Newton methods are considered. The book contains a Chapter on nonsmooth ( $C^{1,1}$ ) nonlinear parametric programs.

Many results in the present report are only available through original articles in journals, in the Lecture Notes edited by Fiacco [20, 21], or in the proceedings of the conferences on "Parametric Optimization and Related Topics" organized regularly by Jürgen Guddat et al., [22, 23, 24, 25, 26].

So, the aim of this report is to make the classical results of Parametric Optimization easier accessible to a broader group of mathematicians and engineers. To that end the exposition contains many illustrative example problems.

The report is organized as follows. After a section with notations we start in Chapter 2 with standard nonparametric optimization problems. We give a brief introduction into properties and optimality conditions of unconstrained and constrained programs. As an application of mathematical programming, Chapter 3 gives a survey of programs occurring in models from traffic engineering.

Chapter 4 starts with parametric problems and presents stability results based on the Implicit Function Theorem (IFT). Chapter 5 contains an introduction into semicontinuity properties of feasible set, value function and solution set of continuous parametric programs  $P(t)$ .

Chapter 6 considers smooth ( $C^2$ ) nonlinear parametric optimization problems  $P(t)$ . Lipschitz stability of the feasible set mapping  $F(t)$  and the value function  $v(t)$  of  $P(t)$  are investigated under Mangasarian Fromovitz Constraint Qualification. Moreover, the stability of local minimizers of  $P(t)$  is proven. Under stronger assumptions directional differentiability of the value function  $v(t)$  and the local minimizer function  $x(t)$  can be shown. We also study Lipschitz stability for minimizers of specially structured problems with linear right-hand side perturbed constraints.

Chapter 7 deals with the directional differentiability of  $v(t)$ . In Chapter 8 we present some applications. In particular we study stability and monotonicity results for programs appearing in the traffic models of Chapter 3. The Appendix in Chapter 9 states some basic facts from analysis used throughout the report.

We start with some important remarks. In this report, for simplicity, we deal with parametric programs (1.1) with only inequality constraints. We could also consider programs with feasible sets defined by equalities and inequalities,

$$(1.2) \quad P(t) : \quad \min_{x \in \mathbb{R}^n} f(x, t) \quad \text{s.t.} \quad \begin{array}{l} h_i(x, t) = 0, \quad i = 1, \dots, q \\ g_j(x, t) \leq 0, \quad j = 1, \dots, m \end{array} .$$

We emphasize that all results in this report remain valid for (1.2) if the LICQ condition for (1.1), that  $\nabla_x g_j(\bar{x}, \bar{t})$ ,  $j \in J_0(\bar{x}, \bar{t})$ , are linearly independent (see

Definition 2.4) is replaced by the LICQ condition,

$$(1.3) \quad \begin{array}{l} \nabla_x h_i(\bar{x}, \bar{t}), \quad i = 1, \dots, q, \quad \text{and} \\ \nabla_x g_j(\bar{x}, \bar{t}), \quad j \in J_0(\bar{x}, \bar{t}) \end{array} \quad \text{are linearly independent,}$$

and if the MFCQ condition for (1.1), that there exists  $\xi$  with  $\nabla_x g_j(\bar{x}, \bar{t})\xi < 0, j \in J_0(\bar{x}, \bar{t})$  (see Definition 2.4), is replaced by the MFCQ condition,

$$(1.4) \quad \begin{array}{l} \nabla_x h_i(\bar{x}, \bar{t}), \quad i = 1, \dots, q, \quad \text{are linearly independent and there exists } \xi \\ \text{such that } \nabla_x h_i(\bar{x}, \bar{t})\xi = 0, \quad i = 1, \dots, q, \quad \nabla_x g_j(\bar{x}, \bar{t})\xi < 0, \quad j \in J_0(\bar{x}, \bar{t}) . \end{array}$$

We also wish to mention that Bonnans, Shapiro in [6] analysed parametric programs of the form: for  $t \in T$  solve

$$(1.5) \quad P(t) : \quad \min_{x \in \mathbb{R}^n} f(x, t) \quad \text{s.t.} \quad G(x, t) \in K ,$$

with an open parameter set  $T \subset \mathbb{R}^p$ ,  $f(x, t) \in C(\mathbb{R}^n \times T, \mathbb{R})$  (or  $f \in C^1, C^2$ ),  $G \in C(\mathbb{R}^n \times T, \mathbb{R}^M)$  and  $K \subset \mathbb{R}^M$  a closed convex set. The constraint qualification at  $(\bar{x}, \bar{t})$ , with  $G(\bar{x}, \bar{t}) \in K$ , for this program reads (see [6, (2.178)]):

$$(1.6) \quad 0 \in \text{int} [G(\bar{x}, \bar{t}) + \nabla_x G(\bar{x}, \bar{t})\mathbb{R}^n - K] .$$

For the special case  $M = q + m$ ,  $G = (G_1, G_2)$ ,  $G_1(x, t) = (h_i(x, t), i = 1, \dots, q)$ ,  $G_2(x, t) = (g_j(x, t), j = 1, \dots, m)$ , and  $K = \{0\} \times \mathbb{R}_-^m$ , the feasibility condition becomes

$$\begin{array}{l} h_i(x, t) = 0, \quad i = 1, \dots, q \\ g_j(x, t) \leq 0, \quad j = 1, \dots, m \end{array}$$

as in (1.2), and the constraint qualification (1.6) splits into

$$\begin{array}{l} 0 \in \text{int} [G_1(\bar{x}, \bar{t}) + \nabla_x G_1(\bar{x}, \bar{t})\mathbb{R}^n] \\ 0 \in \text{int} [G_2(\bar{x}, \bar{t}) + \nabla_x G_2(\bar{x}, \bar{t})\mathbb{R}^n - \mathbb{R}_-^m] \end{array} ,$$

which is equivalent with the MFCQ condition in (1.4).



## 1.2. Notation

We give a list of the notation used in this booklet.

$x \in \mathbb{R}^n$	column vector with $n$ components $x_i, i = 1, \dots, n$
$x^T \in \mathbb{R}^n$	transposed of $x$ , row vector $x^T = (x_1, \dots, x_n)$
$\ x\ $	the Euclidean norm of $x \in \mathbb{R}^n$ , $\ x\  = \sqrt{x_1^2 + \dots + x_n^2}$
$\ x\ _\infty$	the maximum norm of $x \in \mathbb{R}^n$ , $\ x\ _\infty = \max_{1 \leq i \leq n}  x_i $
$\mathbb{R}_+^n (\mathbb{R}_-^n)$	set of $x \in \mathbb{R}^n$ with components $x_i \geq 0, (x_i \leq 0), i = 1, \dots, n$
$A = (a_{ij})$	$A \in \mathbb{R}^{n \times m}$ , a real $(n \times m)$ -matrix
$A_i$	is the $i$ th row of $A$
$A_j$	is the $j$ th column of $A$
$\ A\ $	Frobenius norm, $\ A\  = \sqrt{\sum_{ij} a_{ij}^2}$ . This matrix norm satisfies: $\ Ax\  \leq \ A\  \ x\ $
$\nabla f(x)$	gradient of $f$ , $\nabla f(x) = (\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n})$
$\nabla^2 f(x)$	Hessian of $f$ , $\nabla^2 f(x) = \left( \frac{\partial^2}{\partial x_j \partial x_i} f(x) \right)_{i,j}$
$\nabla_x f(x, t)$	partial derivatives wrt. $x$ , $\nabla_x f(x, t) = (\frac{\partial f(x,t)}{\partial x_1}, \dots, \frac{\partial f(x,t)}{\partial x_n})$
$\nabla_x^2 f(x, t)$	Hessian wrt. $x$ of $f$
$\nabla_{x,t}^2 f(x, t)$	$\left( \frac{\partial^2}{\partial t_j \partial x_i} f(x, t) \right)_{\substack{i=1, \dots, n \\ j=1, \dots, p}}$ , second order mixed partial derivatives
$\nabla x(t; z)$	directional derivative of $x(t)$ at $t$ into direction $z$ , $\ z\  = 1$
$\nabla_+ v(t; z)$	upper directional derivative of $v(t)$ , see Definition 7.1
$\nabla_- v(t; z)$	lower directional derivative of $v(t)$ , see Definition 7.1
$x_l \rightarrow \bar{x}$	given a sequence $x_l \in \mathbb{R}^n$ , $l \in \mathbb{N}$ , by $x_l \rightarrow \bar{x}$ we mean $\lim_{l \rightarrow \infty} x_l = \bar{x}$ or $\lim_{l \rightarrow \infty} \ x_l - \bar{x}\  = 0$
$B_\delta(\bar{x})$	the open $\delta$ ball around the point $\bar{x} \in \mathbb{R}^n$ , $\delta > 0$ , $B_\delta(\bar{x}) = \{x \in \mathbb{R}^n \mid \ x - \bar{x}\  < \delta\}$
$B_\varepsilon(\bar{t})$	the open $\varepsilon$ ball around the parameter $\bar{t} \in \mathbb{R}^p$ , $\varepsilon > 0$
$B_\delta(S)$	for some set $S \subset \mathbb{R}^n$ , it is the open $\delta$ -neighborhood around $S$ , $B_\delta(S) = \{x \in \mathbb{R}^n \mid \text{there exists } \bar{x} \in S \text{ such that } \ x - \bar{x}\  < \delta\}$
$C^0(\mathbb{R}^n, \mathbb{R})$	set of real valued functions $f$ , continuous on $\mathbb{R}^n$ We often simply write $f \in C^0$
$C^1(\mathbb{R}^n, \mathbb{R})$	set of functions $f$ , once continuously differentiable on $\mathbb{R}^n$ We often simply write $f \in C^1$
$C^2(\mathbb{R}^n, \mathbb{R})$	set of functions $f$ , twice continuously differentiable on $\mathbb{R}^n$ We often write $f \in C^2$
$J_0(\bar{x}, \bar{t})$	active index set at $(\bar{x}, \bar{t})$ , $J_0(\bar{x}, \bar{t}) = \{j \in J \mid g_j(\bar{x}, \bar{t}) = 0\}$
$L(x, t, \mu)$	Lagrangian function (locally at $(\bar{x}, \bar{t})$ ): $L(x, t, \mu) = f(x, t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j g_j(x, t)$

$C_{\bar{x}, \bar{t}}$	the cone of critical directions, see Theorem 2.4
$T_{\bar{x}, \bar{t}}$	tangent space of critical directions, see Theorem 4.4
$T_{\bar{x}, \bar{t}}(\bar{\mu})$	extended tangent space of critical directions, see Theorem 4.4
$x(t)$	function of local minimizers of $P(t)$
$P(t)$	parametric program see (2.7), (1.1)
$F(t)$	feasible set of $P(t)$
$v(t)$	value function of $P(t)$
$S(t)$	set of (global) minimizers of $P(t)$
$\text{cl } S$	closure of the set $S$
$\text{int } S$	interior of the set $S$
$\text{dom } F$	domain of the setvalued mapping $F(t)$ , $\text{dom } f = \{t \mid F(t) \neq \emptyset\}$

## CHAPTER 2

### Nonparametric optimization

Before starting with parametric optimization we have to provide an introduction into standard nonparametric programming problems. We start with basic definitions of some type of minimizers.

**DEFINITION 2.1.** *Let be given a function  $f : F \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , continuous on the subset  $F$  of  $\mathbb{R}^n$ .*

(a) *The point  $\bar{x} \in F$  is called a local minimizer of  $f$  on  $F$  if there is some  $\delta > 0$  such that*

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in F \text{ satisfying } \|x - \bar{x}\| < \delta .$$

*If  $f(\bar{x}) \leq f(x) \forall x \in F$ , then we call  $\bar{x}$  a global minimizer. The point  $\bar{x}$  is called a strict local minimizer if*

$$f(\bar{x}) < f(x) \quad \text{for all } x \in F, x \neq \bar{x}, \|x - \bar{x}\| < \delta .$$

(b) *We say that  $\bar{x}$  is a local minimizer of  $f$  on  $F$  of order  $s = 1$  or  $s = 2$  if there are constants  $c, \delta > 0$  such that*

$$f(x) - f(\bar{x}) \geq c\|x - \bar{x}\|^s \quad \forall x \in F, \|x - \bar{x}\| < \delta .$$

(c) *A local minimizer  $\bar{x}$  of  $f$  on  $F$  is called isolated minimizer, if there exists  $\delta > 0$  such that there is no other local minimizer  $\hat{x}$  of  $f$  on  $F$  satisfying  $\|\hat{x} - \bar{x}\| < \delta$ .*

A real symmetric  $(n \times n)$ -matrix  $A$  is called positive semidefinite if

$$\xi^T A \xi \geq 0 \quad \forall \xi \in \mathbb{R}^n .$$

$A$  is called positive definite if  $\xi^T A \xi > 0, \forall \xi \in \mathbb{R}^n \setminus \{0\}$ .

Throughout the report we use the abbreviation  $B_\delta(\bar{x}) = \{x \mid \|x - \bar{x}\| < \delta\}$  where  $\delta > 0$ .

In the following, as usual, we distinguish between unconstrained and constrained optimization.

#### 2.1. Unconstrained minimization problems

In this section we consider unconstrained minimization problems of the form:

$$P : \min_{x \in \mathbb{R}^n} f(x) .$$

Throughout the section we assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^2$ -function.

Here are the well-known optimality conditions.

**THEOREM 2.1.** [Necessary and sufficient optimality conditions]

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^2$ -function.

(a) If  $\bar{x}$  is a local minimizer of  $f$  then

$$\nabla f(\bar{x}) = 0 \quad \text{and} \quad \nabla^2 f(\bar{x}) \succeq 0 \quad (\text{positive semidefinite})$$

(b) If  $\bar{x}$  satisfies

$$\nabla f(\bar{x}) = 0 \quad \text{and} \quad \nabla^2 f(\bar{x}) \succ 0 \quad (\text{positive definite})$$

then  $\bar{x}$  is a strict local minimizer of  $f$  of order 2. More precisely, for any  $c < \beta$  where

$$\beta := \min_{\|d\|=1} \frac{1}{2} d^T \nabla^2 f(\bar{x}) d > 0,$$

there exists  $\delta > 0$  such that

$$f(x) - f(\bar{x}) \geq c \|x - \bar{x}\|^2 \quad \text{for all } x \in B_\delta(\bar{x}).$$

**Proof.** See, e.g., [9, Lemma 11.1, 11.2]. To prove the last statement in (b), assume that it is false. Then there exist  $c < \beta$  and a sequence  $x_l \rightarrow \bar{x}$  such that

$$f(x_l) - f(\bar{x}) < c \|x_l - \bar{x}\|^2.$$

Taylor expansion using  $\nabla f(\bar{x}) = 0$  yields

$$c \|x_l - \bar{x}\|^2 > f(x_l) - f(\bar{x}) = \frac{1}{2} (x_l - \bar{x})^T \nabla^2 f(\bar{x}) (x_l - \bar{x}) + o(\|x_l - \bar{x}\|^2)$$

or after a division by  $\|x_l - \bar{x}\|^2$ ,

$$(2.1) \quad c > \frac{1}{2} \frac{(x_l - \bar{x})^T}{\|x_l - \bar{x}\|} \nabla^2 f(\bar{x}) \frac{(x_l - \bar{x})}{\|x_l - \bar{x}\|} + o(1).$$

Putting  $d_l := \frac{x_l - \bar{x}}{\|x_l - \bar{x}\|}$  we can assume (for some subsequence)  $d_l \rightarrow \bar{d}$ ,  $\|\bar{d}\| = 1$ . Letting  $l \rightarrow \infty$  from (2.1) we obtain

$$c \geq \frac{1}{2} \bar{d}^T \nabla^2 f(\bar{x}) \bar{d} \geq \beta,$$

in contradiction to  $c < \beta$ . □

A point  $\bar{x}$  satisfying

$$\nabla f(\bar{x}) = 0 \quad \text{is called a } \textit{critical point} \text{ of } f.$$

In general, a strict local minimizer needs not be isolated (see Figure 2.1).

**Ex. 2.1.** The minimization problem  $\min f(x) = x^2 + \frac{1}{2}x^2 \sin 1/x$  has a strict (global) minimizer  $\bar{x} = 0$ . But there exist a sequence  $x_l \rightarrow \bar{x}$  of strict local minimizers  $x_l$  of  $f$ . Note that  $f$  is not  $C^2$  at  $\bar{x}$ .

Under the assumptions of Theorem 2.1(b) (in particular  $f \in C^2$ ) however, the strict minimizer  $\bar{x}$  is isolated, i.e., there is a neighborhood  $B_\delta(\bar{x})$  such that  $\bar{x}$  is the unique local minimizer in  $B_\delta(\bar{x})$ .

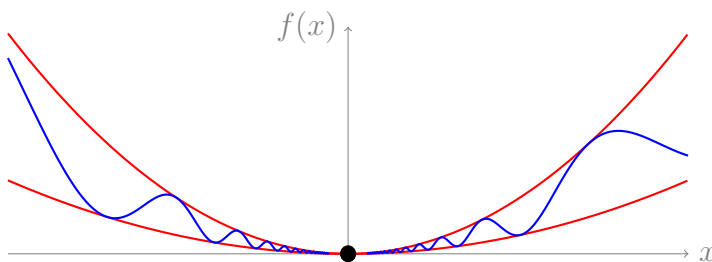


FIGURE 2.1. Sketch of  $f(x) = x^2 + \frac{1}{2}x^2 \sin 1/x$

**LEMMA 2.1.** *Let  $f$  be a  $C^2$ -function and let  $\bar{x}$  satisfy  $\nabla f(\bar{x}) = 0$  and  $\nabla^2 f(\bar{x})$  is positive definite. Then  $\bar{x}$  is a locally unique critical point and thus an isolated strict local minimizer.*

**Proof.** Assume to the contrary that there is a sequence  $x_l \rightarrow \bar{x}$  of critical points  $x_l$ , i.e.,  $\nabla f(x_l) = 0$ . Then by Taylor expansion

$$0 = \nabla f(x_l) - \nabla f(\bar{x}) = \nabla^2 f(\bar{x})(x_l - \bar{x}) + o(\|x_l - \bar{x}\|).$$

Division by  $\|x_l - \bar{x}\|$  yields

$$(2.2) \quad 0 = \nabla^2 f(\bar{x}) \frac{(x_l - \bar{x})}{\|x_l - \bar{x}\|} + o(1).$$

The bounded sequence  $\frac{x_l - \bar{x}}{\|x_l - \bar{x}\|}$  has a convergent subsequence  $\frac{x_{l_\nu} - \bar{x}}{\|x_{l_\nu} - \bar{x}\|} \rightarrow \bar{d}$ ,  $\|\bar{d}\| = 1$ . Taking the limit  $l_\nu \rightarrow \infty$  in (2.2) gives  $0 = \nabla^2 f(\bar{x})\bar{d} = 0$  and thus  $0 = \bar{d}^T \nabla^2 f(\bar{x})\bar{d} = 0$ , contradicting the positive definiteness of  $\nabla^2 f(\bar{x})$ .  $\square$

## 2.2. Convex unconstrained minimization.

Convex functions play an important role in optimization.

**DEFINITION 2.2.** [convex set, convex function]

A set  $D \subset \mathbb{R}^n$  is called convex if

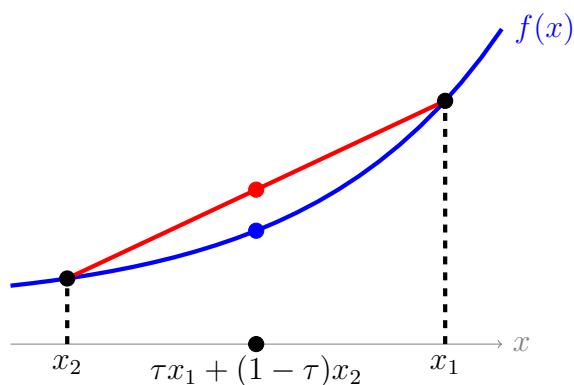
$$\text{for any } x_1, x_2 \in D, \tau \in [0, 1] \text{ we have } \tau x_1 + (1 - \tau)x_2 \in D.$$

A function  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $D$  convex, is called convex on  $D$  if for any  $x_1, x_2 \in D$  and  $\tau \in [0, 1]$  it follows

$$f(\tau x_1 + (1 - \tau)x_2) \leq \tau f(x_1) + (1 - \tau)f(x_2).$$

$f$  is called strictly convex on  $D$  if for any  $x_1, x_2 \in D$ ,  $x_1 \neq x_2$  and  $\tau \in (0, 1)$  it holds

$$f(\tau x_1 + (1 - \tau)x_2) < \tau f(x_1) + (1 - \tau)f(x_2).$$

FIGURE 2.2. Sketch of a convex function  $f(x)$ 

The following is well-known.

**LEMMA 2.2.**

(a) Let  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $D$  convex, be a  $C^1$ -function. Then  $f$  is convex on  $D$  if and only if, for any  $\bar{x} \in D$  we have (cf., Figure 2.3)

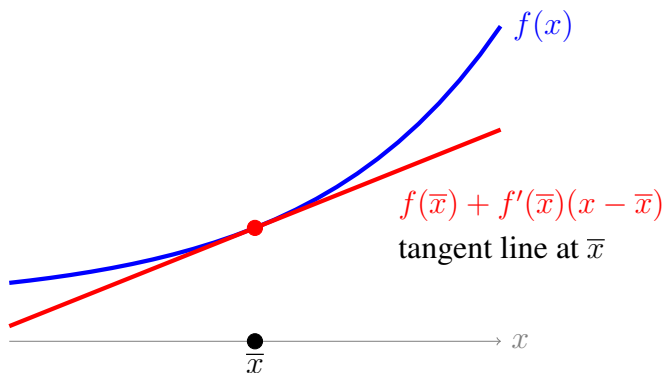
$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) \quad \text{for all } x \in D.$$

(b) Let  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $D$  convex, be a  $C^2$ -function. Then  $f$  is convex on  $D$  if and only if

$$\nabla^2 f(x) \quad \text{is positive semidefinite for all } x \in D.$$

If  $\nabla^2 f(x)$  is positive definite for all  $x \in D$  then  $f$  is strictly convex on  $D$ .

**Proof.** We refer to [9, Th.10.9, Cor.10.4] □

FIGURE 2.3. Sketch of a convex  $C^1$ -function  $f(x)$

**EX. 2.2.** For a  $C^1$ -function  $f : D \rightarrow \mathbb{R}$ ,  $D \subset \mathbb{R}^n$  convex, we have

$$f \text{ is convex on } D \Leftrightarrow (\nabla f(x') - \nabla f(x))(x' - x) \geq 0 \quad \forall x, x' \in D .$$

For a function  $f : [a, b] \rightarrow \mathbb{R}$  this means:

$$f \text{ is convex on } [a, b] \Leftrightarrow f'(x) \text{ is nondecreasing on } [a, b].$$

Note that we call  $f$  nondecreasing if

$$f(y) \leq f(x) \quad \text{for } y \leq x .$$

**Proof.** "⇒:" By Lemma 2.2(a) for  $x', x \in D$  we have

$$\begin{aligned} f(x') - f(x) &\geq \nabla f(x)(x' - x) \\ f(x) - f(x') &\geq \nabla f(x')(x - x') \end{aligned}$$

Adding these relations yields

$$0 \geq (\nabla f(x) - \nabla f(x'))(x' - x)$$

"⇐:" The mean value theorem (see Appendix) implies,

$$f(x') - f(x) = \nabla f(x + \tau(x' - x))(x' - x) ,$$

with some  $0 < \tau < 1$ . According to Lemma 2.2(a) it now suffice to show

$$(2.3) \quad \nabla f(x + \tau(x' - x))(x' - x) \geq \nabla f(x)(x' - x) .$$

By assumption, for  $\tilde{x} = x + \tau(x' - x)$  we have

$$\nabla f(\tilde{x})(\tilde{x} - x) \geq \nabla f(x)(\tilde{x} - x) ,$$

or in view of  $\tilde{x} - x = \tau(x' - x)$ ,

$$\nabla f(\tilde{x})\tau(x' - x) \geq \nabla f(x)\tau(x' - x) ,$$

which after division by  $\tau > 0$  gives (2.3). □

The important fact in convex optimization is that we need not distinguish between local and global minimizers. Moreover, only first order information is required to characterize a minimizer.

**LEMMA 2.3.** Let  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function.

(a) Then any local minimizer of  $f$  on  $D$  is a global minimizer.

If  $f$  is strictly convex on  $D$ , then a (global) minimizer  $\bar{x}$  is unique.

(b) Let  $f$  be a  $C^1$ -function and let  $D$  be open. Then  $\bar{x} \in D$  is a (global) minimizer of  $f$  on  $D$  if and only if  $\nabla f(\bar{x}) = 0$ .

**Proof.** We refer to [9, Th.10.5]. We only prove the uniqueness statement in

(a). Assume  $\bar{x}, x' \in D$ ,  $\bar{x} \neq x'$ , are both global minimizers. Then for  $x_0 := \frac{1}{2}\bar{x} + \frac{1}{2}x' \in D$ , strict convexity implies

$$f(x_0) < \frac{1}{2}f(\bar{x}) + \frac{1}{2}f(x') = f(\bar{x}) ,$$

an obvious contradiction. □

**REMARK 2.1.** Let  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. Then at each interior point  $\bar{x}$  of  $D$  the function  $f$  is locally Lipschitz continuous (cf., Definition 9.1). See [9, Th.10.10] for a proof.

**EX. 2.3.** Show that the set of (global) minimizers of a convex function  $f$  is a convex set.

**EX. 2.4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a  $C^1$ -function. Then we have:

$$\begin{aligned} f'(x) \text{ is strictly increasing on } [a, b] &\Rightarrow f(y) > f(x) + f'(x)(y - x) \quad \forall y \neq x \in [a, b] \\ &\Rightarrow f(x) \text{ is strictly convex on } [a, b]. \end{aligned}$$

**Proof.** Recall that we call  $h : \mathbb{R} \rightarrow \mathbb{R}$  strictly increasing if

$$h(y) < h(x) \quad \text{for } y < x.$$

Let  $f'$  be strictly increasing and assume to the contrary that for some  $x \neq y$  in  $[a, b]$  we have

$$f(y) \leq f(x) + f'(x)(y - x).$$

Then by the mean value theorem it holds for some  $\tau$ ,  $0 < \tau < 1$ ,

$$(2.4) \quad f(y) - f(x) = f'(x + \tau(y - x))(y - x) \leq f'(x)(y - x).$$

But by the monotonicity assumption on  $f'$  in both cases  $x < y$  and  $x > y$  it follows

$$f'(x)(y - x) < f'(x + \tau(y - x))(y - x),$$

contradicting (2.4). To prove the second implication take any  $x, y \in [a, b]$ ,  $x \neq y$  and define  $x_\tau = y + \tau(x - y)$  for  $0 < \tau < 1$ . By assumption we must have

$$\begin{aligned} f(x) &> f(x_\tau) + f'(x_\tau)(x - x_\tau) \\ f(y) &> f(x_\tau) + f'(x_\tau)(y - x_\tau). \end{aligned}$$

Multiplying the first relation with  $\tau > 0$ , the second with  $1 - \tau > 0$  and adding yields

$$\tau f(x) + (1 - \tau)f(y) > f(x_\tau) + f'(x_\tau)(x_\tau - x_\tau) = f(x_\tau).$$

□

### 2.3. Linear programs

Let be given a matrix  $A \in \mathbb{R}^{m \times n}$  with  $m$  rows  $a_j^T$ ,  $j \in J := \{1, \dots, m\}$ , and vectors  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$ . We consider *primal* linear programs (LP) of the form

$$(2.5) \quad P : \quad \max c^T x \quad \text{s.t.} \quad x \in F_P = \{x \in \mathbb{R}^n \mid a_j^T x \leq b_j, j \in J\}.$$



We often write the feasible set in compact form,  $F_P = \{x \mid Ax \leq b\}$ . The program

$$(2.6) \quad D : \quad \min b^T y \quad \text{s.t.} \quad y \in F_D = \{y \in \mathbb{R}^m \mid A^T y = c, y \geq 0\}$$

is called the dual problem. A vector  $x \in F_P$  is called feasible for  $P$  and a vector  $y$  satisfying the feasibility conditions  $A^T y = c, y \geq 0$ , is called feasible for  $D$ . Let  $v_P$ , resp.,  $v_D$  denote the maximum value of  $P$ , resp., minimum value of  $D$ .

**LEMMA 2.4.** [weak duality]

Let  $x$  be feasible for  $P$  and  $y$  be feasible for  $D$ . Then

$$c^T x \leq b^T y \quad \text{and thus} \quad v_P \leq v_D .$$

If  $c^T x = b^T y$  holds, then  $x$  is a maximizer of  $P$  and  $y$  a minimizer of  $D$ .

**Proof.** For  $x \in F_P, y \in F_D$  we obtain using  $y \geq 0, b - Ax \geq 0$

$$b^T y - c^T x = y^T b - (A^T y)^T x = y^T (b - Ax) \geq 0 .$$

□

For  $\bar{x} \in F_P$  we define the active index set  $J_0(\bar{x}) = \{j \in J \mid a_j^T \bar{x} = b_j\}$ . For a subset  $J_0 \subset J$  we denote by  $A_{J_0}$  the submatrix of  $A$  with rows  $a_j^T, j \in J_0$ , and for  $y \in \mathbb{R}^m$  by  $y_{J_0}$  the vector  $(y_j, j \in J_0)$ .

**DEFINITION 2.3.** A feasible point  $\bar{x} \in F_P$  is called a vertex of the polyhedron  $F_P$  if the vectors

$a_j, j \in J_0(\bar{x})$ , form a basis of  $\mathbb{R}^n$  or equivalently if  $A_{J_0(\bar{x})}$  has rank  $n$  .

This implies  $|J_0(\bar{x})| \geq n$ . The vertex  $\bar{x}$  is called nondegenerate if LICQ holds, i.e.,  $a_j, j \in J_0(\bar{x})$ , are linearly independent or equivalently  $A_{J_0(\bar{x})}$  is nonsingular (implying  $|J_0(\bar{x})| = n$ ) (see Figure 2.4).

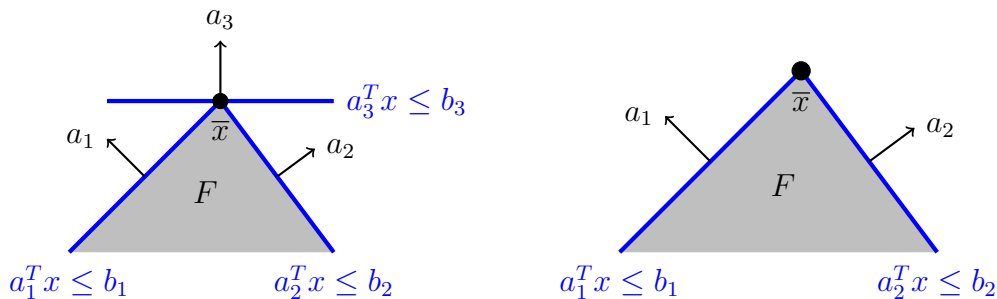


FIGURE 2.4. Vertex  $\bar{x}$  (left) and nondegenerate vertex  $\bar{x}$  (right).

The next theorem states the well-known strong duality results and related optimality conditions in LP, see, e.g., [9, Section 4.1].

**THEOREM 2.2.**

(a) [Existence and strong duality] *If both,  $P$  and  $D$ , are feasible then there exist optimal solutions  $\bar{x}$  of  $P$  and  $\bar{y}$  of  $D$ . Moreover for (any of) these solutions we have*

$$c^T \bar{x} = b^T \bar{y} \quad \text{and thus} \quad v_P = v_D .$$

(b) [Optimality conditions] *A point  $\bar{x} \in F_P$  is a maximizer of  $P$  if and only if there is a corresponding  $\bar{y} \in F_D$  such that the complementarity conditions*

$$\bar{y}^T (b - A\bar{x}) = 0 \quad \text{or} \quad \bar{y}_j (b_j - a_j^T \bar{x}) = 0, \quad \forall j \in J$$

*hold or equivalently if there exist  $\bar{y}_j$ ,  $j \in J_0(\bar{x})$ , such that KKT conditions are satisfied:*

$$\sum_{j \in J_0(\bar{x})} \bar{y}_j a_j = c, \quad \bar{y}_j \geq 0, \quad j \in J_0(\bar{x}) .$$

It appears that the solution of  $P$  arises at a vertex of  $F_P$  (cf. [4, Theorem 2.7]).

**LEMMA 2.5.** *If the polyhedron  $F_P$  has a vertex (at least one) and  $v_P < \infty$ , then the max value  $v_P$  of  $P$  is also attained at some vertex of  $F_P$ .*

**Remark.** The Simplex algorithm for solving  $P$  proceeds from vertex to vertex of  $F_P$  until the optimality conditions in Theorem 2.2(b) are met.

## 2.4. Nonlinear constrained Programs

We now consider *nonlinear programs* of the form

$$(2.7) \quad P : \quad \min_x f(x) \quad \text{s.t.} \quad x \in F := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in J\}$$

with index set  $J = \{1, \dots, m\}$  and *feasible set*  $F$  (omitting equality constraints to avoid technicalities). We again assume  $f, g_j \in C^2$  throughout this section.

For  $\bar{x} \in F$  we introduce the *active index set*

$$J_0(\bar{x}) = \{j \in J \mid g_j(\bar{x}) = 0\}$$

and the *Lagrangian function* (near  $\bar{x}$ )

$$L(x, \mu) = f(x) + \sum_{j \in J_0(\bar{x})} \mu_j g_j(x) .$$

The coefficients  $\mu_j$  are called *Lagrangian multipliers*.

**DEFINITION 2.4.**

(a) We say that the Linear Independence Constraint Qualification (LICQ) is satisfied at  $\bar{x} \in F$  if the vectors

$$\nabla g_j(\bar{x}), j \in J_0(\bar{x}), \text{ are linearly independent .}$$

(b) The (weaker) Mangasarian Fromovitz Constraint Qualification (MFCQ) is satisfied at  $\bar{x} \in F$  if there exists a vector  $\xi$  such that

$$\nabla g_j(\bar{x})\xi < 0, \text{ for all } j \in J_0(\bar{x}) .$$

**EX. 2.5.** Show for  $\bar{x} \in F$ :

$$\text{LICQ holds at } \bar{x} \Rightarrow \text{MFCQ is satisfied at } \bar{x}$$

We now state the well-known necessary optimality conditions.

**THEOREM 2.3.** [first order necessary optimality conditions]

Let  $\bar{x} \in F$  be a local minimizer of  $P$ .

(a) Then there exist multipliers  $\bar{\mu}_0 \geq 0, \bar{\mu}_j \geq 0, j \in J_0(\bar{x})$ , not all zero, such that the Fritz John (FJ) condition holds:

$$(2.8) \quad \bar{\mu}_0 \nabla f(\bar{x}) + \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0 .$$

(b) Let in addition MFCQ be valid at  $\bar{x}$ . Then there exist multipliers  $\bar{\mu}_j \geq 0, j \in J_0(\bar{x})$ , such that the Karush-Kuhn-Tucker (KKT) condition holds:

$$(2.9) \quad \nabla f(\bar{x}) + \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0 .$$

**Proof.** For a proof we refer to [9, Th.12.4, Cor.12.2].  $\square$

A point  $\bar{x} \in F$  satisfying the FJ condition, resp., the KKT condition with corresponding nonnegative multipliers is called FJ point, resp., KKT point.

**REMARK 2.2.** If the constraints of  $P$  are given by only linear equalities/inequalities, then without any constraint qualification assumption at a minimizer  $\bar{x}$  the KKT conditions must hold. For a proof see, e.g., [9, Th.5.4].

**EX. 2.6.** Let the KKT-condition (2.9) hold at  $\bar{x} \in F$ .

(a) Then if LICQ is satisfied at  $\bar{x}$ , the multipliers  $\bar{\mu}_j, j \in J_0(\bar{x})$ , are uniquely determined.

(b) If (only) MFCQ is satisfied at  $\bar{x}$ , then the set of multipliers,

$$M(\bar{x}) := \{ \bar{\mu} = (\bar{\mu}_j, j \in J_0(\bar{x})) \mid \bar{\mu}_j \geq 0, j \in J_0(\bar{x}) \text{ and (2.9) holds} \} ,$$

is bounded (compact).

**Proof.** (b) By MFCQ there exist  $\xi \in \mathbb{R}^n, \sigma < 0$ , such that

$$\nabla g_j(\bar{x})\xi \leq \sigma \quad \forall j \in J_0(\bar{x}) .$$

The KKT condition yields using  $\bar{\mu}_j \geq 0$

$$0 \leq \sum_{j \in J_0(\bar{x})} \bar{\mu}_j |\sigma| \leq - \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) \xi = \nabla f(\bar{x}) \xi ,$$

and thus we obtain the bound for the Lagrangean multipliers:

$$0 \leq \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \leq \frac{1}{|\sigma|} |\nabla f(\bar{x}) \xi| .$$

□

We give an example to show that without some Constraint Qualification the KKT condition need not be satisfied at a local minimizer  $\bar{x}$  of P.

**EX. 2.7.** [MFCQ does not hold] Consider the programs:

$$P_1 : \min x \text{ s.t. } x^2 \leq 0 , \quad P_2 : \min x^2 \text{ s.t. } x^2 \leq 0 .$$

Show that for  $P_1$  at the minimizer  $\bar{x} = 0$  the KKT condition does not hold and that for  $P_2$  at the minimizer  $\bar{x} = 0$  the KKT-condition  $0 + \bar{\mu}0 = 0$  is valid for all multipliers  $\bar{\mu} \geq 0$ .

In (nonconvex) optimization, in general, second order conditions are needed to assure that  $\bar{x} \in F$  is a local minimizer. However in special cases first order conditions are sufficient for optimality.

**THEOREM 2.4.** [Sufficient optimality conditions]

(a) (Order one) Let  $\bar{x} \in F$  satisfy LICQ. Let with multipliers  $\bar{\mu}_j$  the KKT condition

$$(2.10) \quad \nabla_x L(\bar{x}, \bar{\mu}) = \nabla f(\bar{x}) + \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0 , \quad \bar{\mu}_j \geq 0, j \in J_0(\bar{x}) ,$$

be satisfied such that strict complementarity (SC) holds,

$$\bar{\mu}_j > 0, \quad \forall j \in J_0(\bar{x}), \quad (SC)$$

$$\text{and } |J_0(\bar{x})| = n.$$

Then  $\bar{x}$  is a local minimizer of P of order  $s = 1$ .

(b) (Order two) Let at  $\bar{x} \in F$  with multipliers  $\bar{\mu}_j \geq 0, j \in J_0(\bar{x})$ , the KKT condition (2.10) hold and the second order condition

$$SOC : \quad d^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d > 0 \quad \forall d \in C_{\bar{x}} \setminus \{0\},$$

where  $C_{\bar{x}}$  is the cone of critical directions,

$$C_{\bar{x}} = \{d \mid \nabla f(\bar{x}) d \leq 0, \nabla g_j(\bar{x}) d \leq 0, j \in J_0(\bar{x})\} .$$

Then  $\bar{x}$  is a local minimizer of P of order  $s = 2$ . More precisely, for any  $c < \beta$ , where

$$\beta := \min_{d \in C_{\bar{x}}, \|d\|=1} \frac{1}{2} d^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d > 0$$

there exists  $\delta > 0$  such that

$$f(x) - f(\bar{x}) \geq c \|x - \bar{x}\|^2 \quad \text{for all } x \in F, x \in B_\delta(\bar{x}).$$

**Proof.** We refer to [9]. The last statement can be proven by an easy modification of the proof of [9, Th.12.6]. See also the proof of Theorem 2.1.  $\square$

**EX. 2.8.** Let the KKT condition (2.10) hold at  $\bar{x}$ . Show that we have

$$C_{\bar{x}} = \{d \mid \nabla g_j(\bar{x})d = 0, \text{ if } \bar{\mu}_j > 0, \nabla g_j(\bar{x})d \leq 0, \text{ if } \bar{\mu}_j = 0, j \in J_0(\bar{x})\}$$

If moreover SC holds, then  $C_{\bar{x}}$  equals the tangent space:

$$C_{\bar{x}} = T_{\bar{x}} := \{d \mid \nabla g_j(\bar{x})d = 0, j \in J_0(\bar{x})\}.$$

**EX. 2.9.** Show that the point  $\bar{x} = 0$  is the minimizer of order  $s = 1$  of the problem

$$\min x_2 \quad \text{s.t.} \quad g_1(x) := e^{-x_1} - x_2 - 1 \leq 0, \quad g_2(x) := x_1 - x_2 \leq 0$$

Hint: Show that the assumptions of Theorem 2.4(a) are met.

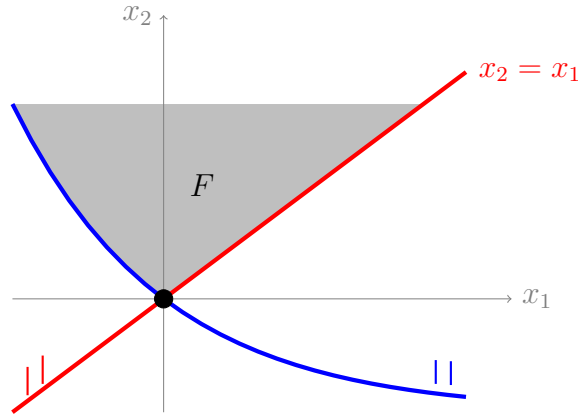


FIGURE 2.5. Feasible set in Ex.2.9

**EX. 2.10.** Show that  $\bar{x} = (\frac{1}{\sqrt{2}}, \frac{1}{2})$  is the unique (global) minimizer of

$$P \quad \min f(x) := x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \quad x_2 \leq x_1^2, x_1 \geq 0.$$

Hint: Show that  $\bar{x}$  satisfies the KKT conditions with  $\mu_1 = 1$  and SOC.

**EX. 2.11.** Show that under the assumptions of Theorem 2.4(a) we have  $C_{\bar{x}} = \{0\}$ , i.e., SOC is automatically fulfilled.

**REMARK 2.3.** *The KKT conditions can also be given in the equivalent (global) form:*

$$(2.11) \quad \begin{aligned} \nabla f(x) + \sum_{j \in J} \mu_j \nabla g_j(x) &= 0 \\ \mu_j \cdot g_j(x) &= 0, \quad j \in J \\ \mu_j, -g_j(x) &\geq 0, \quad j \in J. \end{aligned}$$

We give an example, where a strict local minimizer of  $P$  is not an isolated local minimizer.

**Ex. 2.12.** *Consider the program*

$$P : \quad \min x^2 \quad \text{s.t.} \quad x^6(1 - \sin 1/x) \geq 0.$$

*Show that  $\bar{x} = 0$  is a strict local minimizer but not an isolated one.*

*Note that the problem functions are  $C^2$ , but at  $\bar{x}$  the condition MFCQ is not satisfied.*

Under the assumptions of Theorem 2.4 the local minimizer is isolated (see, e.g., [28, Th.2.3]).

**THEOREM 2.5.** *Let  $f, g_j$  be  $C^2$ -functions. Assume that  $\bar{x} \in F$  satisfies MFCQ and the KKT condition (2.9) with some multiplier  $\hat{\mu} \geq 0$ , as well as SOC (this implies that  $\bar{x}$  is a strict local minimizer of  $P$  of order 2). Then  $\bar{x}$  is an isolated (locally unique) KKT point of  $P$ .*

**Proof.** Suppose to the contrary that there is a sequence of KKT points  $x_l \in F$ ,  $x_l \rightarrow \bar{x}$ , with corresponding Lagrangean multiplier vectors  $\mu_l \geq 0$ . By continuity of  $g_j$  and MFCQ, with some  $\xi$  (see Definition 2.4) and some  $\sigma < 0$  we have for large  $l$ ,

$$J_0(x_l) \subset J_0(\bar{x}) \quad \text{and} \quad \nabla g_j(\bar{x})\xi \leq \sigma, \quad \nabla g_j(x_l)\xi \leq \sigma/2, \quad \forall j \in J_0(\bar{x}).$$

We define with  $\mu = (\mu_j, j \in J_0(\bar{x}))$ ,  $g(x) = (g_j(x), j \in J_0(\bar{x}))$ , the Lagrangean

$$L(x, \mu) = f(x) + \sum_{j \in J_0(\bar{x})} \mu_j g_j(x) = f(x) + \mu^T g(x).$$

By MFCQ the sets of Lagrangean multipliers  $\mu_l$  of  $x_l$  are uniformly bounded (see the proof of Ex 2.6(b)). So we can assume (for some subsequence)  $\mu_l \rightarrow \bar{\mu}$  and since  $J_0(\bar{x})$  is finite, we also can assume that the active index sets of  $x_l$  are the same,  $J_0(x_l) := J_0 \subset J_0(\bar{x})$ . So  $[\mu_l]_j = \bar{\mu}_j = 0$  for all  $j \notin J_0$  and thus

$$0 = \mu_l^T g(x_l) = \bar{\mu}^T g(\bar{x}).$$

Moreover

$$0 = \lim_{l \rightarrow \infty} \nabla_x L(x_l, \mu_l) = \nabla_x L(\bar{x}, \bar{\mu}),$$

i.e.,  $\bar{\mu}$  is a Lagrangean multiplier wrt.  $\bar{x}$ . In view of  $f(x_l) = L(x_l, \mu_l)$ ,  $f(\bar{x}) = L(\bar{x}, \bar{\mu})$  and since by assumption (see Theorem 2.4)  $\bar{x}$  is a second order minimizer, with some  $\kappa > 0$  we have (for large  $l$ )

$$L(x_l, \mu_l) - L(\bar{x}, \bar{\mu}) \geq \kappa \|x_l - \bar{x}\|^2 .$$

In view of  $\mu_l^T g(x_l) = \bar{\mu}^T g(x_l) = 0$  (recall  $\bar{\mu}_j = 0, j \notin J_0$ ) we obtain

$$L(x_l, \mu_l) = f(x_l) + \mu_l^T g(x_l) = f(x_l) + \bar{\mu}^T g(x_l) = L(x_l, \bar{\mu})$$

and thus

$$(2.12) \quad L(x_l, \bar{\mu}) - L(\bar{x}, \bar{\mu}) \geq \kappa \|x_l - \bar{x}\|^2 .$$

From *SOC*, by putting  $\sigma_l := \|x_l - \bar{x}\|$ ,  $d_l := \frac{x_l - \bar{x}}{\|x_l - \bar{x}\|}$ , we find

$$\begin{aligned} L(x_l, \bar{\mu}) - L(\bar{x}, \bar{\mu}) &= \nabla_x L(\bar{x}, \bar{\mu})(x_l - \bar{x}) + \frac{1}{2} \sigma_l^2 d_l^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d_l + o(\sigma_l^2) \\ (2.13) \quad &= \frac{1}{2} \sigma_l^2 d_l^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d_l + o(\sigma_l^2) . \end{aligned}$$

Together, (2.12) and (2.13) yield for large  $l$ ,

$$(2.14) \quad d_l^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d_l \geq 2\kappa + \frac{o(\sigma_l^2)}{\sigma_l^2} > \kappa .$$

Later in Ex 4.8 the relation

$$(2.15) \quad (\mu_l - \bar{\mu})^T (g(x_l) - g(\bar{x})) \geq 0$$

is proven. We also obtain

$$\begin{aligned} \nabla_x L(x_l, \mu_l) &= \nabla f(x_l) + \mu_l^T \nabla g(x_l) \\ &= \nabla f(x_l) + \bar{\mu}^T \nabla g(x_l) + (\mu_l - \bar{\mu})^T \nabla g(x_l) \\ &= \nabla_x L(x_l, \bar{\mu}) + (\mu_l - \bar{\mu})^T [\nabla g(\bar{x}) + O(\|x_l - \bar{x}\|)] \end{aligned}$$

or

$$\nabla_x L(x_l, \mu_l) - \nabla_x L(x_l, \bar{\mu}) = (\mu_l - \bar{\mu})^T [\nabla g(\bar{x}) + O(\|x_l - \bar{x}\|)] .$$

This together with (2.14) now leads to

$$\begin{aligned}
(\mu_l - \bar{\mu})^T (g(x_l) - g(\bar{x})) &= (\mu_l - \bar{\mu})^T (g(x_l) - g(\bar{x})) \\
&\quad - (\nabla_x L(x_l, \mu_l) - \nabla_x L(\bar{x}, \bar{\mu})) (x_l - \bar{x}) \\
&= (\mu_l - \bar{\mu})^T (g(x_l) - g(\bar{x})) \\
&\quad - (\nabla_x L(x_l, \mu_l) - \nabla_x L(x_l, \bar{\mu})) (x_l - \bar{x}) \\
&\quad - (\nabla_x L(x_l, \bar{\mu}) - \nabla_x L(\bar{x}, \bar{\mu})) (x_l - \bar{x}) \\
&= (\mu_l - \bar{\mu})^T [g(x_l) - g(\bar{x}) - \nabla g(\bar{x})(x_l - \bar{x})] \\
&\quad + O(\|\mu_l - \bar{\mu}\| \|x_l - \bar{x}\|^2) \\
&\quad - (\nabla_x L(x_l, \bar{\mu}) - \nabla_x L(\bar{x}, \bar{\mu})) (x_l - \bar{x}) \\
&= (\mu_l - \bar{\mu})^T \left[ \frac{1}{2} (x_l - \bar{x})^T \nabla^2 g(\bar{x}) (x_l - \bar{x}) \right] \\
&\quad + O(\|\mu_l - \bar{\mu}\| \sigma_l^2) \\
&\quad - (x_l - \bar{x})^T \nabla_x^2 L(\bar{x}, \bar{\mu}) (x_l - \bar{x}) + o(\|x_l - \bar{x}\|^2) \\
&= o(\sigma_l^2) - \sigma_l^2 d_l^T \nabla_x^2 L(\bar{x}, \bar{\mu}) d_l < -\frac{\kappa}{2} \sigma_l^2
\end{aligned}$$

if  $l$  is large, contradicting (2.15). □

## 2.5. Convex constrained programs

**DEFINITION 2.5.** *The program  $P$  is called convex if the functions  $f$  and  $g_j$ ,  $j \in J$ , are convex functions on  $\mathbb{R}^n$ .*

**EX. 2.13.** *If the functions  $g_j$ ,  $j \in J$ , are convex, then the feasible set  $F$  of  $P$  is convex.*

In convex constrained optimization, the KKT conditions are sufficient for optimality (cf., also Lemma 2.3).

**LEMMA 2.6.** *Let  $P$  be a convex program and let  $f, g_j$ ,  $j \in J$ , be  $C^1$ -functions. Then if  $\bar{x}$  satisfies the KKT conditions (2.10),  $\bar{x}$  is a global minimizer of  $P$ .*

**Proof.** We refer to [9, Cor.10.5]. □

**LEMMA 2.7.** *Consider the convex program*

$$P_c : \min_x f(x) \quad \text{s.t.} \quad x \in C,$$



with  $C \subset \mathbb{R}^n$ , a closed convex set, and  $f$  a convex  $C^1$ -function on  $C$ . Then  $\bar{x} \in C$  is a (global) minimizer of  $P_c$  if and only if (cf., Figure 2.6)

$$\nabla f(\bar{x})(x - \bar{x}) \geq 0 \quad \text{for all } x \in C .$$

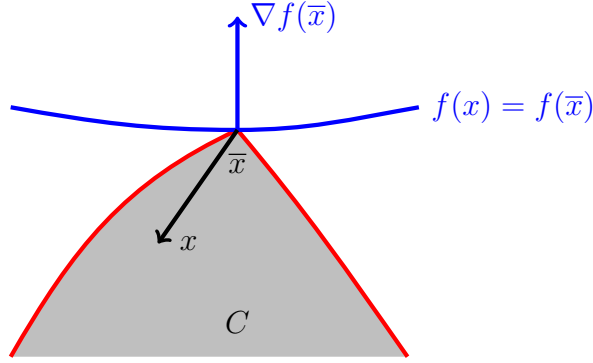


FIGURE 2.6. Sketch of the condition in Lemma 2.7.

**Proof.** "⇒:" Suppose to the contrary that there exists  $x_0 \in C$  satisfying

$$\nabla f(\bar{x})(x_0 - \bar{x}) = \alpha < 0 .$$

Then for  $x_\tau = \bar{x} + \tau(x_0 - \bar{x}) \in C$ ,  $0 < \tau \leq 1$ , we find by Taylor's formula,

$$\begin{aligned} f(x_\tau) &= f(\bar{x}) + \tau \nabla f(\bar{x})(x_0 - \bar{x}) + o(\tau) \\ &= f(\bar{x}) + \tau \alpha + o(\tau) < f(\bar{x}) , \end{aligned}$$

if  $\tau > 0$  is small enough, contradicting the minimality of  $\bar{x}$ .

"⇐:" By convexity of  $f$  (see Lemma 2.2(a)) we obtain for all  $x \in C$ ,

$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) \geq f(\bar{x}) ,$$

i.e.,  $\bar{x}$  is a minimizer of  $f$ . □

Let us look at the set  $M(\bar{x})$  of KKT multiplier vectors of a KKT point  $\bar{x}$  of  $P$ ,

$$M(\bar{x}) = \{0 \leq \mu \in \mathbb{R}^m \mid \nabla f(\bar{x}) + \sum_{j \in J} \mu_j \nabla g_j(\bar{x}) = 0, \mu_j \cdot g_j(\bar{x}) = 0, j \in J\} .$$

For convex programs all KKT points  $\bar{x}$  have the same set  $M(\bar{x})$ .

**LEMMA 2.8.** Let  $\bar{x}, x'$ ,  $\bar{x} \neq x'$ , be two KKT points of the convex program  $P$  (by Lemma 2.6 they are minimizers of  $P$ ). Then  $M(\bar{x}) = M(x')$ .

**Proof.** For convex programs the Lagrangean  $L(x, \mu) = f(x) + \sum_{j \in J} \mu_j g_j(x)$  is a convex function of  $x$  (for  $\mu$  fixed). Let  $\mu \in M(\bar{x})$ . The KKT condition

$\nabla_x L(\bar{x}, \mu) = 0$  for  $\bar{x}$  means that  $\bar{x}$  is a global minimizer of  $L(x, \mu)$ . In view of  $\mu_j g_j(\bar{x}) = 0$ ,  $j \in J$ , we find

$$\begin{aligned} L(x', \mu) = f(x') + \sum_{j \in J} \mu_j g_j(x') &\geq L(\bar{x}, \mu) = f(\bar{x}) + \sum_{j \in J} \mu_j g_j(\bar{x}) \\ &= f(\bar{x}) = f(x') . \end{aligned}$$

Using  $\mu_j \geq 0$ ,  $g_j(x') \leq 0$  this implies  $0 \leq \sum_{j \in J} \mu_j g_j(x') \leq 0$  and thus

$$\mu_j g_j(x') = 0 \quad \text{for all } j \in J .$$

Consequently

$$L(x', \mu) = f(x') = f(\bar{x}) = L(\bar{x}, \mu) ,$$

and  $x'$  is also an unconstrained minimizer of  $L(x, \mu)$ . Therefore  $\nabla_x L(x', \mu) = 0$  must hold, implying  $\mu \in M(x')$  and  $M(\bar{x}) \subset M(x')$ . Interchanging the role of  $x'$  and  $\bar{x}$  gives  $M(x') \subset M(\bar{x})$ , i.e.,  $M(\bar{x}) = M(x')$ .

□

**REMARK 2.4.** Let the constraints of the convex program  $P$  be given only by linear equalities/inequalities. Then (see Remark 2.2) at a minimizer  $\bar{x}$  the KKT conditions must hold and together with Lemma 2.6 we have in this case:

$\bar{x} \in F$  is a global minimizer of  $P \Leftrightarrow$  the KKT condition (2.10) is satisfied at  $\bar{x}$

## CHAPTER 3

### Application: Models from traffic engineering

As an application of optimization, in this chapter, we investigate special models from traffic engineering which are used to predict traffic flows on road networks.

#### 3.1. Static traffic user equilibrium

In this model we have given a traffic network  $N = (V, E)$  with  $V$ , the set of vertices  $i$  (the crossings), and  $E$ , the set of directed edges  $e = (i, j)$ ,  $i, j \in V$  (the road segments see Figure 3.1). Further are given pairs  $(s_w, t_w)$ ,  $w \in W$ , of origin-destination vertices  $s_w, t_w \in V$  (OD-pairs). We denote by

- $\mathcal{P}_w$  the set of all directed paths  $p$  in  $N$  connecting  $s_w$  with  $t_w$ ,
- $\mathcal{P} = \cup_{w \in W} \mathcal{P}_w$ .

Between the OD-pairs  $(s_w, t_w)$  there is a certain traffic demand  $d_w$ , i.e.,  $d_w$  (cars/hour) wish to go from  $s_w$  to  $t_w$ ,  $w \in W$ . This traffic will use certain paths from  $\mathcal{P}_w$  and the corresponding edges  $e$  of the network. We denote by

- $x_e$  the traffic flow (cars/hour) on edges  $e \in E$ ,
- $f_p$  the traffic flow passing through path  $p \in \mathcal{P}$ .

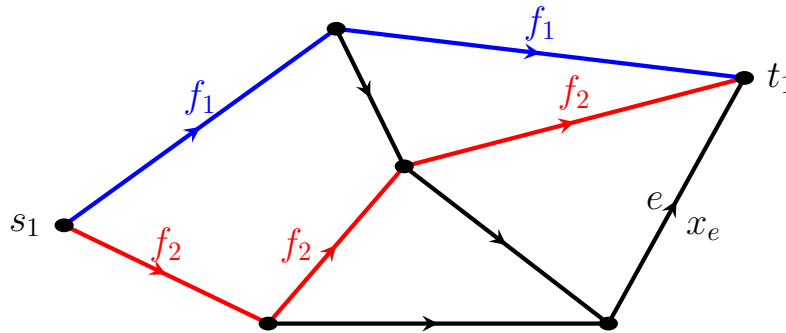


FIGURE 3.1. Example network

Then  $x = (x_e, e \in E)$  is the edge flow vector,  $f = (f_p, p \in \mathcal{P})$  the path flow vector and  $d = (d_w, w \in W)$  the demand vector. In this model we finally are given so-called cost functions

- $c_e(x_e)$ , the edge cost functions, e.g., giving the travel times needed to cross edge  $e$ ,
- $c_p(x) = \sum_{e \in p} c_e(x_e)$ ,  $p \in \mathcal{P}$ , the path cost functions induced by the edge flow  $x$ .

In this report we only consider the model with separated costs, where the costs  $c_e = c_e(x_e)$  only depend on the flow  $x_e$  on the edge  $e$ . We assume

**AT:** The cost functions  $c_e(x_e)$ ,  $c_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $e \in E$ , are continuous, nondecreasing in  $x_e$ , i.e.,

$$c_e(x_e) \leq c_e(x'_e) \quad \text{for } x_e \leq x'_e.$$

According to the definition, a feasible flow  $(x, f) \in \mathbb{R}^{|E|} \times \mathbb{R}^{|\mathcal{P}|}$  must satisfy the conditions

$$(3.1) \quad \begin{aligned} \Lambda f &= d \\ \Delta f - x &= 0 \\ f &\geq 0 \end{aligned}$$

where  $\Lambda = (\Lambda_{wp})$  is the path-demand incidence matrix of the network and  $\Delta = (\Delta_{ep})$  the path-edge incidence matrix:

$$\Lambda_{wp} = \begin{cases} 1 & \text{if } p \in \mathcal{P}_w \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Delta_{ep} = \begin{cases} 1 & \text{if } e \in p \\ 0 & \text{otherwise} \end{cases}.$$

Let denote

- $F_d = \{(x, f) \mid (x, f) \text{ satisfies (3.1)}\}$ , the set of feasible flow for given demand  $d$ .
- $X_d = \{x \mid \text{there exists } f \text{ such that } (x, f) \in F_d\}$ , the projection of  $F_d$  onto the  $x$ -space.

It is easily seen, that the feasible set  $F_d$  is compact (bounded and closed).

**Note:** It is well-known (see, e.g., [9, Th.2.5]) that the projection  $X_d$  of a compact set  $F_d$  given by linear equalities and inequalities (a polyhedron) is again a compact set defined by linear equalities and inequalities (a polyhedron), i.e., (for  $d$  fixed) there are matrices  $A, B$  and vectors  $a, b$  such that

$$(3.2) \quad X_d = \{x \mid Ax = a, Bx \leq b\}.$$

For this network model Wardrop [31] formulated in 1952 his famous equilibrium conditions to predict the traffic flow in the network.

**Wardrop Equilibrium:** A feasible flow  $(x, f) \in F_d$  is called Wardrop equilibrium (WE) wrt.  $d$  if for all  $w \in W$  it holds

$$(3.3) \quad \text{for any } p, q \in \mathcal{P}_w \text{ we have: } f_p > 0 \Rightarrow \begin{cases} c_p(x) = c_q(x) & \text{if } f_q > 0 \\ c_p(x) \leq c_q(x) & \text{if } f_q = 0. \end{cases}$$

Such a Wardrop equilibrium is also called traffic user equilibrium.

**Meaning:** In a traffic user equilibrium  $(x, f)$  according to Wardrop, for any  $w \in W$  it holds that for each used path  $p \in \mathcal{P}_w$  between OD-pairs  $(s_w, t_w)$  the path costs  $c_p(x)$  must be the same.

Beckmann et al. [3] have discovered that the Wardrop equilibrium can be computed as solution of the following optimization problem,

$$(3.4) \quad P_{WE} : \min_{x, f} N(x) := \sum_{e \in E} \int_0^{x_e} c_e(\tau) d\tau \quad \text{s.t.} \quad (x, f) \in F_d .$$

This program has linear constraints and by Ex. 3.1, under AT, the objective  $N(x)$  is convex, and differentiable. By (3.5) it satisfies

$$\nabla N(x) = c(x) .$$

**Ex. 3.1.** Show that by assumption AT the function  $N(x)$  in (3.4) is a convex function of  $x$  in  $X_d$ . If in addition to AT the functions  $c_e(x_e)$  are strictly increasing, i.e.,

$$c_e(x_e) < c_e(x'_e) \quad \text{for } x_e < x'_e .$$

then  $N(x)$  is strictly convex.

**Proof.** By AT according to the fundamental theorem of calculus the function

$$(3.5) \quad n_e(x_e) = \int_0^{x_e} c_e(\tau) d\tau \quad \text{is differentiable with derivative } n'_e(x_e) = c_e(x_e) .$$

The statement then follows from Ex. 2.4.  $\square$

**REMARK 3.1.** By Remark 2.4, under AT, a flow  $(x, f) \in F_d$  is a minimizer of  $P_{WE}$  if and only if the following KKT conditions are satisfied with Lagrangean multipliers  $-\gamma \in \mathbb{R}^{|W|}$  wrt.  $\Lambda f = d$ ,  $\lambda \in \mathbb{R}^{|E|}$  wrt.  $\Delta f - x = 0$  and  $0 \leq \mu \in \mathbb{R}^{|P|}$  wrt.  $-f \leq 0$ ,

$$(3.6) \quad \begin{aligned} c(x) - \lambda &= 0 \\ -\Lambda^T \gamma + \Delta^T \lambda - \mu &= 0 \\ \mu^T f &= 0 \end{aligned}$$

We now obtain the Beckmann characterization of a Wardrop equilibrium.

**THEOREM 3.1.** Let AT hold. Then the following conditions are equivalent for a flow  $(\bar{x}, \bar{f}) \in F_d$ .

- (a) The flow  $(\bar{x}, \bar{f})$  is a Wardrop equilibrium.
- (b) The flow  $(\bar{x}, \bar{f})$  is a minimizer of  $P_{WE}$ .
- (c) We have

$$c(\bar{x})^T(x - \bar{x}) \geq 0 \quad \text{for all } x \in X_d .$$

Moreover, a WE  $(\bar{x}, \bar{f})$  always exists.

**Proof.** (a)  $\Leftrightarrow$  (b): By Remark 3.1, a flow  $(\bar{x}, \bar{f}) \in F_d$  is a minimizer of  $P_{WE}$  if and only if  $(\bar{x}, \bar{f})$  satisfy the KKT conditions (3.6) with multipliers  $\bar{\gamma}, \bar{\lambda}, \bar{\mu}$ . In view of  $[\Delta^T c(\bar{x})]_p = \sum_{e \in p} c_e(\bar{x}_e) = c_p(\bar{x})$  these conditions can equivalently be written as  $c(\bar{x}) = \bar{\lambda}, \bar{\mu}^T \bar{f} = 0$  and then as

$$\Lambda^T \bar{\gamma} = \Delta^T c(\bar{x}) - \bar{\mu} \quad \text{or} \quad \bar{\gamma}_w = c_p(\bar{x}) - \bar{\mu}_p \begin{cases} = c_p(\bar{x}) & \text{if } \bar{f}_p > 0 \\ \leq c_p(\bar{x}) & \text{if } \bar{f}_p = 0 \end{cases}$$

which coincide with the WE conditions.

(b)  $\Leftrightarrow$  (c) Obviously  $(\bar{x}, \bar{f}) \in F_d$  is a minimizer of  $P_{WE}$  if and only if  $\bar{x}$  solves the program in  $x$ :

$$\min_x N(x) \quad \text{s.t.} \quad x \in X_d.$$

In view of  $\nabla N(x) = c(x)$  the claim follows by Lemma 2.7.

To prove the existence of a WE we only have to note that the feasible set  $F_d$  of  $P_{WE}$  (for fixed  $d$ ) is compact and  $N(x)$  is continuous. Then, according to Weierstrass's theorem 9.1 the program  $P_{WE}$  has a minimizer (at least one), which is a WE. □

**REMARK 3.2.** According to the proof of the preceding theorem, the equilibrated costs  $c_p(\bar{x})$  of used paths  $p \in \mathcal{P}_w$  of a WE  $(\bar{x}, \bar{f})$  coincide with the Lagrangean multiplier  $\bar{\gamma}_w$ , i.e., for all  $w$  and  $p \in \mathcal{P}_w$  it holds:

$$\bar{f}_p > 0 \Rightarrow c_p(\bar{x}) = \bar{\gamma}_w.$$

The next theorem establishes uniqueness results.

**THEOREM 3.2.** *Let AT hold.*

(a) *Then for any two Wardrop equilibria  $(\bar{x}, \bar{f}), (x', f')$  the costs are the same:*

$$c_e(\bar{x}_e) = c_e(x'_e) \quad \text{for all } e \in E.$$

(b) *If in addition to AT the costs  $c_e$  are strictly increasing then the edge flow part  $\bar{x}$  of a Wardrop equilibrium  $(\bar{x}, \bar{f})$  is uniquely determined.*

**Proof.** (a): By AT the functions  $c_e$  are nondecreasing, i.e.,

$$c_e(x_e) \leq c_e(\tilde{x}_e) \quad \text{if and only if } x_e \leq \tilde{x}_e.$$

This is equivalent with

$$(3.7) \quad (c_e(x_e) - c_e(\tilde{x}_e))(x_e - \tilde{x}_e) \geq 0 \quad \text{for all } x_e, \tilde{x}_e, e \in E,$$

and implies

$$(c(\bar{x}) - c(x'))^T (\bar{x} - x') = \sum_{e \in E} (c_e(\bar{x}_e) - c_e(x'_e)) (\bar{x}_e - x'_e) \geq 0.$$

Since  $(\bar{x}, \bar{f}), (x', f')$  are Wardrop equilibria, by Theorem 3.1(c) it holds,

$$c(\bar{x})^T (x' - \bar{x}) \geq 0 \quad \text{and} \quad c(x')^T (\bar{x} - x') \geq 0,$$

and thus

$$-(c(\bar{x}) - c(x'))^T (\bar{x} - x') \geq 0 .$$

Altogether we find

$$(c(\bar{x}) - c(x'))^T (\bar{x} - x') = \sum_{e \in E} (c_e(\bar{x}_e) - c_e(x'_e)) (\bar{x}_e - x'_e) = 0 ,$$

which in view of (3.7) gives

$$(c_e(\bar{x}_e) - c_e(x'_e)) (\bar{x}_e - x'_e) = 0 .$$

For  $\bar{x}_e \neq x'_e$  this yields

$$c_e(\bar{x}_e) - c_e(x'_e) = 0 .$$

But also for  $\bar{x}_e = x'_e$  this equality is true.

(b): If the  $c_e$ 's are strictly increasing, by Ex. 3.1, the objective  $N(x)$  of  $P_{WE}$  is strictly convex in  $x$ . The uniqueness statement then follows from Lemma 2.3 (a).  $\square$

Under the assumption of Theorem 3.2(b), the edge flow part  $\bar{x}$  of a WE  $(\bar{x}, \bar{f})$  is uniquely determined. This does not hold for the path flow part  $\bar{f}$ . We present an illustrative example.

**Ex. 3.2.** Consider the example network in Figure 3.2,

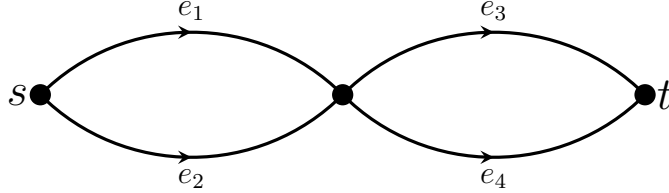


FIGURE 3.2. Example network

with 4 edges  $e_j, j = 1, \dots, 4$ . Let the strictly increasing edge costs be

$$c_{e_j}(x_{e_j}) = 1 + x_{e_j} \quad j = 1, \dots, 4 .$$

We further have an  $s$ - $t$ -demand of  $d = 2$  (thousand cars/hour). The unique  $x$  part of a WE is  $\bar{x} = (1, 1, 1, 1)$ . There are 4 paths from  $s$  to  $t$ :

$$p_1 = (e_1, e_3), \quad p_2 = (e_1, e_4), \quad p_3 = (e_2, e_3), \quad p_4 = (e_2, e_4) .$$

There are infinitely many possible WE path flows  $f = (f_{p_1}, f_{p_2}, f_{p_3}, f_{p_4})$ , corresponding to  $\bar{x}$  namely any convex combination  $\tau f + (1 - \tau) f'$ ,  $\tau \in [0, 1]$  of

$$f = (1, 0, 0, 1) \quad \text{and} \quad f' = (0, 1, 1, 0) .$$

Hence for certain aspects it is convenient to write the Wardrop equilibrium problem as a program only in the  $x$  variable (as we did before in Theorem 3.1(c)). This program reads (cf., (3.2))

$$(3.8) \quad P_{WE}^x : \quad \min_x N(x) = \sum_{e \in E} \int_0^{x_e} c_e(\tau) d\tau \quad \text{s.t.} \quad x \in X_d .$$

**Note.** Given a minimizer  $\bar{x}$  of  $P_{WE}^x$ , then any solution  $\bar{f}$  of

$$\begin{aligned} \Lambda f &= d \\ \Delta f - \bar{x} &= 0 \\ f &\geq 0 \end{aligned}$$

yields a minimizer  $(\bar{x}, \bar{f})$  of  $P_{WE}$ , i.e., a WE.

For  $P_{WE}^x$  we can say more about the minimizer  $\bar{x}$ , i.e., about the WE edge flow  $\bar{x}$ .

**COROLLARY 3.1.** *Assume that in addition to AT the costs  $c_e$  are continuously differentiable with*

$$c'_e(x_e) > 0 \quad \text{for all } x_e \geq 0, e \in E .$$

*Then a minimizer  $\bar{x}$  of  $P_{WE}^x$  is a unique strict minimizer of order 2.*

**Proof.** By Lemma 2.2(b), the positivity assumption on the functions  $c'_e$  implies that the functions  $n_e(x_e) = \int_0^{x_e} c_e(\tau) d\tau$  are strictly convex with  $n''_e(x_e) = c'_e(x_e)$ . Hence, the Hessian

$$(3.9) \quad \nabla^2 N(x) = \begin{pmatrix} c'_{e_1}(x_{e_1}) & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & c'_{e_{|E|}}(x_{e_{|E|}}) \end{pmatrix} \quad \text{is positive definite .}$$

Theorem 3.2(b) assures uniqueness of the minimizer  $\bar{x}$  of  $P_{WE}^x$ . To prove the second order property, by Theorem 2.4(b), we only have to show that the Lagrangean of  $P_{WE}^x$  (cf., (3.2)),

$$L(x, \lambda, \mu) = N(x) + (Ax - a)^T \lambda + (Bx - b)^T \mu ,$$

satisfies at the minimizer  $\bar{x}$ , with corresponding multipliers  $\bar{\lambda}, 0 \leq \bar{\mu}$ , the relation

$$d^T \nabla_x^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}) d > 0 \quad \text{for all } d \in C_{\bar{x}} \setminus \{0\} .$$

But this inequality trivially holds, since the Hessian  $\nabla_x^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \nabla^2 N(\bar{x})$  is positive definite, i.e., we have

$$d^T \nabla_x^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}) d > 0 \quad \text{for all } d \neq 0 .$$

□

Later in Section 8.5 we will consider such traffic models as a parametric problem depending on perturbations of the demand  $d$  and/or the costs  $c_e$ .



### 3.2. System optimum

The WE traffic flow is characterized by the selfish behavior for the network users: each driver will only use a path with minimal path costs.

A public authority on the other hand would prefer another flow, e.g., a flow that minimizes the so-called social cost

$$(3.10) \quad S(x) = \sum_{e \in E} x_e c_e(x_e) .$$

This function can be seen as a measure for the total travel time of all users.

The so-called system optimum  $x^s \in X_d$  is a minimizer of the program,

$$(3.11) \quad P_s^x \quad \min_x S(x) = \sum_{e \in E} x_e c_e(x_e) \quad \text{s.t.} \quad x \in X_d .$$

**Ex. 3.3.** *Let AT hold and assume that the functions  $c_e(x_e), e \in E$ , are twice continuously differentiable on  $\mathbb{R}_+$  with*

$$c'_e(x_e) > 0, \quad c''_e(x_e) \geq 0 \quad \text{for } x_e \geq 0 .$$

*Then the function  $S(x)$  is strictly convex on  $\mathbb{R}_+^{|E|}$ .*

**Proof.** For  $s_e(x_e) := x_e c_e(x_e)$  we obtain  $s'_e(x_e) = x_e c'_e(x_e) + c_e(x_e)$  and the assumptions imply

$$s''_e(x_e) = x_e c''_e(x_e) + 2c'_e(x_e) > 0 .$$

The claim then follows from Lemma 2.2(b). □

**THEOREM 3.3.** *Let the costs  $c_e(x_e), e \in E$ , satisfy the assumptions of Ex. 3.3. Then the system optimum  $x^s$ , i.e., the minimizer of  $P_s^x$ , exists and is uniquely determined.*

**Proof.** Since  $X_d$  is bounded (compact) the existence of a minimizer of  $P_s^x$  is assured by the Weierstrass Theorem 9.1 and in view of Ex. 3.3 the uniqueness of  $x^s$  follows by Lemma 2.3(a). □

### 3.3. Elastic demand model

Now we examine a traffic model where the demand  $d_w$  is "elastic" in the sense that  $d_w$  depends on the (equilibrated) costs (travel time)  $\gamma_w$  on the paths  $p \in \mathcal{P}_w$ ,  $w \in W$  (cf., Remark 3.2). So we assume that we have given separated demand functions

$$(3.12) \quad d_w(\gamma_w) : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \text{continuous and strictly decreasing} .$$

Then the inverse functions

$$\gamma_w := g_w(d_w) = d_w^{-1}(d_w)$$

exist, are continuous, and strictly decreasing (see Figure 3.3).

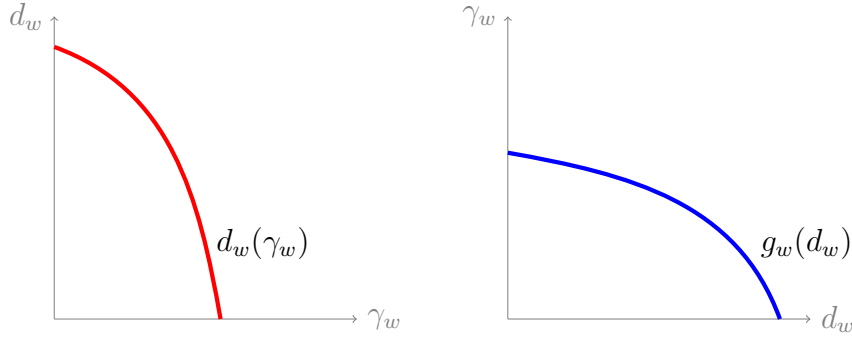


FIGURE 3.3. Sketch of a function  $d_w(\gamma_w)$  (left) and  $\gamma_w(d_w)$  (right)

**Wardrop equilibrium for elastic demand.** A flow  $(x, f)$  and corresponding demand  $d \geq 0$  with  $(x, f) \in F_d$  is called an elastic Wardrop equilibrium (eWE) wrt.  $g_w(d_w)$ ,  $w \in W$ , if for all  $w$  the relations hold: For all  $p \in \mathcal{P}_w$  we have

$$\begin{aligned} f_p > 0 &\Rightarrow c_p(x) = g_w(d_w) \\ f_p = 0 &\Rightarrow c_p(x) \geq g_w(d_w) \end{aligned} .$$

As we shall see, the Beckmann program corresponding to this model is:

$$(3.13) \quad \begin{aligned} P_{eWE} : \min_{x, f, d} M(x, d) &:= \sum_{e \in E} \int_0^{x_e} c_e(\tau) d\tau - \sum_{w \in W} \int_0^{d_w} g_w(\tau) d\tau \quad \text{s.t.} \\ &\Lambda f = d \\ &\Delta f - x = 0 \\ &f \geq 0 \\ &d \geq 0 \end{aligned}$$

with linear constraints and convex objective function  $M$ . The gradient of  $M(x, d)$  is given by

$$\nabla_x M(x, d) = c(x) \quad \text{and} \quad \nabla_d M(x, d) = -g(d) .$$

Note that the integral function  $m_w(d_w) = -\int_0^{d_w} g_w(\tau) d\tau$  is continuously differentiable with  $m'_w(d_w) = -g_w(d_w)$ .

The KKT conditions for a minimizer  $(\bar{x}, \bar{f}, \bar{d})$  of  $P_{eWE}$  read: with Lagrangean multipliers  $-\bar{\gamma}$  wrt.  $\Lambda f = d$ ,  $\bar{\lambda}$  wrt.  $\Delta f - x = 0$ ,  $\bar{\mu} \geq 0$  wrt.  $-f \leq 0$  and  $\bar{\sigma} \geq 0$

wrt.  $-d \leq 0$  we must have,

$$(3.14) \quad \begin{aligned} c(\bar{x}) - \bar{\lambda} &= 0, \\ -\Lambda^T \bar{\gamma} + \Delta^T \bar{\lambda} - \bar{\mu} &= 0, \quad \bar{\mu}^T \bar{f} = 0 \\ -g(\bar{d}) + \bar{\gamma} - \bar{\sigma} &= 0, \quad \bar{\sigma}^T \bar{d} = 0. \end{aligned}$$

These conditions can equivalently be written as,

$$\Lambda^T \bar{\gamma} = \Delta^T c(\bar{x}) - \bar{\mu}, \quad \bar{\gamma} = g(\bar{d}) + \bar{\sigma}, \quad \bar{\mu}^T \bar{f} = 0, \quad \bar{\sigma}^T \bar{d} = 0,$$

or componentwise as (use  $\bar{\sigma}_w = 0$  if  $\bar{d}_w > 0$ ,  $\bar{\mu}_p = 0$  if  $\bar{f}_p > 0$ ): for all  $w \in W$  and  $p \in P_w$ ,

$$\bar{\gamma}_w \begin{cases} = g_w(\bar{d}_w) & \text{if } \bar{d}_w > 0 \\ \geq g_w(\bar{d}_w) & \text{if } \bar{d}_w = 0 \end{cases} \quad \text{and} \quad \bar{\gamma}_w \begin{cases} = c_p(\bar{x}) & \text{if } \bar{f}_p > 0 \\ \leq c_p(\bar{x}) & \text{if } \bar{f}_p = 0 \end{cases}$$

which coincide with the conditions for an elastic WE. Summarizing, as in Theorem 3.1 and Theorem 3.2(b) we obtain

**THEOREM 3.4.** *Let AT and (3.12) hold. Then the following conditions are equivalent for a flow  $(\bar{x}, \bar{f}, \bar{d})$  feasible for  $P_{eWE}$ .*

- (a) *The point  $(\bar{x}, \bar{f}, \bar{d})$  is an elastic Wardrop equilibrium.*
- (b) *The point  $(\bar{x}, \bar{f}, \bar{d})$  is a minimizer of  $P_{eWE}$ .*
- (c) *We have*

$$c(\bar{x})^T (x - \bar{x}) - g(\bar{d})^T (d - \bar{d}) \geq 0 \quad \text{for all } (x, d) \in XD,$$

where  $XD$  is the projection

$$XD = \{(x, d) \mid \text{with some } f, (x, f, d) \text{ is feasible for } P_{eWE}\}.$$

- (d) *Let in addition, the costs  $c_e(x_e)$  be strictly increasing. Then the  $(\bar{x}, \bar{d})$  part of an elastic Wardrop equilibrium  $(\bar{x}, \bar{f}, \bar{d})$  is uniquely determined.*



## CHAPTER 4

### Parametric optimization

In this chapter we start with unconstrained and constrained parametric programs. We analyse the dependence of minimizers on the parameter under assumptions such that the Implicit Function Theorem (IFT) can be applied.

The next section gives a short introduction into parametric equations and states the Implicit Function Theorem which can be seen as the most important basic theorem in parametric optimization.

#### 4.1. Parametric equations and the Implicit Function Theorem

We begin with an illustrative example and consider the non-parametric equation: Solve for  $x \in \mathbb{R}$

$$H(x) := x^2 - 2x - 1 = 0 \quad \text{or equivalently} \quad (x - 1)^2 - 2 = 0$$

with solutions  $x_{1,2} = 1 \pm \sqrt{2}$ .

The parametric version is: For parameter  $t \in \mathbb{R}$  find a solution  $x = x(t)$  of

$$(4.1) \quad H(x, t) := x^2 - 2t^2x - t^4 = 0 \quad \text{or equivalently} \quad (x - t^2)^2 - 2t^4 = 0 .$$

The solutions are (see Figure 4.1)

$$x_{1,2}(t) = t^2 \pm \sqrt{2}t^2 = t^2(1 \pm \sqrt{2})$$

Obviously the solution curves bifurcate at  $(\bar{x}, \bar{t}) = (0, 0)$ . At this point we find for the partial derivative of  $H$

$$\nabla_x H(\bar{x}, \bar{t}) = 2x - 2t^2 \Big|_{\bar{x}, \bar{t}} = 0 .$$

**Rule.** For  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $H \in C^1$ , the solution set of a one-parametric equation  $H(x, t) = 0$  is “normally” locally given by a *one-dimensional*  $C^1$  solution curve  $(x(t), t)$ . However at points  $(\bar{x}, \bar{t})$  where  $\nabla_x H(\bar{x}, \bar{t}) = 0$  holds, the solution set may show a *singularity* (such as a bifurcation or nonsmoothness).

#### The Implicit Function Theorem

More generally we consider systems of  $n$  equations in  $n + p$  variables:

$$H(x, t) = 0 \quad \text{where} \quad H : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^p .$$

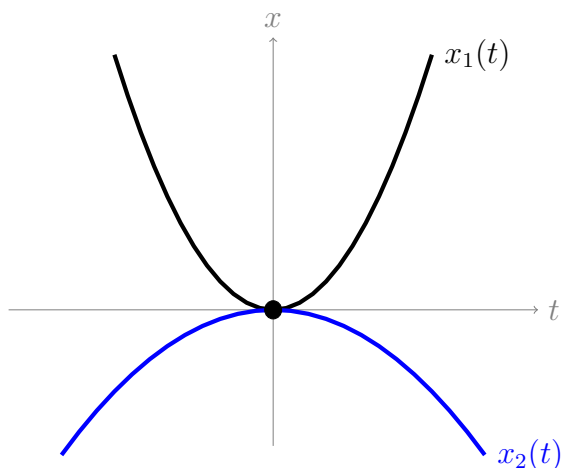


FIGURE 4.1. Sketch of the solution curves  $x_1(t), x_2(t)$

The Implicit Function Theorem (IFT) makes a statement on the structure of the solution set of this equation in the “normal” situation.

**THEOREM 4.1.** [General version of the IFT]

Let  $H : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  be a  $C^1$ -function  $H(x, t)$ . Suppose for  $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^p$  we have  $H(\bar{x}, \bar{t}) = 0$  and the matrix

$$\nabla_x H(\bar{x}, \bar{t}) \quad \text{is nonsingular.}$$

Then there is a neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , of  $\bar{t}$  and a  $C^1$ -function  $x : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^n$  satisfying  $x(\bar{t}) = \bar{x}$  such that near  $(\bar{x}, \bar{t})$  the solution set  $S(H) := \{(x, t) \mid H(x, t) = 0\}$  is described by

$$(4.2) \quad \{(x(t), t) \in \mathbb{R}^n \times \mathbb{R}^p \mid t \in B_\varepsilon(\bar{t})\}, \quad \text{i.e.,} \quad H(x(t), t) = 0 \text{ for } t \in B_\varepsilon(\bar{t}).$$

So, locally near  $(\bar{x}, \bar{t})$ , the set  $S(H)$  is a  $p$  dimensional  $C^1$ -manifold. Moreover, the gradient  $\nabla x(t)$  is given by

$$\nabla x(t) = -[\nabla_x H(x(t), t)]^{-1} \nabla_t H(x(t), t) \quad \text{for } t \in B_\varepsilon(\bar{t}).$$

**Proof.** See e.g., [29]. Note that if  $x(t)$  is a  $C^1$ -function satisfying  $H(x(t), t) = 0$ , then differentiation wrt.  $t$  yields by applying the chain rule,

$$\nabla_x H(x(t), t) \nabla x(t) + \nabla_t H(x(t), t) = 0.$$

□

**Pathfollowing in practice**

We shortly discuss how a solution curve  $(x(t), t)$  of a one-parametric equation

$$H(x, t) = 0 \quad H : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad t \in \mathbb{R},$$

can be followed numerically.

The basic idea is to use some sort of Newton procedure. Recall that the classical Newton method is the most fundamental approach for solving a system of  $n$  equations in  $n$  unknowns:

$$H(x) = 0 \quad H : \mathbb{R}^n \rightarrow \mathbb{R}^n .$$

The famous Newton iteration for computing a solution is to start with some (appropriate) starting point  $x^0$  and to iterate according to

$$x^{k+1} = x^k - [\nabla H(x^k)]^{-1} H(x^k), \quad k = 0, 1, \dots .$$

It is well-known (see, e.g., [9, Th.11.4]) that this iteration converges quadratically to a solution  $\bar{x}$  of  $H(\bar{x}) = 0$  if

- $x^0$  is chosen close enough to  $\bar{x}$  and if
- $\nabla H(\bar{x})$  is a nonsingular matrix.

The simplest way to follow approximately a solution curve  $x(t)$  of  $H(x, t) = 0$ , i.e.,  $H(x(t), t) = 0$ , on an interval  $t \in [a, b]$  is to discretize  $[a, b]$  by

$$t_\ell = a + \ell \frac{b-a}{N}, \quad \ell = 0, \dots, N ,$$

(for some  $N \in \mathbb{N}$ ) and to compute for any  $\ell = 0, \dots, N$ , a solution  $x_\ell = x(t_\ell)$  of  $H(x, t_\ell) = 0$  by a Newton iteration,

$$x_\ell^{k+1} = x_\ell^k - [\nabla_x H(x_\ell^k, t_\ell)]^{-1} H(x_\ell^k, t_\ell), \quad k = 0, 1, \dots ,$$

starting with  $x_\ell^0 = x_{\ell-1} + \frac{(b-a)}{N} x'(t_{\ell-1})$  (for  $\ell \geq 1$ ). The derivative  $x'(t_{\ell-1})$  can be computed with the formula in Theorem 4.1. We refer the reader to the book [1] for details e.g., on:

- How to perform pathfollowing efficiently?
- How to deal with branching points  $(\bar{x}, \bar{t})$  where different solution curves intersect?

## 4.2. Parametric unconstrained minimization problems

Let  $f(x, t)$  be a  $C^2$ -function,  $f : \mathbb{R}^n \times T \rightarrow \mathbb{R}$ , where the parameter set  $T \subset \mathbb{R}^p$  is open. We consider the *parametric problem*: for parameter  $t \in T$  find a local or global minimizer  $x = x(t)$  of

$$(4.3) \quad P(t) : \quad \min_{x \in \mathbb{R}^n} f(x, t)$$

We recall that if for any  $t \in T$  the function  $f(x, t)$  is convex in  $x$  then a (local) minimizer  $x(t)$  is a global minimizer of  $P(t)$  (cf., Lemma 2.3). In this case we can define the (global) value function of  $P(t)$  by

$$v(t) = \inf_{x \in \mathbb{R}^n} f(x, t) .$$

In the general nonlinear case,  $x(t)$  is thought to be a local minimizer of  $P(t)$  and the value function is defined locally by

$$v(t) = f(x(t), t) .$$

To solve this problem  $P(t)$ , for  $t \in T$ , we have to find solutions  $x$  of the *critical point equation*

$$(4.4) \quad H(x, t) := \nabla_x f(x, t) = 0 .$$

The next examples show the possible bad behavior in case the regularity condition, that  $\nabla_x H(\bar{x}, \bar{t}) = \nabla_x^2 f(\bar{x}, \bar{t})$  is nonsingular, does not hold.

Consider e.g. the parametric program

$$P(t) \quad \min_{x \in \mathbb{R}} f(x, t) = tx^2, \quad \text{for } t \in \mathbb{R} ,$$

near  $\bar{t} = 0$ . Here for the critical point equation  $H(x, t) = 2tx = 0$  we find at  $\bar{t} = 0$ ,  $\nabla_x H(x, \bar{t}) = 2\bar{t} = 0$ . So the regularity condition fails at all solutions  $x \in \mathbb{R}$  of  $H(x, \bar{t}) = 0$ . The set  $S(t)$ , of global minimizers is given by:

$$S(t) = \begin{cases} \{0\} & \text{for } t > 0 \\ \mathbb{R} & \text{for } t = 0 \\ \emptyset & \text{for } t < 0 \end{cases} .$$

**EX. 4.1.** For

$$P(t) : \quad \min_x f(x, t) := \frac{1}{3}x^3 - t^2x$$

the minimizers are given by  $x(t) = |t|$  with minimal value  $v(t) := f(x(t), t) = -\frac{2}{3}|t|^3$ .

**EX. 4.2.** Show that for the parametric minimization problems

$$P_1(t) : \quad \min_x f_1(x, t) := \frac{1}{3}tx^3 - tx$$

$$P_2(t) : \quad \min_x f_2(x, t) := \frac{1}{3}t^3x^3 - tx$$

we obtain for the minimizers  $x(t)$  and the value function  $v(t)$ :

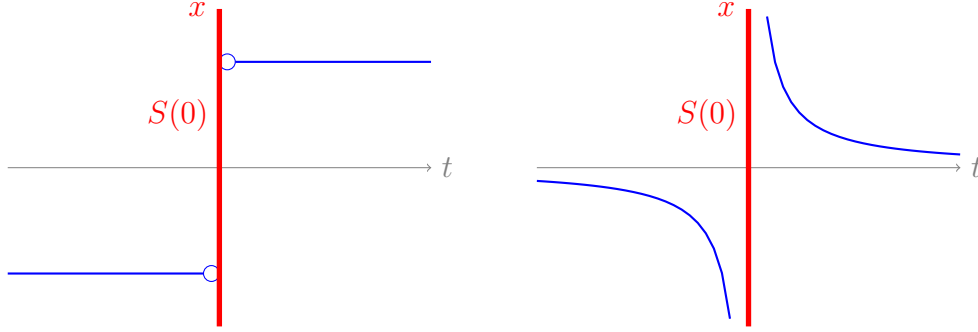
$$\text{for } P_1(t): \quad x(t) = \begin{cases} -1 & t < 0 \\ \mathbb{R} & t = 0 \\ 1 & t > 0 \end{cases}, \quad v(t) = \begin{cases} \frac{2}{3}t^3 & t < 0 \\ -\frac{2}{3}t^3 & t > 0 \end{cases}$$

$$\text{for } P_2(t): \quad x(t) = \begin{cases} -\frac{1}{|t|} & t < 0 \\ \mathbb{R} & t = 0 \\ \frac{1}{|t|} & t > 0 \end{cases}, \quad v(t) = \begin{cases} \frac{2}{3}t & t < 0 \\ -\frac{2}{3}t & t > 0 \end{cases}$$

Note that in both cases the set  $S(0)$  of minimizers of  $P_1(0), P_2(0)$  is  $S(0) = \mathbb{R}$  (see Figure 4.2)

The next theorem describes the solution set of (4.4) near a (non-singular) minimizer  $(\bar{x}, \bar{t})$  of (4.3), where the IFT can be applied.



FIGURE 4.2. Minimizers of  $P_1(t)$  and minimizers of  $P_2(t)$ **THEOREM 4.2.** [local stability result based on IFT]

Let  $f(x, t)$  be a  $C^2$ -function. Suppose,  $\bar{x}$  is a (local) minimizer of  $P(\bar{t})$ ,  $\bar{t} \in T$ , such that

$$\nabla_x f(\bar{x}, \bar{t}) = 0 \quad \text{and} \quad \nabla_x^2 f(\bar{x}, \bar{t}) \succ 0 \quad (\text{positive definite}).$$

(According to Theorem 2.1, Lemma 2.1,  $\bar{x}$  is an isolated strict local minimizer of order 2.)

Then there exists a neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , of  $\bar{t}$  and a  $C^1$ -function  $x : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^n$  such that  $x(\bar{t}) = \bar{x}$  and for any  $t \in B_\varepsilon(\bar{t})$ ,  $x(t)$  is an (isolated) strict local minimizer of  $P(t)$ . Moreover for  $t \in B_\varepsilon(\bar{t})$ ,

$$\nabla x(t) = -[\nabla_x^2 f(x(t), t)]^{-1} \nabla_{xt}^2 f(x(t), t),$$

and the value function  $v(t) := f(x(t), t)$  is a  $C^2$ -function with

$$\nabla v(t) = \nabla_t f(x(t), t) \quad \text{and} \quad \nabla^2 v(t) = \nabla_{tx}^2 f(x(t), t) \nabla x(t) + \nabla_t^2 f(x(t), t).$$

**Proof.** Apply the IFT to the critical point equation  $\nabla_x f(x, t) = 0$ .  $\square$

**Remark.** Note that for  $f \in C^2$  the solution function  $x(t)$  is  $C^1$  but the value function  $v(t)$  is  $C^2$ .

**Ex. 4.3.** For

$$P(t) : \quad \min f(x, t) := \frac{1}{3}x^3 - t^2x^2 - t^4x$$

the critical points are given by the curves  $x_{1,2}(t) = t^2(1 \pm \sqrt{2})$  and the minimizer by  $x_1(t)$  (maximizer by  $x_2(t)$ ) (cf., Figure 4.1). At all points of the solution curves  $x_1(t), x_2(t)$  the IFT can be applied except at the branching point at  $\bar{t} = 0$ .

**Rule.** The following appears:

- The value function  $v(t) = f(x(t), t)$  behaves “smoother” than the minimizer function  $x(t)$ .

- A singular behavior may appear at solution points  $(\bar{x}, \bar{t})$  of  $\nabla_x f(x, t) = 0$  where the matrix  $\nabla_x^2 f(\bar{x}, \bar{t})$  is singular.

**Ex. 4.4.** Check the singular behavior for the examples Ex. 4.1 and Ex. 4.2.

### 4.3. Parametric linear programs

In a parametric linear program (LP) we have given a  $C^2$ -matrix function  $A(t) : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times n}$  with  $m$  rows  $a_j^T(t), j \in J := \{1, \dots, m\}$ ,  $C^2$ -vector functions  $b(t) : \mathbb{R}^p \rightarrow \mathbb{R}^m, c(t) : \mathbb{R}^p \rightarrow \mathbb{R}^n$  and an open parameter set  $T \subset \mathbb{R}^p$ : For any  $t \in T$  we wish to solve the primal program

$$(4.5) \quad P(t) : \max c(t)^T x \quad \text{s.t.} \quad x \in F_P(t) = \{x \mid A(t)x \leq b(t)\} .$$

The corresponding dual reads

$$(4.6) \quad D(t) : \min b(t)^T y \quad \text{s.t.} \quad y \in F_D(t) = \{y \mid A(t)^T y = c(t), y \geq 0\} .$$

For  $\bar{t} \in T$  and  $\bar{x} \in F_P(\bar{t})$  the active index set is  $J_0(\bar{x}, \bar{t}) = \{j \in J \mid a_j(\bar{t})^T \bar{x} = b_j(\bar{t})\}$ .

Suppose for  $\bar{t} \in T$  the point  $\bar{x} \in F_P(\bar{t})$  is a vertex solution of  $P(\bar{t})$ . To find for  $t$  near  $\bar{t}$  solutions  $x(t)$  of  $P(t)$  we have to find feasible solutions  $x$  and  $y$  of the system of optimality conditions,

$$(4.7) \quad \text{for } P(t) : \quad A_{J_0(\bar{x}, \bar{t})}(t)x = b_{J_0(\bar{x}, \bar{t})}(t)$$

$$(4.8) \quad \text{for } D(t) : \quad A_{J_0(\bar{x}, \bar{t})}(t)^T y_{J_0(\bar{x}, \bar{t})} = c(t), \quad y_{J_0(\bar{x}, \bar{t})} \geq 0$$

If  $\bar{x}$  is a nondegenerate vertex of  $P(\bar{t})$ , i.e., if  $A_{J_0(\bar{x}, \bar{t})}(\bar{t})$  is nonsingular, and the minimizer  $\bar{y}$  of  $D(\bar{t})$  satisfies the strict complementarity (SC) condition  $\bar{y}_j > 0, j \in J_0(\bar{x}, \bar{t})$ , this is possible by applying the IFT to these systems.

**THEOREM 4.3.** [Local stability result]

Let  $\bar{x} \in F_P(\bar{t})$  be a vertex maximizer of  $P(\bar{t})$  with corresponding dual solution  $\bar{y}_{J_0(\bar{x}, \bar{t})}$  such that

- (1)  $\bar{x}$  is a nondegenerate vertex, i.e., LICQ holds,
- (2)  $\bar{y}_j > 0, \forall j \in J_0(\bar{x}, \bar{t})$ . (SC)

(According to Theorem 2.4(a),  $\bar{x}$  is a maximizer of  $P(\bar{t})$  of order  $s = 1$ .) Then there exist a neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , and  $C^1$ -functions  $x : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^n, y_{J_0(\bar{x}, \bar{t})} : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^{|J_0(\bar{x}, \bar{t})|}$  such that  $x(\bar{t}) = \bar{x}, y_{J_0(\bar{x}, \bar{t})}(\bar{t}) = \bar{y}_{J_0(\bar{x}, \bar{t})}$  and for any  $t \in B_\varepsilon(\bar{t})$  the point  $x(t)$  is a vertex maximizer of  $P(t)$  (of order  $s=1$ ) with corresponding multiplier  $y_{J_0(\bar{x}, \bar{t})}(t)$ . Moreover, for  $t \in B_\varepsilon(\bar{t})$  the derivatives of  $x(t)$  and the value function  $v(t) = c(t)^T x(t)$  are given by

$$\nabla x(t) = [A_{J_0(\bar{x}, \bar{t})}(t)]^{-1} (\nabla b_{J_0(\bar{x}, \bar{t})}(t) - \nabla A_{J_0(\bar{x}, \bar{t})}(t)x(t))$$

and

$$\nabla v(t) = \nabla c(t)^T x(t) + [y_{J_0(\bar{x}, \bar{t})}(t)]^T [\nabla b_{J_0(\bar{x}, \bar{t})}(t) - \nabla A_{J_0(\bar{x}, \bar{t})}(t)x(t)] .$$

**Proof.** Since the vertex  $\bar{x}$  is nondegenerate, the matrix  $A_{J_0(\bar{x}, \bar{t})}(\bar{t})$  is nonsingular (see Definition 2.3). So the IFT can be applied to the systems (4.7), (4.8), yielding in a neighborhood  $B_\varepsilon(\bar{t})$  vertex solutions  $x(t)$  of  $P(t)$  and  $y_{J_0(\bar{x}, \bar{t})}(t) > 0$  (since  $\bar{y}_{J_0(\bar{x}, \bar{t})} = y_{J_0(\bar{x}, \bar{t})}(\bar{t}) > 0$ ) of  $D(t)$ . Differentiating the relation

$$A_{J_0(\bar{x}, \bar{t})}(t)x(t) = b_{J_0(\bar{x}, \bar{t})}(t)$$

wrt.  $t$  directly leads to the formulas for  $\nabla x(t)$  and  $\nabla v(t)$ .  $\square$

### **Linear production model and shadow prices**

We discuss a production model to present a simple application of the result in Theorem 4.3. Assume a factory produces  $n$  different products  $P_1, \dots, P_n$ . The production relies on material coming from  $m$  different resources  $R_1, \dots, R_m$  in such a way that the production of 1 unit of a product  $P_i$  requires  $a_{ji}$  units of resource  $R_j$ .

Suppose we can sell our production for the price of  $c_i$  per 1 unit of  $P_i$  and that  $b_j$  units of each resource  $R_j$  are available for the total production. How many units  $x_i$  of each product  $P_i$  should we produce in order to maximize the total receipt from the sales?

An optimal production plan  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T$  corresponds to an optimal solution of the linear program

$$(4.9) \quad P : \quad \max c^T x \quad \text{s.t.} \quad Ax \leq b, \quad x \geq 0 .$$

Here  $A = (a_{ji})$  is the matrix with the elements  $a_{ji}$ . Let  $\bar{x}$  be a maximizer with corresponding minimizer  $\bar{y}$  of the dual problem

$$(4.10) \quad D : \quad \min b^T y \quad \text{s.t.} \quad A^T y \geq c, \quad y \geq 0 .$$

and maximum profit  $\bar{z} = c^T \bar{x} = b^T \bar{y}$ .

Could we possibly increase the profit by spending money on increasing the resource capacity  $b$  and adjusting the production plan? If so, how much would we be willing to pay for 1 more unit of resource  $R_j$ ?

Let us increase for fixed  $j_0 \in J_0(\bar{x})$  only the capacity of  $R_{j_0}$  as follows. For fixed  $b$  we consider the vector  $b(t)$  given componentwise by

$$(4.11) \quad b_{j_0}(t) = b_{j_0} + t, \quad b_j(t) = b_j, \quad j \neq j_0 ,$$

depending on the parameter  $t \in \mathbb{R}$  near  $\bar{t} = 0$ . So we have given the parametric linear programs

$$P(t) : \quad \max c^T x \quad \text{s.t.} \quad \begin{array}{l} Ax \leq b(t) \\ x \geq 0 \end{array} , \quad D(t) : \quad \min b(t)^T y \quad \text{s.t.} \quad \begin{array}{l} A^T y \geq c \\ y \geq 0 \end{array}$$

for  $t$  near  $\bar{t} = 0$ . We assume that the optimal solutions  $\bar{x}$  of  $P(\bar{t})$  and  $\bar{y}$  of  $D(\bar{t})$  satisfy the assumptions in Theorem 4.3. Then, since  $c, A$  do not depend on  $t$  and

(see (4.11))

$$\frac{d}{dt}b_{j_0}(\bar{t}) = 1 \quad \text{and} \quad \frac{d}{dt}b_j(\bar{t}) = 0, \quad j \neq j_0,$$

for the derivative of the value function  $v(t)$  in Theorem 4.3 we find

$$\frac{d}{dt}v(\bar{t}) = \sum_{j \in J_0(\bar{x})} \bar{y}_j \frac{d}{dt}b_j(\bar{t}) = \bar{y}_{j_0}.$$

This means that near  $\bar{t} = 0$  the value function  $v(t)$  has a first order approximation

$$v(t) = v(\bar{t}) + \frac{d}{dt}v(\bar{t})t + o(t) = b^T \bar{y} + \bar{y}_{j_0}t + o(t)$$

and thus (note that by assumption  $\bar{y}_{j_0} > 0$ ) we would not want to pay more than  $t \cdot \bar{y}_{j_0}$  for  $t$  more units of  $R_{j_0}$ . In this sense, the coefficients  $\bar{y}_j$  of the dual optimal solution  $\bar{y}$  can be interpreted as the *shadow prices* of the resources  $R_j$ .

**REMARK.** The notion of shadow prices furnishes also an intuitive interpretation of complementary slackness. If the slack  $\bar{s}_i = b_i - \sum_{j=1}^n a_{ij}\bar{x}_j$  is strictly positive at the optimal production  $\bar{x}$  (and thus  $\bar{y}_i = 0$ ), we do not use resource  $R_i$  to its full capacity. Therefore, we would expect no gain from an increase of  $R_i$ 's capacity.

#### 4.4. Parametric nonlinear constrained programs

Let  $T \subset \mathbb{R}^p$  be some open parameter set. We consider *nonlinear parametric programs* of the form (omitting equality constraints): For  $t \in T$  find local minimizers  $x = x(t)$  of

$$(4.12) \quad P(t) : \min_x f(x, t) \quad \text{s.t.} \quad x \in F(t) = \{x \in \mathbb{R}^n \mid g_j(x, t) \leq 0, \quad j \in J\}.$$

We assume throughout this section,  $f, g_j \in C^2(\mathbb{R}^n \times T, \mathbb{R})$ .

We emphasise again, that we have to distinguish between convex and nonconvex programs. If for any  $t \in T$  the functions  $f(x, t)$ ,  $g_j(x, t)$ ,  $j \in J$ , are convex in  $x$  then for any  $t$  the program  $P(t)$  is convex (cf., Definition 2.5) and any (local) minimizer  $x(t)$  is a global one. In this case we can define the (global) value function of  $P(t)$  by

$$v(t) = \inf_{x \in F(t)} f(x, t),$$

( $v(t) = \infty$ , if  $F(t) = \emptyset$ ) and the set  $S(t)$  of global minimizers:

$$S(t) = \{x \in F(t) \mid f(x, t) = v(t)\}.$$

In the general nonlinear case, we consider local minimizer  $x(t)$  of  $P(t)$  and the corresponding local minimum value function,

$$v(t) = f(x(t), t).$$

We extend the notation of Section 2.4 to the parametric context. For  $\bar{t} \in T$  and feasible  $\bar{x} \in F(\bar{t})$  we denote by  $J_0(\bar{x}, \bar{t})$  the active index set,

$$J_0(\bar{x}, \bar{t}) = \{j \in J \mid g_j(\bar{x}, \bar{t}) = 0\} ,$$

and by  $L(x, t, \mu)$  the Lagrangean function (near  $(\bar{x}, \bar{t})$ ),

$$L(x, t, \mu) = f(x, t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j g_j(x, t) .$$

Strict Complementarity (SC), second order conditions (SOC) and the cone of critical directions  $C_{\bar{x}, \bar{t}}$  are defined accordingly (cf., Theorem 2.4). To find, near  $(\bar{x}, \bar{t})$ , local minimizers  $x$  of  $P(t)$  we are looking for solutions  $(x, t, \mu) = (x(t), t, \mu(t))$  of the KKT equations with  $\mu_j \geq 0, j \in J_0(\bar{x}, \bar{t})$ ,

$$(4.13) \quad H(x, t, \mu) := \begin{aligned} & \nabla_x f(x, t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_x g_j(x, t) = 0, \\ & g_j(x, t) = 0, \quad j \in J_0(\bar{x}, \bar{t}) . \end{aligned}$$

From the sufficient optimality conditions in Theorem 2.4 we obtain the following basic stability result by applying IFT to the KKT-system (4.13).

**THEOREM 4.4.** *[Local stability result based on IFT]*

Let  $\bar{x} \in F(\bar{t})$ . Suppose that with multipliers  $\bar{\mu}_j$  the KKT condition  $\nabla_x L(\bar{x}, \bar{t}, \bar{\mu}) = 0$  is satisfied such that

- (1) LICQ holds at  $(\bar{x}, \bar{t})$ .
- (2)  $\bar{\mu}_j > 0, \forall j \in J_0(\bar{x}, \bar{t})$  (SC)

and either

$$(3a) \text{ (order one condition) } |J_0(\bar{x}, \bar{t})| = n$$

or

$$(3b) \text{ (order two condition)}$$

$$d^T \nabla_x^2 L(\bar{x}, \bar{t}, \bar{\mu}) d > 0 \quad \forall d \in T_{\bar{x}, \bar{t}} \setminus \{0\}$$

where  $T_{\bar{x}, \bar{t}}$  is the tangent space  $T_{\bar{x}, \bar{t}} = \{d \mid \nabla_x g_j(\bar{x}, \bar{t}) d = 0, j \in J_0(\bar{x}, \bar{t})\}$ .

(According to Theorem 2.4,  $\bar{x}$  is a local minimizer of  $P(\bar{t})$  of order  $s = 1$  in case (3a) and of order  $s = 2$  in case (3b).)

Then there exist a neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , of  $\bar{t}$  and  $C^1$ -functions  $x : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^n$ ,  $\mu : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^{|J_0(\bar{x}, \bar{t})|}$  such that  $x(\bar{t}) = \bar{x}$ ,  $\mu(\bar{t}) = \bar{\mu}$  and for any  $t \in B_\varepsilon(\bar{t})$  the point  $x(t)$  is a strict local minimizer of  $P(t)$  (of order 1 in case (3a) and of order 2 in case (3b)) with corresponding multiplier vector  $\mu(t)$ . Moreover, for  $t \in B_\varepsilon(\bar{t})$  the derivative of the value function  $v(t) = f(x(t), t)$  is

$$\nabla v(t) = \nabla_t L(x(t), t, \mu(t)) .$$

The derivative of the solution function  $x(t)$  is given in case (3a) by

$$\nabla x(t) = - \begin{pmatrix} \nabla_x g_j(x(t), t), j \in J_0(\bar{x}, \bar{t}) \\ \vdots \end{pmatrix}^{-1} \begin{pmatrix} \nabla_t g_j(x(t), t), j \in J_0(\bar{x}, \bar{t}) \\ \vdots \end{pmatrix}$$

and in case (3b) by (cf., (4.13))

$$\begin{pmatrix} \nabla x(t) \\ \nabla \mu(t) \end{pmatrix} = -[\nabla_{(x,\mu)} H(x(t), t, \mu(t))]^{-1} \nabla_t H(x(t), t, \mu(t)) .$$

**Proof.** In case (3a), by LICQ and  $|J_0(\bar{x}, \bar{t})| = n$  the system (4.13) splits into the system of  $n$  equations in  $n$  variables,

$$g_j(x, t) = 0, \quad j \in J_0(\bar{x}, \bar{t}),$$

for  $x = x(t)$  with  $\nabla x(t)$  given as solution of (derivatives wrt.  $t$ )

$$(4.14) \quad \nabla_x g_j(x(t), t) \nabla x(t) + \nabla_t g_j(x(t), t) = 0, \quad j \in J_0(\bar{x}, \bar{t}),$$

and the equations

$$(4.15) \quad \nabla_x f(x(t), t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_x g_j(x(t), t) = 0,$$

for the components  $\mu_j = \mu_j(t)$ ,  $j \in J_0(\bar{x}, \bar{t})$  ( $\mu_j(t) = 0$ ,  $j \in J \setminus J_0(\bar{x}, \bar{t})$ ). For the value function  $v(t) = f(x(t), t)$ , by differentiation wrt.  $t$ , we find using (4.14) and (4.15)

$$\begin{aligned} \nabla v(t) &= \nabla_x f(x(t), t) \nabla x(t) + \nabla_t f(x(t), t) \\ &= - \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j(t) \nabla_x g_j(x(t), t) \nabla x(t) + \nabla_t f(x(t), t) \\ &= \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j(t) \nabla_t g_j(x(t), t) + \nabla_t f(x(t), t) = \nabla_t L(x(t), t, \mu(t)). \end{aligned}$$

In case (3b) we have to apply the IFT to the coupled system (4.13) and to make use of Ex. 4.5. to show that the matrix  $\nabla_{(x,\mu)} H(\bar{x}, \bar{t}, \bar{\mu})$  is nonsingular, where

$$\nabla_{(x,\mu)} H(\bar{x}, \bar{t}, \bar{\mu}) = \begin{pmatrix} \nabla_x^2 L(\bar{x}, \bar{t}, \bar{\mu}) & B \\ B^T & 0 \end{pmatrix} \text{ with } B = (\nabla_x g_j(\bar{x}, \bar{t})^T, j \in J_0(\bar{x}, \bar{t})).$$

□

**REMARK 4.1.** Note that the statement of Theorem 4.3 for linear programs represents a special case of Theorem 4.4, case (3a).

**EX. 4.5.** Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix and  $B \in \mathbb{R}^{n \times m}$  ( $n \geq m$ ). Suppose the matrix  $B$  has full rank  $m$  and the following holds:

$$d^T A d \neq 0 \quad \forall d \in \mathbb{R}^n \setminus \{0\} \text{ such that } B^T d = 0 .$$

Show that then the following matrix is nonsingular:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

**Ex. 4.6.** Let the KKT conditions (4.13) be satisfied at  $(\bar{x}, \bar{t})$ . Then (see Ex. 2.8)

$$SC \quad \Rightarrow \quad C_{\bar{x}, \bar{t}} = T_{\bar{x}, \bar{t}}.$$

We add an exercise on a parametric program as in Theorem 4.4 but with only equality constraints which will be used in Chapter 7.

**Ex. 4.7.** Consider

$$P_=(t) : \min_{x \in \mathbb{R}^n} f(x, t) \quad \text{s.t.} \quad x \in F_=(t) = \{x \mid g_j(x, t) = 0, j = 1, \dots, m\},$$

where  $f, g_j \in C^2(\mathbb{R}^n \times T, \mathbb{R})$ . We assume that LICQ holds at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F_=(\bar{t})$ , i.e.,  $\nabla_x g_j(\bar{x}, \bar{t})$ ,  $j = 1, \dots, m$ , are linearly independent, as well as the KKT condition

$$\nabla_x L(\bar{x}, \bar{t}, \bar{\lambda}) := \nabla f(\bar{x}, \bar{t}) + \sum_{j=1}^m \bar{\lambda}_j \nabla_x g_j(\bar{x}, \bar{t}) = 0$$

with multipliers  $\bar{\lambda}_j \in \mathbb{R}$  (not necessarily  $\geq 0$ ). Suppose the second order condition holds:

$$d^T \nabla_x^2 L(\bar{x}, \bar{t}, \bar{\lambda}) d > 0 \quad \forall d \in T_{\bar{x}, \bar{t}}^- \setminus \{0\},$$

where  $T_{\bar{x}, \bar{t}}^- = \{d \mid \nabla_x g_j(\bar{x}, \bar{t}) d = 0, j = 1, \dots, m\}$ . Then,  $\bar{x}$  is a local minimizer of  $P_=(\bar{t})$  of order 2 and there exist a neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , differentiable functions  $x(t) : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^n$ ,  $\lambda(t) : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^m$ ,  $x(\bar{t}) = \bar{x}$ ,  $\lambda(\bar{t}) = \bar{\lambda}$  such that for  $t \in B_\varepsilon(\bar{t})$  the point  $x(t)$  is a local minimizer of  $P_=(t)$  and with corresponding multiplier  $\lambda(t)$  the KKT relation holds with second order condition.

**Proof.** For the proof that  $\bar{x}$  is local minimizer of  $P_=(\bar{t})$  of order 2 we refer to [9, Th.10.6]. To prove the further statements we have to apply the IFT to the system of KKT equations:

$$\begin{aligned} \nabla f(x, t) + \sum_{j=1}^m \lambda_j \nabla_x g_j(x, t) &= 0 \\ g_j(x, t) &= 0, \quad j = 1, \dots, m. \end{aligned}$$

□

**Remark.** Many further (often difficult) results are dealing with generalizations of these stability results in Theorem 4.4, Ex. 4.7 under weaker assumptions, i.e., LICQ and/or SC does not hold; see e.g., [2], [6] and Chapters 6,7.

The next example presents a monotonicity result needed later on.

**Ex. 4.8.** With  $g(x, t) = (g_j(x, t), j \in J)$ ,  $(x, t) \in \mathbb{R}^n \times T$ , consider the feasible set  $F(t)$  of  $P(t)$ ,

$$F(t) = \{x \mid g(x, t) \leq 0\} .$$

Let be given  $t_1, t_2 \in T$  and corresponding feasible points  $x_1 \in F(t_1)$ ,  $x_2 \in F(t_2)$  satisfying with multiplier vectors  $\mu_1, \mu_2 \geq 0$  the complementarity conditions

$$\mu_1^T g(x_1, t_1) = 0 , \quad \mu_2^T g(x_2, t_2) = 0 .$$

Then the monotonicity relation

$$(\mu_2 - \mu_1)^T (g(x_2, t_2) - g(x_1, t_1)) \geq 0$$

holds. In particular this relation is true if  $g = g(x)$  does not depend on  $t$ .

**Proof.** In view of the complementarity conditions and  $\mu_1^T g(x_2, t_2) \leq 0$ ,  $\mu_2^T g(x_1, t_1) \leq 0$  we find

$$\begin{aligned} (\mu_2 - \mu_1)^T (g(x_2, t_2) - g(x_1, t_1)) &= \mu_2^T g(x_2, t_2) + \mu_1^T g(x_1, t_1) \\ &\quad - \mu_2^T g(x_1, t_1) - \mu_1^T g(x_2, t_2) \\ &\geq 0 . \end{aligned}$$

□

A similar monotonicity property holds for the following right-hand side perturbed convex program,

$$(4.16) \quad P(t) : \quad \min f(x) \quad \text{s.t.} \quad \begin{array}{l} Ax = t \\ x \geq 0 \end{array}$$

where  $f$  is a  $C^1$  convex function on  $\mathbb{R}_+^n$  and  $A \in \mathbb{R}^{m \times n}$ ,  $t \in \mathbb{R}^m$ . A minimizer  $\bar{x} = x(\bar{t})$  of  $P(\bar{t})$  satisfies the KKT conditions with multipliers  $\bar{\lambda} \in \mathbb{R}^m$ ,  $0 \leq \bar{\mu} \in \mathbb{R}^n$  (cf., Remark 2.4):

$$(4.17) \quad \begin{array}{l} \nabla f(\bar{x})^T + A^T \bar{\lambda} - \bar{\mu} = 0 \\ \bar{\mu}^T \bar{x} = 0 \\ A \bar{x} = \bar{t} \\ \bar{x} \geq 0 \end{array}$$

**LEMMA 4.1.** Let  $f \in C^1$  be convex on  $\mathbb{R}_+^n$ . Let further  $\tilde{x}, \bar{x}$  be (global) minimizers of  $P(\tilde{t}), P(\bar{t})$  with corresponding multipliers  $\tilde{\lambda}, \tilde{\mu}$  and  $\bar{\lambda}, \bar{\mu}$  in (4.17). Then the monotonicity relation holds:

$$(\tilde{\lambda} - \bar{\lambda})^T (\tilde{t} - \bar{t}) \leq -\tilde{\mu}^T \bar{x} - \bar{\mu}^T \tilde{x} \leq 0 .$$

**Proof.** By convexity of  $f$  (see Ex. 2.2(a)) we have

$$(\nabla f(\tilde{x}) - \nabla f(\bar{x}))(\tilde{x} - \bar{x}) \geq 0 .$$



Using the KKT relations for  $\tilde{x}, \bar{x}$  leads to,

$$(\tilde{\mu} - A^T \tilde{\lambda} - \bar{\mu} + A^T \bar{\lambda})^T (\tilde{x} - \bar{x}) \geq 0 ,$$

or in view of  $\tilde{\mu}^T \tilde{x} = \bar{\mu}^T \bar{x} = 0$ ,

$$\begin{aligned} -(\tilde{\lambda} - \bar{\lambda})^T A(\tilde{x} - \bar{x}) &= -(\tilde{\lambda} - \bar{\lambda})^T (\tilde{t} - \bar{t}) \\ &\geq -(\tilde{\mu} - \bar{\mu})^T (\tilde{x} - \bar{x}) = \tilde{\mu}^T \bar{x} + \bar{\mu}^T \tilde{x} \geq 0 . \end{aligned}$$

Multiplying by  $-1$  proves the statement.  $\square$



## CHAPTER 5

# General continuity properties of parametric programs

### 5.1. Introduction

In the present section we analyze the parametric behavior of a parametric program under weaker assumptions, where the Implicit Function Theorem is not applicable. We try to keep the introduction of new concepts to a minimum and motivate the results by examples.

Consider again the parametric optimization problem

$$(5.1) \quad P(t) : \min_x f(x, t) \text{ s.t. } x \in F(t) = \{x \in \mathbb{R}^n \mid g_j(x, t) \leq 0, j \in J\},$$

depending on the parameter  $t \in T$ , where  $T \subset \mathbb{R}^p$  is an open parameter set and again  $J = \{1, \dots, m\}$ . All functions  $f, g_j$  are assumed to be (at least) continuous everywhere.

Recall that  $v(t) = \min_{x \in F(t)} f(x, t)$  denotes the (global) minimal value of  $P(t)$  ( $v(t) = \infty$  if  $F(t) = \emptyset$ ) and  $S(t)$  is the set of global minimizers,  $S(t) = \{x \in F(t) \mid f(x, t) = v(t)\}$ .

The mappings  $F : T \rightrightarrows \mathbb{R}^n$  and  $S : T \rightrightarrows \mathbb{R}^n$  are so-called setvalued mappings.  $\text{dom } F$  denotes the domain of the setvalued mapping  $F$ ,  $\text{dom } F = \{t \in T \mid F(t) \neq \emptyset\}$ .

In this section we mostly consider (semi) continuity properties of  $F(t), v(t), S(t)$ . Lipschitz continuity and differentiability properties are discussed later on in Chapters 6, 7.

**Main questions of parametric optimization:** How do the value function  $v(t)$  and the mappings  $F(t), S(t)$  change with  $t$ . Continuously, smoothly?

**DEFINITION 5.1.** Let  $v : T \rightarrow \mathbb{R}_\infty$  be given, where  $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty, \infty\}$ .

(a) The function  $v$  is called *upper semicontinuous* (usc) at  $\bar{t} \in T$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$v(t) \leq v(\bar{t}) + \varepsilon \quad \text{for all } \|t - \bar{t}\| < \delta.$$

(b) The function  $v$  is called *lower semicontinuous* (lsc) at  $\bar{t} \in T$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$v(t) \geq v(\bar{t}) - \varepsilon \quad \text{for all } \|t - \bar{t}\| < \delta .$$

We shall see that the lower- and upper semicontinuity of the value function  $v(t)$  depend on different assumptions. Obviously, to assure the upper- (lower) semicontinuity of  $v$  at  $\bar{t}$  the feasible set  $F(t)$  should not become essentially smaller (larger) by a small perturbation  $t$  of  $\bar{t}$ . To avoid an "explosion" of  $F(t)$  we will need some compactness assumptions for  $F(t)$  and to prevent an "implosion" a Constraint Qualification will be needed.

**DEFINITION 5.2.** Let the setvalued mapping  $F : T \rightrightarrows \mathbb{R}^n$  be given.

(a)  $F$  is called *closed* at  $\bar{t} \in T$  if for any sequences  $t_l, x_l, l \in \mathbb{N}$ , with  $t_l \rightarrow \bar{t}$ ,  $x_l \in F(t_l)$ , the condition  $x_l \rightarrow \bar{x}$  implies  $\bar{x} \in F(\bar{t})$ .

(b) [no explosion of  $F(t)$  after perturbation of  $t = \bar{t}$ ]

$F$  is called *outer semicontinuous* (osc) at  $\bar{t} \in T$  if for any sequences  $t_l, x_l, l \in \mathbb{N}$ , with  $t_l \rightarrow \bar{t}$ ,  $x_l \in F(t_l)$ , there exists  $\bar{x}_l \in F(\bar{t})$  (for large  $l$ ) such that  $\|x_l - \bar{x}_l\| \rightarrow 0$  for  $l \rightarrow \infty$ .

(c) [no implosion of  $F(t)$  after perturbation of  $t = \bar{t}$ ]

$F$  is *inner semicontinuous* (isc) at  $\bar{t} \in T$  if for any  $\bar{x} \in F(\bar{t})$  and sequence  $t_l \rightarrow \bar{t}$ , there exists a sequence  $x_l$  such that  $x_l \in F(t_l)$  (for large  $l$ ), and  $\|x_l - \bar{x}\| \rightarrow 0$  for  $l \rightarrow \infty$ .

The mapping  $F$  is called *continuous* at  $\bar{t}$  if it is both osc and isc at  $\bar{t}$ .

**Remark.** There are other concepts of semicontinuity for mappings  $F : T \rightrightarrows \mathbb{R}^n$  (see e.g., [2]):  $F$  is called *upper semicontinuous* (usc) at  $\bar{t} \in T$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$F(t) \subset B_\varepsilon(F(\bar{t})) \quad \text{for all } t \in B_\delta(\bar{t}) .$$

$F$  is called *lower semicontinuous* (lsc) at  $\bar{t} \in T$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$F(\bar{t}) \subset B_\varepsilon(F(t)) \quad \text{for all } t \in B_\delta(\bar{t}) .$$

One can show (as Ex.)

$$\begin{aligned} F \text{ is usc at } \bar{t} \in T & \text{ if and only if } F \text{ is osc at } \bar{t} \in T \\ F \text{ is lsc at } \bar{t} \in T & \text{ if and only if } F \text{ is isc. at } \bar{t} \in T \end{aligned}$$

We give an example to show that even nicely behaving setvalued mappings  $F(t)$  need not be osc if  $F(t)$  does not satisfy some compactness condition.

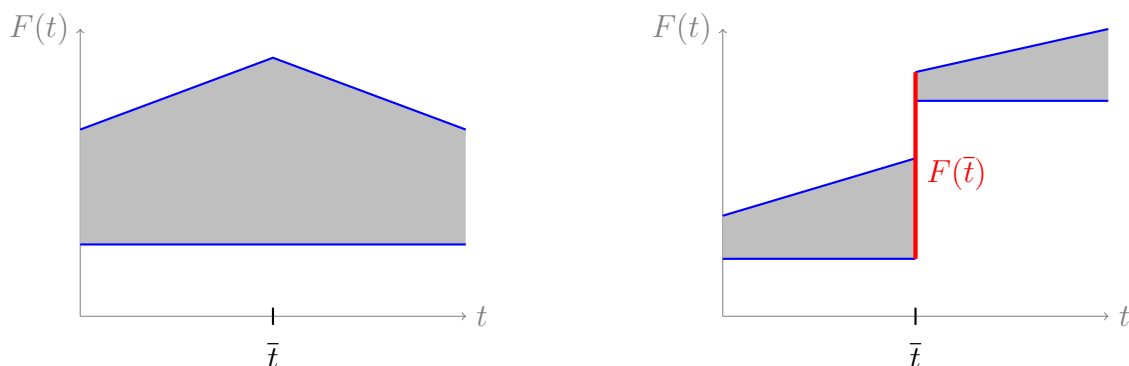


FIGURE 5.1.  $F$  is closed, osc, isc       $F$  is closed, osc, not isc

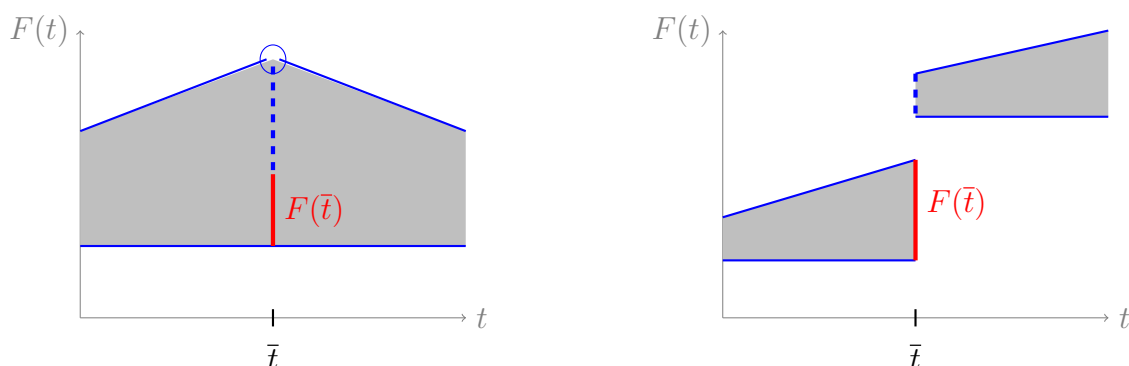


FIGURE 5.2.  $F$  not closed, not osc, isc       $F$  not closed, not osc, not isc

**Ex. 5.1.** Let  $F(t) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq tx_1\}$ ,  $t \in \mathbb{R}$ . Show that  $F$  is not osc at  $\bar{t} = 0$  (at any  $t \in \mathbb{R}$ ) but it is isc.

*Solution:* Take  $t_\ell \searrow 0$  and  $x_\ell = (\frac{1}{t_\ell^2}, \frac{1}{t_\ell}) \in F(t_\ell)$ . The nearest point in  $F(0)$  is  $\bar{x}_\ell = (\frac{1}{t_\ell^2}, 0)$  with  $\|\bar{x}_\ell - x_\ell\| = \frac{1}{t_\ell} \rightarrow \infty$ .

## 5.2. Closedness and outer semicontinuity of $F$

By continuity of the functions  $g_j$  the feasible set mapping  $F(t)$  of  $P(t)$  is always closed.

**Ex. 5.2.** Show for  $F(t) = \{x \in \mathbb{R}^n \mid g_j(x, t) \leq 0, j \in J\}$ ,  $t \in T$ , ( $g_j$  continuous) that the mapping  $F : T \rightrightarrows \mathbb{R}^n$  is closed on  $T$ .

**Ex. 5.3.**  $[F(\bar{t}), S(\bar{t})$  compact, LC (see below) does not hold and  $v$  is not lsc and  $F(t)$  is not osc at  $\bar{t}$ .] Consider with  $\bar{t} = 0$  the problem

$$P(t) : \min x_2 - x_1 \quad \text{s.t.} \quad x_2 \leq tx_1 - \frac{1}{2}, \quad x_2 \leq -tx_1, \quad x_2 \geq x_2(x_1),$$

where  $x_2(x_1), x_1 \in \mathbb{R}$ , is piecewise linear with  $x_2(0) = -1$ ,  $x_2(t) = -\frac{1}{4}$  for  $|x_1| \geq 1$ . Show that  $F(t)$  is not osc at  $\bar{t}$ . Moreover show:

$$v(t) = \begin{cases} -1 - \frac{1}{6-8t} & \text{if } t \leq 0 \\ -\frac{1}{4}(1 + \frac{1}{t}) & \text{if } 0 < t \leq \frac{1}{4} \end{cases} \quad S(t) = \begin{cases} \{(\frac{4}{6-8t}, \frac{3}{6-8t} - 1)\} & \text{if } t \leq 0 \\ \{\frac{1}{4}(\frac{1}{t}, -1)\} & \text{if } 0 < t \leq \frac{1}{4} \end{cases}.$$

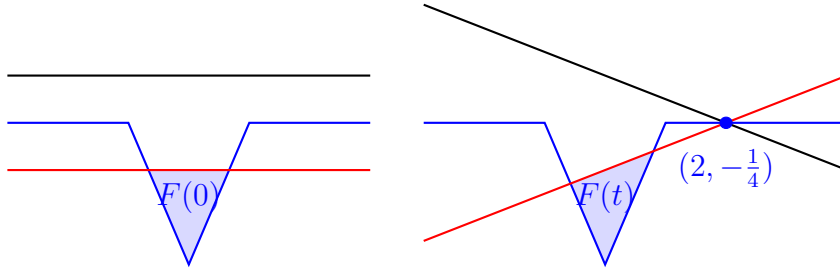


FIGURE 5.3. Feasible set  $F(0)$  and  $F(t)$  for  $t = 1/8$  in Ex. 5.3.

Definition 5.2 suggests a strong relation between closedness and osc of a mapping  $F$ . Indeed, under some extra compactness condition closedness of  $F$  implies osc.

**LC.** [local compactness of  $F$  at  $\bar{t}$ ]

Let  $F : T \rightrightarrows \mathbb{R}^n$ ,  $\bar{t} \in T$ . We assume that there exists  $\delta > 0$  and a compact set  $C_0$  such that

$$\bigcup_{\|t-\bar{t}\| \leq \delta} F(t) \subset C_0.$$

**LEMMA 5.1.** Let for  $F : T \rightrightarrows \mathbb{R}^n$  the local compactness condition LC be fulfilled at  $\bar{t}$ . Then if the mapping  $F$  is closed at  $\bar{t}$  it is osc at  $\bar{t}$ .

**Proof.** Assume that  $F$  is not osc at  $\bar{t}$ , i.e., there exist sequences  $t_l, x_l, t_l \rightarrow \bar{t}, x_l \in F(t_l)$  and  $\varepsilon > 0$  such that for all  $l$

$$(5.2) \quad \|x_l - x\| \geq \varepsilon \quad \text{for all } x \in F(\bar{t}).$$

LC implies  $x_l \in C_0$ , with  $C_0$  compact, and there exists a convergent subsequence  $x_{l_\nu} \rightarrow \tilde{x}$ . By closedness of  $F$  it follows  $\tilde{x} \in F(\bar{t})$  contradicting (5.2).  $\square$

### 5.3. Inner semicontinuity of $F(t)$

To obtain isc of the feasible set mapping  $F(t) = \{x \mid g_j(x, t) \leq 0, j \in J\}$  we need some other condition.

**DEFINITION 5.3.** *The Constraint Qualification CQ is said to hold for the feasible set map  $F(t)$  at  $(\bar{x}, \bar{t})$  with  $\bar{x} \in F(\bar{t})$ , if there is a sequence  $x_\nu \rightarrow \bar{x}$  such that*

$$g_j(x_\nu, \bar{t}) < 0 \quad \text{for all } j \in J .$$

$F(t)$  satisfies a global CQ at  $\bar{t}$  if CQ holds at  $(\bar{x}, \bar{t})$  for all  $\bar{x} \in F(\bar{t})$ .

**Ex. 5.4.** *Let CQ hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ . Show that  $F(t) \neq \emptyset$  holds for all  $t$  with small  $\|t - \bar{t}\|$ .*

**Proof.** For fixed  $\nu$ , we have  $g_j(x_\nu, \bar{t}) < 0, j \in J$ . By continuity also  $g_j(x_\nu, t) < 0, j \in J$ , is satisfied for small  $\|t - \bar{t}\|$ .  $\square$

The preceding Ex. in particular shows that under CQ at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ , after a small perturbation of  $\bar{t}$  the feasible set  $F(t)$  still contains points near  $\bar{x}$ .

**LEMMA 5.2.** *Let  $F(t)$  satisfy the global CQ at  $\bar{t} \in T$  with  $F(\bar{t}) \neq \emptyset$ . Then  $F(t)$  is isc at  $\bar{t}$ . In particular,  $F(t) \neq \emptyset$  for all  $t$ , with  $\|t - \bar{t}\|$  small.*

**Proof.** Assume to the contrary that  $F$  is not isc at  $\bar{t}$ . This means that there exist  $\bar{x} \in F(\bar{t})$ , a sequence  $t_l \rightarrow \bar{t}$  and  $\varepsilon > 0$  such that for all  $l$  (large)

$$(5.3) \quad \|x_l - \bar{x}\| \geq \varepsilon \quad \text{for all } x_l \in F(t_l) .$$

By CQ there is a sequence  $x_\nu \rightarrow \bar{x}$  with  $g_j(x_\nu, \bar{t}) < 0$  for  $j \in J$ . Choosing  $\nu_0$  large such that  $\|x_{\nu_0} - \bar{x}\| < \varepsilon$ , by continuity of  $g_j$  and using  $g_j(x_{\nu_0}, \bar{t}) < 0, j \in J$ , we find  $g_j(x_{\nu_0}, t_l) < 0$  for  $l$  large enough, and thus  $x_{\nu_0} \in F(t_l)$  in contradiction to (5.3).  $\square$

**Ex. 5.5.** *[CQ is not satisfied at  $\bar{t}$ ]*

*Consider  $F(t) = \{x \in \mathbb{R} \mid g(x, t) := |x| - t \leq 0\}$ ,  $t \in \mathbb{R}$ . Here,  $F(t) = \emptyset$  for  $t < 0$ ,  $F(0) = \{0\}$  and CQ does not hold at  $\bar{t} = 0$ . Show that  $F$  is not isc at  $\bar{t} = 0$ .*

### 5.4. Lower semicontinuity of $v(t)$

Lower semicontinuity of  $v$  depends on the following technical condition, combining a weak outer semicontinuity and a compactness assumption, saying that for  $t$  near  $\bar{t}$  at least one point  $x_t \in S(t)$  can be approached by elements in a compact subset of  $F(\bar{t})$ .

**AL.** There exists  $\delta > 0$ , and a compact set  $C \subset \mathbb{R}^n$  such that for any  $t$ ,

$\|t - \bar{t}\| < \delta$  with  $F(t) \neq \emptyset$  there is some  $x_t \in S(t) \cap C$  and some  $\bar{x}_t \in F(\bar{t}) \cap C$  satisfying  $\|x_{t_l} - \bar{x}_{t_l}\| \rightarrow 0$  for each sequence  $t_l \rightarrow \bar{t}$ .

**LEMMA 5.3.**

(a) *Let AL be satisfied at  $\bar{t}$ . Then  $v$  is lsc at  $\bar{t}$ .*

(b) *Let the local compactness condition LC be fulfilled at  $\bar{t}$ . Then AL is satisfied, i.e.,  $v$  is lsc at  $\bar{t}$ .*

**Proof.** (a) Choose any  $t$ ,  $\|t - \bar{t}\| < \delta$ . For  $F(t) = \emptyset$  it follows  $\infty = v(t) \geq v(\bar{t})$ . In case  $F(t) \neq \emptyset$  let  $x_t \in S(t)$ ,  $\bar{x}_t \in F(\bar{t})$  satisfy the conditions in AL. Then by the uniform continuity of  $f$  on  $C \times \{t \mid \|t - \bar{t}\| \leq \delta\}$  we find for  $t_l \rightarrow \bar{t}$ :

$$v(t_l) - v(\bar{t}) \geq f(x_{t_l}, t_l) - f(\bar{x}_{t_l}, \bar{t}) \rightarrow 0.$$

(b) Note that by LC, for  $\|t - \bar{t}\| < \delta$  and  $F(t) \neq \emptyset$  the feasible set is contained in a compact set  $C_0$ . So  $S(t) \neq \emptyset$  and we can choose elements  $x_t \in S(t) \cap C_0$ . By Lemma 5.1 the feasible set mapping  $F(t) \subset C_0$  is osc at  $\bar{t}$ . So in particular the weaker osc condition of AL is satisfied with  $C = C_0$ . □

To assure lower semicontinuity of  $v$  at  $\bar{t}$  we (minimally) need compactness of  $F(\bar{t})$  (even in the case that  $F(t)$  behaves continuously). This fact is illustrated by the next

**Ex. 5.6.** [Linear problem,  $F(t) \equiv F$  constant, not bounded, and  $v$  is not lsc.]

For the problem  $\min x_2 - tx_1$  s.t.  $x_1 \geq 0$ ,  $x_2 \geq 0$  we find

$$v(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ -\infty & \text{for } t > 0 \end{cases} \quad \text{and} \quad S(t) = \begin{cases} \{(0, 0)\} & \text{for } t < 0 \\ \{(x_1, 0) \mid x_1 \geq 0\} & \text{for } t = 0 \\ \emptyset & \text{for } t > 0 \end{cases}$$

### 5.5. Upper semicontinuity of $v(t)$

Upper semicontinuity of  $v$  depends on a constraint qualification which assures that no implosion of  $F(t)$  near  $\bar{t}$  can occur. The local compactness condition LC is not sufficient and not necessary.

**Ex. 5.7.** For the problem  $\min x_1$  s.t.  $\sqrt{x_1^2 + x_2^2} \leq -t$ , which obviously satisfies LC at  $\bar{t} = 0$ , we find

$$F(t) = \begin{cases} \{\sqrt{x_1^2 + x_2^2} \leq |t|\} & \text{for } t < 0 \\ \{(0, 0)\} & \text{for } t = 0 \\ \emptyset & \text{for } t > 0 \end{cases} \quad \text{and} \quad v(t) = \begin{cases} t & \text{for } t \leq 0 \\ \infty & \text{for } t > 0 \end{cases}$$

$F(0) = \{(0, 0)\}$  does not satisfy a constraint qualification and  $F(t)$  is not isc at  $\bar{t}$ . So  $v$  is lsc but not usc at  $\bar{t}$ .



The problem in this Ex. is that the mapping  $F$  is not isc at  $\bar{t}$ . To assure upper semicontinuity of  $v$  at  $\bar{t}$  we only need an inner semicontinuity condition at one point  $\bar{x} \in S(\bar{t})$ .

**AU.** There exist a minimizer  $\bar{x} \in S(\bar{t})$  and  $\delta > 0$  such that for all  $t$ ,  $\|t - \bar{t}\| < \delta$ , there is some  $x_t \in F(t)$  satisfying  $\|x_t - \bar{x}\| \rightarrow 0$  for each sequence  $t_l \rightarrow \bar{t}$ .

A natural condition to force the assumption AU is a *Constraint Qualification*.

**LEMMA 5.4.**

(a) Let AU be fulfilled at  $\bar{t}$ . Then  $v$  is usc at  $\bar{t}$ .

(b) Let CQ be satisfied at  $(\bar{x}, \bar{t})$  for (at least) one point  $\bar{x} \in S(\bar{t})$ . Then the condition AU is satisfied, i.e.,  $v$  is usc at  $\bar{t}$ .

**Proof.** (a) Consider any sequence  $t_l \rightarrow \bar{t}$  and the corresponding points  $x_{t_l} \in F(t_l)$ , satisfying  $\|x_{t_l} - \bar{x}\| \rightarrow 0$  according to AU. By continuity, for any  $\varepsilon > 0$ , the inequality  $|f(x_{t_l}, t_l) - f(\bar{x}, \bar{t})| < \varepsilon$  holds for  $l$  large enough. So

$$v(t_l) \leq f(x_{t_l}, t_l) \leq f(\bar{x}, \bar{t}) + \varepsilon = v(\bar{t}) + \varepsilon ,$$

proving that  $v$  is usc at  $\bar{t}$ .

(b) Assume to the contrary that AU is not satisfied for  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in S(\bar{t})$ . This means that there exist a sequence  $t_l \rightarrow \bar{t}$  and  $\varepsilon > 0$  such that for all  $l$  (large)

$$(5.4) \quad \|x_l - \bar{x}\| \geq \varepsilon \quad \text{for all } x_l \in F(t_l) .$$

By CQ there is a sequence  $x_\nu \rightarrow \bar{x}$  with  $g_j(x_\nu, \bar{t}) < 0$  for  $j \in J$ . Choosing  $\nu_0$  large such that  $\|x_{\nu_0} - \bar{x}\| < \varepsilon$ , by continuity of  $g_j$  and using  $g_j(x_{\nu_0}, \bar{t}) < 0$ , we find  $g_j(x_{\nu_0}, t_l) < 0$ ,  $j \in J$ , for  $l$  large enough and thus  $x_{\nu_0} \in F(t_l)$  in contradiction to (5.4). □

## 5.6. The behavior of $S(t)$

Let us study the continuity properties of the mapping  $S(t)$ . Obviously the inner semicontinuity of  $S$  is stronger than the condition AU. So if  $S$  is isc the function  $v$  is usc (see Lemma 5.4). However the inner semicontinuity of  $S(t)$  is a strong condition. Even if the local compactness condition LC and the Constraint Qualification CQ hold it need not be satisfied. We give an example of a simple parametric LP.

**Ex. 5.8.** [LC and CQ holds but  $S$  is not isc.]

$\min x_2 - tx_1$  s.t.  $|x_1| \leq 1$ ,  $|x_2| \leq 1$ . Then near  $\bar{t} = 0$  we obtain

$$S(t) = \begin{cases} \{(-1, -1)\} & \text{for } t < 0 \\ \{(x_1, -1) \mid |x_1| \leq 1\} & \text{for } t = 0 \\ \{(1, -1)\} & \text{for } t > 0 \end{cases} \quad \text{and} \quad v(t) = -1 - |t| .$$

We now discuss closedness and outer semicontinuity of  $S$ .

**LEMMA 5.5.** *Let CQ be satisfied at  $(\bar{x}, \bar{t})$  for (at least) one point  $\bar{x} \in S(\bar{t})$ . Then  $S$  is closed at  $\bar{t}$ .*

**Proof.** Assume to the contrary that  $S$  is not closed. This means that there are sequences  $t_l \rightarrow \bar{t}$ ,  $x_l \rightarrow \bar{x}$  with  $x_l \in S(t_l)$  but  $\bar{x} \notin S(\bar{t})$ . By closedness of  $F$  it follows  $\bar{x} \in F(\bar{t})$ . So,  $\bar{x} \notin S(\bar{t})$  yields

$$v(\bar{t}) < f(\bar{x}, \bar{t}) = \lim_{l \rightarrow \infty} f(x_l, t_l) = \lim_{l \rightarrow \infty} v(t_l).$$

Consequently,  $v$  is not usc at  $\bar{t}$  in contradiction to Lemma 5.4(b).  $\square$

**LEMMA 5.6.** *Let LC be satisfied at  $\bar{t}$  and let CQ hold at  $(\bar{x}, \bar{t})$  for (at least) one  $\bar{x} \in S(\bar{t})$ . Then there exists an  $\varepsilon$ -neighborhood  $B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , of  $\bar{t}$  such that for all  $t \in B_\varepsilon(\bar{t})$  the set  $S(t)$  is nonempty and contained in a compact set  $C_0$ . Moreover the mapping  $S$  is osc at  $\bar{t}$ .*

**Proof.** By LC and CQ, in some  $\varepsilon$ -neighborhood  $B_\varepsilon(\bar{t})$  of  $\bar{t}$  the sets  $F(t)$  are nonempty (cf. Ex. 5.4) and contained in a compact set  $C_0$ . So  $\emptyset \neq S(t) \subset C_0$  for  $t \in B_\varepsilon(\bar{t})$ . Assume now that  $S$  is not osc at  $\bar{t}$ , i.e., there exist  $t_l \rightarrow \bar{t}$ ,  $x_l \in S(t_l)$ ,  $\delta > 0$ , such that

$$(5.5) \quad \|x_l - x\| \geq \delta \quad \text{for all } x \in S(\bar{t}).$$

By LC (taking a subsequence if necessary) we can assume  $x_l \rightarrow \tilde{x}$ . By Lemma 5.5 it follows  $\tilde{x} \in S(\bar{t})$  contradicting (5.5).  $\square$

## 5.7. Parametric convex programs

By Ex.5.3, in the general (non-convex) case, the condition  $\emptyset \neq F(\bar{t})$  compact is not sufficient to assure LC and the outer semicontinuity of  $F$ . Moreover, the condition  $\emptyset \neq S(\bar{t})$  compact does not imply the lower semicontinuity of  $v$ . Under the following convexity assumptions the situation is changed.

**AC<sub>g</sub>.** For each (fixed)  $t \in T$  the functions  $g_j(x, t)$  are convex in  $x$  for all  $j \in J$ .

**AC.** In addition to AC<sub>g</sub>, for each (fixed)  $t \in T$  the function  $f(x, t)$  is convex in  $x$ , i.e., the problems  $P(t)$  are convex programs.

**LEMMA 5.7.** *Let the convexity condition AC<sub>g</sub> hold and assume  $\emptyset \neq F(\bar{t})$  is bounded (compact). Then the local compactness condition LC is satisfied. In particular, the feasible set map  $F$  is osc and  $v$  is lsc at  $\bar{t}$  (cf. Lemma 5.3).*

**Proof.** Assume to the contrary that LC doesn't hold, *i.e.*, there exists  $t_l \rightarrow \bar{t}$ ,  $x_l \in F(t_l)$  with  $\|x_l\| \rightarrow \infty$ . Since  $F(\bar{t})$  is bounded, we can choose some  $\rho > 0$  satisfying

$$(5.6) \quad \|x\| < \rho \quad \text{for all } x \in F(\bar{t}) .$$

Given some  $\bar{x} \in F(\bar{t})$  determine  $0 \leq \tau_l < 1$  such that  $\tilde{x}_l := \bar{x} + \tau_l(x_l - \bar{x})$  satisfies  $\|\tilde{x}_l\| = \rho$ . Selecting some subsequence, we can assume  $\tilde{x}_l \rightarrow \tilde{x}$  with  $\|\tilde{x}\| = \rho$ . By convexity of  $g_j$  using  $x_l \in F(t_l)$  we find for all  $j \in J$ ,

$$g_j(\tilde{x}_l, t_l) \leq (1 - \tau_l)g_j(\bar{x}, t_l) + \tau_l g_j(x_l, t_l) \leq (1 - \tau_l)g_j(\bar{x}, t_l) .$$

The condition  $\|\tilde{x}\| = \rho$ ,  $\|x_l\| \rightarrow \infty$  implies  $\tau_l = \frac{\|\tilde{x}_l - \bar{x}\|}{\|x_l - \bar{x}\|} \rightarrow 0$  and then

$$g_j(\tilde{x}, \bar{t}) = \lim_{l \rightarrow \infty} g_j(\tilde{x}_l, t_l) \leq \lim_{l \rightarrow \infty} (1 - \tau_l)g_j(\bar{x}, t_l) = g_j(\bar{x}, \bar{t}) \leq 0 .$$

So  $\tilde{x} \in F(\bar{t})$  and  $\|\tilde{x}\| = \rho$  in contradiction to (5.6). □

In the convex case even the boundedness of  $S(\bar{t})$  is sufficient to assure that  $v$  is lsc at  $\bar{t}$ .

**THEOREM 5.1.** *Let the convexity assumption AC hold and let  $S(\bar{t})$  be nonempty and bounded (compact). Then  $v$  is lsc at  $\bar{t}$ .*

**Proof.** Consider the lower level set  $\tilde{F}(t) := \{x \in F(t) \mid f(x, t) \leq v(\bar{t})\}$ . Then the set  $\tilde{F}(\bar{t}) = S(\bar{t})$  is nonempty and bounded and the constraint set  $\tilde{F}(t)$  satisfies the assumption of Lemma 5.7. So the mapping  $\tilde{F}$  has the local boundedness property LC, *i.e.*, with some  $\tilde{C}_0$  and  $\varepsilon > 0$ ,

$$\bigcup_{\|t - \bar{t}\| \leq \varepsilon} \tilde{F}(t) \subset \tilde{C}_0 .$$

Consider now any  $t$ ,  $\|t - \bar{t}\| \leq \varepsilon$ . If  $\tilde{F}(t) = \emptyset$  it directly follows  $v(t) \geq v(\bar{t})$ . In case  $\tilde{F}(t) \neq \emptyset$  the set  $\tilde{F}(t)$  is a (nonempty) compact lower level feasible set for  $P(t)$ . This means that the problem

$$\tilde{P}(t) : \quad \min_x f(t, x) \quad \text{s.t.} \quad x \in \tilde{F}(t)$$

with value  $\tilde{v}(t)$  has a nonempty solution set  $\tilde{S}(t)$  and  $S(t) = \tilde{S}(t)$  implying  $v(t) = \tilde{v}(t)$ . Since LC holds for  $\tilde{F}$  at  $\bar{t}$  by Lemma 5.7 the function  $\tilde{v}(t)$  and thus  $v(t)$  is lsc at  $\bar{t}$ . □

If we assume convexity in the variable  $(x, t)$  then we can even say more.

**AC<sup>x,t</sup>.** Let the functions  $f(x, t)$ ,  $g_j(x, t)$ ,  $j \in J$ , be convex in  $(x, t)$  on  $\mathbb{R}^n \times T$ ,  $T$  convex.

**EX. 5.9.** Let  $AC^{x,t}$  be satisfied. Then  $\text{dom } F$  is convex.

**Proof.** For  $t_1, t_2 \in T$ ,  $t_1, t_2 \in \text{dom } F$  take  $x_1 \in F(t_1), x_2 \in F(t_2)$  and define  $x_\tau = \tau x_1 + (1 - \tau)x_2$ ,  $t_\tau = \tau t_1 + (1 - \tau)t_2$ ,  $\tau \in [0, 1]$ . Then for all  $j \in J$  we find

$$g_j(x_\tau, t_\tau) \leq \tau g_j(x_1, t_1) + (1 - \tau)g_j(x_2, t_2) \leq 0,$$

i.e.,  $x_\tau \in F(t_\tau)$  and  $t_\tau \in \text{dom } F$ .

**THEOREM 5.2.** Let  $AC^{x,t}$  hold and assume  $S(t) \neq \emptyset$  for all  $t \in T$ . Then the function  $v(t)$  is convex on  $T$ . In particular  $v$  is Lipschitz continuous on each point  $\bar{t} \in \text{int } T$ .

**Proof.** Let  $x_1 \in S(t_1)$ ,  $x_2 \in S(t_2)$ ,  $t_1, t_2 \in T$  and define  $x_\tau = \tau x_1 + (1 - \tau)x_2$ ,  $t_\tau = \tau t_1 + (1 - \tau)t_2$ ,  $\tau \in [0, 1]$ . Then for all  $j \in J$  we find

$$g_j(x_\tau, t_\tau) \leq \tau g_j(x_1, t_1) + (1 - \tau)g_j(x_2, t_2) \leq 0,$$

i.e.,  $x_\tau \in F(t_\tau)$ . We moreover obtain

$$\begin{aligned} v(t_\tau) \leq f(x_\tau, t_\tau) &\leq \tau f(x_1, t_1) + (1 - \tau)f(x_2, t_2) \\ &= \tau v(t_1) + (1 - \tau)v(t_2). \end{aligned}$$

The last statement follows from the fact that any convex function on  $T$  (convex) is Lipschitz continuous at any interior point  $\bar{t}$  of  $T$  (cf., Remark 2.1). □

## 5.8. Parametric linear inequalities, Lipschitz continuity of $F(t)$

We start with the famous Hoffman Lemma [16] (cf., also [18, Lemma 2.7] for a proof).

**LEMMA 5.8.** Let  $A \in \mathbb{R}^{m \times n}$  and  $F(t) = \{x \mid Ax \leq t\}$ ,  $t \in \mathbb{R}^m$ . Then there is some  $L_0 > 0$  (only depending on  $A$ ) such that for each  $x \in \mathbb{R}^n$ , and  $t \in \mathbb{R}^m$  with  $F(t) \neq \emptyset$ , the following holds: there exists  $x_t \in F(t)$  satisfying

$$(5.7) \quad \|x - x_t\| \leq L_0 \max_{1 \leq j \leq m} (A_j \cdot x - t_j)^+,$$

where  $A_j$  is the  $j$ th row of  $A$ , and for  $\alpha \in \mathbb{R}$ ,  $\alpha^+ := \max\{0, \alpha\}$ .

As an application of this lemma we obtain global Lipschitz continuity of the feasible set mapping

$$F(t) = \{x \mid Ax \leq t\}.$$

**COROLLARY 5.1.** Let  $F(t) = \{x \mid Ax \leq t\}$ ,  $t \in \mathbb{R}^m$ . Then there exists  $L_0 > 0$  such that for any  $t, t' \in \text{dom } F$  it holds: to any  $x \in F(t)$  we can find a point  $x' \in F(t')$  such that

$$\|x' - x\| \leq L_0 \|t' - t\|.$$

**Proof.** Let be given  $t, t' \in \text{dom } F$ , and  $x \in F(t)$ , i.e.,  $Ax \leq t$  or  $Ax - t' \leq t - t'$ . By Lemma 5.8 there exists  $x' \in F(t')$  such that

$$\begin{aligned} \|x' - x\| &\leq L_0 \max_{1 \leq j \leq m} (A_j \cdot x - t'_j)^+ \\ &\leq L_0 \max_{1 \leq j \leq m} (t_j - t'_j)^+ \leq L_0 \|t' - t\| . \end{aligned}$$

□

According to Corollary 5.1, for the right-hand perturbation, a global Lipschitz result holds for  $F(t)$  even if  $F(t)$  is not bounded. Ex. 5.1 shows that this is no more the case if in a feasibility condition  $A(t)x \leq b(t)$  also  $A(t)$  depends on the parameter  $t$ . We give an illustrative example:

$$(5.8) \quad F(t) = \{(x_1, x_2) \mid \pm x_1 \leq 1, \pm x_2 \leq 1, tx_1 \leq 0\} .$$

Obviously

$$tx_1 \leq 0 \Leftrightarrow \begin{cases} 0 \leq 0 & \text{for } t = 0 \\ x_1 \leq 0 & \text{for } t > 0 \\ x_1 \geq 0 & \text{for } t < 0 \end{cases} .$$

So for  $t$  near  $\bar{t} = 0$  the feasible set  $F(t)$  behaves as sketched in Figure 5.4. From

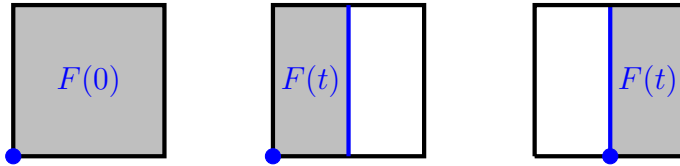


FIGURE 5.4. Sketch of the sets  $F(0)$ , and  $F(t)$  for  $t > 0$  and  $t < 0$ .

this picture it is clear that

$$F(t) \text{ is osc at } \bar{t} = 0 \text{ but not isc .}$$

The problem here is that the functions  $g_j$  describing  $F(0)$  do not fulfill a constraint qualification (see Definition 5.3). Indeed for the constraint  $g(x, t) = tx_1$  there is no  $x \in F(0)$  satisfying

$$g(x, 0) = 0 < 0 .$$

**Ex. 5.10.** Consider the parametric LP,

$$P(t) \quad \min_{x \in \mathbb{R}^2} (1, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{s.t.} \quad x \in F(t) ,$$

with  $F(t)$  in (5.8). Show that the vertex minimizers of order 1 are given by (see Figure 5.4):

$$x(t) = \begin{cases} (-1, -1) & \text{for } t \leq 0 \\ (0, -1) & \text{for } t > 0 \end{cases} .$$

The preceding discussion shows that without some constraint qualification we cannot expect a feasible set  $F(t) = \{x \mid A(t)x \leq b(t)\}$  to be isc but only osc. To assure the latter only some compactness assumption is needed. Based on Hoffman's Lemma we can prove Lipschitz osc of  $F(t)$  (see also [27]).

**LEMMA 5.9.** [*Lipschitz outer semicontinuity of the solution set of linear inequalities*] *Let*

$$F(t) = \{x \mid A(t)x \leq b(t)\}, \quad t \in \mathbb{R}^p,$$

*and suppose the functions  $A(t), b(t)$  are Lipschitz continuous at  $\bar{t} \in \mathbb{R}^p$ . Further assume that  $\emptyset \neq F(\bar{t})$  is bounded. Then  $F(t)$  is Lipschitz osc at  $\bar{t}$ , i.e., there exist  $B_\varepsilon(\bar{t})$  and  $L > 0$  such that for any  $x(t) \in F(t)$ ,  $t \in B_\varepsilon(\bar{t})$ , there exists some  $\bar{x}(t) \in F(\bar{t})$  with*

$$\|x(t) - \bar{x}(t)\| \leq L\|t - \bar{t}\|.$$

**Proof.** We first show that there exist  $\varepsilon > 0, K > 0$ , such that the boundedness relation holds:

$$\|x(t)\| \leq K \quad \text{for all } x(t) \in F(t), t \in B_\varepsilon(\bar{t}).$$

Assume to the contrary that there are sequences  $t_l \rightarrow \bar{t}$ ,  $x(t_l) \in F(t_l)$ , satisfying  $\|x(t_l)\| \rightarrow \infty$  for  $l \rightarrow \infty$ . Then (for some subsequence)  $\frac{x(t_l)}{\|x(t_l)\|} \rightarrow d$ ,  $\|d\| = 1$  and so, the inequality

$$A(t_l) \frac{x(t_l)}{\|x(t_l)\|} \leq \frac{b(t_l)}{\|x(t_l)\|} \text{ yields for } l \rightarrow \infty \text{ the relation } A(\bar{t})d \leq 0.$$

Thus, with  $\bar{x} \in F(\bar{t})$  the whole ray  $\{\bar{x} + \tau d \mid \tau \geq 0\}$  is contained in  $F(\bar{t})$  contradicting the boundedness of  $F(\bar{t})$ .

Now take any  $x(t) \in F(t)$ ,  $t \in B_\varepsilon(\bar{t})$ , i.e.,  $A(t)x(t) - b(t) \leq 0$ . Lipschitz continuity at  $\bar{t}$  for  $A(t), b(t)$  means that there exists  $L_1 > 0$  such that for all  $j$  we have

$$\|A_j(t) - A_j(\bar{t})\| \leq L_1\|t - \bar{t}\|, \quad |b_j(t) - b_j(\bar{t})| \leq L_1\|t - \bar{t}\|.$$

Thus we find

$$\begin{aligned} A_j(\bar{t})x(t) - b_j(\bar{t}) &\leq [A_j(\bar{t})x(t) - b_j(\bar{t})] - [A_j(t)x(t) - b_j(t)] \\ &= [(A_j(\bar{t}) - A_j(t))x(t)] + [b_j(t) - b_j(\bar{t})] \\ &\leq \|A_j(\bar{t}) - A_j(t)\|\|x(t)\| + |b_j(t) - b_j(\bar{t})| \\ &\leq L_1\|t - \bar{t}\|K + L_1\|t - \bar{t}\| \\ &= L_1(K + 1)\|t - \bar{t}\|. \end{aligned}$$

According to Hoffman's Lemma with  $A = A(\bar{t})$ , there exists some  $L_0 > 0$  such that for  $x(t) \in F(t)$  there exists  $\bar{x}(t) \in F(\bar{t})$  with

$$\|x(t) - \bar{x}(t)\| \leq L_0L_1(K + 1)\|t - \bar{t}\|$$

and the lemma is proven with  $L = L_0L_1(K + 1)$ . □

**REMARK 5.1.** Since an equality  $Ax = t$  can be written equivalently as

$$Ax \leq t \quad \text{and} \quad -Ax \leq -t ,$$

Hoffman's Lemma remains valid if (in addition to inequalities) equality constraints are present. For the equalities we only have to replace (5.7) by

$$\|x - x_t\| \leq L_0 \max_{1 \leq j \leq m} |A_j \cdot x - t_j| .$$

Consequently also Lemma 5.9 remains true for linear equalities or combinations of linear equalities and inequalities.

### 5.9. LP with right-hand side perturbation, Lipschitz continuity of $v(t)$

As a special case of Theorem 5.2 we consider linear programs (LP), where the parameter  $t$  only appears in the right-hand side of the feasibility condition:

$$(5.9) \quad P(t) : \quad \max c^T x \quad \text{s.t.} \quad x \in F(t) = \{x \mid Ax \leq t\} .$$

Recall that the dual of  $P(t)$  reads:

$$D(t) : \quad \min t^T y \quad \text{s.t.} \quad A^T y = c, \quad y \geq 0 .$$

Corollary 5.1 leads to a global Lipschitz result for the value function  $v(t)$  of  $P(t)$  in (5.9), cf., also Theorem 5.2.

**COROLLARY 5.2.** *Assume that the dual  $D(t)$  of  $P(t)$  has a feasible solution  $A^T y = c, y \geq 0$ . Then there exists  $L > 0$  such that for any  $t_1, t_2 \in \text{dom } F$  we have*

$$|v(t_1) - v(t_2)| \leq L \|t_1 - t_2\| .$$

**Proof.** Take  $t_1, t_2 \in \text{dom } F$ . Note that by Ex. 5.9  $\text{dom } F$  is convex. By LP duality, since the dual of  $P(t)$  is feasible, maximizer  $x_1, x_2$  of  $P(t_1), P(t_2)$  exist. By Corollary 5.1, for  $x_1$  there exists  $\tilde{x}_2 \in F(t_2)$  such that  $\|x_1 - \tilde{x}_2\| \leq L_0 \|t_1 - t_2\|$ . So

$$v(t_2) - v(t_1) \leq c^T \tilde{x}_2 - c^T x_1 \leq \|c\| \|\tilde{x}_2 - x_1\| \leq L_0 \|c\| \|t_2 - t_1\| .$$

The same can be done by interchanging the role of  $x_1, x_2$ . This shows the result for  $L = L_0 \|c\|$ .

□





## CHAPTER 6

### Local stability in nonlinear parametric optimization

We consider again for some open parameter set  $T \subset \mathbb{R}^p$  the parametric program: for  $t \in T$  solve

$$(6.1) \quad P(t) : \min_x f(x, t) \quad \text{s.t.} \quad x \in F(t) = \{x \in \mathbb{R}^n \mid g_j(x, t) \leq 0, j \in J\},$$

with  $J = \{1, \dots, m\}$ . We assume throughout this section that  $f, g_j, j \in J$ , are (twice) continuously differentiable on  $\mathbb{R}^n \times T$ , notation  $f, g_j \in C^1$  ( $\in C^2$ ).

In this chapter (except for Sections 6.5, 6.6) we deal with general nonlinear parametric programs. So, we consider local minimizers  $x(t)$  of  $P(t)$  with corresponding local value function

$$v(t) = f(x(t), t).$$

(Except for Lemma 6.2(b)).

In what follows, we are interested in local stability (continuity) of the feasible set map  $F(t)$  and a function of local minimizers  $x(t)$  of  $P(t)$  for  $t$  near  $\bar{t}$ . Appropriate constraint qualifications and compactness assumptions will directly lead to Lipschitz continuity results for the feasible set mapping  $F(t)$  and the value function  $v(t)$ . In this section we do not distinguish between Lipschitz outer and inner continuity for  $F(t)$ .

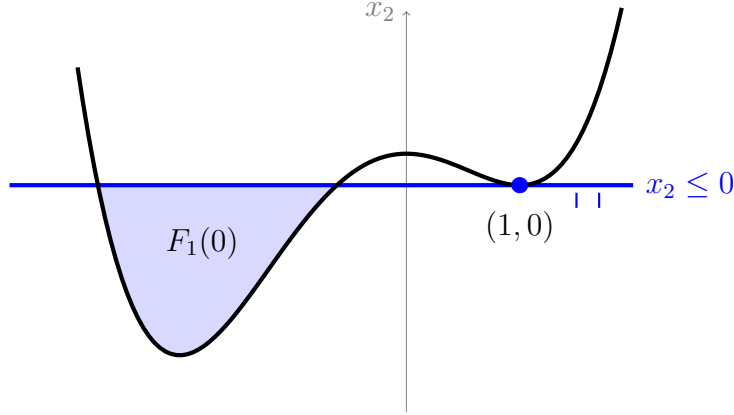
#### 6.1. Local Lipschitz stability of $F(t)$

Without a constraint qualification we cannot expect that  $F(t)$  behaves continuously. Components of the feasible set may disappear after perturbations of  $\bar{t}$ . We start with an example.

**Ex. 6.1.** Let  $F(t) \subset \mathbb{R}^2$  be given by

$$F(t) : \begin{array}{l} x_2 \geq h(x_1) \\ x_2 \leq t \end{array} \quad \text{with} \quad h(x_1) := \frac{1}{4}x_1^4 + \frac{1}{3}x_1^3 - x_1^2 + \frac{5}{12}, \quad t \in \mathbb{R},$$

and  $\bar{t} = 0$ . Then  $F(0)$  (see Figure 6.1) consist of a full dimensional set  $F_1(0)$  and an isolated point  $F_2(0) = \{(1, 0)\}$ . For  $t < 0$  (small)  $F_1(t)$  changes continuously but the component  $F_2(t)$  disappears. Note, that at the isolated point of  $F_2(0)$  no constraint qualification is satisfied.

FIGURE 6.1. Feasible set  $F(0)$  of Ex. 6.1

So, in order to assure that some component of  $F(t)$  behaves stably (continuously) we should assume some constraint qualification, e.g., a Mangasarian Fromovitz Constraint Qualification (MFCQ).

Let again  $F(t) = \{x \in \mathbb{R}^n \mid g_j(x, t) \leq 0, j \in J\}$  with  $g_j \in C^1$ . Recall that the MFCQ is said to hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ , if there exists some  $\xi$  such that

$$\nabla_x g_j(\bar{x}, \bar{t})\xi < 0 \quad \forall j \in J_0(\bar{x}, \bar{t}).$$

We say that the global MFCQ holds for  $F(\bar{t})$  if MFCQ holds at  $(\bar{x}, \bar{t})$  for all points  $\bar{x} \in F(\bar{t})$ .

Under MFCQ the feasible set behaves (locally) Lipschitz stable.

**LEMMA 6.1.** *Let  $g_j$  be  $C^1$ -functions and let MFCQ hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$  with a vector  $\xi$ .*

(a) *Then there exists  $B_{\delta_1}(\bar{x}), B_{\varepsilon_1}(\bar{t})$ ,  $\delta_1, \varepsilon_1 > 0$ , such that MFCQ is satisfied with  $\xi$  at all  $(x, t) \in B_{\delta_1}(\bar{x}) \times B_{\varepsilon_1}(\bar{t})$  and  $\text{int } F(t) \cap B_{\delta_1}(\bar{x}) \neq \emptyset$  holds for all  $t \in B_{\varepsilon_1}(\bar{t})$ . More precisely, with some  $\rho > 0$ , for any  $\delta > 0$ ,  $\delta \leq \delta_1$ , there exists  $\varepsilon = \varepsilon(\delta) := \frac{\delta}{\rho \|\xi\|} \leq \varepsilon_1$  such that*

$$\text{int } F(t) \cap \{x \mid \|x - \bar{x}\| = \delta\} \neq \emptyset \quad \text{for all } t \in B_{\varepsilon(\delta)}(\bar{t}).$$

(b) *There exist  $B_{\delta_2}(\bar{x}), B_{\varepsilon_2}(\bar{t})$ ,  $\delta_2, \varepsilon_2 > 0$ , such that  $F(t) \cap B_{\delta_2}(\bar{x})$  is Lipschitz continuous in  $B_{\varepsilon_2}(\bar{t})$ . More precisely, there exists  $L > 0$  such that for any  $t_1, t_2 \in B_{\varepsilon_2}(\bar{t})$  it holds: for any  $x_1 \in F(t_1) \cap B_{\delta_2}(\bar{x})$  we can find some  $x_2 \in F(t_2) \cap B_{2\delta_2}(\bar{x})$  satisfying*

$$\|x_1 - x_2\| \leq L \|t_1 - t_2\|.$$

**Proof.** (a) By MFCQ and continuity of  $g_j, \nabla_x g_j, j \in J$ , we can choose  $B_{\delta_0}(\bar{x}), B_{\varepsilon_0}(\bar{t})$  and some  $\sigma < 0$  such that

$$(6.2) \quad J_0(x, t) \subset J_0(\bar{x}, \bar{t}) \text{ and } \nabla_x g(x, t)\xi \leq \sigma \quad \forall (x, t) \in B_{\delta_0}(\bar{x}) \times B_{\varepsilon_0}(\bar{t}).$$

We now define  $x(t) := \bar{x} + \rho\xi\|t - \bar{t}\|$  and the Lipschitz constant  $c_2 := \max_{j \in J_0(\bar{x}, \bar{t})} \sup_{(x,t) \in B_{\delta_0}(\bar{x}) \times B_{\varepsilon_0}(\bar{t})} \|\nabla_t g_j(x, t)\|$ . We will determine  $\rho > 0$  appropriately as follows. By applying the Mean Value Theorem (cf., Appendix), for any  $j \in J_0(\bar{x}, \bar{t})$ , with some  $\tau = \tau_j$ ,  $0 < \tau < 1$ , we find assuming  $(x(t), t) \in B_{\delta_0}(\bar{x}) \times B_{\varepsilon_0}(\bar{t})$ :

$$\begin{aligned}
(6.3) \quad g_j(x(t), t) &= g_j(x(t), t) - g_j(\bar{x}, \bar{t}) \\
&= \nabla_{(x,t)} g_j(\bar{x} + \tau(x(t) - \bar{x}), \bar{t} + \tau(t - \bar{t}))(x(t) - \bar{x}, t - \bar{t}) \\
&= \rho \nabla_x g_j(\bar{x} + \tau(x(t) - \bar{x}), \bar{t} + \tau(t - \bar{t}))\xi\|t - \bar{t}\| \\
&\quad + \nabla_t g_j(\bar{x} + \tau(x(t) - \bar{x}), \bar{t} + \tau(t - \bar{t}))(t - \bar{t}) \\
&\leq (\sigma\rho + c_2)\|t - \bar{t}\|.
\end{aligned}$$

We firstly fix  $\rho$  such that  $\sigma\rho + c_2 < 0$  or  $\rho > \frac{c_2}{-\sigma} > 0$ . For  $x(t)$  we consider the conditions

$$(6.4) \quad \|x(t) - \bar{x}\| = \rho\|\xi\|\|t - \bar{t}\| \begin{cases} = \\ \leq \end{cases} \delta \quad \text{or} \quad \|t - \bar{t}\| \begin{cases} = \\ \leq \end{cases} \frac{\delta}{\rho\|\xi\|}.$$

We now choose  $\delta_1, \varepsilon_1$  such that

$$\delta_1 < \delta_0 \quad \text{and} \quad \varepsilon_1 < \varepsilon_0$$

as follows. In case  $\frac{\delta_0}{2\rho\|\xi\|} < \varepsilon_0$  we put

$$\delta_1 := \frac{\delta_0}{2} \quad \text{and} \quad \varepsilon_1 := \frac{\delta_0}{2\rho\|\xi\|} = \frac{\delta_1}{\rho\|\xi\|}.$$

If  $\frac{\delta_0}{2\rho\|\xi\|} \geq \varepsilon_0$  we define  $\varepsilon_1 := \frac{\varepsilon_0}{2}$  and determine  $\delta_1$  via  $\frac{\delta_1}{\rho\|\xi\|} = \varepsilon_1$  or  $\delta_1 := \rho\|\xi\|\varepsilon_1$ , in which case  $\delta_1 = \rho\|\xi\|\frac{\varepsilon_0}{2} < \rho\|\xi\|\varepsilon_0 < \delta_0$ , i.e.,  $\delta_1 < \delta_0$ . In both cases,  $\delta_1, \varepsilon_1$  are related via

$$\varepsilon_1 = \frac{\delta_1}{\rho\|\xi\|}.$$

This construction in mind, for any  $0 < \delta \leq \delta_1$  we define

$$(6.5) \quad \varepsilon(\delta) = \frac{\delta}{\rho\|\xi\|}.$$

In view of (6.3), (6.4) the points  $y(\delta) := \bar{x} + \rho\xi\varepsilon(\delta) = \bar{x} + \delta\frac{\xi}{\|\xi\|}$ ,  $\delta \leq \delta_1$ , satisfy

$$y(\delta) \in \text{int } F(t) \cap \{x \mid \|x - \bar{x}\| = \delta\} \quad \forall t \in B_{\varepsilon(\delta)}(\bar{t}).$$

Indeed, as in (6.3) we obtain for  $j \in J_0(\bar{x}, \bar{t})$ ,

$$\begin{aligned}
g_j(y(\delta), t) &\leq \rho\sigma\varepsilon(\delta) + c_2\|t - \bar{t}\| \\
&< (\rho\sigma + c_2)\varepsilon(\delta) < 0 \quad \forall t \in B_{\varepsilon(\delta)}(\bar{t}).
\end{aligned}$$

We also have for  $x(t) = \bar{x} + \rho\xi\|t - \bar{t}\|$

$$x(t) \in \text{int } F(t) \quad \forall t \in B_{\varepsilon(\delta)}(\bar{t}), \quad t \neq \bar{t}.$$

(b) Let as in (a),  $B_{\delta_1}(\bar{x}), B_{\varepsilon_1}(\bar{t})$  be such that (see (6.5))  $\varepsilon_1 = \frac{\delta_1}{\rho\|\xi\|}$  and

$$J_0(x, t) \subset J_0(\bar{x}, \bar{t}) \quad \text{and} \quad \nabla_x g(x, t)\xi \leq \sigma \quad \forall (x, t) \in B_{\delta_1}(\bar{x}) \times B_{\varepsilon_1}(\bar{t}).$$

We now choose

$$\delta_2 := \frac{\delta_1}{2} \quad , \quad \varepsilon_2 := \frac{1}{4} \frac{\delta_1}{\rho \|\xi\|} = \frac{\varepsilon_1}{4} .$$

For given  $t_1, t_2 \in B_{\varepsilon_2}(\bar{t})$ ,  $x_1 \in B_{\delta_2}(\bar{x})$  we define  $x_2 = x_1 + \rho \xi \|t_2 - t_1\|$  where  $\rho$  is determined as in the proof of (a). Using  $x_1 \in F(t_1)$ , we find for any  $j \in J_0(\bar{x}, \bar{t})$  as in (6.3),

$$\begin{aligned} g_j(x_2, t_2) &= g_j(x_2, t_2) - g_j(x_1, t_1) + g_j(x_1, t_1) \\ &\leq g_j(x_2, t_2) - g_j(x_1, t_1) \\ &\leq (\rho\sigma + c_2) \|t_2 - t_1\| < 0 . \end{aligned}$$

It only remains to check  $x_2 \in B_{\delta_1}(\bar{x}) = B_{2\delta_2}(\bar{x})$ . Indeed, in view of  $x_1 \in B_{\delta_2}(\bar{x})$ ,  $t_1, t_2 \in B_{\varepsilon_2}(\bar{t})$  we obtain

$$\begin{aligned} \|x_2 - \bar{x}\| &\leq \|x_1 - \bar{x}\| + \|x_2 - x_1\| \\ &\leq \|x_1 - \bar{x}\| + \rho \|\xi\| \|t_2 - t_1\| < \delta_2 + \rho \|\xi\| 2\varepsilon_2 \\ &= \frac{\delta_1}{2} + \rho \|\xi\| \frac{\varepsilon_1}{2} = \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 = 2\delta_2 . \end{aligned}$$

□

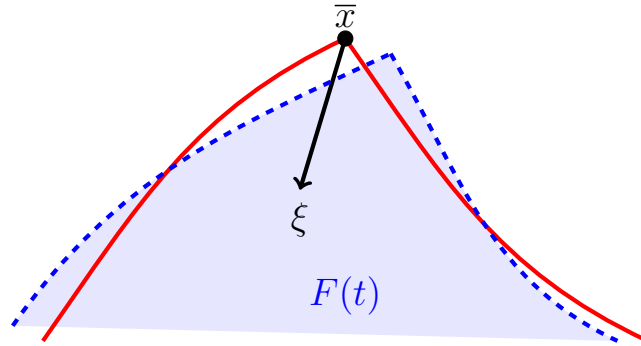


FIGURE 6.2. Sketch of feasible sets  $F(0)$  (red) and  $F(t)$  (blue, dashed) for small  $t$  under MFCQ at  $\bar{x}$ .

## 6.2. Stability of local minimizers and Lipschitz continuity of

$$v(t)$$

We first study the stability (existence) of local minimizers. Suppose that  $\bar{x}$  is a strict local minimizer of  $P(\bar{t})$  in (6.1). Then only MFCQ suffices to assure the existence of local minimizers  $x(t)$  for  $t$  near  $\bar{t}$ .

**THEOREM 6.1.** *Let  $f, g_j$  be  $C^1$ -functions. Let  $\bar{x}$  be a strict local minimizer of  $P(\bar{t})$  such that MFCQ holds at  $(\bar{x}, \bar{t})$ . Then there are  $\delta, \varepsilon > 0$  such that for any  $t \in B_\varepsilon(\bar{t})$  there exists a local minimizer  $x(t) \in B_\delta(\bar{x})$  of  $P(t)$  (at least one) which*

is a global minimizer in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$ . Moreover these minimizers satisfy  $x(t) \rightarrow \bar{x}$  for  $t \rightarrow \bar{t}$ . More precisely, for any  $\delta_1$ ,  $0 < \delta_1 \leq \delta$ , there exists some  $\varepsilon_1$ ,  $0 < \varepsilon_1 \leq \varepsilon$ , such that we have

$$x(t) \in B_{\delta_1}(\bar{x}) \quad \text{for all } t \in B_{\varepsilon_1}(\bar{t}).$$

**Proof.** Let  $\bar{x}$  be a strict local minimizer of  $P(\bar{t})$  such that

$$(6.6) \quad f(\bar{x}) < f(x) \quad \text{for all } x \in F(\bar{t}) \cap \text{cl } B_{\delta_0}(\bar{x}), \quad x \neq \bar{x},$$

for some  $\delta_0 > 0$ . Let  $\delta'_1$  be the minimum of  $\delta_1, \delta_2$  from Lemma 6.1(a),(b) and put  $\delta_2 = \frac{1}{2} \min\{\delta_0, \delta'_1\}$ . Then we have (cf., Lemma 6.1(a)) for all  $\delta_3$ ,  $0 < \delta_3 \leq \delta_2$ ,

$$(6.7) \quad \text{int } F(t) \cap \{x \mid \|x - \bar{x}\| = \delta_3\} \neq \emptyset \quad \forall t \in B_{\varepsilon(\delta_3)}(\bar{t}),$$

where  $\varepsilon(\delta) = \frac{\delta}{\rho \|\xi\|}$ . By continuity of  $f$ , in view of (6.6), there exist  $\alpha > 0$ , a small  $\sigma > 0$ , and some  $\delta_3$ ,  $0 < \delta_3 < \delta_2 - \sigma$ , such that with min-value  $m := f(\bar{x}, \bar{t})$  it holds:

$$f(x, \bar{t}) \leq m + \frac{\alpha}{2} \quad \forall x \in B_{\delta_3}(\bar{x}),$$

$$(6.8) \quad f(x, \bar{t}) \geq m + 4\alpha \quad \forall x \in F(\bar{t}), \quad \delta_2 - \sigma \leq \|x - \bar{x}\| \leq \delta_2 + \sigma.$$

By making  $\delta_3$  and  $\varepsilon(\delta_3) = \frac{\delta_3}{\rho \|\xi\|}$  smaller, if necessary, due to the continuity of  $f$ , we have

$$(6.9) \quad f(x, t) \leq m + \alpha \quad \forall x \in B_{\delta_3}(\bar{x}), t \in B_{\varepsilon(\delta_3)}(\bar{t})$$

$$(6.10) \quad f(x, t) \geq m + 2\alpha \quad \forall x \in F(t), \quad \|x - \bar{x}\| = \delta_2, \quad t \in B_{\varepsilon(\delta_3)}(\bar{t}).$$

The first inequality is clear. For the second let us assume to the contrary, that there exist sequences  $t_l \rightarrow \bar{t}$ ,  $x_{t_l} \in F(t_l)$  such that

$$(6.11) \quad \|x_{t_l} - \bar{x}\| = \delta_2 \quad \text{but} \quad f(x_{t_l}, t_l) < m + 2\alpha.$$

By Lemma 6.1(b) we can chose  $\hat{x}_{t_l} \in F(\bar{t}) \cap B_{2\delta_2}$  satisfying  $\|x_{t_l} - \hat{x}_{t_l}\| \leq L\|t_l - \bar{t}\| < \sigma$  (for large  $l$ ). With a Lipschitz constant  $c_1$  for  $f$  wrt.  $(x, t)$  near  $(\bar{x}, \bar{t})$  we obtain

$$\begin{aligned} |f(x_{t_l}, t_l) - f(\hat{x}_{t_l}, \bar{t})| &\leq c_1 (\|x_{t_l} - \hat{x}_{t_l}\| + \|t_l - \bar{t}\|) \\ &= O(\|t_l - \bar{t}\|) \end{aligned}$$

and thus by (6.8) using  $\delta_2 - \sigma \leq \|\hat{x}_{t_l} - \bar{x}\| \leq \delta_2 + \sigma$  (recall  $\|x_{t_l} - \hat{x}_{t_l}\| < \sigma$ ),

$$\begin{aligned} f(x_{t_l}, t_l) &\geq f(\hat{x}_{t_l}, \bar{t}) - O(\|t_l - \bar{t}\|) \\ &\geq m + 4\alpha - O(\|t_l - \bar{t}\|). \end{aligned}$$

Letting  $t_l \rightarrow \bar{t}$  this contradicts (6.11).

The relations (6.9) and (6.10) together assure that for all  $t \in B_{\varepsilon(\delta_3)}(\bar{t})$  there must exist a global minimizer  $x(t)$  on the compact set  $F(t) \cap \{x \mid \|x - \bar{x}\| \leq \delta_2\}$  satisfying  $\|x(t) - \bar{x}\| < \delta_2$ . Thus,  $x(t)$  is a local minimizer of  $P(t)$  and the existence statement is proven with  $\varepsilon := \varepsilon(\delta_3) = \frac{\delta_3}{\rho \|\xi\|}$  and  $\delta := \delta_2$ .

To prove the continuity condition let us assume to the contrary that there is some  $\delta_1 > 0$ ,  $\delta_1 \leq \delta$ , such that there exists a sequence  $t_l \rightarrow \bar{t}$  and for the global

minimizers  $x(t_l)$  of  $P(t_l)$  in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$  it holds  $\|x(t_l) - \bar{x}\| \geq \delta_1$ . We then have (for some subsequence)  $x(t_l) \rightarrow \hat{x} \neq \bar{x}$ ,  $\hat{x} \in B_\delta(\bar{x})$ . Closedness of  $F(t)$  (see Ex. 5.2) yields  $\hat{x} \in F(\bar{t})$ . But  $\hat{x} \neq \bar{x}$  means that  $\hat{x}$  is not a global minimizer of  $P(\bar{t})$  in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$ , i.e., with some  $\rho_1 > 0$  we have

$$f(\hat{x}, \bar{t}) \geq f(\bar{x}, \bar{t}) + \rho_1 .$$

In view of Lemma 6.1(b) there are points  $y(t_l) \in F(t_l)$  such that  $\|y(t_l) - \bar{x}\| = O(\|t_l - \bar{t}\|)$ . For these points we then obtain (note,  $y(t_l) \in B_\delta(\bar{x})$  for large  $l$ )  $f(y(t_l), t_l) \geq f(x(t_l), t_l)$  and thus

$$f(\bar{x}, \bar{t}) + O(\|t_l - \bar{t}\|) = f(y(t_l), t_l) \geq f(x(t_l), t_l) \rightarrow f(\hat{x}, \bar{t}) = f(\bar{x}, \bar{t}) + \rho_1 ,$$

leading to the contradiction

$$O(\|t_l - \bar{t}\|) \geq \rho_1 > 0 \quad \text{for all large } l .$$

□

**EX. 6.2.** *Let  $\bar{x}$  be an isolated local minimizer of  $P(\bar{t})$ , then  $\bar{x}$  is also a strict local minimizer.*

**Proof.** Assume  $\bar{x}$  is a local minimizer but not a strict local minimizer, i.e., there exist  $x_l \rightarrow \bar{x}$  with  $f(x_l, \bar{t}) = f(\bar{x}, \bar{t})$ . This means that  $\bar{x}$  is not an isolated local minimizer. □

MFCQ together with some compactness condition on  $F(t)$  implies Lipschitz continuity of the value function. We give a local and a global version.

**LEMMA 6.2.** *Let  $f, g_j$  be  $C^1$ -functions.*

(a) *Let  $\bar{x}$  be a strict local minimizer of  $P(\bar{t})$  such that MFCQ holds at  $(\bar{x}, \bar{t})$ . Then with the local minimizers of  $P(t)$ ,  $x(t) \in B_\delta(\bar{x})$  (see Theorem 6.1), and some  $L_1 > 0$ , for the minimum values  $f(x(t), t)$  we have*

$$|f(x(t), t) - f(\bar{x}, \bar{t})| \leq L_1 \|t - \bar{t}\| \quad \text{for all } t \in B_\varepsilon(\bar{t}) .$$

(b) *Let MFCQ hold at  $(\bar{x}, \bar{t})$  for any  $\bar{x} \in S(\bar{t})$ . Assume there exists a neighborhood  $B_{\varepsilon_0}(\bar{t})$ ,  $\varepsilon_0 > 0$ , such that  $\cup_{t \in B_{\varepsilon_0}(\bar{t})} F(t) \subset C_0$ , with  $C_0$  compact. In particular, this assures the existence of a global minimizer (at least one) of  $P(t)$  for all  $t \in B_{\varepsilon_0}(\bar{t})$ . Then there exists some  $\varepsilon > 0$ ,  $L_2 > 0$ , such that for the minimum value  $v(t)$  of  $P(t)$  we have*

$$|v(t) - v(\bar{t})| \leq L_2 \|t - \bar{t}\| \quad \text{for all } t \in B_\varepsilon(\bar{t}) .$$

**Proof.** (a) Let  $\bar{x}$  be such that  $\bar{x}$  is strict global minimizer in  $\text{cl } B_{2\delta}(\bar{x})$ ,  $\delta > 0$ . For  $t \in B_\varepsilon(\bar{t})$  let the point  $x(t) \in B_\delta(\bar{x})$  be the local minimizer of  $P(t)$  (global in  $\text{cl } B_\delta(\bar{x})$ ) according to Theorem 6.1. By Lemma 6.1(b) (choosing  $\delta, \varepsilon > 0$

smaller if necessary) to any  $t \in B_\varepsilon(\bar{t})$  there exists  $y(t) \in F(\bar{t}) \cap B_{2\delta}(\bar{x})$  such that with  $L > 0$  we have

$$(6.12) \quad \|y(t) - x(t)\| \leq L\|t - \bar{t}\| .$$

So, in view of  $f(\bar{x}, \bar{t}) \leq f(y(t), \bar{t})$ , with Lipschitz constants

$$c_1 := \max_{(x,t) \in \text{cl}(B_\delta(\bar{x}) \times B_\varepsilon(\bar{t}))} \|\nabla_x f(x, t)\| , \quad c_2 := \max_{(x,t) \in \text{cl}(B_\delta(\bar{x}) \times B_\varepsilon(\bar{t}))} \|\nabla_t f(x, t)\|$$

we find using (6.12),

$$\begin{aligned} f(\bar{x}, \bar{t}) - f(x(t), t) &\leq f(y(t), \bar{t}) - f(x(t), t) \\ &= \nabla_x f(x(t), t)(y(t) - x(t)) + \nabla_t f(x(t), t)(\bar{t} - t) \\ &\quad + o(\|y(t) - x(t)\|) + o(\|t - \bar{t}\|) \\ &\leq c_1\|y(t) - x(t)\| + c_2\|t - \bar{t}\| + o(\|t - \bar{t}\|) \\ &\leq (c_1L + c_2)\|t - \bar{t}\| + o(\|t - \bar{t}\|) \leq 2(c_1L + c_2)\|t - \bar{t}\| , \end{aligned}$$

if  $\|t - \bar{t}\|$  is small. In the same way choosing  $\tilde{x}(t) \in F(t)$  such that  $\|\tilde{x}(t) - \bar{x}\| \leq L\|t - \bar{t}\|$  (see Lemma 6.1(b)) we get

$$\begin{aligned} f(x(t), t) - f(\bar{x}, \bar{t}) &\leq f(\tilde{x}(t), t) - f(\bar{x}, \bar{t}) \\ &\leq 2(c_1L + c_2)\|t - \bar{t}\| \end{aligned}$$

and the statement is true with  $L_1 = 2(c_1L + c_2)$ .

(b) Let  $\bar{x} \in S(\bar{t})$  be a global minimizer of  $P(\bar{t})$ . By MFCQ at  $(\bar{x}, \bar{t})$  (see Lemma 6.1(b)) in some neighborhood  $B_\varepsilon(\bar{t})$  there exist  $x(t) \in F(t)$  such that  $\|x(t) - \bar{x}\| \leq L\|t - \bar{t}\|$ . In view of  $v(t) \leq f(x(t), t)$  we obtain as in (a) with some  $L_1 > 0$

$$v(t) - v(\bar{t}) \leq f(x(t), t) - f(\bar{x}, \bar{t}) \leq L_1\|t - \bar{t}\| .$$

The compactness condition  $\cup_{t \in B_{\varepsilon_0}(\bar{t})} F(t) \subset C_0$  assures that for all  $t \in B_\varepsilon(\bar{t})$  (with  $0 < \varepsilon < \varepsilon_0$ ) the set  $S(t)$  of global minimizers is nonempty and contained in the compact set  $C_0$ .

We now show that there exists  $c_2 > 0$  such that for all  $t \in B_\varepsilon(\bar{t})$  we have:

$$(6.13) \quad \begin{aligned} &\text{for any } x(t) \in S(t) \text{ there is some } \bar{x}(t) \in F(\bar{t}) \\ &\text{with } \|x(t) - \bar{x}(t)\| \leq c_2\|t - \bar{t}\| . \end{aligned}$$

If this holds then using  $v(\bar{t}) \leq f(\bar{x}(t), \bar{t})$ , for  $x(t) \in S(t)$  we obtain with some Lipschitz constant  $L'_1 > 0$ ,

$$\begin{aligned} v(\bar{t}) - v(t) &\leq f(\bar{x}(t), \bar{t}) - f(x(t), t) \\ &\leq \nabla_x f(x(t), t)(\bar{x}(t) - x(t)) \\ &\quad + \nabla_t f(x(t), t)(\bar{t} - t) + o(\|t - \bar{t}\|) \\ &\leq L'_1\|t - \bar{t}\| . \end{aligned}$$

Now it only remains to prove (6.13). Assume to the contrary that there exists sequences  $t_l \rightarrow \bar{t}$ ,  $x_l \in S(t_l)$ , such that

$$(6.14) \quad \min_{x \in F(\bar{t})} \frac{\|x_l - x\|}{\|t_l - \bar{t}\|} \rightarrow \infty \quad \text{for } l \rightarrow \infty .$$

By compactness condition (for a subsequence) we have  $x_l \rightarrow \hat{x}$ . By closedness of  $S(t)$  at  $\bar{t}$  (in view of MFCQ and thus CQ at  $(\bar{x}, \bar{t})$  for any  $\bar{x} \in S(\bar{t})$ , see Lemma 5.5) we must have  $\hat{x} \in S(\bar{t}) \subset F(\bar{t})$ . Again by MFCQ at  $(\hat{x}, \bar{t})$  (see Lemma 6.1(b)) to  $x_l$  (near  $\hat{x}$ ) there exists  $\bar{x}_l \in F(\bar{t})$  with  $\|x_l - \bar{x}_l\| \leq c_2 \|t_l - \bar{t}\|$  for some  $c_2 > 0$ . This contradicts (6.14).  $\square$

### 6.3. Lipschitz and Hölder stability of local minimizers of $P(t)$

In this section we are interested in the question whether a strict local minimizer  $\bar{x}$  of  $P(\bar{t})$  of order 2 behaves Lipschitz stable for  $t \approx \bar{t}$ .

#### Constant feasible set $F(t) = F_0$ .

We firstly consider the special case that the feasible set  $\emptyset \neq F_0$  does not depend on  $t$ , i.e.,  $g_j = g_j(x)$ ,  $j \in J$ , and only  $f(x, t)$  depends on  $t$ . In this case we obtain a general Lipschitz stability result (cf., also [6, Prop.4.36]).

**THEOREM 6.2.** *Assume  $f, g_j \in C^2$ . Let the feasible set  $\emptyset \neq F_0$  of  $P(t)$  be constant.*

(a) *Let  $\bar{x} = x(\bar{t})$  be a (strict) local minimizer of  $P(\bar{t})$  of order 2. Then there are  $\varepsilon, \delta, L > 0$  such that for all  $t \in B_\varepsilon(\bar{t})$  there exists a local minimizer  $x(t) \in B_\delta(\bar{x})$  of  $P(t)$  (at least one) and for each such local minimizer we have*

$$\|x(t) - \bar{x}\| \leq L \|t - \bar{t}\| .$$

(b) *If  $\bar{x}$  is a local minimizer of order 1 then we have for some  $\varepsilon > 0$ ,*

$$x(t) = \bar{x} \quad \text{for all } t \in B_\varepsilon(\bar{t}) .$$

**Proof.** (a) Since  $\bar{x}$  is a strict local maximizer of order 2 (see Definition 2.1) with some  $c > 0, \delta_1 > 0$  it holds:

$$(6.15) \quad f(x, \bar{t}) - f(\bar{x}, \bar{t}) \geq c \|x - \bar{x}\|^2 \quad \forall x \in B_{\delta_1}(\bar{x}) \cap F_0 .$$

The existence of (at least) one local minimizer  $x(t)$  of  $P(t)$  near  $\bar{x}$  follows easier than in the proof of Theorem 6.1. Since  $\bar{x}$  is a strict local minimizer with min-value  $m := f(\bar{x}, \bar{t})$ , by continuity, there exist numbers  $\varepsilon, \alpha > 0, \delta > 0, \delta < \delta_1$ , such that

$$(6.16) \quad f(\bar{x}, t) \leq m + \frac{\alpha}{2} \quad \forall t \in B_\varepsilon(\bar{t}) ,$$

$$f(x, \bar{t}) \geq m + 2\alpha \quad \forall x \in F_0, \|x - \bar{x}\| = \delta$$

$$(6.17) \quad f(x, t) \geq m + \alpha \quad \forall x \in F_0, \|x - \bar{x}\| = \delta, t \in B_\varepsilon(\bar{t}) .$$



The existence of a local minimizer  $x(t)$  with  $\|x(t) - \bar{x}\| < \delta$  for  $t \in B_\varepsilon(\bar{t})$  follows from (6.16) and (6.17) using the fact that  $\bar{x} \in F_0$  is feasible for  $P(t)$ .

For a local minimizer  $x := x(t) \in B_\delta(\bar{x})$  we find  $f(x, t) - f(\bar{x}, t) \leq 0$  and then

$$\begin{aligned} f(x, \bar{t}) - f(\bar{x}, \bar{t}) &= [f(x, \bar{t}) - f(x, t)] - [f(\bar{x}, \bar{t}) - f(\bar{x}, t)] + [f(x, t) - f(\bar{x}, t)] \\ &\leq [f(x, \bar{t}) - f(x, t)] - [f(\bar{x}, \bar{t}) - f(\bar{x}, t)] \\ &= \nabla_x [f(\bar{x} + \tau(x - \bar{x}), \bar{t}) - f(\bar{x} + \tau(x - \bar{x}), t)](x - \bar{x}) \end{aligned}$$

with some  $0 < \tau < 1$ . In the last equality we have applied the mean value theorem wrt.  $x$  to the function  $f(x, \bar{t}) - f(x, t)$ . By using  $\nabla_x [f(x, \bar{t}) - f(x, t)] = \nabla_{xt}^2 f(x, \bar{t})(t - \bar{t}) + o(\|t - \bar{t}\|)$  we find

$$f(x, \bar{t}) - f(\bar{x}, \bar{t}) \leq \max_{z \in \text{cl } B_\delta(\bar{x})} [ \|\nabla_{xt}^2 f(z, \bar{t})\| + o(1) ] \|t - \bar{t}\| \|x - \bar{x}\| .$$

Letting  $\gamma := \max_{z \in \text{cl } B_\delta(\bar{x})} [ \|\nabla_{xt}^2 f(z, \bar{t})\| + 1 ]$ , with (6.15) we obtain (recall  $x = x(t)$ )

$$c\|x - \bar{x}\|^2 \leq \gamma\|t - \bar{t}\| \cdot \|x - \bar{x}\| \quad \text{or} \quad \|x(t) - \bar{x}\| \leq \frac{\gamma}{c}\|t - \bar{t}\| ,$$

and the Lipschitz continuity result is valid with  $L := \gamma/c$ . Note that in the last step we tacitly assumed  $\|x(t) - \bar{x}\| \neq 0$ .

(b) Assume that there is a sequence  $t_l \rightarrow \bar{t}$  such that  $x(t_l) \neq \bar{x}$  for local minimizers  $x(t_l)$  of  $P(t_l)$  in  $B_\delta(\bar{x})$  (see (a)). By definition of a local minimizer of order 1, with some  $c, \delta_1 > 0$  we have

$$f(x, \bar{t}) - f(\bar{x}, \bar{t}) \geq c\|x - \bar{x}\| \quad \forall x \in B_{\delta_1}(\bar{x}) \cap F_0 .$$

Then arguing further as in (a) we obtain for  $x = x(t_l)$

$$c\|x(t_l) - \bar{x}\| \leq \gamma\|t_l - \bar{t}\| \cdot \|x(t_l) - \bar{x}\| \quad \text{or} \quad 1 \leq \frac{\gamma}{c}\|t_l - \bar{t}\| \quad \forall l .$$

which in view of  $t_l \rightarrow \bar{t}$  is a contradiction. □

We give an example showing that under the assumptions of Theorem 6.2(a) a strict local minimizer  $\bar{x}$  of  $P(\bar{t})$  (of order 2) may be perturbed into a whole set of local minimizers, i.e., the local uniqueness of the local minimizer is lost.

**Ex. 6.3.** We look at the parametric program:

$$P(t) : \quad \min -\frac{1}{2}x^T \begin{pmatrix} 1-2t & 1-t & 1-t \\ 1-t & 0 & -t \\ 1-t & -t & 0 \end{pmatrix} x \quad \text{s.t.} \quad x \in F_0$$

where  $F_0 = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1, x_i \geq 0, i = 1, 2, 3\}$ . Then for  $\bar{t} = 0$  the point  $\bar{x} = (1, 0, 0)$  is a strict minimizer of order 2 such that LICQ, SOC holds but not SC (two of the active multipliers are zero). It can be shown that for small  $t > 0$ , the program  $P(t)$  has the (nonstrict) minimizers  $x_\tau(t) = \tau(1-t, 0, t) + (1-\tau)(1-t, t, 0)$ ,  $\tau \in [0, 1]$ . So the strict local minimizer is lost.

We will see later that even when SC does not hold under a stronger second order condition the strict local minimizer behaves locally Lipschitz stable (see Theorem 6.4).

### Feasible set $F(t)$ depending on $t$

Now we allow also the functions  $g_j(x, t)$  and thus  $F(t)$  to depend on  $t$ . Under the same further conditions as in Theorem 6.2 we only obtain a Hölder stability result with exponent  $1/2$  for the local minimizer. We start with an example.

**Ex. 6.4.** Consider for  $t \in \mathbb{R}$  the program

$$(6.18) \quad P(t) \quad \min_x f(x) := x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \quad x_2 \leq x_1^2, \quad x_2 \leq t.$$

It is not difficult to see that the minimizers are given by (see Figure 6.3)

$$x(t) = \begin{cases} (0, t) & t \leq 0 \\ (\pm\sqrt{|t|}, t) & 0 \leq t \leq \frac{1}{2} \\ (\pm\frac{1}{4}, \frac{1}{2}) & t \geq \frac{1}{2} \end{cases}$$

In view of the relation

$$\|x(t) - x(0)\| = \sqrt{t + t^2} = \sqrt{1+t} \cdot \sqrt{t} \quad \text{for } 0 \leq t \leq 1/2,$$

the solution functions are (only) Hölder continuous at  $\bar{t} = 0$ . In this example, at the strict local minimizer  $\bar{x} = (0, 0)$  of  $P(\bar{t})$ , MFCQ is satisfied and the KKT condition holds with SOC (but not with SOC2, see Definition 6.1). Note that at  $\bar{t} = 0$  a bifurcation of the minimizer curve occurs.

*Hint for the proof:* On the set  $x_2 = x_1^2$  the objective function takes the values

$$h(x_2) := f(x_1, x_2) = x_2 + (x_2 - 1)^2 = x_2^2 - x_2 + 1.$$

This function  $h(x_2)$  is strictly decreasing for  $0 \leq x_2 \leq \frac{1}{2}$ . Note that for the above program  $P(t)$  at  $t < 0$ , only the constraint  $x_2 \leq t$  is active. At  $\bar{t} = 0$  the other constraint  $x_2 \leq x_1^2$  becomes active and for the solution  $(0, 0)$  of  $P(0)$  the condition LICQ is not satisfied such that around  $t = 0$  the IFT cannot be applied (to the KKT conditions; otherwise the solution function  $x(t)$  would be a  $C^1$ -function und thus Lipschitz continuous).

We now present the Hölder stability result for the local minimizers. Such a result has firstly been presented in [11], however under the stronger second order conditions SSOC in Definition 6.2.

**THEOREM 6.3.** Let  $P(t)$  be given with  $f, g_j \in C^2$ .

(a) Let MFCQ hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ , and let  $\bar{x}$  be a local minimizer of order 2 of  $P(\bar{t})$ . (This holds, e.g., if the KKT conditions are satisfied at  $\bar{x}$  with SOC, cf., Theorem 2.4(b).)

Then there are neighborhoods  $B_\delta(\bar{x}), B_\varepsilon(\bar{t})$ ,  $\delta, \varepsilon > 0$ , such that for all  $t \in B_\varepsilon(\bar{t})$  there exists a local minimizer  $x(t) \in B_\delta(\bar{x})$  of  $P(t)$  (at least one) and for each such local minimizer we have with some  $c > 0$ ,

$$\|x(t) - \bar{x}\| \leq c\|t - \bar{t}\|^{1/2} \quad \text{for all } t \in B_\varepsilon(\bar{t}).$$

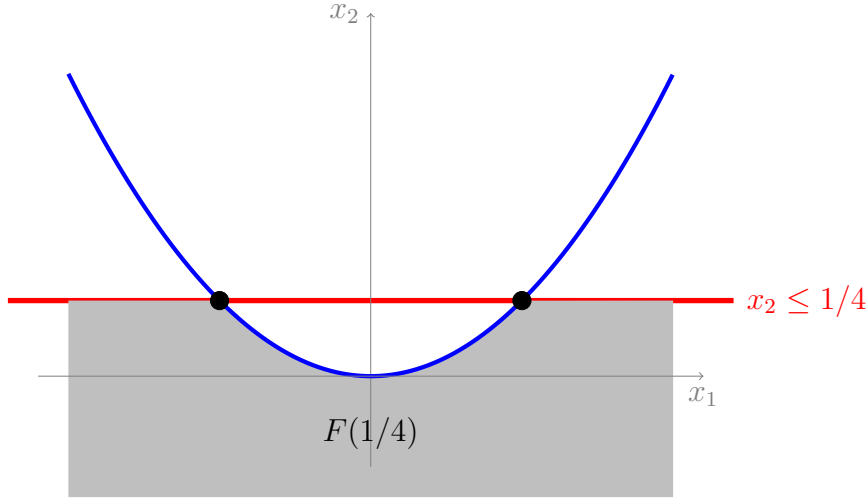


FIGURE 6.3. Feasible set of Ex. 6.4

(b) Let  $\bar{x}$  be a local minimizer of order 1. Then we have with some  $c, \varepsilon > 0$

$$\|x(t) - \bar{x}\| \leq c\|t - \bar{t}\| \quad \text{for all } t \in B_\varepsilon(\bar{t}).$$

**Proof.** (a) By assumption (see Theorem 2.4(b)) there exists  $B_{\delta_0}(\bar{x})$  and some  $\kappa > 0$  such that

$$(6.19) \quad f(x, \bar{t}) - f(\bar{x}, \bar{t}) \geq \kappa\|x - \bar{x}\|^2 \quad \forall x \in F(\bar{t}) \cap B_{\delta_0}(\bar{x}).$$

By Lemma 6.1(b) there exist  $\delta, \varepsilon > 0$ ,  $\delta < \delta_0$ , and  $L > 0$ , such that for any  $t \in B_\varepsilon(\bar{t})$  we can find some  $y(t) \in F(t) \cap B_\delta(\bar{x})$  with

$$(6.20) \quad \|y(t) - \bar{x}\| \leq L\|t - \bar{t}\|.$$

The existence of a (at least one) local minimizer  $x(t) \in B_{\delta_1}(\bar{x})$  (global in  $\text{cl } B_{\delta_1}(\bar{x})$ ) for some  $\delta_1$ ,  $0 < 2\delta_1 \leq \delta$  for all  $t \in B_{\varepsilon_1}(\bar{t})$  ( $0 < \varepsilon_1 \leq \varepsilon$ ) was proven in Theorem 6.1.

By Lemma 6.1(b) we also can choose some  $z(t) \in F(\bar{t}) \cap B_{2\delta_1}(\bar{x})$  satisfying

$$(6.21) \quad \|z(t) - x(t)\| \leq L\|t - \bar{t}\|.$$

With a Lipschitz constant  $c_2 > 0$  for  $f$  (wrt.  $(x, t)$ ) we find using (6.20) and (6.21),

$$\begin{aligned} f(\bar{x}, \bar{t}) \leq f(z(t), \bar{t}) &\leq f(x(t), t) + c_2(\|x(t) - z(t)\| + \|t - \bar{t}\|) \\ &\leq f(x(t), t) + c_2(L + 1)\|t - \bar{t}\| \\ &\leq f(y(t), t) + c_2(2L + 1)\|t - \bar{t}\| \\ &\leq f(\bar{x}, \bar{t}) + c_2(3L + 2)\|t - \bar{t}\| \end{aligned}$$

and thus

$$0 \leq f(z(t), t) - f(\bar{x}, \bar{t}) \leq c_2(3L + 2)\|t - \bar{t}\|.$$

Combining this with (6.21) and (6.19) we finally obtain for small  $\|t - \bar{t}\|$ ,

$$\begin{aligned}
(6.22) \quad \|x(t) - \bar{x}\| &\leq \|x(t) - z(t)\| + \|z(t) - \bar{x}\| \\
&\leq L\|t - \bar{t}\| + \left( \frac{f(z(t), \bar{t}) - f(\bar{x}, \bar{t})}{\kappa} \right)^{1/2} \\
&\leq \left( \frac{2c_2(3L+2)}{\kappa} \right)^{1/2} \|t - \bar{t}\|^{1/2}.
\end{aligned}$$

(b) In this case, with some  $\delta_0, c > 0$  we have

$$f(x, \bar{t}) - f(\bar{x}, \bar{t}) \geq \kappa\|x - \bar{x}\| \quad \forall x \in F(\bar{t}) \cap B_{\delta_0}(\bar{x}).$$

Then arguing further as in (a) instead of (6.22) we find

$$\begin{aligned}
\|x(t) - \bar{x}\| &\leq L\|t - \bar{t}\| + \left( \frac{f(z(t), \bar{t}) - f(\bar{x}, \bar{t})}{\kappa} \right) \\
&\leq \left( \frac{L + c_2(3L+2)}{\kappa} \right) \|t - \bar{t}\|.
\end{aligned}$$

□

In the proof of the next main theorem we need an auxiliary result concerning the Lagrangean multipliers  $\mu(t) \in \mathbb{R}^{|J_0(\bar{x}, \bar{t})|}$  of the local minimizers  $x(t)$  of  $P(t)$  in Theorem 6.3. Recall that the multiplier set  $M(x(t), t)$  corresponding to the KKT point  $x(t)$  is defined by

$$\begin{aligned}
(6.23) \quad M(x(t), t) &= \{ \mu \in \mathbb{R}^{|J_0(\bar{x}, \bar{t})|} \mid \nabla_x f(x(t), t) \\
&\quad + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_x g_j(x(t), t) = 0, \quad \mu \geq 0 \}.
\end{aligned}$$

**LEMMA 6.3.** *Let the assumption of Theorem 6.1 hold. Then the following osc condition is valid for  $M(x(t), t)$  at  $\bar{t}$ : There exist  $B_\varepsilon(\bar{t}), \varepsilon > 0$ , and  $L_1, L_2 > 0$ , such that for any  $\mu(t) \in M(x(t), t)$ ,  $t \in B_\varepsilon(\bar{t})$ , there is some  $\bar{\mu}(t) \in M(\bar{x}, \bar{t})$  satisfying*

$$\|\mu(t) - \bar{\mu}(t)\| \leq L_1\|x(t) - \bar{x}\| + L_2\|t - \bar{t}\|.$$

**Proof.** According to (6.23) the multiplier set  $M(x(t), t)$  is defined as a solution set of the linear equalities and inequalities for  $\mu_j$ ,  $j \in J_0(\bar{x}, \bar{t})$ ,

$$\begin{aligned}
\sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_x g_j(x(t), t) &= -\nabla_x f(x(t), t) \\
-\mu_j &\leq 0.
\end{aligned}$$

Putting  $J_0 := J_0(\bar{x}, \bar{t})$  and introducing the matrix function  $A(t) \in \mathbb{R}^{n \times |J_0|}$  with columns  $\nabla_x g_j(x(t), t)^T$ ,  $j \in J_0$ , and the vector function  $b(t) = -\nabla_x f(x(t), t)^T$ , this system becomes

$$A(t)\mu = b(t), \quad \text{and} \quad -I\mu \leq 0.$$

Let  $c_{1j}$ , resp.,  $c_{2j}$  be Lipschitz constants with respect to  $(x, t)$  for the rows  $(\frac{\partial}{\partial x_j} g_i(x(t), t), i \in J_0)$  of  $A(t)$ , resp., components  $-\frac{\partial}{\partial x_j} f(x(t), t)$  of  $b(t)$ . Then with  $L_0 := \max_{0 \leq j \leq n} \max\{c_{1j}, c_{2j}\}$  we find using  $\|(x, t)\| \leq \|x\| + \|t\|$ ,

$$\begin{aligned} \|A_j(\cdot)(t) - A_j(\cdot)(\bar{t})\| &\leq L_0 \|(x(t), t) - (\bar{x}, \bar{t})\| \leq L_0 (\|x(t) - \bar{x}\| + \|t - \bar{t}\|) \\ |b_j(t) - b_j(\bar{t})| &\leq L_0 \|(x(t), t) - (\bar{x}, \bar{t})\| \leq L_0 (\|x(t) - \bar{x}\| + \|t - \bar{t}\|). \end{aligned}$$

Recalling that by MFCQ the set  $M(\bar{x}, \bar{t})$  is nonempty and compact, the statement then follows from Lemma 5.9 (see also the proof) and Remark 5.1. We emphasize that  $\mu$  corresponds to  $x$  in Lemma 5.9 and that  $t$  in Lemma 5.9 is now  $(x(t), t)$ .  $\square$

If in comparison to Theorem 6.3 we impose a stronger second order condition we even obtain Lipschitz stability for the local minimizers.

**DEFINITION 6.1.** *We say that at a KKT point  $\bar{x}$  of  $P(\bar{t})$  the second order condition SOC2 is satisfied, if with the multiplier set*

$$M(\bar{x}, \bar{t}) := \{\mu \in \mathbb{R}^{|J_0(\bar{x}, \bar{t})|} \mid \nabla_x L(\bar{x}, \bar{t}, \mu) = 0, \mu \geq 0\}$$

it holds with some  $\beta > 0$ :

$$d^T \nabla_x^2 L(\bar{x}, \bar{t}, \mu) d \geq \beta \|d\|^2 \quad \forall d \in C_{\bar{x}, \bar{t}}, \text{ and } \forall \mu \in M(\bar{x}, \bar{t}).$$

**THEOREM 6.4.** *Let the assumptions of Theorem 6.3 be satisfied with SOC replaced by SOC2. Then for the local minimizers  $x(t)$ , with some  $\varepsilon, c_0 > 0$  it holds:*

$$\|x(t) - \bar{x}\| \leq c_0 \|t - \bar{t}\| \quad \text{for all } t \in B_\varepsilon(\bar{t}).$$

**Proof.** Assume to the contrary that there exists sequences  $t_l \rightarrow \bar{t}$ ,  $x_l := x(t_l) \rightarrow \bar{x}$ , with local minimizers  $x_l$  of  $P(t_l)$  such that

$$(6.24) \quad \frac{\|x_l - \bar{x}\|}{\|t_l - \bar{t}\|} \rightarrow \infty \quad \text{for } l \rightarrow \infty.$$

By MFCQ the Lagrangean multiplier vectors  $\mu_l \in M(x(t_l), t_l)$  corresponding to the KKT-points  $x_l$  are uniformly bounded (see Ex. 2.6), and by Lemma 6.3 there exist  $\mu'_l \in M(\bar{x}, \bar{t})$  with

$$(6.25) \quad \|\mu_l - \mu'_l\| \leq O(\|x_l - \bar{x}\|) + O(\|t_l - \bar{t}\|).$$

We put

$$\sigma_l := \|x_l - \bar{x}\|, \quad \tau_l := \|t_l - \bar{t}\|, \quad d_l := \frac{x_l - \bar{x}}{\sigma_l}.$$

Taylor expansion yields for  $j \in J_0(\bar{x}, \bar{t})$ ,

$$0 \geq g_j(x_l, t_l) - g_j(\bar{x}, \bar{t}) = \sigma_l \nabla_x g_j(\bar{x}, \bar{t}) d_l + o(\sigma_l) + O(\tau_l).$$

Since by Lemma 6.2(a) the local minimum values are Lipschitz continuous at  $\bar{x}$ , we also find

$$O(\tau_l) \geq f(x_l, t_l) - f(\bar{x}, \bar{t}) = \sigma_l \nabla_x f(\bar{x}, \bar{t}) d_l + o(\sigma_l) + O(\tau_l).$$

So, after division by  $\sigma_l$  we get

$$\nabla_x g_j(\bar{x}, \bar{t}) d_l \leq o(1) + O\left(\frac{\tau_l}{\sigma_l}\right), \quad \nabla_x f(\bar{x}, \bar{t}) d_l \leq o(1) + O\left(\frac{\tau_l}{\sigma_l}\right).$$

In view of (6.24) using  $d_l \rightarrow \bar{d}$ ,  $\|\bar{d}\| = 1$  (for some subsequence), we find

$$\nabla_x g_j(\bar{x}, \bar{t}) \bar{d} \leq 0, \quad j \in J_0(\bar{x}, \bar{t}), \quad \nabla_x f(\bar{x}, \bar{t}) \bar{d} \leq 0.$$

Thus  $0 \neq \bar{d} \in C_{\bar{x}, \bar{t}}$ . We now define  $g := (g_j, j \in J_0(\bar{x}, \bar{t}))$ ,  $z_l := (x_l, \mu_l)$ ,  $z'_l := (\bar{x}, \mu'_l)$  and

$$H(z, t) := \begin{pmatrix} \nabla_x L(x, t, \mu) \\ -g(x, t) \end{pmatrix} \quad \text{for } z = (x, \mu)$$

with Jacobian

$$\nabla_z H(z, t) = \begin{pmatrix} \nabla_x^2 L(x, t, \mu) & \nabla_x g(x, t) \\ -\nabla_x g(x, t)^T & 0 \end{pmatrix}.$$

Note that we have

$$\begin{aligned} (x, \mu)^T \nabla_z H(z, t) \begin{pmatrix} x \\ \mu \end{pmatrix} &= x^T \nabla_x^2 L(x, t, \mu) x + x^T \nabla_x g(x, t) \mu - \mu^T \nabla_x g(x, t)^T x \\ (6.26) \qquad \qquad \qquad &= x^T \nabla_x^2 L(x, t, \mu) x. \end{aligned}$$

We recall the monotonicity condition in Ex. 4.8,

$$-(\mu_l - \mu'_l)^T (g(x_l, t_l) - g(\bar{x}, \bar{t})) \leq 0.$$

Together with the KKT condition for  $x_l, \bar{x}$  this yields

$$\begin{aligned} A_1 &:= (H(z_l, t_l) - H(z'_l, \bar{t})) (z_l - z'_l) \\ &= (\nabla_x L(x_l, t_l, \mu_l) - \nabla_x L(\bar{x}, \bar{t}, \mu'_l)) (x_l - \bar{x}) \\ &\quad - (\mu_l - \mu'_l)^T (g(x_l, t_l) - g(\bar{x}, \bar{t})) \\ &\leq 0. \end{aligned}$$

On the other hand, by the Mean Value Theorem in Lemma 9.1, for the vector-valued function  $H$ , and using SOC2 as well as (6.25), (6.26) and  $d_l \rightarrow \bar{d} \in C_{\bar{x}, \bar{t}}$ ,  $\|\bar{d}\| = 1$ , we get for large  $l$

$$\begin{aligned} A_2 &:= (H(z_l, \bar{t}) - H(z'_l, \bar{t})) (z_l - z'_l) \\ &= (z_l - z'_l)^T \int_0^1 \nabla_z H(z'_l + \tau(z_l - z'_l), \bar{t}) (z_l - z'_l) d\tau \\ &= \sigma_l^2 \int_0^1 d_l^T \nabla_x^2 L(\bar{x} + \tau(x_l - \bar{x}), \bar{t}, \mu'_l + \tau(\mu_l - \mu'_l)) d_l d\tau \\ &= \sigma_l^2 \left[ \bar{d}^T [\nabla_x^2 L(\bar{x}, \bar{t}, \mu'_l) + o(1)] \bar{d} + o(1) \right] \\ &\geq \sigma_l^2 \beta + \sigma_l^2 o(1) \geq \frac{1}{2} \sigma_l^2 \beta. \end{aligned}$$

Altogether, again using (6.25) and the boundedness of  $\mu_l$ , we finally obtain

$$\begin{aligned} -\frac{1}{2}\beta\sigma_l^2 &\geq A_1 - A_2 = (H(z_l, t_l) - H(z_l, \bar{t}))(z_l - z'_l) \\ &= (\nabla_x L(x_l, t_l, \mu_l) - \nabla_x L(x_l, \bar{t}, \mu_l))(x_l - \bar{x}) \\ &\quad - (\mu_l - \mu'_l)^T (g(x_l, t_l) - g(x_l, \bar{t})) \\ &= O(\sigma_l \tau_l) + [O(\sigma_l) + O(\tau_l)]O(\tau_l) = O(\sigma_l \tau_l) + O(\tau_l^2). \end{aligned}$$

Division by  $\sigma_l \tau_l$  yields

$$-\frac{1}{2}\beta\frac{\sigma_l}{\tau_l} \geq O(1) + O\left(\frac{\tau_l}{\sigma_l}\right),$$

which in view of (6.24) is a contradiction. □

We give an example showing that under the assumptions of Theorem 6.4 at  $(\bar{x}, \bar{t})$ , still a bifurcation of the minimizer curve is possible.

**EX. 6.5.** [cf. [28, p.213]]

$$P(t) : \quad \min \frac{1}{2}(x_1^2 - x_2^2) - tx_1 \quad \text{s.t.} \quad \begin{cases} -x_1 + 2x_2 \leq 0 \\ -x_1 - 2x_2 \leq 0 \end{cases}.$$

Show that at the minimizer  $\bar{x} = (0, 0)$  of  $P(\bar{t})$ ,  $\bar{t} = 0$ , LICQ and SOC2 hold. The global minimizers of  $P(t)$  are given by

$$x(t) = \begin{cases} (0, 0) & \text{for } t \leq 0, \\ t\left(\frac{4}{3}, \pm\frac{2}{3}\right) & \text{for } t \geq 0, \end{cases}$$

with a bifurcation at  $\bar{t} = 0$ . For  $t > 0$  there is also a saddle point (KKT-point)  $\tilde{x}(t) = (t, 0)$ .

Under a stronger second order condition such a bifurcation is excluded.

**DEFINITION 6.2.** We say that the strong second order condition (SSOC) holds at a KKT point  $\bar{x}$  of  $P(\bar{t})$  if with the multiplier set  $M(\bar{x}, \bar{t}) := \{0 \leq \mu \in \mathbb{R}^{|J_0(\bar{x})|} \mid \nabla_x L(\bar{x}, \bar{t}, \mu) = 0\}$  it holds for some  $\beta > 0$ :

$$d^T \nabla_x^2 L(\bar{x}, \bar{t}, \mu) d \geq \beta \|d\|^2 \quad \forall d \in T_{\bar{x}, \bar{t}}(\mu) \text{ and } \forall \mu \in M(\bar{x}, \bar{t}),$$

where  $T_{\bar{x}, \bar{t}}(\mu)$  is the tangent space  $T_{\bar{x}, \bar{t}}(\mu) = \{d \mid \mu_j \nabla_x g_j(\bar{x}, \bar{t}) d = 0, j \in J_0(\bar{x}, \bar{t})\}$ .

Note that  $C_{\bar{x}, \bar{t}} \subset T_{\bar{x}, \bar{t}}(\mu)$  holds with  $T_{\bar{x}, \bar{t}}(\mu) = C_{\bar{x}, \bar{t}}$  if the SC-condition  $\mu > 0$  is satisfied.

The following result goes back to Robinson [28].

**THEOREM 6.5.** Let  $f, g_j \in C^2$  and let MFCQ hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ . Assume the KKT condition is satisfied at  $\bar{x}$  with SSOC. Then there exist neighborhoods  $B_\delta(\bar{x}), B_\varepsilon(\bar{t})$ ,  $\delta, \varepsilon > 0$ , such that for all  $t \in B_\varepsilon(\bar{t})$  there is a unique KKT point

$x(t)$  of  $P(t)$  in  $B_\delta(\bar{x})$ , and this point  $x(t)$  is thus an isolated local minimizer of  $P(t)$ . This minimizer function  $x(t)$  is continuous in  $B_\varepsilon(\bar{t})$  and by Theorem 6.4 satisfies with some  $c_0 > 0$ ,

$$\|x(t) - \bar{x}\| \leq \|t - \bar{t}\| \quad \text{for all } t \in B_\varepsilon(\bar{t}) .$$

**Proof.** The existence of (at least) one local minimizer  $x(t) \in B_\delta(\bar{x})$  for all  $B_\varepsilon(\bar{t})$  is shown in Theorem 6.1. By Lemma 6.1(a) MFCQ holds at all these minimizers  $x(t), t \in B_\varepsilon(\bar{t})$ . So these points are KKT points. Assume now to the contrary, that the KKT points  $x(t)$  are not unique in some (possibly smaller)  $B_\varepsilon(\bar{t})$ . Then there must exist a sequence  $t_l \rightarrow \bar{t}$  and KKT points  $y_l$  of  $P(t_l)$  in  $B_\delta(\bar{x})$  with  $x_l := x(t_l), x_l \neq y_l$ . By continuity of  $g_j$  we have

$$J_0(x_l, t_l), J_0(y_l, t_l) \subset J_0(\bar{x}, \bar{t}) ,$$

and by taking an appropriate subsequence, we can assume that

$$f(y_l, t_l) - f(x_l, t_l) \leq 0 \quad \forall l \quad \text{or} \quad f(y_l, t_l) - f(x_l, t_l) \geq 0 \quad \forall l$$

holds. Wlog. we suppose

$$(6.27) \quad f(y_l, t_l) - f(x_l, t_l) \leq 0 \quad \forall l .$$

For the Lagrangean multipliers  $\mu^{x_l} := (\mu_j^{x_l}, j \in J_0(\bar{x}, \bar{t}))$ ,  $\mu^{y_l} := (\mu_j^{y_l}, j \in J_0(\bar{x}, \bar{t}))$  wrt.  $x_l, y_l$  we have

$$\nabla_x L(x_l, t_l, \mu^{x_l}) = \nabla_x L(y_l, t_l, \mu^{y_l}) = 0 .$$

Since  $x_l, y_l \in B_\delta(\bar{x})$  and by the boundedness of the Lagrangean multipliers due to MFCQ (see Ex 2.6) (by taking a subsequence if necessary) we can assume

$$x_l \rightarrow \tilde{x} \in \text{cl } B_\delta(\bar{x}), \quad y_l \rightarrow \tilde{y} \in \text{cl } B_\delta(\bar{x}), \quad \text{and} \quad \mu^{x_l} \rightarrow \mu^{\tilde{x}}, \quad \mu^{y_l} \rightarrow \mu^{\tilde{y}} .$$

Since the mapping of KKT points of  $P(t)$  is closed at  $\bar{t}$ , with the unique KKT point  $\bar{x}$  of  $P(\bar{t})$  in  $B_\delta(\bar{x})$  (see Theorem 2.5), we have

$$\tilde{x} = \tilde{y} = \bar{x} \quad \text{and} \quad \mu^{\tilde{x}} \in M(\bar{x}, \bar{t}) .$$

By putting  $\sigma_l := \|y_l - x_l\|$ ,  $d_l := \frac{y_l - x_l}{\sigma_l}$  we can assume  $d_l \rightarrow \bar{d}$ ,  $\|\bar{d}\| = 1$ . A Taylor expansion using  $\mu_j^{x_l} > 0 \Rightarrow g_j(x_l, t_l) = 0$ , yields for  $j \in J_0(\bar{x}, \bar{t})$  with  $\mu_j^{x_l} > 0$ :

$$\begin{aligned} 0 &\geq g_j(y_l, t_l) - g_j(x_l, t_l) \\ &= \sigma_l \nabla_x g_j(x_l, t_l) d_l + \frac{1}{2} \sigma_l^2 d_l^T \nabla_x^2 g_j(x_l, t_l) d_l + o(\sigma_l^2) \end{aligned}$$

and see (6.27)

$$\begin{aligned} 0 &\geq f(y_l, t_l) - f(x_l, t_l) \\ &= \sigma_l \nabla_x f(x_l, t_l) d_l + \frac{1}{2} \sigma_l^2 d_l^T \nabla_x^2 f(x_l, t_l) d_l + o(\sigma_l^2) . \end{aligned}$$



Division by  $\sigma_l$  leads to

$$(6.28) \quad 0 \geq \nabla_x g_j(x_l, t_l) d_l + \frac{1}{2} \sigma_l d_l^T \nabla_x^2 g_j(x_l, t_l) d_l + o(\sigma_l) \quad \text{if } \mu_j^{x_l} > 0$$

$$(6.29) \quad 0 \geq \nabla_x f(x_l, t_l) d_l + \frac{1}{2} \sigma_l d_l^T \nabla_x^2 f(x_l, t_l) d_l + o(\sigma_l) .$$

Letting  $l \rightarrow \infty$  using  $d_l \rightarrow \bar{d}$  gives in view of  $\mu_j^{\tilde{x}} > 0 \Rightarrow \mu_j^{x_l} > 0$  (for  $l$  large)

$$\nabla_x g_j(\tilde{x}, \bar{t}) \bar{d} \leq 0 \quad \text{if } \mu_j^{\tilde{x}} > 0, \quad \nabla_x f(\tilde{x}, \bar{t}) \bar{d} \leq 0 .$$

By the KKT condition  $\nabla_x f(\tilde{x}, \bar{t}) \bar{d} = -\sum_{j \in J_0(\tilde{x}, \bar{t})} \mu_j^{\tilde{x}} \nabla_x g_j(\tilde{x}, \bar{t}) \bar{d} \geq 0$ , this implies  $\nabla_x f(\tilde{x}, \bar{t}) \bar{d} = 0$  and then

$$(6.30) \quad \nabla_x g_j(\tilde{x}, \bar{t}) \bar{d} = 0 \quad \text{if } \mu_j^{\tilde{x}} > 0 \quad \text{and thus } \bar{d} \in T_{\tilde{x}, \bar{t}}(\mu^{\tilde{x}}) .$$

Multiplication of (6.28) by  $\mu_j^{\tilde{x}}$  and adding to (6.29) finally gives

$$0 \geq \nabla_x L(x_l, t_l, \mu^{x_l}) d_l + \frac{1}{2} \sigma_l d_l^T \nabla_x^2 L(x_l, t_l, \mu^{x_l}) d_l + o(\sigma_l) ,$$

or in view of  $\nabla_x L(x_l, t_l, \mu^{x_l}) = 0$ ,

$$\frac{1}{2} \sigma_l d_l^T \nabla_x^2 L(x_l, t_l, \mu^{x_l}) d_l \leq o(\sigma_l) .$$

Division by  $\sigma_l$  and letting  $l \rightarrow \infty$  yields in view of  $\tilde{x} = \bar{x}$ ,

$$\bar{d}^T \nabla_x^2 L(\bar{x}, \bar{t}, \mu^{\tilde{x}}) \bar{d} \leq 0 \quad \text{with } \bar{d} \in T_{\bar{x}, \bar{t}}(\mu^{\tilde{x}}), \quad \|\bar{d}\| = 1 ,$$

contradicting SSOC.

We finally prove that in (a possible smaller)  $B_\varepsilon(\bar{t})$  the function  $x(t)$  is continuous. Assume that this is not the case. Then for any small  $\varepsilon > 0$  there exists  $t_\varepsilon, \|t_\varepsilon - \bar{t}\| < \varepsilon$  such that  $x(t)$  is not continuous in  $t_\varepsilon$ . This means that there is some sequence  $t_l \rightarrow t_\varepsilon$ , and some  $\alpha > 0$  such that

$$\|x(t_l) - x(t_\varepsilon)\| \geq \alpha \quad \forall l .$$

Now we can argue as in the proof of Lemma 6.1. By construction,  $x(t_l), x(t_\varepsilon) \in B_\delta(\bar{x})$  are global minimizers of  $P(t_l), P(t_\varepsilon)$  in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$ . So we can assume (for some subsequence)  $x(t_l) \rightarrow \hat{x} \neq x(t_\varepsilon), \hat{x} \in B_\delta(\bar{x})$ . Closedness of  $F(t)$  (see Ex. 5.2) yields  $\hat{x} \in F(t_\varepsilon)$ . But  $\hat{x} \neq x(t_\varepsilon)$  means that  $\hat{x}$  is not a global minimizer of  $P(t_\varepsilon)$  in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$ , i.e., with some  $\rho_1 > 0$  we have

$$f(\hat{x}, \bar{t}) \geq f(x(t_\varepsilon), t_\varepsilon) + \rho_1 .$$

In view of Lemma 6.1(b) there are points  $y(t_l) \in F(t_l)$  such that  $\|y(t_l) - x(t_\varepsilon)\| = O(\|t_l - t_\varepsilon\|)$ . For these points we then have (note  $y(t_l) \in B_\delta(\bar{x})$  for large  $l$ )  $f(y(t_l), t_l) \geq f(x(t_l), t_l)$  and thus

$$\begin{aligned} f(x(t_\varepsilon), t_\varepsilon) + O(\|t_l - t_\varepsilon\|) &= f(y(t_l), t_l) \\ &\geq f(x(t_l), t_l) \rightarrow f(\hat{x}, t_\varepsilon) \geq f(x(t_\varepsilon), t_\varepsilon) + \rho_1 , \end{aligned}$$

leading to the contradiction

$$O(\|t_l - t_\varepsilon\|) \geq \rho_1 > 0 \quad \text{for all large } l .$$

□

## 6.4. Stability of local minimizers under LICQ and SSOC

In this section we analyse stability properties under the stronger assumptions that at a KKT point  $\bar{x}$  of  $P(\bar{t})$ , LICQ and SSOC is satisfied but not SC. In this situation we can sharpen Theorem 6.4. The result goes back to Jittorntrum [17]. We start with an auxiliary lemma on the stability of tangent spaces.

**LEMMA 6.4.** *Let  $A(t) \in \mathbb{R}^{n \times k}$ ,  $t \in B_\varepsilon(\bar{t})$ ,  $\varepsilon > 0$ , be a continuous matrix function such that for all  $t \in B_\varepsilon(\bar{t})$  the matrix  $A(t)$  has full rank  $k$  ( $k \leq n$ ). Let*

$$T(t) := \{d \in \mathbb{R}^n \mid A(t)^T d = 0\} \quad \text{and} \quad C(t) := T(t) \cap \{d \mid \|d\| = 1\} .$$

*Then for any  $d(t) \in C(t)$ ,  $t \in B_\varepsilon(\bar{t})$ , there exists  $\bar{d}(t) \in C(\bar{t})$  such that*

$$\|d(t) - \bar{d}(t)\| \rightarrow 0 \quad \text{for } t \rightarrow 0 .$$

**Proof.** We firstly construct a continuous basis of the tangent spaces  $T(t)$  as follows. By applying Gauss elimination (see, e.g., [9, Section 2.1]) there exists some  $\varepsilon > 0$ , a permutation matrix  $P \in \mathbb{R}^{k \times k}$ , continuous nonsingular lower triangular matrices  $L(t) \in \mathbb{R}^{k \times k}$ , and continuous upper triangular matrices  $U(t) \in \mathbb{R}^{k \times n}$  of rank  $k$ , such that

$$L(t)PA(t)^T = U(t) \quad \text{for all } t \in B_\varepsilon(\bar{t}) .$$

So  $A(t)^T d = 0$  is equivalent with the triangular system  $U(t)d = 0$ . By a standard procedure, solving this triangular system recursively, we obtain a basis,

$$d_1(t), d_2(t), \dots, d_{n-k}(t) ,$$

of  $T(t)$ , continuously depending on  $t$ . The elements  $d(t)$  of  $T(t)$  are given as combinations

$$d(t) = \sum_{j=1}^{n-k} u_j d_j(t) = B(t)u ,$$

if  $B(t) \in \mathbb{R}^{n \times n-k}$  is the matrix with columns  $d_j(t)$ ,  $j = 1, \dots, n-k$ , and  $u = (u_1, \dots, u_{n-k})$ . Since  $B(t)$  has full rank  $n-k$ , the matrices  $B(t)^T B(t)$  are positive definite with smallest eigenvalue  $\lambda_1(t) > 0$  (see (8.2) later on),

$$\lambda_1(t) = \min_{u \neq 0} \frac{u^T B(t)^T B(t) u}{u^T u} .$$

It is well-known that the eigenvalues of a continuous symmetric matrix function are continuous in  $t$ . So, by making  $\varepsilon > 0$  smaller (if necessary), we can define

$$\lambda_1 := \min_{t \in \text{cl } B_\varepsilon(\bar{t})} \lambda_1(t) > 0 .$$

For any  $d = B(t)u \in T(t)$  with  $\|d\| = 1$  we then find

$$1 = d^T d = u^T B(t)^T B(t)u \geq \lambda_1 u^T u, \quad t \in B_\varepsilon(\bar{t}),$$

and thus

$$(6.31) \quad \|B(t)u\| = 1 \quad \Rightarrow \quad \|u\| \leq \frac{1}{\sqrt{\lambda_1}}.$$

Now for any  $d(t) = B(t)u \in T(t)$ ,  $\|d(t)\| = 1$ , i.e.,  $d(t) \in C(t)$ , we can define  $\tilde{d}(t) = B(\bar{t})u \in T(\bar{t})$  and  $\bar{d}(t) = \frac{\tilde{d}(t)}{\|\tilde{d}(t)\|} \in C(\bar{t})$ . Obviously, using Cauchy Schwarz inequality, in view of (6.31) we find

$$\begin{aligned} \|d(t) - \tilde{d}(t)\| &= \|(B(t) - B(\bar{t}))u\| = \left\| \sum_{j=1}^{n-k} u_j (d_j(t) - d_j(\bar{t})) \right\| \\ &\leq \sum_{j=1}^{n-k} |u_j| \|d_j(t) - d_j(\bar{t})\| \\ &\leq \left( \sum_{j=1}^{n-k} u_j^2 \right)^{1/2} \cdot \left( \sum_{j=1}^{n-k} \|d_j(t) - d_j(\bar{t})\|^2 \right)^{1/2} \\ &\leq \frac{1}{\sqrt{\lambda_1}} \cdot \left( \sum_{j=1}^{n-k} \|d_j(t) - d_j(\bar{t})\|^2 \right)^{1/2}. \end{aligned}$$

So recalling  $d_j(t) \rightarrow d_j(\bar{t})$  for  $t \rightarrow \bar{t}$ , we have  $\|d(t) - \tilde{d}(t)\| \rightarrow 0$  and  $\|\tilde{d}(t)\| \rightarrow \|d(t)\| = 1$ . This proves  $\|d(t) - \bar{d}(t)\| \rightarrow 0$  for  $t \rightarrow \bar{t}$ .  $\square$

We now analyse stability under the stronger assumptions that at a KKT point  $\bar{x}$  of  $P(\bar{t})$  the conditions LICQ and SSOC are satisfied but not SC. Recall that  $\bar{x}$  is then an isolated local minimizer of order 2. Under these conditions we can sharpen the statements of Theorem 6.5.

**THEOREM 6.6.** *Assume  $f, g_j \in C^2$ . Let LICQ hold at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ , as well as the KKT condition with (unique) multipliers  $\bar{\mu}_j \geq 0, j \in J_0(\bar{x}, \bar{t})$ , such that SSOC is satisfied (see, Definition 6.2).*

*Then there exist neighborhoods  $B_\delta(\bar{x}), B_\varepsilon(\bar{t})$ ,  $\delta, \varepsilon > 0$ , and functions  $x(t) : B_\varepsilon(\bar{t}) \rightarrow B_\delta(\bar{x})$ ,  $\mu(t) : B_\varepsilon(\bar{t}) \rightarrow \mathbb{R}^{|J_0(\bar{x}, \bar{t})|}$ , continuous on  $B_\varepsilon(\bar{t})$ , such that  $x(t)$  is unique KKT point of  $P(t)$  in  $B_\delta(\bar{x})$  with unique multiplier vector  $\mu(t)$ . Moreover,  $x(t)$  satisfies LICQ and for  $x(t), \mu(t)$  SSOC holds for all  $t \in B_\varepsilon(\bar{t})$ . In particular, the points  $x(t)$  are isolated local minimizers of  $P(t)$  of order 2.*

**Proof.** Recall that LICQ implies MFCQ at  $(\bar{x}, \bar{t})$  (see Ex. 2.5). So, the assumptions of Theorem 6.5 are satisfied, assuring the existence of the continuous local

minimizer function  $x(t)$  in  $B_\delta(\bar{x})$ . Since  $\nabla g_j(\bar{x}, \bar{t}), j \in J_0(\bar{x}, \bar{t})$ , are linearly independent, by continuity in (a possibly smaller)  $B_\varepsilon(\bar{t})$ , also the gradients

$$\nabla_x g_j(x(t), t), \quad j \in J_0(\bar{x}, \bar{t}), \quad \text{are linearly independent.}$$

Also recall, that by continuity we have

$$J_0(x(t), t) \subset J_0(\bar{x}, \bar{t}),$$

so that LICQ also holds at  $(x(t), t)$ . Consequently for  $t \in B_\varepsilon(\bar{t})$  there is a unique multiplier vector  $\mu(t) \in \mathbb{R}^{|J_0(\bar{x}, \bar{t})|}$  for the KKT point  $x(t)$ . We now prove the

continuity of  $\mu(t)$ : By defining the continuous functions  $b(t) = -\nabla_x f(x(t), t)$ ,  $A(t) = (\nabla_x g_j(x(t), t))^T, j \in J_0(\bar{x}, \bar{t})$  the KKT condition for  $x(t)$  reads

$$A(t)\mu = b(t),$$

with unique solution  $\mu(t) \geq 0$ . This solution must satisfy

$$A(t)^T A(t)\mu = A(t)^T b(t).$$

By LICQ the matrix  $A(t)^T A(t)$  is positive definite (and thus nonsingular), so that also the latter system has a unique solution which must coincide with the unique multiplier  $\mu(t)$ . So  $\mu(t)$  is explicitly given by

$$\mu(t) = (A(t)^T A(t))^{-1} A(t)^T b(t),$$

which is a continuous function of  $t$ . Next we prove

SSOC for  $x(t), \mu(t)$ : We show (see the SSOC condition in Definition 6.2),

$$(6.32) \quad d^T \nabla_x^2 L(x(t), t, \mu(t)) d \geq \frac{\beta}{2} \quad \text{for all } d \in T_{x(t), t}(\mu(t)), \quad \|d\| = 1,$$

where  $T_{x(t), t}(\mu(t))$  is the tangent space

$$T_{x(t), t}(\mu(t)) = \{d \mid \nabla_x g_j(x(t), t)d = 0 \quad \forall j \in J_0(\bar{x}, \bar{t}) \text{ with } \mu_j(t) > 0\},$$

and  $L$  is the (local near  $(\bar{x}, \bar{t})$ ) Lagrangean function

$$L(x, t, \mu) = f(x, t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j g_j(x, t).$$

Recall that by continuity we have for small  $\|t - \bar{t}\|$ ,

$$(6.33) \quad J_0(x(t), t) \subset J_0(\bar{x}, \bar{t}) \quad \text{and} \quad \mu_j(\bar{t}) > 0 \Rightarrow \mu_j(t) > 0.$$

If we define

$$T_{x(t), t}(\mu(\bar{t})) = \{d \mid \nabla_x g_j(x(t), t)d = 0 \quad \forall j \in J_0(\bar{x}, \bar{t}) \text{ with } \mu_j(\bar{t}) > 0\},$$

we thus have

$$(6.34) \quad T_{x(t), t}(\mu(t)) \subset T_{x(t), t}(\mu(\bar{t})).$$

In Lemma 6.4 we have proven that for any  $d(t) \in T_{x(t), t}(\mu(\bar{t}))$ ,  $\|d(t)\| = 1$ , there exists  $\bar{d}(t) \in T_{\bar{x}, \bar{t}}(\mu(\bar{t}))$ ,  $\|\bar{d}(t)\| = 1$ , such that  $\|d(t) - \bar{d}(t)\| \rightarrow 0$  for  $t \rightarrow \bar{t}$ .

Therefore, by continuity, the SSOC condition for  $(\bar{x}, \bar{t})$  implies that for  $t \in B_\varepsilon(\bar{t})$  (with possibly smaller  $\varepsilon$ ) the condition

$$d^T \nabla_x^2 L(x(t), t, \mu(t)) d \geq \frac{\beta}{2} \quad \text{for all } d \in T_{x(t), t}(\mu(\bar{t})), \|d(t)\| = 1,$$

is satisfied. In view of (6.34) this proves the SSOC condition (6.32) for  $(x(t), \mu(t))$ .  $\square$

We now will prove that the functions  $x(t), \mu(t)$  in Theorem 6.6 possess directional derivatives. Recall that the directional derivative, e.g., of  $x(t)$  at  $t$  into the direction  $z, \|z\| = 1$  is defined by

$$\nabla x(t; z) = \lim_{\tau \downarrow 0} \frac{x(t + \tau z) - x(t)}{\tau}.$$

So for fixed  $t \in B_\varepsilon(\bar{t})$  and  $z$  we have to consider the program  $P(t + \tau z)$  for small  $\tau \geq 0$ . Recall that by continuity, for the local minimizer  $x(t + \tau z)$  and corresponding multiplier vector  $\mu(t + \tau z)$  for small  $\tau > 0$  we have

$$(6.35) \quad g_j(x(t + \tau z), t + \tau z) \begin{cases} = 0 & \text{if } \mu_j(t) > 0 \\ \leq 0 & \text{if } \mu_j(t) = 0 \end{cases}, \quad j \in J_0(\bar{x}, \bar{t}).$$

So, for fixed  $t \in B_\varepsilon(\bar{t})$ , and fixed  $z$ , the program  $P(t + \tau z)$  with parameter  $\tau \geq 0$  reads:

$$P(t + \tau z) : \quad \min_x \quad f(x, t + \tau z) \quad \text{s.t.} \\ g_j(x, t + \tau z) \begin{cases} = 0 & \text{if } \mu_j(t) > 0 \\ \leq 0 & \text{if } \mu_j(t) = 0 \end{cases}, \quad j \in J_0(\bar{x}, \bar{t}).$$

These programs have unique local minimizers  $x(t + \tau z) \in B_\delta(\bar{x})$  with unique multiplier  $\mu(t + \tau z)$ . Note that for  $j \in J_0(\bar{x}, \bar{t})$  with  $\mu_j(t) = 0$ , for small  $\tau > 0$ , we may have

$$g_j(x(t + \tau z), t + \tau z) = 0 \quad \text{or} \quad g_j(x(t + \tau z), t + \tau z) < 0.$$

By continuity we have ( $t$  fixed)

$$J_0(x(t + \tau z), t + \tau z) \subset J_0(x(t), t) \subset J_0(\bar{x}, \bar{t}).$$

So, if we define

$$J_1(x(t), t) = \{j \in J_0(x(t), t) \mid \mu_j(t) > 0\}, \\ J_2(x(t), t) = \{j \in J_0(x(t), t) \mid \mu_j(t) = 0\},$$

for any index set  $R, J_1(x(t), t) \subset R \subset J_1(x(t), t) \cup J_2(x(t), t)$  we can look at the parametric program with parameter  $\tau \geq 0$  ( $t, z$  fixed):

$$(6.36) \quad P_R(t + \tau z) : \quad \min_x f(x, t + \tau z) \quad \text{s.t.} \quad g_j(x, t + \tau z) = 0, \quad j \in R.$$

Under the assumption of Theorem 6.6, this program satisfies LICQ and the second order condition in Ex.4.7. Consequently, the IFT can be applied, and for each  $R$  there exist local minimizer functions, differentiable in  $\tau$ , ( $t, z$  fixed) (see Ex. 4.7)

$$x_R(t + \tau z) \text{ of } P_R(t + \tau z) \text{ with unique multiplier } \mu_R(t + \tau z) .$$

**Note:** By construction, for each  $\tau \geq 0$  there exists some  $R = R(\tau)$ ,  $J_1(x(t), t) \subset R \subset J_1(x(t), t) \cup J_2(x(t), t)$  such that the local minimizer (in  $B_\delta(\bar{x})$ )

$$x(t + \tau z) \text{ of } P(t + \tau z) \text{ coincides with } x_R(t + \tau z) .$$

**THEOREM 6.7.** *Assume  $f, g_j \in C^2$ . Let the assumptions of Theorem 6.6 be satisfied at  $(\bar{x}, \bar{t})$ ,  $\bar{x} \in F(\bar{t})$ . Then, in addition to the statement in Theorem 6.6, for any  $t \in B_\varepsilon(\bar{t})$ , the local minimizer function  $x(t) \in B_\delta(\bar{x})$  and corresponding multiplier function  $\mu(t)$  have directional derivatives  $\nabla x(t; z)$ ,  $\nabla \mu(t; z)$  in any direction  $z$ ,  $\|z\| = 1$ . Moreover, in  $B_\varepsilon(\bar{t})$ , the value function  $v(t) := f(x(t), t)$  is differentiable with derivatives  $\nabla v(t) = \nabla_t L(x(t), t, \mu(t))$  and the second directional derivatives  $\nabla^2 v(t; z)$  exist.*

**Proof.** We start proving that for any  $t \in B_\varepsilon(\bar{t})$  the directional derivatives  $\nabla x(t; z)$ ,  $\nabla \mu(t; z)$  exist. We only treat the case  $t = \bar{t}$  since the arguments for any other fixed  $t$  is the same. To do so, for fixed  $z$ ,  $\|z\| = 1$ , we analyse the local minimizers  $x(\bar{t} + \tau z)$  of  $P(\bar{t} + \tau z)$  and multipliers  $\mu(\bar{t} + \tau z)$  for small  $\tau \geq 0$ . Let us introduce the abbreviations  $J_0 := J_0(\bar{x}, \bar{t})$  and

$$J_1 := \{j \in J_0 \mid \mu_j(\bar{t}) > 0\}, \quad J_2 := \{j \in J_0 \mid \mu_j(\bar{t}) = 0\} .$$

The following is known from the proof of Theorem 6.6 and the discussion above: there is some  $\rho > 0$  such that

- $x(\bar{t} + \tau z), \mu(\bar{t} + \tau z)$  are continuous in  $\tau$ , for  $0 \leq \tau \leq \rho$ ,
- $g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) = 0, \forall j \in J_1$  for  $\tau, 0 \leq \tau \leq \rho$ .
- For each  $\tau, 0 \leq \tau \leq \rho$ , there exists  $R = R(\tau)$ ,  $J_1 \subset R \subset J_1 \cup J_2$ , such that  $x(\bar{t} + \tau z)$  coincides with the differentiable minimizer function  $x_R(\bar{t} + \tau z)$  of  $P_R(\bar{t} + \tau z)$  in (6.36).

So, only for  $j \in J_2$ , depending on  $\tau > 0$ , there are two possible cases,

$$\begin{aligned} g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) &= 0 \quad \text{or} \\ g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) &< 0 \quad \text{and then } \mu_j(\bar{t} + \tau z) = 0 . \end{aligned}$$

We now introduce the two index sets,

$$\begin{aligned} R_1 &= \{j \in J_0 \mid g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) = 0 \forall 0 < \tau \leq \rho_1(j) \text{ for some } \rho_1(j) > 0\} \\ R_2 &= \{j \in J_2 \mid g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) < 0 \forall 0 < \tau \leq \rho_2(j) \text{ for some } \rho_2(j) > 0\} \end{aligned}$$

and put  $\rho_3 := \min\{\min_{j \in R_1} \rho_1(j), \min_{j \in R_2} \rho_2(j)\}$ ,  $R_3 := J_0 \setminus (R_1 \cup R_2)$ . Note that we have  $R_1 \cap R_2 = \emptyset$  and  $J_1 \subset R_1$ .

case  $R_3 = \emptyset$ : Then for all  $\tau, 0 < \tau < \rho_3$ , the minimizer  $x(\bar{t} + \tau z)$  coincides with the (differentiable) minimizer  $x_{R_1}(\bar{t} + \tau z)$  of  $P_{R_1}(\bar{t} + \tau z)$ .

case  $R_3 \neq \emptyset$ : For  $j \in R_3$  by definition the following holds:

if  $g_j(x(\bar{t} + \tau_1 z), \bar{t} + \tau_1 z) < 0$  for some  $0 < \tau_1 \Rightarrow$  there is some  $\tau_2$ ,  
 $0 < \tau_2 < \tau_1$ , with  $g_j(x(\bar{t} + \tau_2 z), \bar{t} + \tau_2 z) = 0$   
 if  $g_j(x(\bar{t} + \tau_2 z), \bar{t} + \tau_2 z) = 0$  for some  $0 < \tau_2 \Rightarrow$  there is some  $\tau_3$ ,  
 $0 < \tau_3 < \tau_2$ , with  $g_j(x(\bar{t} + \tau_3 z), \bar{t} + \tau_3 z) < 0$ .

So, for each  $j \in R_3$  and  $\tau \downarrow 0$  the values

$$(6.37) \quad g_j(x(\bar{t} + \tau z), \bar{t} + \tau z) \text{ alternate infinitely many times} \\ \text{between values } 0 \text{ and } < 0.$$

Assume for the moment that  $R_3 = \{j_0\}$  contains only one index. Then for any  $\tau$ ,  $0 < \tau < \rho_3$ , the local minimizer  $x(\bar{t} + \tau z)$  coincides with one of the differentiable local minimizer functions

$$x_{R_1}(\bar{t} + \tau z) \text{ of } P_{R_1}(\bar{t} + \tau z) \quad \text{or} \quad x_{R_1 \cup \{j_0\}}(\bar{t} + \tau z) \text{ of } P_{R_1 \cup \{j_0\}}(\bar{t} + \tau z).$$

Since  $j \in R_3$ , in view of (6.37), for  $\tau \downarrow 0$ , the continuous minimizer function  $x(\bar{t} + \tau z)$  will switch infinitely many times between the curves  $x_{R_1}(\bar{t} + \tau z)$  and  $x_{R_1 \cup \{j_0\}}(\bar{t} + \tau z)$ . Consequently, these curves intersect infinitely many times for  $\tau \downarrow 0$ . Both curves are differentiable and satisfy  $x_{R_1}(\bar{t}) = x_{R_1 \cup \{j_0\}}(\bar{t}) = x(\bar{t}) = \bar{x}$ . Thus the following limits exist and are equal:

$$\lim_{\tau \downarrow 0} \frac{x_{R_1}(\bar{t} + \tau z) - x_{R_1}(\bar{t})}{\tau} = \lim_{\tau \downarrow 0} \frac{x_{R_1 \cup \{j_0\}}(\bar{t} + \tau z) - x_{R_1 \cup \{j_0\}}(\bar{t})}{\tau},$$

and because  $x(\bar{t} + \tau z)$  coincides with either of these functions  $x_{R_1}$  or  $x_{R_1 \cup \{j_0\}}$  we must have

$$\begin{aligned} \nabla x(\bar{t}; z) &= \lim_{\tau \downarrow 0} \frac{x(\bar{t} + \tau z) - x(\bar{t})}{\tau} = \lim_{\tau \downarrow 0} \frac{x_{R_1}(\bar{t} + \tau z) - x_{R_1}(\bar{t})}{\tau} \\ &= \lim_{\tau \downarrow 0} \frac{x_{R_1 \cup \{j_0\}}(\bar{t} + \tau z) - x_{R_1 \cup \{j_0\}}(\bar{t})}{\tau}. \end{aligned}$$

Let now  $R_3$  consist of several indices. Then, there are subsets  $R'_3 \subset R_3$  with the property:

$$(6.38) \quad \text{for any } \rho > 0 \text{ there exists } \tau, 0 < \tau < \rho \text{ such that } x(\bar{t} + \tau z) \\ \text{coincides with the minimizer } x_{R_1 \cup R'_3}(\bar{t} + \tau z) \text{ of } P_{R_1 \cup R'_3}(\bar{t} + \tau z).$$

Now we define the set  $S := \{R'_3 \subset R_3 \mid (6.38) \text{ is satisfied}\}$ . Then for  $\tau \downarrow 0$  the curve  $x(\bar{t} + \tau z)$  switches infinitely many times between all curves  $x_{R_1 \cup R'_3}(\bar{t} + \tau z)$ ,  $R'_3 \in S$ . So, as for the case  $R_3 = \{j_0\}$  before, we can conclude that the following limits exist and are equal,

$$\nabla x(\bar{t}; z) = \lim_{\tau \downarrow 0} \frac{x(\bar{t} + \tau z) - x(\bar{t})}{\tau} = \lim_{\tau \downarrow 0} \frac{x_{R_1 \cup R'_3}(\bar{t} + \tau z) - x_{R_1 \cup R'_3}(\bar{t})}{\tau} \quad \forall R'_3 \in S.$$

Similar arguments can be used to show that the directional derivatives  $\nabla \mu(\bar{t}; z)$  exist.

To prove the statements on the directional derivatives of the value function, we take a fixed  $t \in B_\varepsilon(\bar{t})$  and small  $\tau \geq 0$ , and we look at the value function  $v(t +$

$\tau z) = f(x(t+\tau z), t+\tau z)$ . Recall that the minimizer  $x(t+\tau z)$  of  $P(t+\tau z)$  satisfy for small  $\tau \geq 0$  with corresponding multipliers  $\mu_j(t+\tau z)$  the KKT conditions

$$(6.39) \quad \nabla_x f(x(t+\tau z), t+\tau z) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j(t+\tau z) \nabla_x g_j(x(t+\tau z), t+\tau z) = 0$$

and

$$g_j(x(t+\tau z), t+\tau z) = 0 \quad \text{for } j \in J_0(\bar{x}, \bar{t}) \text{ with } \mu_j(t) > 0 .$$

Differentiation with respect to  $\tau$  yields at  $\tau = 0$ ,

$$(6.40) \quad \nabla_x g_j(x(t), t) \nabla x(t; z) + \nabla_t g_j(x(t), t) z = 0 \quad \begin{array}{l} \text{for } j \in J_0(\bar{x}, \bar{t}) \\ \text{with } \mu_j(t) > 0 . \end{array}$$

So we find using (6.39) and (6.40)

$$\begin{aligned} \nabla v(t; z) &= \frac{d}{d\tau} f(x(t+\tau z), t+\tau z)|_{\tau=0} \\ &= \nabla_x f(x(t), t) \nabla x(t; z) + \nabla_t f(x(t), t) z \\ &= - \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j(t) \nabla_x g_j(x(t), t) \nabla x(t; z) + \nabla_t f(x(t), t) z \\ &= \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j(t) \nabla_t g_j(x(t), t) z + \nabla_t f(x(t), t) z \\ &= \nabla_t L(x(t), t, \mu(t)) z . \end{aligned}$$

Since  $\nabla_t L(x(t), t, \mu(t)) z$  is linear in  $z$ , the value function is differentiable with gradient  $\nabla v(t) = \nabla_t L(x(t), t, \mu(t))$ . For the second directional derivative we obtain the formula

$$\begin{aligned} \nabla^2 v(t; z) &= \lim_{\tau \downarrow 0} \frac{\nabla v(t+\tau z; z) - \nabla v(t; z)}{\tau} = \frac{d}{d\tau} \nabla v(t+\tau z; z)|_{\tau=0} \\ &= \frac{d}{d\tau} \nabla_t L(x(t+\tau z), t+\tau z, \mu(t+\tau z)) z|_{\tau=0} \\ &= z^T \nabla_{tx}^2 L(x(t), t, \mu(t)) \nabla x(t; z) + z^T \nabla_t^2 L(x(t), t, \mu(t)) z \\ &\quad + \sum_{j \in J_0(\bar{x}, \bar{t})} \nabla \mu_j(t; z) \nabla_t g_j(x(t), t) z . \end{aligned}$$

□

**Remark.** In [17] it is stated that the directional derivative  $\nabla x(t; z)$  can be computed as unique minimizer of the quadratic program,

$$\begin{aligned} Q(z) : \quad \min_{\xi} \frac{1}{2} \xi^T \nabla_x^2 L(x(t), t, \mu(t)) \xi + z^T \nabla_{tx}^2 L(x(t), t, \mu(t)) \xi \quad \text{s.t.} \\ \nabla_x g_j(x(t), t) \xi + \nabla_t f(x(t), t) z \begin{cases} = 0 & \text{if } \mu_j(t) > 0 \\ \leq 0 & \text{if } \mu_j(t) = 0 \end{cases} , \quad j \in J_0(\bar{x}, \bar{t}) . \end{aligned}$$



The proof of this fact relies on results in [5], in particular, [5, Theorem(ii)]. For the proof of this Theorem the authors refer to [7, Theorem 2.2, p.539]. This reference however seems to be incorrect.

## 6.5. Stability without CQ, linear right-hand side perturbed system

For the local stability results above, the MFCQ condition played an essential role. We now consider a special class of parametric programs with parameters  $(t, r)$  where  $t \in T \subset \mathbb{R}^p$ ,  $T$  open, and  $r \in R \subset \mathbb{R}^m$ ,  $R$  open:

$$(6.41) \quad P(t, r) : \min_x f(x, t) \quad \text{s.t.} \quad x \in F(r) := \{x \mid Ax \leq r\}$$

where  $A \in \mathbb{R}^{m \times n}$  and  $f \in C^2(\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R})$ .

In this section we do not assume any constraint qualification (such as Slater's condition, see Ex 7.1). Such a constraint qualification will be replaced by the following (minimal) assumption.

**A<sub>ex</sub>:** Let  $(\bar{x}, \bar{r})$  with  $\bar{x} \in F(\bar{r})$ . We assume that  $F(\bar{r})$  is not a singleton and that there is some  $\delta_r > 0$  such that

$$(6.42) \quad F(r) \neq \emptyset, \quad \forall r \in B_{\delta_r}(\bar{r}).$$

We firstly are interested in the existence of local minimizers of  $P(t, r)$  near a strict local minimizer  $\bar{x}$  of  $P(\bar{t}, \bar{r})$ . Similar to Theorem 6.1 we obtain

**THEOREM 6.8.** *Let  $f$  be a  $C^1$ -function and let  $\bar{x}$  be a strict local minimizer of  $P(\bar{t}, \bar{r})$  such that **A<sub>ex</sub>** holds at  $(\bar{x}, \bar{r})$ . Then there are  $\delta, \varepsilon > 0$  such that for any  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  there exists a local minimizer  $x(t, r) \in B_\delta(\bar{x})$  of  $P(t, r)$  (at least one) which is a global minimizer in  $\{x \mid \|x - \bar{x}\| \leq \delta\}$ . Moreover these minimizers satisfy  $x(t, r) \rightarrow \bar{x}$  for  $(t, r) \rightarrow (\bar{t}, \bar{r})$ . More precisely, for any  $\delta_1$ ,  $0 < \delta_1 \leq \delta$ , there exists some  $\varepsilon_1$ ,  $0 < \varepsilon_1 \leq \varepsilon$ , such that we have*

$$x(t, r) \in B_{\delta_1}(\bar{x}) \quad \text{for all } (t, r) \in B_{\varepsilon_1}(\bar{t}, \bar{r}).$$

**Proof.** Let  $\bar{x}$  be a strict local minimizer of  $P(\bar{t}, \bar{r})$  such that

$$f(\bar{x}) < f(x) \quad \text{for all } x \in F(\bar{t}) \cap \text{cl } B_{\delta_0}(\bar{x}), \quad x \neq \bar{x},$$

for some  $\delta_0 > 0$ . By **A<sub>ex</sub>** there exists some  $\bar{x}_0 \neq \bar{x}$ ,  $\bar{x}_0 \in F(\bar{r})$ . Note that by convexity also the whole line segment  $[\bar{x}_0, \bar{x}]$  between  $\bar{x}$  and  $\bar{x}_0$  is contained in  $F(\bar{r})$ . Put  $\delta_2 = \frac{1}{2} \min\{\delta_0, \|\bar{x}_0 - \bar{x}\|\}$  and take the point  $\hat{x}_0 \in [\bar{x}_0, \bar{x}]$  such that  $\|\hat{x}_0 - \bar{x}\| = \delta_2$ . By assumption  $f(\bar{x}) < f(\hat{x}_0)$ . The continuity of  $f$  assures that there exist  $\alpha > 0$ , a small  $\sigma > 0$ , and  $\delta_3, 0 < \delta_3 < \delta_2 - \sigma$ , such that with min-value  $m := f(\bar{x}, \bar{t})$  it holds:

$$(6.43) \quad \begin{aligned} f(x, \bar{t}) &\leq m + \frac{\alpha}{2} \quad \forall x \in B_{\delta_3}(\bar{x}), \\ f(x, \bar{t}) &\geq m + 4\alpha \quad \forall x \in F(\bar{r}), \quad \delta_2 - \sigma \leq \|x - \bar{x}\| \leq \delta_2 + \sigma. \end{aligned}$$

By making  $\delta_3$  smaller, if necessary, due to the continuity of  $f$ , we can choose some  $\varepsilon > 0$  such that

$$(6.44) \quad f(x, t) \leq m + \alpha \quad \forall x \in B_{\delta_3}(\bar{x}), t \in B_\varepsilon(\bar{t})$$

$$(6.45) \quad f(x, t) \geq m + 2\alpha \quad \forall x \in F(r), \|x - \bar{x}\| = \delta_2, (t, r) \in B_\varepsilon(\bar{t}, \bar{r}) .$$

The first inequality is clear. For the second let us assume to the contrary, that there exist sequences  $(t_l, r_l) \rightarrow (\bar{t}, \bar{r})$ ,  $x_l \in F(r_l)$  such that

$$(6.46) \quad \|x_l - \bar{x}\| = \delta_2 \quad \text{but} \quad f(x_{t_l}, t_l) < m + 2\alpha .$$

By Corollary 5.1 using  $A_{ex}$  we can choose  $\hat{x}_l \in F(\bar{r}) \cap B_{\delta_2}(\bar{x})$  satisfying  $\|x_l - \hat{x}_l\| \leq L_0 \|r_l - \bar{r}\|$  (for large  $l$ ) with some  $L_0 > 0$ . With a Lipschitz constant  $c_1$  for  $f$  wrt.  $(x, t)$  near  $(\bar{x}, \bar{t})$  we obtain

$$\begin{aligned} |f(x_l, t_l) - f(\hat{x}_l, \bar{t})| &\leq c_1 (\|x_l - \hat{x}_l\| + \|t_l - \bar{t}\|) \\ &= O(\|(t_l, r_l) - (\bar{t}, \bar{r})\|) \end{aligned}$$

and thus by (6.43) for large  $l$ ,

$$\begin{aligned} f(x_l, t_l) &\geq f(\hat{x}_l, \bar{t}) - O(\|(t_l, r_l) - (\bar{t}, \bar{r})\|) \\ &\geq m + 4\alpha - O(\|(t_l, r_l) - (\bar{t}, \bar{r})\|) . \end{aligned}$$

Letting  $(t_l, r_l) \rightarrow (\bar{t}, \bar{r})$  this contradicts (6.46). From Corollary 5.1 we also can deduce that (for  $\varepsilon$  small enough) we must have

$$F(r) \cap B_{\delta_3}(\bar{x}) \neq \emptyset \quad \forall r \in B_\varepsilon(\bar{r}) .$$

The relations (6.44) and (6.45) together assure that for all  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  there must exist a global minimizer  $x(t, r)$  on the compact set  $F(r) \cap \{x \mid \|x - \bar{x}\| \leq \delta_2\}$  satisfying  $\|x(t, r) - \bar{x}\| < \delta_2$ . Thus,  $x(t, r)$  is a local minimizer of  $P(t, r)$  and the existence statement is proven with  $\varepsilon$  and  $\delta := \delta_2$ .

The proof of the continuity statement is the same as in the proof of Theorem 6.1.  $\square$

We also obtain a Lipschitz condition for the value function of our special program.

**LEMMA 6.5.** *Let  $f$  be a  $C^1$ -function and suppose,  $\bar{x}$  is a strict local minimizer of  $P(\bar{t}, \bar{r})$  such that  $A_{ex}$  holds at  $(\bar{x}, \bar{r})$ . Then with the local minimizers of  $P(t, r)$ ,  $x(t, r) \in B_\delta(\bar{x})$  in Theorem 6.8, and some  $\varepsilon, L_1 > 0$ , for the minimum values  $f(x(t, r), t)$  we have*

$$|f(x(t, r), t) - f(\bar{x}, \bar{t})| \leq L_1 \|(t, r) - (\bar{t}, \bar{r})\| \quad \text{for all } (t, r) \in B_\varepsilon(\bar{t}, \bar{r}) .$$

**Proof.** The proof follows with the same arguments as in the proof of Lemma 6.2, with the difference that instead of Theorem 6.1 we use Theorem 6.8 and instead of Lemma 6.1(b) under  $A_{ex}$  we have to apply Corollary 5.1.  $\square$

Concerning a stability result for the minimizer function  $x(t, r)$  of  $P(t, r)$  we could transfer the proof of the Hölder stability statement of Theorem 6.3 to our program (6.41). However for this special problem we expect generally a Lipschitz stability result as in Theorem 6.4. To do so we need at least some boundedness result for the Lagrangean multipliers as in Lemma 6.3. This is achieved with a trick which avoids the MFCQ assumption. We simply chose appropriate Lagrangean multipliers  $\mu(t, r)$  corresponding to  $x(t, r)$ .

So we again consider a strict local minimizer  $\bar{x}$  of  $P(\bar{t}, \bar{r})$  and the local minimizers  $x(t, r)$  of  $P(t, r)$  for  $(t, r)$  near  $(\bar{t}, \bar{r})$  in Theorem 6.8. Since  $P(t, r)$  has linear constraints (see Remark 2.2) the minimizers must satisfy the KKT condition

$$(6.47) \quad \nabla_x f(x(t, r), t) + \sum_{j \in J_0} \mu_j a_j = 0, \quad \mu_j \geq 0, \quad j \in J_0,$$

where  $a_j^T$  are the rows of  $A$  (see (6.41)), and  $J_0 := J_0(\bar{x}, \bar{r}) = \{j \in J \mid a_j^T \bar{x} = \bar{r}_j\}$  ( $J = \{1, \dots, m\}$ ) is the active index set of  $\bar{x}$ . Recall that by continuity for small  $\|(t, r) - (\bar{t}, \bar{r})\|$  we must have

$$J_0(x(t, r), r) \subset J_0.$$

To obtain bounded multipliers we chose  $\mu = \mu(t, r) \in \mathbb{R}^{|J_0|}$  as solutions of the parametric program

$$(6.48) \quad Q(t, r) : \min_{\mu} \|\mu\| \quad \text{s.t.} \quad \mu \text{ satisfies (6.47)}.$$

Note that the following problem has the same solutions:

$$(6.49) \quad \hat{Q}(t, r) : \min_{\mu} \frac{1}{2} \|\mu\|^2 \quad \text{s.t.} \quad \mu \text{ satisfies (6.47)}.$$

Since under  $A_{ex}$  for each  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  the feasible set of  $Q(t, r)$  (or  $\hat{Q}(t, r)$ ) is nonempty, by the Weierstrass Theorem 9.1 solutions of these programs exist. Indeed, with a feasible point  $\mu_0(t, r)$  ( $t, r$  fixed) the lower level set

$$\{\mu \mid \mu \text{ is feasible and } \|\mu\| \leq \|\mu_0(t, r)\|\}$$

is compact. Moreover, due to strict convexity of the objective  $\frac{1}{2} \mu^T \mu$  of  $\hat{Q}(t, r)$ , the solutions  $\mu(t, r)$  of  $\hat{Q}(t, r)$  and thus of  $Q(t, r)$  are uniquely determined. The next lemma proves a result for  $P(t, r)$  as in Lemma 6.3.

**LEMMA 6.6.** *Let  $A_{ex}$  hold at  $(\bar{x}, \bar{r})$ . Let  $f \in C^2$  and suppose,  $x(t, r)$  are the minimizers of  $P(t, r)$  for  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  ( $\bar{x} = x(\bar{t}, \bar{r})$ ) according to Theorem 6.8. Then for the unique solutions  $\mu(t, r)$  of  $Q(t, r)$  we have for  $(t, r) \rightarrow (\bar{t}, \bar{r})$ :*

$$(6.50) \quad \|\mu(t, r) - \mu(\bar{t}, \bar{r})\| = O(\|x(t, r) - \bar{x}\| + \|(t, r) - (\bar{t}, \bar{r})\|).$$

Since  $x(t, r) \rightarrow \bar{x}$  it follows  $\|\mu(t, r) - \mu(\bar{t}, \bar{r})\| \rightarrow 0$  for  $(t, r) \rightarrow (\bar{t}, \bar{r})$  and in particular the multipliers  $\mu(t, r)$  are uniformly bounded in  $B_\varepsilon(\bar{t}, \bar{r})$ .

**Proof. step 1.** Let  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  be fixed. With the local minimizer  $x(t, r)$  of  $P(t, r)$  and  $\bar{x}$  of  $P(\bar{t}, \bar{r})$  we define

$$b^1 := -\nabla_x f(x(t, r), t) \quad , \quad b^0 := -\nabla_x f(\bar{x}, \bar{t}) .$$

Then with a Lipschitz constant  $c_1$  for  $\nabla f$  wrt.  $(x, t)$  near  $(\bar{x}, \bar{t})$  we obtain

$$\begin{aligned} \|b^1 - b^0\| &= \|\nabla_x f(x(t, r), t) - \nabla_x f(\bar{x}, \bar{t})\| \\ &\leq c_1(\|x(t, r) - \bar{x}\| + \|t - \bar{t}\|) \\ (6.51) \quad &\leq c_1(\|x(t, r) - \bar{x}\| + \|(t, r) - (\bar{t}, \bar{r})\|) . \end{aligned}$$

Now we introduce the parametric program depending on parameter  $\mu \in \mathbb{R}^{|J_0|}$  (see (6.49)),

$$(6.52) \quad \tilde{Q}(b) : \quad \min_{\mu} \frac{1}{2} \mu^T \mu \quad \text{s.t.} \quad B_0 \mu = b, \quad -\mu \leq 0 ,$$

where  $B_0 := [a_j, j \in J_0]$  (with  $a_j$  the  $j$ th row of  $A$  in (6.41)), and again  $J_0 := J_0(\bar{x}, \bar{r})$ . Note that

$$\tilde{Q}(b^1) = \hat{Q}(t, r) \quad \text{and} \quad \tilde{Q}(b^0) = \hat{Q}(\bar{t}, \bar{r}) ,$$

and since  $\tilde{Q}(b)$  is a linearly constrained program,  $\mu$  is the (unique) solution of  $\tilde{Q}(b)$  if and only if  $\mu$  with corresponding Lagrangean multipliers  $\lambda \in \mathbb{R}^n$ ,  $\sigma \geq 0$  is a feasible point of  $F(b)$ ,

$$F(b) = \{(\mu, \lambda, \sigma) \mid (\mu, \lambda, \sigma) \text{ solves the system (6.53)}\} ,$$

$$(6.53) \quad \begin{aligned} B_0 \mu &= b , & -\mu &\leq 0 \\ B_0^T \lambda - \mu &= 0 , & \sigma &\leq 0 \\ \sigma_i \cdot \mu_i &= 0 , & i &\in J_0 . \end{aligned}$$

These feasibility conditions are linear except for the complementarity constraints  $\sigma_i \cdot \mu_i = 0$ ,  $i \in J_0$ . We can split up and simplify the system as follows: For any (fixed) index set  $I \subset J_0$  we introduce the feasible sets,

$$F_I(b) = \{(\mu, \lambda, \sigma) \mid (\mu, \lambda, \sigma) \text{ solves the system (6.54)}\} ,$$

$$(6.54) \quad \begin{aligned} B_0 \mu &= b , & -\mu &\leq 0 \\ B_0^T \lambda - \mu &= 0 , & \sigma &\leq 0 \\ \sigma_i &= 0 , & -\mu_i &\leq 0 , & i &\in I \\ -\sigma_i &\leq 0 , & \mu_i &= 0 , & i &\in J_0 \setminus I \end{aligned}$$

We emphasize that for each fixed  $b$  and for a corresponding point  $(\mu, \lambda, \sigma) \in F(b)$  there exists some  $I = I(b)$  such that  $(\mu, \lambda, \sigma) \in F_I(b)$ .

**step 2.** We now apply Corollary 5.1 to  $F_I(b)$  and conclude that for each  $I$  there exists  $L_I > 0$  such that for all  $b, b' \in \text{dom } F_I$  we have: to any  $(\mu, \lambda, \sigma) \in F_I(b)$  we can find  $(\mu', \lambda', \sigma') \in F_I(b')$  satisfying

$$(6.55) \quad \|\mu - \mu'\| \leq \|(\mu, \lambda, \sigma) - (\mu', \lambda', \sigma')\| \leq L_I \|b - b'\| .$$

By defining the line segment between  $b^1, b^0$ ,

$$\ell(\tau) = (1 - \tau)b^1 + \tau b^0, \quad \tau \in [0, 1],$$

we notice that by convexity, with  $F(b^1), F(b^0) \neq \emptyset$ , it follows  $F(\ell(\tau)) \neq \emptyset$  for all  $\tau \in [0, 1]$ . We introduce the sets

$$T(I) = \{\tau \in [0, 1] \mid F_I(\ell(\tau)) \neq \emptyset\}$$

and apply arguments as in the proof of [19, 3.1 Lemma] to show: there exists a partition  $0 \leq \tau_0 < \tau_1 < \dots < \tau_N = 1$  and index sets  $I_i \subset J_0$  such that for  $1 \leq i \leq N$  it holds,

$$\tau_{i-1} \in T(I_i) \quad \text{and} \quad \tau_i \in T(I_i).$$

To see this, note that the sets  $T(I)$  are closed. Indeed by continuity, for any  $I_1 \subset J_0$  and sequence  $\tau_l \rightarrow \bar{t}$  the relation

$$\tau_l \in T(I_1) \quad \forall l \quad \Rightarrow \quad \bar{t} \in T(I_1)$$

is true. Since there are only finitely many sets  $I_1$ , for  $\bar{t} = 0$  there must exist such an index set  $I_1$  and we can find such an  $I_1$  and some  $\tau_1, 0 < \tau_1 \leq 1$  with

$$0 \in T(I_1) \quad \text{and} \quad \tau_1 \in T(I_1).$$

We can chose  $I_1$  in such a way that  $\tau_1$  is maximal. This means either that  $\tau_1 = 1$  and we are finished, or that there is no  $\tau, \tau_1 < \tau \leq 1$  with  $\tau \in T(I_1)$ , i.e..  $I_1$  can no more appear in the further construction. In case  $\tau_1 < 1$ , we proceed in the same way with  $\tau_1$  and generate a maximal  $\tau_2, \tau_1 < \tau_2 \leq 1$  and corresponding set  $I_2 \neq I_1$  such that

$$\tau_1 \in T(I_2) \quad \text{and} \quad \tau_2 \in T(I_2),$$

and no  $\tau, \tau_2 < \tau \leq 1$  can exists with  $\tau \in T(I_2) \cup T(I_1)$ . After finitely many steps we end up with  $\tau_{N-1} < 1$  and  $I_N$  satisfying

$$\tau_{N-1} \in T(I_N) \quad \text{and} \quad 1 \in T(I_N).$$

Suppose to the contrary that we do not end up at  $\tau = 1$ . This means that the construction generates an infinite sequence  $\tau_l \rightarrow \bar{\tau} < 1$  with corresponding new sets  $I_l$ . But this cannot happen due to the fact that there are finitely many subsets  $I_l \subset J_0$  and in each step  $l - 1 \rightarrow l$  the new  $I_l$  can no more reappear.

final step. Now take the partition  $\tau_i$  and sets  $I_i, i = 1, \dots, N$ , in step 2, together with feasible points  $(\mu_{i-1}, \lambda_{i-1}, \sigma_{i-1}), (\mu_i, \lambda_i, \sigma_i)$  in  $F_{I_i}(\ell(\tau_i))$ . In view of (6.55) we find for the corresponding (unique)  $\mu_{i-1}, \mu_i$  the inequality

$$\|\mu_i - \mu_{i-1}\| \leq L_{I_i} \|\ell(\tau_i) - \ell(\tau_{i-1})\| = L_{I_i} (\tau_i - \tau_{i-1}) \|b^1 - b^0\|.$$

Defining  $L = \max_{1 \leq i \leq N} L_{I_i}$ , summing up, and recalling  $\mu_0 = \mu(t, r)$ ,  $\mu_N = \mu(\bar{t}, \bar{r})$  we find using (6.51)

$$\begin{aligned} \|\mu(t, r) - \mu(\bar{t}, \bar{r})\| &\leq \sum_{i=1}^N \|\mu_{i-1} - \mu_i\| \\ &\leq L \sum_{i=1}^N (\tau_i - \tau_{i-1}) \|b^1 - b^0\| \leq L \|b^1 - b^0\| \\ &\leq Lc_1 (\|x(t, r) - \bar{x}\| + \|(t, r) - (\bar{t}, \bar{r})\|) \end{aligned}$$

as stated. □

To obtain a result as in Theorem 6.4 we consider again the local minimizer  $\bar{x}$  of  $P(\bar{t}, \bar{r})$  and corresponding Lagrangean multiplier  $\bar{\mu} = \mu(\bar{t}, \bar{r})$ . For our linearly constrained program the Hessian of the Lagrangean function  $L$  reduces to

$$\nabla_x^2 L(\bar{x}, \bar{t}, \bar{r}, \bar{\mu}) = \nabla_x^2 f(\bar{x}, \bar{t}),$$

and the second order condition SOC2 in Definition 6.1 becomes: with some  $\beta > 0$ ,

$$(6.56) \quad d^T \nabla_x^2 f(\bar{x}, \bar{t}) d \geq \beta \|d\|^2 \quad \forall d \in C_{\bar{x}, \bar{t}, \bar{r}}.$$

We now can prove the Lipschitz stability of the local minimizers  $x(t, r)$  of  $P(t, r)$ .

**THEOREM 6.9.** *Let  $f \in C^2$  and let for the local minimizer  $\bar{x}$  of  $P(\bar{t}, \bar{r})$  the condition  $A_{ex}$  hold as well as the second order condition (6.56). Suppose,  $x(t, r)$  are the minimizers of  $P(t, r)$  for  $(t, r) \in B_\varepsilon(\bar{t}, \bar{r})$  ( $\bar{x} = x(\bar{t}, \bar{r})$ ) according to Theorem 6.8, with unique multipliers  $\mu(t, r)$  (cf., Lemma 6.6). Then for the local minimizers with some  $c_0 > 0$  we have:*

$$\|x(t, r) - \bar{x}\| \leq c_0 \|(t, r) - (\bar{t}, \bar{r})\| \quad \text{for all } (t, r) \in B_\varepsilon(\bar{t}, \bar{r}).$$

**Proof.** We can go step for step through the proof of Theorem 6.4. Assume to the contrary that there are sequences  $(t_l, r_l) \rightarrow (\bar{t}, \bar{r})$ ,  $x_l = x(t_l, r_l) \rightarrow \bar{x}$  with

$$(6.57) \quad \frac{\|x_l - \bar{x}\|}{\|(t_l, r_l) - (\bar{t}, \bar{r})\|} \rightarrow \infty \quad \text{for } l \rightarrow \infty.$$

Put  $\sigma_l = \|x_l - \bar{x}\|$ ,  $\tau_l = \|(t_l, r_l) - (\bar{t}, \bar{r})\|$ , and  $d_l = \frac{x_l - \bar{x}}{\sigma_l}$ . Here, we have given (unique) multipliers  $\mu_l = \mu(t_l, r_l)$  with the properties (see Lemma 6.6)

$$(6.58) \quad \|\mu(t, r) - \mu(\bar{t}, \bar{r})\| = O(\|x(t, r) - \bar{x}\| + \|(t, r) - (\bar{t}, \bar{r})\|).$$

Then as in the proof of Theorem 6.4 we find

$$d_l \rightarrow \bar{d} \in C_{\bar{x}, \bar{t}, \bar{r}}, \quad \|\bar{d}\| = 1,$$

and further (with  $\mu' = \mu(\bar{t}, \bar{r})$ )

$$-\frac{1}{2}\beta\frac{\sigma_l}{\pi} \geq O(1) + O\left(\frac{\pi_l}{\sigma_l}\right) \quad \text{for } l \rightarrow \infty ,$$

which with regard to (6.57) and  $\beta > 0$  yields a contradiction.  $\square$

**REMARK 6.1.** In this section we did not assume any constraint qualification. Since an equality  $Bx = r$  can be written equivalently as

$$Bx \leq r, \quad -Bx \leq -r ,$$

all results of this section are also valid for programs of the form (with  $r = (r^1, r^2)$ )

$$P(t, r) : \min_x f(x, t) \quad \text{s.t.} \quad Ax \leq r^1, \quad Bx = r^2 .$$

**REMARK 6.2.** Ex. 5.10 shows that the stability results of this section are no more valid for parametric linear constraints of the form  $A(t)x \leq r$  where also the matrix  $A(t)$  depends on a parameter.

## 6.6. Right-hand side perturbed quadratic problems

In this section we study special parametric programs with strictly convex quadratic objective and linear constraints depending on the parameters  $b \in \mathbb{R}^n$  and  $r \in \mathbb{R}^p$ ,

$$(6.59) \quad Q^=(b, r) : \min q(x) = \frac{1}{2}x^T Bx - b^T x \quad \text{s.t.} \quad x \in F^=(r) = \{x \mid A_0 x = r\}$$

or

$$(6.60) \quad Q(b, r) : \min q(x) = \frac{1}{2}x^T Bx - b^T x \quad \text{s.t.} \quad x \in F(r) = \{x \mid Ax \leq r\}$$

with fixed positive definite (symmetric) matrix  $B$ , and fixed matrices  $A_0, A \in \mathbb{R}^{p \times p}$ . We will show that the solutions  $x(b, r)$  of  $Q^=(b, r)$  are linear functions of the parameters  $(b, r) \in \mathbb{R}^n \times \mathbb{R}^p$  and the solutions of  $Q(b, r)$  are piecewise linear. We start with a lemma which assures the existence of a solution of these programs.

**LEMMA 6.7.** *Let be given the program*

$$Q : \min q(x) = \frac{1}{2}x^T Bx - b^T x \quad \text{s.t.} \quad x \in D ,$$

where  $B$  is positive definite,  $b \in \mathbb{R}^n$ , and  $D \subset \mathbb{R}^n$  is a nonempty, closed, convex set. Then a minimizer  $\bar{x}$  of  $Q$  exists and is uniquely determined.

**Proof.** By Lemma 2.2(b) the objective is strictly convex, so that according to Lemma 2.3(a) a minimizer  $\bar{x}$  of  $Q$  (if existent) is uniquely determined. To show the existence of a minimizer, take a point  $x_0$  of  $D$  and consider the lower level set of  $q$

$$\ell(x_0, q) = \{x \in D \mid q(x) \leq q(x_0)\} .$$

It is well-known that the smallest eigenvalue of the positive definite matrix  $B$  satisfies (see (8.2) later on)

$$0 < \lambda_0 = \min_{\|x\|=1} x^T B x .$$

This implies the inequality

$$q(x) \geq \frac{1}{2} \lambda_0 \|x\|^2 - \|b\| \|x\| \quad \forall x \in \mathbb{R}^n ,$$

and thus for all  $x \in \ell(x_0, q)$  we must have

$$\frac{1}{2} \lambda_0 \|x\|^2 - \|b\| \|x\| \leq q(x_0) .$$

By solving this quadratic inequality wrt.  $\|x\|$  we find that for all  $x \in \ell(x_0, q)$  it holds

$$\|x\| \leq \frac{2\|b\|}{\lambda_0} + \sqrt{\frac{2q(x_0)}{\lambda_0} + \frac{4\|b\|^2}{\lambda_0^2}} .$$

So the lower level set  $\ell(x_0, q)$  is compact (bounded) and a minimizer of  $Q$  exists according to the Weierstrass Theorem 9.1.  $\square$

In view of this lemma, for any  $r$  such that  $F^=(r) \neq \emptyset$  (or  $F(r) \neq \emptyset$ ), a solution of  $Q^=(b, r)$  (or  $Q(b, r)$ ) exists and is uniquely determined.

With regard to Remark 2.4 a point  $x(b, r)$  is (unique) minimizer of  $Q^=(b, r)$  if and only  $x = x(b, r)$  with a corresponding multiplier  $\lambda \in \mathbb{R}^p$  solves the system of KKT equations

$$(6.61) \quad \begin{array}{rcl} Bx + A_0^T \lambda & = & b \\ A_0 x & = & r \end{array} .$$

This characterisation leads to the following result.

**THEOREM 6.10.** *For  $r \in \text{dom } F^=$  the unique solutions  $x(b, r)$  of  $Q^=(b, r)$  are linear functions of  $b, r$ , i.e., there are matrices  $M^1 \in \mathbb{R}^{n \times n}$ ,  $M^2 \in \mathbb{R}^{n \times p}$  such that*

$$x(b, r) = M^1 b + M^2 r .$$

**Proof.** The unique solutions of  $Q^=(b, r)$  are given as unique solutions  $x = x(b, r)$  of (6.61) with corresponding (possibly nonunique) multipliers  $\lambda$ . It is well-known that the solution  $(x, \lambda)$  of (6.61) with smallest Euclidean norm is given by (see, e.g., [12, Section 5.5.4])

$$(6.62) \quad \begin{pmatrix} x(b, r) \\ \lambda(b, r) \end{pmatrix} = \begin{pmatrix} M^1 & M^2 \\ (M^2)^T & M^3 \end{pmatrix} \begin{pmatrix} b \\ r \end{pmatrix} ,$$

where

$$M^+ = \begin{pmatrix} M^1 & M^2 \\ (M^2)^T & M^3 \end{pmatrix} \quad \text{is the pseudoinverse of} \quad M = \begin{pmatrix} B & A_0^T \\ A_0 & 0 \end{pmatrix} .$$

$\square$



We now come to the inequality constrained program  $Q(b, r)$  and define the index set  $J = \{1, \dots, p\}$ . By Lemma 6.7 also here for all  $r \in \text{dom } F$  the minimizer  $x(b, r)$  exists and is uniquely determined as the solution  $x = x(b, r)$  with corresponding multiplier  $\mu \in \mathbb{R}^p$  of the KKT system

$$(6.63) \quad \begin{array}{rcl} Bx + A^T \mu & = & b, \quad \mu_j(r_j - A_j \cdot x) = 0, \quad j \in J \\ Ax & \leq & r \quad \quad \quad -\mu \leq 0 \end{array} .$$

We denote the solution set of this complementarity constrained system by  $S(b, r)$ . We can split up this solution set as follows. For any subset  $I \subset J$  we define the sets  $S_I(b, r) = \{(x, \mu) \mid (x, \mu) \text{ are solutions of (6.64)}\}$ , where

$$(6.64) \quad \begin{array}{rcl} Bx + A_I^T \mu_I & = & b, \quad A_j \cdot x < r_j, \quad j \in J \setminus I \\ A_{i \cdot} x & = & r_i, \quad i \in I \quad \quad \quad -\mu_I \leq 0 \end{array} .$$

Here,  $A_I$  denotes the submatrix of  $A$  containing the rows  $A_i$  with  $i \in I$ , and  $\mu_I$  is the corresponding subvector of  $\mu$ . By construction, for each solution  $(x, \mu)$  of (6.63) there exists some  $I \subset J$ ,  $I = I(x, \mu)$  such that  $(x, \mu) \in S_I(b, r)$ . More precisely we get a partition

$$(6.65) \quad S(b, r) = \cup_{I \subset J} S_I(b, r) \quad \text{and} \quad \text{dom } S = \cup_{I \subset J} \text{dom } S_I ,$$

where as usual the domains are defined as  $\text{dom } S = \{(b, r) \mid S(b, r) \neq \emptyset\}$  and  $\text{dom } S_I = \{(b, r) \mid S_I(b, r) \neq \emptyset\}$ . As projections of systems of linear equalities/inequalities the sets  $\text{dom } S_I$  are polyhedra (solution sets of linear inequalities). Moreover, on  $\text{dom } S_I$  the solutions  $x(b, r)$  and corresponding multipliers  $\mu$  of (6.64) are solutions of the system of equations

$$(6.66) \quad \begin{array}{rcl} Bx + A_I^T \mu_I & = & b \\ A_I x & = & r_I \end{array} .$$

So by the arguments above (cf., (6.61)), on  $\text{dom } S_I$  the unique solution  $x(b, r)$  of  $Q(b, r)$  is also unique solution of the program

$$Q_I^-(b, r) : \quad \min q(x) = \frac{1}{2} x^T B x - b^T x \quad \text{s.t.} \quad A_I x = r_I$$

and by Theorem 6.10 on  $\text{dom } S_I$  the minimizer function  $x(b, r)$  is given by

$$x(b, r) = M_I^1 b + M_I^2 r .$$

where

$$M_I^+ = \begin{pmatrix} M_I^1 & M_I^2 \\ (M_I^2)^T & M_I^3 \end{pmatrix} \quad \text{is the pseudoinverse of} \quad M_I = \begin{pmatrix} B & A_I^T \\ A_I & 0 \end{pmatrix} .$$

Summarizing we have proven the following piecewise linearity of the solution function  $x(b, r)$ .

**THEOREM 6.11.** *The domain  $\text{dom } S$  is the union of finitely many polyhedra  $\text{dom } S_I$ ,  $I \subset J$ . The (unique) solution function  $x(b, r)$  of  $Q(b, r)$  is continuous on  $\text{dom } S$  and linear on the polyhedra  $\text{dom } S_I$  :*

$$x(b, r) = M_I^1 b + M_I^2 r .$$

*The corresponding value function  $v(b, r) = \frac{1}{2}x(b, r)^T B x(b, r) - b^T x(b, r)$  is piecewise quadratic on dom  $S$ .*

## CHAPTER 7

### Directional differentiability of the value function

In this Chapter we analyse directional differentiability of the (global) value function  $v(t)$  of  $P(t)$  in (6.1),

$$v(t) = \inf_{x \in F(t)} f(x, t) .$$

As before,  $S(t)$  denotes the set of global minimizers of  $P(t)$ ,

$$S(t) = \{x \in F(t) \mid f(x, t) = v(t)\} .$$

**DEFINITION 7.1.** *Let  $v$  be a function  $v : T \rightarrow \mathbb{R} \cup \{\infty\}$ , where  $T$  is an open subset of  $\mathbb{R}^p$ . For  $t \in T$  and a direction vector  $z \in \mathbb{R}^n$ ,  $\|z\| = 1$  the limits*

$$\begin{aligned} \nabla_+ v(t; z) &= \limsup_{\tau \downarrow 0} \frac{v(t + \tau z) - v(t)}{\tau} , \\ \nabla_- v(t; z) &= \liminf_{\tau \downarrow 0} \frac{v(t + \tau z) - v(t)}{\tau} , \end{aligned}$$

are called *upper and lower directional derivatives of  $v$  at  $t$  into direction  $z$* . The value  $\nabla v(t; z) = \lim_{\tau \downarrow 0} \frac{v(t + \tau z) - v(t)}{\tau}$  (provided the limit exists) is the *directional derivative*.

Note that  $\limsup_{\tau \downarrow 0} g(\tau)$  and  $\liminf_{\tau \downarrow 0} g(\tau)$  are defined as

$$\limsup_{\tau \downarrow 0} g(\tau) = \lim_{\varepsilon \downarrow 0} \sup_{0 < \tau < \varepsilon} g(\tau) \quad \text{and} \quad \liminf_{\tau \downarrow 0} g(\tau) = \lim_{\varepsilon \downarrow 0} \inf_{0 < \tau < \varepsilon} g(\tau)$$

#### 7.1. Upper bound for $\nabla_+ v(t; z)$

Let be given  $\bar{t} \in T$ ,  $\bar{x} \in S(\bar{t})$  and a direction  $z$ ,  $\|z\| = 1$ . Assume MFCQ holds at  $(\bar{x}, \bar{t})$ . Recall that by MFCQ the minimizer  $\bar{x}$  must satisfy the KKT conditions (see Theorem 2.3). So, by Ex. 2.6(b) the set  $M(\bar{x}, \bar{t})$  of multipliers,

$$M(\bar{x}, \bar{t}) = \{\mu \in \mathbb{R}_+^{|J_0(\bar{x}, \bar{t})|} \mid \nabla_x L(\bar{x}, \bar{t}, \mu) = 0\} ,$$

is compact and nonempty. Here again  $L(x, t, \mu) = f(x, t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j g_j(x, t)$  denotes the (local) Lagrangean.

For perturbations  $t = \bar{t} + \tau z$ ,  $\tau > 0$ , we now consider the point  $x_\tau = \bar{x} +$

$\tau\xi$ , where we are going to determine a MFCQ-vector  $\xi$  in an "optimal" way. Assuming  $x_\tau \in F(\bar{t} + \tau z)$ , a Taylor expansion yields

$$(7.1) \quad \begin{aligned} v(\bar{t} + \tau z) - v(\bar{t}) &\leq f(x_\tau, \bar{t} + \tau z) - f(\bar{x}, \bar{t}) \\ &= \tau[\nabla_x f(\bar{x}, \bar{t})\xi + \nabla_t f(\bar{x}, \bar{t})z] + o(\tau). \end{aligned}$$

The feasibility condition  $x_\tau \in F(\bar{t} + \tau z)$  requires for any  $j \in J_0(\bar{x}, \bar{t})$ :

$$(7.2) \quad g_j(x_\tau, \bar{t} + \tau z) - g_j(\bar{x}, \bar{t}) = \tau[\nabla_x g_j(\bar{x}, \bar{t})\xi + \nabla_t g_j(\bar{x}, \bar{t})z] + o(\tau) \leq 0.$$

To minimize the upper bound in (7.1) we determine  $\xi$  as a minimizer of the linear program depending on the parameter  $z$ ,

$$\begin{aligned} P_{\bar{x}, \bar{t}}^+(z) : \quad \min_{\xi} \quad & [\nabla_x f(\bar{x}, \bar{t})\xi + \nabla_t f(\bar{x}, \bar{t})z] \quad \text{s.t.} \\ & [\nabla_x g_j(\bar{x}, \bar{t})\xi + \nabla_t g_j(\bar{x}, \bar{t})z] \leq 0, \quad j \in J_0(\bar{x}, \bar{t}). \end{aligned}$$

Let  $v_{\bar{x}, \bar{t}}^+(z)$  denote the value of this program. If we write the objective as  $\max_{\xi} -[\nabla_x f(\bar{x}, \bar{t})\xi + \nabla_t f(\bar{x}, \bar{t})z]$  the dual becomes

$$\begin{aligned} \min_{\mu \geq 0} \quad & - \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_t g_j(\bar{x}, \bar{t})z - \nabla_t f(\bar{x}, \bar{t})z \quad \text{s.t.} \\ & \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j \nabla_x g_j(\bar{x}, \bar{t}) = -\nabla_x f(\bar{x}, \bar{t}). \end{aligned}$$

So, the dual of  $P_{\bar{x}, \bar{t}}^+(z)$  is given by

$$D_{\bar{x}, \bar{t}}^+(z) : \quad \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

Obviously, MFCQ at  $(\bar{x}, \bar{t})$  implies that the program  $P_{\bar{x}, \bar{t}}^+(z)$  is feasible and due to  $M(\bar{x}, \bar{t}) \neq \emptyset$ , the dual is feasible as well. By LP duality (see Theorem 2.2) it follows that both programs have optimal solutions with the same value

$$v_{\bar{x}, \bar{t}}^+(z) = \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

These preparations now lead to the following upper bound.

**THEOREM 7.1.** *Assume  $f, g_j \in C^1$ . Let MFCQ hold at  $(\bar{x}, \bar{t})$  for some  $\bar{x} \in S(\bar{t})$ . Then for any direction  $z$ ,  $\|z\| = 1$  we have,*

$$\nabla_+ v(\bar{t}; z) \leq \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

*If MFCQ is satisfied at  $(\bar{x}, \bar{t})$  for all  $\bar{x} \in S(\bar{t})$  then*

$$\nabla_+ v(\bar{t}; z) \leq \inf_{\bar{x} \in S(\bar{t})} \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

**Proof.** Let at  $(\bar{x}, \bar{t})$  with  $\bar{x} \in S(\bar{t})$  MFCQ hold with  $\bar{\xi}$ . Take a minimizer  $\xi_z$  of  $P_{\bar{x}, \bar{t}}^+(z)$  and consider for  $\varepsilon > 0$  the points  $x_\tau := \bar{x} + \tau(\xi_z + \varepsilon\bar{\xi})$ . For  $j \in J_0(\bar{x}, \bar{t})$  we find for small  $\tau > 0$  (see (7.2)),

$$g_j(x_\tau, \bar{t} + \tau z) = \tau[\nabla_x g_j(\bar{x}, \bar{t})\xi_z + \nabla_t g_j(\bar{x}, \bar{t})z] + \tau\varepsilon\nabla_x g_j(\bar{x}, \bar{t})\bar{\xi} + o(\tau) < 0,$$

i.e.,  $x_\tau$  is feasible for  $P(\bar{t} + \tau z)$ . So by (7.1) we obtain

$$\begin{aligned} v(\bar{t} + \tau z) - v(\bar{t}) &\leq f(x_\tau, \bar{t} + \tau z) - f(\bar{x}, \bar{t}) \\ &= \tau[\nabla_x f(\bar{x}, \bar{t})\xi_z + \nabla_t f(\bar{x}, \bar{t})z] + \tau\varepsilon\nabla_x f(\bar{x}, \bar{t})\bar{\xi} + o(\tau) \end{aligned}$$

and thus

$$\frac{v(\bar{t} + \tau z) - v(\bar{t})}{\tau} \leq v_{\bar{x}, \bar{t}}^+(z) + \varepsilon\nabla_x f(\bar{x}, \bar{t})\bar{\xi} + o(1).$$

Since  $\varepsilon > 0$  can be chosen arbitrarily small, the first statement follows. The second result is obvious.  $\square$

## 7.2. Lower bound for $\nabla_-v(\bar{t}; z)$

This bound depends on a constraint qualification and a compactness assumption. Let again be given  $\bar{t} \in T$ ,  $\bar{x} \in S(\bar{t})$  and a direction  $z$ ,  $\|z\| = 1$ . Assume that LC holds for  $F(t)$  at  $\bar{t}$  and MFCQ at  $(\bar{x}, \bar{t})$ . For perturbations  $t = \bar{t} + \tau z$ ,  $\tau > 0$ ,  $\tau$  small, we consider corresponding minimizers  $\bar{x}_\tau \in S(\bar{t} + \tau z)$ , which exist due to LC. For the moment let us assume  $\bar{x}_\tau \rightarrow \bar{x} \in S(\bar{t})$ , for  $\tau \rightarrow 0$ . With a vector  $\xi$  we define

$$(7.3) \quad \tilde{x}_\tau = \bar{x}_\tau + \tau\xi.$$

Assuming  $\tilde{x}_\tau \in F(\bar{t})$  and using  $\bar{x}_\tau \rightarrow \bar{x}$  we find

$$\begin{aligned} v(\bar{t} + \tau z) - v(\bar{t}) &\geq f(\bar{x}_\tau, \bar{t} + \tau z) - f(\tilde{x}_\tau, \bar{t}) \\ &= f(\bar{x}_\tau, \bar{t}) + \tau\nabla_t f(\bar{x}_\tau, \bar{t})z + o(\tau) - f(\tilde{x}_\tau, \bar{t}) \\ &= \tau[-\nabla_x f(\bar{x}_\tau, \bar{t})\xi + \nabla_t f(\bar{x}_\tau, \bar{t})z] + o(\tau) \\ (7.4) \quad &= \tau[-\nabla_x f(\bar{x}, \bar{t})\xi + \nabla_t f(\bar{x}, \bar{t})z] + o(\tau). \end{aligned}$$

The feasibility condition  $\bar{x}_\tau \in F(\bar{t} + \tau z)$ , using  $g_j(\bar{x}_\tau, \bar{t} + \tau z) \leq 0$  for any  $j \in J_0(\bar{x}, \bar{t})$ , leads to the relation

$$\begin{aligned} g_j(\tilde{x}_\tau, \bar{t}) &= g_j(\tilde{x}_\tau, \bar{t}) - g_j(\bar{x}_\tau, \bar{t} + \tau z) + g_j(\bar{x}_\tau, \bar{t} + \tau z) \\ &\leq g_j(\tilde{x}_\tau, \bar{t}) - g_j(\bar{x}_\tau, \bar{t} + \tau z) \\ &= g_j(\tilde{x}_\tau, \bar{t}) - g_j(\tilde{x}_\tau, \bar{t} + \tau z) + g_j(\tilde{x}_\tau, \bar{t} + \tau z) - g_j(\bar{x}_\tau, \bar{t} + \tau z) \\ &= -\tau\nabla_t g_j(\tilde{x}_\tau, \bar{t})z + \tau\nabla_x g_j(\bar{x}_\tau, \bar{t} + \tau z)\xi + o(\tau) \\ (7.5) \quad &= \tau[\nabla_x g_j(\bar{x}, \bar{t})\xi - \nabla_t g_j(\bar{x}, \bar{t})z] + o(\tau). \end{aligned}$$

To maximize the lower bound in (7.4), now, we determine  $\xi$  as a maximizer of the linear program

$$P_{\bar{x}, \bar{t}}^-(z) : \quad \max_{\xi} \quad [-\nabla_x f(\bar{x}, \bar{t})\xi + \nabla_t f(\bar{x}, \bar{t})z] \quad \text{s.t.} \\ [\nabla_x g_j(\bar{x}, \bar{t})\xi - \nabla_t g_j(\bar{x}, \bar{t})z] \leq 0, \quad j \in J_0(\bar{x}, \bar{t}),$$

with optimal value  $v_{\bar{x}, \bar{t}}^-(z)$  and dual

$$D_{\bar{x}, \bar{t}}^-(z) : \quad \min_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

As before in Section 7.1, MFCQ at  $(\bar{x}, \bar{t})$  implies that both programs are solvable with the same value

$$v_{\bar{x}, \bar{t}}^-(z) = \min_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

This leads to the following lower bound.

**THEOREM 7.2.** *Assume  $f, g_j \in C^1$ . Let LC be satisfied at  $\bar{t}$  as well as MFCQ at  $(\bar{x}, \bar{t})$  for all  $\bar{x} \in S(\bar{t})$ . Then for any direction  $z$ ,  $\|z\| = 1$  we have,*

$$\nabla_- v(\bar{t}; z) \geq \inf_{\bar{x} \in S(\bar{t})} \min_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

**Proof.** By definition of  $\nabla_- v(\bar{t}; z)$  there exists a sequence  $t_l = \bar{t} + \tau_l z$ ,  $\tau_l \downarrow 0$ , such that

$$\nabla_- v(\bar{t}; z) = \lim_{l \rightarrow \infty} \frac{v(\bar{t} + \tau_l z) - v(\bar{t})}{\tau_l}.$$

In view of LC the sets  $S(t_l)$  are nonempty, i.e., we can choose a sequence  $\bar{x}_l \in S(t_l)$ . By LC we can assume  $\bar{x}_l \rightarrow \bar{x}$  (for some subsequence). Closedness of  $S$  (see Lemma 5.5) yields  $\bar{x} \in S(\bar{t})$ . Now take a solution  $\xi_z$  of  $P_{\bar{x}, \bar{t}}^-(z)$ , a MFCQ-vector  $\bar{\xi}$  (at  $(\bar{x}, \bar{t})$ ) and define for  $\varepsilon > 0$

$$x_l = \bar{x}_l + \tau_l(\xi_z + \varepsilon \bar{\xi}).$$

As in (7.5) we find for  $j \in J_0(\bar{x}, \bar{t})$ ,

$$g_j(x_l, \bar{t}) = \tau_l[\nabla_x g_j(\bar{x}, \bar{t})\xi_z - \nabla_t g_j(\bar{x}, \bar{t})z] + \tau_l \varepsilon \nabla_x g_j(\bar{x}, \bar{t})\bar{\xi} + o(\tau_l) \\ \leq \tau_l[\varepsilon \nabla_x g_j(\bar{x}, \bar{t})\bar{\xi} + o(1)] < 0,$$

if  $\tau_l > 0$  is small. So  $x_l$  is feasible for  $P(\bar{t})$  and we obtain as in (7.4)

$$v(\bar{t} + \tau_l z) - v(\bar{t}) \geq f(\bar{x}_l, t_l) - f(x_l, \bar{t}) \\ = \tau_l[v_{\bar{x}, \bar{t}}^-(z) + \varepsilon \nabla_x f(\bar{x}, \bar{t})\bar{\xi} + o(1)].$$

After division by  $\tau_l > 0$ , recalling that  $\varepsilon > 0$  is arbitrary, we conclude

$$\nabla_- v(\bar{t}; z) \geq v_{\bar{x}, \bar{t}}^-(z) \geq \inf_{\bar{x} \in S(\bar{t})} v_{\bar{x}, \bar{t}}^-(z) = \inf_{\bar{x} \in S(\bar{t})} \min_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu)z.$$

□

### 7.3. The convex case, existence of $\nabla v(\bar{t}; z)$

For convex programs  $P(t)$  in (6.1) the above results can be sharpened. We show that under some constraint qualification the directional derivatives  $\nabla v(\bar{t}; z)$  exist.

**THEOREM 7.3.** *Assume  $f, g_j \in C^1$ . Let the convexity conditions AC be satisfied for  $P(t)$  and let  $F(\bar{t})$  be compact. Suppose that the Slater condition is valid, i.e., there is some  $\tilde{x} \in F(\bar{t})$  such that*

$$g_j(\tilde{x}, \bar{t}) < 0 \quad \text{for all } j \in J .$$

*This implies that MFCQ holds at  $(x, \bar{t})$ , for all  $x \in F(\bar{t})$ , see Ex. 7.1. Then  $v$  is directional differentiable at  $\bar{t}$  into any direction  $z, \|z\| = 1$  with*

$$\nabla v(\bar{t}; z) = \min_{\bar{x} \in S(\bar{t})} \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu) z .$$

**Proof.** In view of Theorem 7.2 it suffices to show

$$\nabla_- v(\bar{t}; z) \geq \inf_{\bar{x} \in S(\bar{t})} \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu) z .$$

Since  $F(\bar{t})$  is compact, by Lemma 5.7 the condition LC is satisfied. As in the proof of Theorem 7.2 we can argue that there exist sequences  $t_l = \bar{t} + \tau_l z, \tau_l \downarrow 0, x_l \in S(t_l), x_l \rightarrow \bar{x} \in S(\bar{t})$ , such that

$$(7.6) \quad \nabla_- v(\bar{t}; z) = \lim_{l \rightarrow \infty} \frac{v(\bar{t} + \tau_l z) - v(\bar{t})}{\tau_l} .$$

By MFCQ for  $\bar{x} \in S(\bar{t})$ , the KKT condition holds at  $\bar{x}$  and  $M(\bar{x}, \bar{t})$  is nonempty, compact, and by definition  $\nabla_x L(\bar{x}, \bar{t}, \mu) = 0$  is satisfied for all  $\mu \in M(\bar{x}, \bar{t})$ .

For the convex function  $L(x, \bar{t}, \mu)$  wrt.  $x$ , the condition  $\nabla_x L(\bar{x}, \bar{t}, \mu) = 0$  means that for any (fixed)  $\mu \in M(\bar{x}, \bar{t})$  the point  $\bar{x}$  is a global minimizer (in  $x$ ) of  $L(x, \bar{t}, \mu)$ . So, by using  $\mu_j \geq 0, g_j(x_l, t_l) \leq 0$  and  $g_j(\bar{x}, \bar{t}) = 0, j \in J_0(\bar{x}, \bar{t})$ , we find for each  $\mu \in M(\bar{x}, \bar{t})$  the relation

$$\begin{aligned} L(x_l, t_l, \mu) - L(x_l, \bar{t}, \mu) &\leq L(x_l, t_l, \mu) - L(\bar{x}, \bar{t}, \mu) \\ &= f(x_l, t_l) - f(\bar{x}, \bar{t}) + \sum_{j \in J_0(\bar{x}, \bar{t})} \mu_j (g_j(x_l, t_l) - g_j(\bar{x}, \bar{t})) \\ &\leq f(x_l, t_l) - f(\bar{x}, \bar{t}) = v(t_l) - v(\bar{t}) . \end{aligned}$$

In view of  $x_l \rightarrow \bar{x}$  this leads to

$$v(t_l) - v(\bar{t}) \geq L(x_l, t_l, \mu) - L(x_l, \bar{t}, \mu) = \tau_l \nabla_t L(\bar{x}, \bar{t}, \mu) z + o(\tau_l) ,$$

and thus (cf., (7.6))  $\nabla_- v(\bar{t}; z) \geq \nabla_t L(\bar{x}, \bar{t}, \mu) z$ . Recalling that this holds for any  $\mu$  in the compact set  $M(\bar{x}, \bar{t})$  (compact by CQ, see Ex. 2.6(b)) we finally conclude

$$\nabla_- v(\bar{t}; z) \geq \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu) z \geq \inf_{\bar{x} \in S(\bar{t})} \max_{\mu \in M(\bar{x}, \bar{t})} \nabla_t L(\bar{x}, \bar{t}, \mu) z .$$

According to the remark below the inf over  $S(\bar{x})$  can be replaced by min.  $\square$

**Remark.** By Lemma 2.8, under the convexity condition AC, for each  $x \in S(\bar{t})$  the multiplier set  $M(x, \bar{t}) = M_0$  is the same. Consequently the function

$$\varphi(x) = \max_{\mu \in M_0} \nabla_t L(x, \bar{t}, \mu)z$$

is continuous in  $x$  (proof as Ex.) and attains its minimum value on the compact set  $S(\bar{t})$ .

**Ex. 7.1.** Show that the following constraint qualifications are equivalent for  $C^1$  convex programs  $P(\bar{t})$ :

- (a) *The Slater condition: for some  $\tilde{x} \in F(\bar{t})$  we have  $g_j(\tilde{x}, \bar{t}) < 0, \forall j \in J$*
- (b) *The MFCQ condition holds at  $(x, \bar{t})$ , for all  $x \in F(\bar{t})$ .*



## CHAPTER 8

### Applications

#### 8.1. Interior point method

The basic idea of the interior point method for solving a non-parametric program

$$P : \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_j(x) \leq 0, \quad j \in J,$$

is simply as follows. Consider the perturbed KKT system (cf., Remark 2.3)

$$(8.1) \quad H(x, \tau, \mu) := \begin{cases} \nabla f(x) + \sum_{j \in J} \mu_j \nabla g_j(x) & = 0 \\ -\mu_j \cdot g_j(x) & = \tau, \quad j \in J, \\ \mu_j, -g_j(x) & \geq 0, \quad j \in J, \end{cases}$$

where  $\tau > 0$  is a perturbation parameter. The idea is to find solutions  $x(\tau)$  and  $\mu_j(\tau)$  of this system (satisfying  $-g_j(x(\tau)), \mu_j(\tau) > 0$ ) and to let  $\tau \searrow 0$ . We expect that  $x(\tau)$  for  $\tau \searrow 0$  converges to a solution  $\bar{x}$  of  $P$ . Under sufficient optimality conditions this procedure is well-defined.

**THEOREM 8.1.** *Let  $\bar{x}$  be a local minimizer of  $P$  such that the sufficient optimality conditions of Theorem 2.4(a) or (b) are fulfilled with a multiplier  $\bar{\mu}$  such that  $\bar{\mu}_j > 0$  holds for all  $j \in J_0(\bar{x})$ . Then there exist some  $\alpha > 0$  and  $C^1$ -functions  $x : (-\alpha, \alpha) \rightarrow \mathbb{R}^n$ ,  $\mu_j : (-\alpha, \alpha) \rightarrow \mathbb{R}$ ,  $j \in J_0(\bar{x})$ , satisfying  $x(0) = \bar{x}$ ,  $\mu_j(0) = \bar{\mu}_j$ , and  $x(\tau), \mu_j(\tau)$  are locally unique solutions of (8.1).*

**Proof.** The statement follows by applying the IFT to the equation (8.1). By Ex. 4.5, the Jacobian

$$\nabla H(x, \mu) = \begin{pmatrix} \nabla_x^2 L(x, \mu) & \nabla^T g(x) \\ -\text{diag}(\mu) \nabla g(x) & -\text{diag}(g(x)) \end{pmatrix}$$

is invertible at  $(x, \mu) = (\bar{x}, \bar{\mu})$ . Here,  $g(x) := (g_j(x), j \in J)$ , and  $\text{diag}(a)$  denotes the diagonal matrix with diagonal elements  $a_j, j \in J$ . □

#### 8.2. A parametric location problem

We consider a concrete location problem in Germany (see [14] for details). Suppose some good is produced at 5 existing plants at location  $s^j = (s_1^j, s_2^j)$ ,  $j = 1, \dots, 5$  ( $s_1^j$  longitude,  $s_2^j$  latitude in Germany), and a sixth new plant has to be build at location  $t = (t_1, t_2)$  (to be determined) to satisfy the demands of  $V_i$  units

of goods in 99 towns  $i$  at location  $\ell^i = (\ell_1^i, \ell_2^i)$ ,  $i = 1, \dots, 99$ . Suppose (for simplicity) that the transportation cost  $c_{ij}$  from plant  $j$  to town  $i$  (per unit of the good) is (proportionally to) the Euclidian distance

$$c_{ij} = \sqrt{(s_1^j - \ell_1^i)^2 + (s_2^j - \ell_2^i)^2}, \quad c_{i6}(t) = \sqrt{(t_1 - \ell_1^i)^2 + (t_2 - \ell_2^i)^2} .$$

Suppose further that the total demand  $V = \sum_i V_i$  will be produced in the 6 plants with a production of  $p_j$  units of the good in plant  $j$  where

$$p_1 = V \frac{10}{100}, \quad p_2 = V \frac{30}{100}, \quad p_3 = V \frac{10}{100}, \quad p_4 = V \frac{15}{100}, \quad p_5 = V \frac{15}{100}, \quad p_6 = V \frac{20}{100} .$$

The problem now is to find the location  $t = (t_1, t_2)$  of the new plant such that the total transportation costs are minimized. For any fixed location  $t$  the optimal transportation strategy is given by the solution of the transportation problem,

$$\begin{aligned} P(t) : \quad v(t) &:= \min_{y_{ij}} \sum_{j=1}^5 \sum_i c_{ij} y_{ij} + \sum_i c_{i6}(t) y_{i6} \quad \text{s.t.} \\ &\sum_i y_{ij} = p_j, \quad j = 1, \dots, 6, \\ &\sum_j y_{ij} = V_i, \quad i = 1, \dots, 99. \end{aligned}$$

Here  $y_{ij}$  is the number of units of the good to be transported from plant  $j$  to town  $i$ . The problem is now to find a (global or local) minimizer of  $P(t)$ . Most local minimization algorithms are based on the computation of the (negative) gradient  $d = -\nabla v(t^k)$  of the objective function at some actual iteration point  $t = t^k$ .

Suppose that  $y_{ij}^k$  is the unique optimal vertex solution of  $P(t^k)$ . Then a result as in Theorem 7.3 can be proven for parametric linear programs (with linear equalities/inequalities) without a CQ assumption. Consequently the gradient  $\nabla v(t^k)$  can be computed via the formula

$$\nabla v(t^k) = \nabla_t L(y^k, t^k, \lambda^k) = \sum_{i=1}^{99} y_{i6}^k \frac{1}{\sqrt{(t_1^k - \ell_1^i)^2 + (t_2^k - \ell_2^i)^2}} \begin{pmatrix} (t_1^k - \ell_1^i) \\ (t_2^k - \ell_2^i) \end{pmatrix},$$

and the new iterate  $t^{k+1}$  is given by solving the line minimization problem

$$\min_{\tau \geq 0} v(t^k - \tau \nabla v(t^k)) .$$

### 8.3. Maximum eigenvalue function of a symmetric matrix

We present an example of a parametric program which can easily be analyzed. We denote the linear space of real symmetric  $n \times n$ -matrices by  $\mathbb{S}^n$ . Recall that a function  $f$  is called concave if  $-f$  is convex.

Let  $\lambda_+(A)$ , resp.,  $\lambda_-(A)$  denote the maximum, resp., minimum eigenvalue of the

symmetric matrix  $A \in \mathbb{S}^n$ . The following representations are well-known (see, e.g., [9, (2.29) in Section 2.2]):

$$(8.2) \quad \lambda_+(A) = \max_{x^T x=1} x^T A x, \quad \lambda_-(A) = \min_{x^T x=1} x^T A x$$

The maximizer  $x_+ = x_+(A)$ , resp., minimizer  $x_- = x_-(A)$  are the corresponding unit eigenvectors.

**LEMMA 8.1.** *The function  $\lambda_+(A)$  is a convex function and  $\lambda_-(A)$  a concave function of the parameter  $A \in \mathbb{S}^n$ .*

*This implies (see Remark 2.1) that the functions  $\lambda_+(A)$ ,  $\lambda_-(A)$  are locally Lipschitz continuous at any matrix  $\bar{A} \in \mathbb{S}^n$ .*

**Proof.** For  $A_1, A_2 \in \mathbb{S}^n$  and  $\tau \in [0, 1]$  we find,

$$\begin{aligned} \lambda_+(\tau A_1 + (1 - \tau)A_2) &= \max_{x^T x=1} x^T [\tau A_1 + (1 - \tau)A_2]x \\ &= \max_{x^T x=1} [\tau x^T A_1 x + (1 - \tau)x^T A_2 x] \\ &\leq \max_{x^T x=1} [\tau x^T A_1 x] + \max_{x^T x=1} [(1 - \tau)x^T A_2 x] \\ &= \tau \lambda_+(A_1) + (1 - \tau) \lambda_+(A_2). \end{aligned}$$

The concavity of  $\lambda_-(A)$  is proven similarly. □

In the same way we can prove the following generalization. Let for  $A \in \mathbb{S}^n$  with ordered eigenvalues

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$$

be defined

$$f_k(A) = \sum_{j=1}^k \lambda_j(A), \quad k = 1, \dots, n.$$

Fan [8] has proven the relation

$$f_k(A) = \max_{X \in S_k} \text{tr}(X^T A X) \quad \text{where } S_k = \{X \in \mathbb{R}^{n \times k} \mid X^T X = I_k\}.$$

$I_k$  denotes the  $k \times k$ -unit matrix, and  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$  is the trace of  $A = (a_{ij})$ .

**LEMMA 8.2.** *For any  $k$ ,  $1 \leq k \leq n$  the function  $f_k(A)$  is a convex function of the parameter  $A \in \mathbb{S}^n$ .*

**Proof.** The trace function is obviously linear. So, for  $A_1, A_2 \in \mathbb{S}^n$  and  $\tau \in [0, 1]$  we obtain,

$$\begin{aligned}
f_k(\tau A_1 + (1 - \tau)A_2) &= \max_{X \in S_k} \operatorname{tr} (X^T [\tau A_1 + (1 - \tau)A_2] X) \\
&= \max_{X \in S_k} \operatorname{tr} (\tau X^T A_1 X + (1 - \tau) X^T A_2 X) \\
&= \max_{X \in S_k} (\operatorname{tr} [\tau X^T A_1 X] + \operatorname{tr} [(1 - \tau) X^T A_2 X]) \\
&\leq \tau \max_{X \in S_k} \operatorname{tr} [X^T A_1 X] + (1 - \tau) \max_{X \in S_k} \operatorname{tr} [X^T A_2 X] \\
&= \tau f_k(A_1) + (1 - \tau) f_k(A_2)
\end{aligned}$$

□

### 8.4. Penalty method, exact penalty method

We discuss the so-called penalty method for solving a standard constrained program

$$(8.3) \quad P : \min_x f(x) \quad \text{s.t.} \quad x \in F = \{x \mid g_j(x) \leq 0, j \in J\},$$

with  $J = \{1, \dots, m\}$ ,  $f \in C^0$ , and  $g_j \in C^1$ . In this method the constrained problem (8.3) is replaced by the unconstrained problem:

$$P_\rho : \min_x f(x) + \rho p(x),$$

where  $p(x)$  is a penalty function for  $F$  and  $\rho > 0$  is a weight parameter.

**DEFINITION 8.1.** A function  $p(x)$ , continuous on  $\mathbb{R}^n$ , is called *penalty function* for the feasible set  $F$  of  $P$  if

$$p(x) = 0 \quad \text{for } x \in F \quad \text{and} \quad p(x) > 0 \quad \text{for } x \notin F.$$

Note that the term  $\rho p(x)$  in  $P_\rho$  penalizes the deviation of  $x$  from feasibility in  $P$ . We expect that for large  $\rho$  the minimizer of  $P_\rho$  yields an approximation of a minimizer of  $P$ .

As we shall see below, the penalty method is closely related to the parametric program with parameter  $t \in \mathbb{R}$ ,  $t \geq 0$ :

$$(8.4) \quad P(t) : \min_x f(x) \quad \text{s.t.} \quad x \in F(t) = \{x \mid g_j(x) \leq t, j \in J\}.$$

We again introduce the vector function  $g(x) = (g_j(x), j \in J)^T$  and consider the penalty functions:

$$\begin{aligned}
p(x) &= \sum_{j \in J} g_j^+(x)^2 = \|g^+(x)\|^2 \\
p(x) &= \max_{j \in J} g_j^+(x) = \|g^+(x)\|_\infty
\end{aligned}$$

where

$$g_j^+(x) = \max\{0, g_j(x)\}, \quad g^+(x) = (g_j^+(x), j \in J)^T.$$

Recall that for  $y \in \mathbb{R}^m$ ,  $\|y\|$  denotes the Euclidean norm of  $y$  and  $\|y\|_\infty = \max_{1 \leq j \leq m} |y_j|$  the maximum norm.

We emphasize that typically the penalty function  $\|g^+(x)\|_\infty$  is not differentiable at points  $\bar{x}$  where  $\max_j g_j(\bar{x}) = 0$ . On the other hand, the penalty function  $\|g^+(x)\|^2$  is  $C^1$  everywhere (recall  $g(x) \in C^1$ ).

**Penalty method with  $p(x) = \|g^+(x)\|^2$ , local version.**

In this local version we deal with local minimizers and assume that  $\bar{x}$  is a local minimizer of  $P$ . We therefore restrict  $p_\rho$  to a neighborhood  $B_\delta(\bar{x})$  with  $\delta > 0$ , chosen so that

$$(8.5) \quad f(x) - f(\bar{x}) \geq 0 \quad \text{for all } x \in F \cap B_{2\delta}(\bar{x}).$$

So our penalty problem is

$$P_\rho : \quad v_\rho := \min_{x \in \text{cl } B_\delta(\bar{x})} f_\rho(x) := f(x) + \rho \|g^+(x)\|^2.$$

The next theorem gives an error bound for the difference of the values  $v = f(\bar{x})$  of  $P$  and  $v_\rho$  of  $P_\rho$ .

**THEOREM 8.2.** *Let  $\bar{x}$  be a local minimizer of  $P$  such that MFCQ holds at  $\bar{x}$ . Then there exists  $\alpha > 0$  such that for large  $\rho > 0$  we have,*

$$0 \leq v - v_\rho \leq \alpha \frac{1}{\rho}.$$

**Proof.** We can assume that for at least one index  $j_0$  the minimizer  $\bar{x}$  is active,  $g_{j_0}(\bar{x}) = 0$ . Indeed if not, we have  $g(\bar{x}) < 0$  and thus  $g^+(x) = 0$  for all  $x$  in some neighborhood  $B_\delta(\bar{x})$ . This implies that  $\bar{x}$  is an unconstrained local minimizer and thus  $\bar{x}$  is a local minimizer of  $P_\rho$  for any  $\rho \geq 0$ .

For  $x \in F$  we have  $g^+(x) = 0$  and thus  $f_\rho(x) = f(x)$ . So we trivially obtain

$$v_\rho \leq v \quad \text{or} \quad 0 \leq v - v_\rho.$$

Now let  $x_\rho$  be a minimizer of  $P_\rho$  in  $\text{cl } B_\delta(\bar{x})$ . Then using  $g^+(\bar{x}) = 0$  and  $\rho \|g^+(x_\rho)\|^2 \geq 0$  we find

$$(8.6) \quad f(x_\rho) \leq v_\rho = f(x_\rho) + \rho \|g^+(x_\rho)\|^2 \leq f(\bar{x}) + \rho \|g^+(\bar{x})\|^2 = f(\bar{x})$$

and thus

$$(8.7) \quad 0 \leq \|g^+(x_\rho)\|^2 \leq \frac{f(\bar{x}) - f(x_\rho)}{\rho}.$$

Now we look at  $P(t)$  (cf., (8.4)). By Lemma 6.1(b) there exist  $B_\delta(\bar{x})$  (with possibly smaller  $\delta > 0$ ),  $B_\varepsilon(0)$ ,  $L_1 > 0$ , such that for any  $t \in B_\varepsilon(0)$  the following holds:

$$(8.8) \quad \begin{array}{l} \text{if } x_\rho \in F(t) \cap \text{cl } B_\delta(\bar{x}), \text{ we can find } \bar{x}_\rho \in F(0) \cap B_{2\delta}(\bar{x}) \\ \text{satisfying } \|x_\rho - \bar{x}_\rho\| \leq L_1 |t|. \end{array}$$

Since  $x_\rho \in \text{cl } B_\delta(\bar{x})$  (bounded) the values  $f(\bar{x}) - f(x_\rho)$  are bounded,  $f(\bar{x}) - f(x_\rho) \leq M$ . So by (8.7), for large  $\rho$  we must have

$$0 \leq g_j^+(x_\rho) \leq \|g^+(x_\rho)\| \leq \sqrt{\frac{M}{\rho}} < \varepsilon \quad \text{for all } j \in J.$$

In view of  $x_\rho \in F(\|g^+(x_\rho)\|_\infty) \cap \text{cl } B_\delta(\bar{x})$ , using (8.8), we find ( $\|y\|_\infty \leq \|y\|$ )

$$\|x_\rho - \bar{x}_\rho\| \leq L_1 \|g^+(x_\rho)\|_\infty \leq L_1 \|g^+(x_\rho)\|.$$

By (8.7) it follows

$$(8.9) \quad \|x_\rho - \bar{x}_\rho\|^2 \leq L_1^2 \|g^+(x_\rho)\|^2 \leq L_1^2 \frac{f(\bar{x}) - f(x_\rho)}{\rho}.$$

On the other hand, with a Lipschitz constant  $L_3$  for  $f$  in  $B_{2\delta}(\bar{x})$  we obtain (recall  $f(\bar{x}) \leq f(\bar{x}_\rho)$ )

$$(8.10) \quad f(\bar{x}) - f(x_\rho) \leq f(\bar{x}_\rho) - f(x_\rho) \leq L_3 \|x_\rho - \bar{x}_\rho\|.$$

Combined with (8.9) this yields

$$\|x_\rho - \bar{x}_\rho\|^2 \leq L_1^2 L_3 \|x_\rho - \bar{x}_\rho\| \frac{1}{\rho}$$

or putting  $\beta := L_1^2 L_3$ ,

$$(8.11) \quad \|x_\rho - \bar{x}_\rho\| \leq \beta \frac{1}{\rho},$$

which finally, in view of  $f(\bar{x}_\rho) \geq f(\bar{x}) = v$  and (8.10), leads to

$$\begin{aligned} v_\rho &= f(x_\rho) + \rho \|g^+(x_\rho)\|^2 \geq f(x_\rho) \\ &= f(\bar{x}_\rho) + f(x_\rho) - f(\bar{x}_\rho) \\ &\geq v - L_3 \|x_\rho - \bar{x}_\rho\| \geq v - \beta L_3 \frac{1}{\rho}, \end{aligned}$$

and with  $\alpha := \beta L_3$ ,

$$0 \leq v - v_\rho \leq \alpha \frac{1}{\rho},$$

as stated. □

Under stronger assumptions we also obtain an error bound for  $\|x_\rho - \bar{x}\|$ .

**THEOREM 8.3.** *Let  $\bar{x}$  be a local minimizer of  $P$  of order  $s = 1$  or  $s = 2$ , and suppose MFCQ is valid at  $\bar{x}$ . Then, for the (local) minimizers  $x_\rho$  of  $P_\rho$  in  $B_\delta(\bar{x})$ , with some  $c > 0$  we have for large  $\rho > 0$ ,*

$$\|x_\rho - \bar{x}\| \leq c \frac{1}{\rho^{1/s}}.$$

**Proof.** Recall from the proof of Theorem 8.2 with the Lipschitz constant  $L_3$  for  $f$  the relation (see, (8.10))

$$f(\bar{x}_\rho) - f(x_\rho) \leq L_3 \|x_\rho - \bar{x}_\rho\| .$$

By assumption, with some  $\delta > 0$ ,  $\kappa > 0$ , we have

$$f(x) - f(\bar{x}) \geq \kappa \|x - \bar{x}\|^s \quad \forall x \in F \cap B_{2\delta}(\bar{x}) .$$

So, for  $x = \bar{x}_\rho \in F \cap B_{2\delta}(\bar{x})$  (cf., the proof of Theorem 8.2, (8.8)) we obtain using  $v_\rho - v \leq 0$ ,  $f(x_\rho) \leq v_\rho$  (see, (8.6) and (8.11)),

$$\begin{aligned} \|\bar{x}_\rho - \bar{x}\|^s &\leq \frac{1}{\kappa} (f(\bar{x}_\rho) - f(\bar{x})) = \frac{1}{\kappa} (f(x_\rho) - f(\bar{x}) + f(\bar{x}_\rho) - f(x_\rho)) \\ &\leq \frac{1}{\kappa} (v_\rho - v + L_3 \|x_\rho - \bar{x}_\rho\|) \\ &\leq \frac{L_3}{\kappa} \|x_\rho - \bar{x}_\rho\| \\ &\leq \frac{\beta L_3}{\kappa} \frac{1}{\rho} , \end{aligned}$$

and then

$$\|\bar{x}_\rho - \bar{x}\| \leq \left( \frac{\beta L_3}{\kappa} \right)^{1/s} \frac{1}{\rho^{1/s}} .$$

This finally leads to (see (8.11)),

$$\begin{aligned} \|x_\rho - \bar{x}\| &\leq \|x_\rho - \bar{x}_\rho\| + \|\bar{x}_\rho - \bar{x}\| \\ &\leq \beta \frac{1}{\rho} + \left( \frac{\beta L_3}{\kappa} \right)^{1/s} \frac{1}{\rho^{1/s}} \\ &\leq c \frac{1}{\rho^{1/s}} , \end{aligned}$$

if we put  $c = \beta + \left( \frac{\beta L_3}{\kappa} \right)^{1/s}$ .

□

We now deal with the so-called exact penalty method and present a local- and a global version.

**Exact penalty method,  $p(x) = \|g^+(x_\rho)\|_\infty$ , local version.**

Here we again assume that  $\bar{x}$  is a local minimizer of  $P$  satisfying (8.5). So again, we will restrict the problems  $P_\rho, P(t)$  to  $\text{cl } B_\delta(\bar{x})$ . The next theorem states that for  $\rho$  large enough the local minimizer  $\bar{x}$  of  $P$  can be computed exactly as a local minimizer of  $P_\rho$ . This fact explains the name exact penalty method. The proof is based on a Lipschitz lower semicontinuity result for the value function  $v(t)$  of  $P(t)$ . As we will prove below, for some neighborhood  $B_\varepsilon(0)$ , with some  $L > 0$ , the following relation is valid,

$$(8.12) \quad v(t) - v(0) \geq -L|t| \quad \text{for all } t \in B_\varepsilon(0) .$$

Note that for  $t \geq 0$  we have  $F(0) \subset F(t)$  implying  $v(t) - v(0) \leq 0$  for all  $t \geq 0$ .

**THEOREM 8.4.** *Let  $\bar{x}$  be a local minimizer of  $P = P(0)$  such that MFCQ holds at  $\bar{x}$ . Then for any  $\rho \geq L$ , with  $L$  in (8.12),  $\bar{x}$  is also a local minimizer of  $P_\rho$ . Moreover, for the (local) minimal values we have  $v_\rho = v(0) = f(\bar{x})$ .*

**Proof.** Recall that we can assume that for at least one index  $j_0$  the minimizer  $\bar{x}$  is active,  $g_{j_0}(\bar{x}) = 0$ . Indeed if not we have  $g(\bar{x}) < 0$  and thus  $g^+(x) = 0$  for all  $x$  in some neighborhood  $B_\delta(\bar{x})$  implying that  $\bar{x}$  is an unconstrained local minimizer and thus  $\bar{x}$  is a local minimizer of  $P_\rho$  for any  $\rho \geq 0$ .

We first show that in an appropriate neighborhood  $B_\varepsilon(0)$  of  $\bar{t} = 0$  an inequality (8.12) is satisfied.

We can choose  $\delta > 0$  such that  $\bar{x}$  is a global minimizer of  $P$  in  $\text{cl } B_{2\delta}(\bar{x}) = \{x \mid \|x - \bar{x}\| \leq 2\delta\}$ . By continuity of  $f$  and MFCQ there exists  $B_\varepsilon(0)$  such that for all  $t \in B_\varepsilon(0)$  the program  $P(t)$  has a global minimizer  $\bar{x}_t$  on the compact set  $\text{cl } B_\delta(\bar{x})$ . According to Lemma 6.1(b) with some  $L_1 > 0$ , for any  $t \in B_\varepsilon(0)$  ( $\varepsilon > 0$  possibly smaller) we can find  $x_t^0 \in F(0) \cap B_{2\delta}(\bar{x})$  satisfying

$$\|\bar{x}_t - x_t^0\| \leq L_1|t - 0|.$$

Using  $v(0) = f(\bar{x}) \leq f(x_t^0)$ , with a Lipschitz constant  $L_2$  for  $f$  on  $B_{2\delta}(\bar{x})$ , we obtain for  $t \in B_\varepsilon(0)$ ,

$$\begin{aligned} v(t) - v(0) &\geq f(\bar{x}_t) - f(x_t^0) \geq -L_2\|\bar{x}_t - x_t^0\| \\ &\geq -L_2L_1|t|. \end{aligned}$$

Putting  $L = L_1L_2$  this shows (8.12).

To prove the theorem, we assume to the contrary that for some  $\rho \geq L$  the point  $\bar{x}$  is not a local minimizer of  $P_\rho$ . Then, for any  $\delta_1 > 0$ ,  $\delta_1 < \delta$ , there is some  $x' \in B_{\delta_1}(\bar{x})$  such that (use  $g^+(\bar{x}) = 0$ )

$$f(x') + \rho\|g^+(x')\|_\infty < f(\bar{x}) + \rho\|g^+(\bar{x})\|_\infty = f(\bar{x})$$

or

$$(8.13) \quad f(x') - f(\bar{x}) < -\rho\|g^+(x')\|_\infty.$$

By continuity of  $g(x)$  we can choose  $\delta_1 > 0$  such that with  $\varepsilon$  above we have (recall  $g^+(\bar{x}) = 0$ )

$$\|g^+(x)\|_\infty \leq \varepsilon \quad \text{for all } x \in B_{\delta_1}(\bar{x}).$$

By definition  $f(x') \geq v(\|g^+(x')\|_\infty)$ , which together with  $f(\bar{x}) = v(0)$  and (8.12) yields

$$\begin{aligned} f(x') - f(\bar{x}) &\geq v(\|g^+(x')\|_\infty) - v(0) \\ &\geq -L\|g^+(x')\|_\infty \geq -\rho\|g^+(x')\|_\infty \end{aligned}$$

in contradiction to (8.13). Further, since for  $\rho \geq L$  the point  $\bar{x}$  is local minimizer of  $P_\rho$ , it follows,

$$v_\rho = f(\bar{x}) + \rho\|g^+(\bar{x})\|_\infty = f(\bar{x}) = v(0).$$



□

**Ex. 8.1.** Show the following modification of Theorem 8.4: If  $\bar{x}$  is a strict local minimizer of  $P = P(0)$  such that MFCQ holds at  $\bar{x}$ . Then, for any  $\rho > L$  in (8.12),  $\bar{x}$  is also a strict local minimizer of  $P_\rho$ .

*Hint:* Argue as in the proof of Theorem 8.4 with a few modifications: For any  $\delta_1 > 0$ ,  $\delta_1 < \delta$  there is some  $x' \in B_{\delta_1}(\bar{x})$  such that (use  $g^+(\bar{x}) = 0$ )

$$f(x') + \rho \|g^+(x')\|_\infty \leq f(\bar{x}) + \rho \|g^+(\bar{x})\|_\infty = f(\bar{x})$$

or

$$f(x') - f(\bar{x}) \leq -\rho \|g^+(x')\|_\infty .$$

Then later we obtain the contradiction: For  $\rho > L$

$$\begin{aligned} f(x') - f(\bar{x}) &\geq v(\|g^+(x')\|_\infty) - v(0) \\ &\geq -L\|g^+(x')\|_\infty > -\rho\|g^+(x')\|_\infty . \end{aligned}$$

**Exact penalty method,  $p(x) = \|g^+(x_\rho)\|_\infty$ , global version.**

Here we assume that  $\bar{x}$  is a global minimizer of  $P$ . In this case we need some global compactness assumption. To obtain this, from a practical point of view it is useful to restrict all problems to some (large) ball  $B_R(\bar{x})$ ,  $R > 0$ . So we consider the programs:

$$\begin{aligned} (8.14) \quad P &: \min_{x \in B_R(\bar{x})} f(x) \quad \text{s.t.} \quad g_j(x) \leq 0, \quad j \in J, \\ P_\rho &: \min_{x \in B_R(\bar{x})} f(x) + \rho \|g^+(x)\|_\infty, \\ P(t) &: \min_{x \in B_R(\bar{x})} f(x) \quad \text{s.t.} \quad g_j(x) \leq t, \quad j \in J. \end{aligned}$$

Then we introduce the following assumption

**A<sub>global</sub>:** Suppose that for each value  $t \in T_0$ ,

$$T_0 := \{t \mid 0 \leq t \leq \max_{x \in B_R(\bar{x})} \|g^+(x')\|_\infty + 1\},$$

MFCQ holds at  $(\bar{x}_t, t)$  for all  $\bar{x}_t \in S(t)$ .

**THEOREM 8.5.** Let  $\bar{x}$  be a global minimizer of  $P = P(0)$  in (8.14) such that  $A_{global}$  is valid. Then for any  $\rho \geq L$ , with  $L$  in (8.15),  $\bar{x}$  is also a global minimizer of  $P_\rho$ . Moreover, for the minimal values we have  $v_\rho = v(0) = f(\bar{x})$ .

**Proof.** Since the minimization is restricted to the compact set  $B_R(\bar{x})$ , the programs  $P(t)$  have global minimizers  $\bar{x}_t$  for  $t \in T_0$ . Now we can argue as in the proof of Theorem 8.4. In view of the assumption  $A_{global}$ , according to Lemma 6.2(b), the value function  $v(t)$  of  $P(t)$  is locally Lipschitz continuous on  $T_0$ . In particular,  $v(t)$  is continuous on  $T_0$  and there exist  $\varepsilon, L_0 > 0$ , such that

$$v(t) - v(0) \geq -L_0|t| \quad \text{for all } |t| \leq \varepsilon .$$

Now we can define

$$(8.15) \quad M := \min_{t \in T_0} \{v(t) - v(0)\} \quad \text{and} \quad L := \max \left\{ L_0, \frac{-M}{\varepsilon} \right\}$$

Note, that by  $F(0) \subset F(t)$  for  $t \geq 0$ , we have  $v(t) - v(0) \leq 0$  and thus  $M \leq 0$ . Then we find

$$(8.16) \quad v(t) - v(0) \geq \begin{cases} -L_0|t| & \text{if } |t| \leq \varepsilon \\ \frac{M}{\varepsilon}t & \text{if } t \geq \varepsilon \end{cases} \geq -L|t| \quad \forall t \in T_0 .$$

Then, assuming that for some  $\rho \geq L$  the point  $\bar{x}$  is not a global minimizer of  $P_\rho$  we can argue as in the proof of Theorem 8.4: There is some  $x' \in B_R(\bar{x})$  with

$$f(x') + \rho \|g^+(x')\|_\infty < f(\bar{x}) + \rho \|g^+(\bar{x})\|_\infty = f(\bar{x})$$

or

$$f(x') - f(\bar{x}) < -\rho \|g^+(x')\|_\infty .$$

Using  $f(x') \geq v(\|g^+(x')\|_\infty)$  and  $f(\bar{x}) = v(0)$  the relation (8.16) leads to the contradiction

$$\begin{aligned} f(x') - f(\bar{x}) &\geq v(\|g^+(x')\|_\infty) - v(0) \\ &\geq -L \|g^+(x')\|_\infty \geq -\rho \|g^+(x')\|_\infty . \end{aligned}$$

Again, since for  $\rho \geq L$  the point  $\bar{x}$  is a global minimizer of  $P_\rho$  we find

$$v_\rho = f(\bar{x}) + \rho \|g^+(\bar{x})\|_\infty = f(\bar{x}) = v(0) .$$

□

## 8.5. Sensitivity and monotonicity results for traffic equilibria

In this section we study the Wardrop equilibria introduced in Section 3 as parametric problems depending on changes in the cost functions and/or changes in the traffic demand. In the general case we have given costs  $c_e(x_e, t)$  depending on the parameter  $t \in T$ ,  $T \subset \mathbb{R}^p$  open, and a convex parameter set  $D \subset \mathbb{R}^{|W|}$  of demands  $d$ . We assume throughout this chapter

**AT<sub>c</sub>.** Let for each fixed  $t \in T$  the functions  $c_e(x_e, t)$ ,  $e \in E$ , be continuously differentiable in  $x_e$  (see AT in Section 3) with partial derivatives  $\frac{\partial}{\partial x_e} c_e(x_e, t) > 0$  for all  $x_e \geq 0$ . Let further  $c_e(x_e, t)$ ,  $e \in E$ , be locally Lipschitz continuous in the variables  $(x_e, t)$  at each  $(x_e, t)$ ,  $x_e \geq 0$ ,  $t \in T$ .

### General sensitivity results for the Wardrop equilibrium

By Theorems 3.1, 3.2(b), for each (fixed)  $(t, d) \in T \times D$ , a Wardrop equilibrium  $(x(t, d), f(t, d))$  exists and the edge flow part  $x(t, d)$  is uniquely determined as solution of the following Beckmann program depending on the parameter  $(t, d)$ :

$$(8.17) \quad P_{WE}(t, d) : \quad \min_{x, f} N(x, t) = \sum_{e \in E} \int_0^{x_e} c(\tau, t) d\tau \quad \text{s.t.} \quad (x, f) \in F_d ,$$

where  $F_d$  is defined by,

$$F_d = \{(x, f) \mid \Lambda f = d, \Delta f - x = 0, f \geq 0\} .$$

We now wish to examine how the solutions of  $P_{WE}(t, d)$  change with changes in  $t$  and/or  $d$ .

Since in general the path flow part  $\bar{f}$  of a WE  $(\bar{x}, \bar{f})$  (for fixed  $(\bar{t}, \bar{d})$ ) is not uniquely determined we again consider the problem reduced to the  $x$  variable. To do so, we have to examine the structure of the projection  $X_d$  of  $F_d$  in dependence on  $d$ ,

$$(8.18) \quad X_d = \{x \mid \text{there exists } f \text{ such that } (x, f) \in F_d\} .$$

**LEMMA 8.3.**

(a) *There exist matrices  $B$  and  $C$ , only depending on the network matrices  $\Lambda, \Delta$  (cf., Section 3.1), such that*

$$X_d = \{x \mid Ax \leq Cd\} \quad \text{for all } d \geq 0.$$

(b) *Any  $x \in X_d$  is bounded by*

$$\|x\| \leq \sqrt{|E|} \cdot \sum_{w \in W} d_w .$$

(c) *There is a constant  $L_1 > 0$  such that for any  $d, d' \geq 0$  and  $x \in X_d$  there exists  $x' \in X_{d'}$  satisfying,*

$$\|x' - x\| \leq L_1 \|d' - d\| .$$

**Proof.** (a) The feasibility conditions for  $F_d$  can equivalently be written as a system of inequalities,

$$\begin{array}{rcl} \Lambda f & \leq & d \\ -\Lambda f & \leq & -d \\ \Delta f - x & \leq & 0 \\ -\Delta f + x & \leq & 0 \\ -f & \leq & 0 \end{array} .$$

To find  $X_d$ , we can apply the Fourier Motzkin algorithm to eliminate the variable  $f$ , see [9, Section 2.4]. This elimination process yields a system of say  $r$  linear inequalities in  $x$ , where each inequality is given by a (nonnegative) linear combination of the original inequalities. So, the  $j$ th relation has the form

$$b_j^T x \leq c_j^T d \quad \text{with some } b_j \in \mathbb{R}^{|E|}, c_j \in \mathbb{R}^{|W|}, j = 1, \dots, r .$$

By defining  $B \in \mathbb{R}^{r \times |E|}$  with rows  $b_j^T$ , and  $C \in \mathbb{R}^{r \times |W|}$  with rows  $c_j^T$ , the statement is proven.

(b) Obviously for each edge  $e$  the flow  $x_e$  of  $x \in X_d$  is bounded by  $x_e \leq \omega := \sum_{w \in W} d_w$ . Hence, for  $x \in X_d$  we find

$$\|x\| = \sqrt{\sum_{e \in E} x_e^2} \leq \sqrt{|E|} \omega = \sqrt{|E|} \omega .$$

(c) Applying Corollary 5.1 (with  $t = Cd$ ) to  $Bx \leq Cd$  which defines  $X_d$ , for any  $d, d' \geq 0$ , and  $x \in X_d$ , we can find some  $x' \in X_{d'}$  such that with  $L_0 > 0$  we have

$$\|x' - x\| \leq L_0 \|Cd' - Cd\| \leq L_0 \|C\| \|d' - d\| ,$$

and with  $L_1 = L_0 \|C\|$  the claim is proven.  $\square$

The parametric program  $P_{WE}(t, d)$  (cf., (8.17)) now reduces to the equivalent program in  $x$ :

$$(8.19) \quad P_{WE}^x(t, d) : \quad \min_x N(x, t) = \sum_{e \in E} \int_0^{x_e} c(\tau, t) d\tau \quad \text{s.t.} \quad x \in X_d .$$

The next theorem states that the minimizer of  $P_{WE}^x(t, d)$  is continuously depending on the parameter  $(t, d)$  and the value function  $v(t, d)$  is locally Lipschitz continuous.

**THEOREM 8.6.** *Let the assumption  $AT_c$  hold and let  $x(t, d)$  denote the unique minimizer of  $P_{WE}^x(t, d)$ .*

(a) *Then at each  $(\bar{t}, \bar{d}) \in T \times D$  the minimizer function  $x(t, d)$  is continuous, i.e., for any sequence  $(t_l, d_l) \rightarrow (\bar{t}, \bar{d})$ ,  $l \in \mathbb{N}$ ,  $(t_l, d_l) \in T \times D$ , we have*

$$x(t_l, d_l) \rightarrow x(\bar{t}, \bar{d}) .$$

(b) *At each  $(\bar{t}, \bar{d}) \in T \times D$  the value function  $v(t, d) = N(x(t, d), t)$  is locally Lipschitz continuous, i.e., there is some  $B_\varepsilon(\bar{t}, \bar{d})$ ,  $\varepsilon > 0$ , and  $L > 0$  ( $L = L(\bar{t}, \bar{d})$ ), such that*

$$|v(t, d) - v(\bar{t}, \bar{d})| \leq L \|(t, d) - (\bar{t}, \bar{d})\| \quad \text{for all } (t, d) \in B_\varepsilon(\bar{t}, \bar{d}) \cap T \times D .$$

**Proof.** (a) Consider any sequence  $(t_l, d_l) \rightarrow (\bar{t}, \bar{d})$  and  $x_l = x(t_l, d_l)$ . By Lemma 8.3(b) the sequence  $x_l \in X_{d_l}$  is bounded and thus (for some subsequence) we can assume  $x_l \rightarrow \bar{x}$ . The relation  $d_l \rightarrow \bar{d}$  implies  $\bar{x} \in X_{\bar{d}}$ . Let us assume  $\bar{x} \neq x(\bar{t}, \bar{d})$ , i.e.,  $\bar{x}$  is not the minimizer of  $P_{WE}^x(\bar{t}, \bar{d})$ . By Theorem 3.1(c) there must exist some  $\tilde{x} \in X_{\bar{d}}$  satisfying

$$(8.20) \quad c(\bar{x}, \bar{t})^T (\tilde{x} - \bar{x}) = \alpha < 0 ,$$

and since  $x_l$  are WE wrt.  $(t_l, d_l)$  we have

$$(8.21) \quad c(x_l, t_l)^T (x - x_l) \geq 0 \quad \text{for all } x \in X_{d_l} .$$

In view of Lemma 8.3(c), corresponding to  $\tilde{x} \in X_{\bar{d}}$ , we can choose points  $\tilde{x}_l \in X_{d_l}$  such that

$$\|\tilde{x}_l - \tilde{x}\| \leq L_1 \|d_l - \bar{d}\| .$$

From (8.21) we find

$$c(x_l, t_l)^T (\tilde{x}_l - x_l) \geq 0 \quad \text{for all } l ,$$

and the continuity of  $c$ , using  $t_l \rightarrow \bar{t}$ ,  $d_l \rightarrow \bar{d}$ ,  $x_l \rightarrow \bar{x}$ ,  $\tilde{x}_l \rightarrow \tilde{x}$ , yields

$$c(\bar{x}, \bar{d})^T (\tilde{x} - \bar{x}) \geq 0 ,$$

contradicting (8.20).

(b) We can follow the arguments in the proof of Lemma 6.2(b). Applying Lemma 8.3(c), corresponding to  $\bar{x} := x(\bar{t}, \bar{d})$  there exist flows  $\tilde{x}_d \in X_d$  such that

$$\|\tilde{x}_d - \bar{x}\| \leq L_1 \|d - \bar{d}\| .$$

In view of  $v(t, d) \leq N(\tilde{x}_d, t)$ , with a Lipschitz constant  $c_0$  for  $N$  in  $B_\sigma(\bar{t}, \bar{d})$  for some  $\sigma > 0$ , we obtain using  $\|(x, t)\| \leq \|x\| + \|t\| \leq \sqrt{2}\|(x, t)\|$ ,

$$\begin{aligned} v(t, d) - v(\bar{t}, \bar{d}) &\leq N(\tilde{x}_d, t) - N(\bar{x}, \bar{t}) \\ &\leq c_0 \|(\tilde{x}_d, t) - (\bar{x}, \bar{t})\| \\ &\leq c_0 (\|\tilde{x}_d - \bar{x}\| + \|t - \bar{t}\|) \\ &\leq c_0 (L_1 \|d - \bar{d}\| + \|t - \bar{t}\|) \\ &\leq c_0 \max\{L_1, 1\} (\|d - \bar{d}\| + \|t - \bar{t}\|) \\ &\leq c_0 \max\{L_1, 1\} \sqrt{2} \|(t, d) - (\bar{t}, \bar{d})\| . \end{aligned}$$

According to Lemma 8.3(b), choosing  $\varepsilon > 0$ , for all  $d \in B_\varepsilon(\bar{d})$  the flows  $x_d \in X_d$  are bounded by (use  $|d_w - \bar{d}_w| \leq \|d - \bar{d}\| \leq \varepsilon$ )

$$\|x_d\| \leq \sqrt{|E|} \sum_{w \in W} d_w \leq \sqrt{|E|} \cdot \sum_{w \in W} (\bar{d}_w + \varepsilon) =: \kappa .$$

In view of Lemma 8.3(c), to the minimum flow  $x(t, d) \in X_d$  there exist  $\bar{x}_{\bar{d}} \in X_{\bar{d}}$  such that

$$\|x(t, d) - \bar{x}_{\bar{d}}\| \leq L_1 \|d - \bar{d}\| ,$$

and with regard to  $v(\bar{t}, \bar{d}) \leq N(\bar{x}_{\bar{d}}, \bar{t})$  we obtain (with Lipschitz constant  $c_0$  for  $N$ ) as before

$$\begin{aligned} v(\bar{t}, \bar{d}) - v(t, d) &\leq N(\bar{x}_{\bar{d}}, \bar{t}) - N(x(t, d), t) \\ &\leq c_0 \|(\bar{x}_{\bar{d}}, \bar{t}) - (x(t, d), t)\| \\ &\leq c_0 \max\{L_1, 1\} \sqrt{2} \|(t, d) - (\bar{t}, \bar{d})\| . \end{aligned}$$

So, the claim is true if we put  $L = c_0 \max\{L_1, 1\} \sqrt{2}$ . □

By applying the results of Section 6.5 we even can prove a Lipschitz stability result for  $P_{WE}^x(t, d)$ .

**COROLLARY 8.1.** *Let  $AT_c$  hold and assume that  $c_e(x_e, t)$ ,  $e \in E$ , are  $C^1$ -functions on  $\mathbb{R}_+ \times T$ . Let  $\bar{t} \in T$ ,  $\bar{d} \in D$ ,  $\bar{d} > 0$ , be given. Then there exist  $\varepsilon > 0$ ,  $L > 0$ , such that for the unique minimizers  $x(t, d)$  of  $P_{WE}^x(t, d)$  we have*

$$\|x(t, d) - x(\bar{t}, \bar{d})\| \leq L \|(t, d) - (\bar{t}, \bar{d})\| \quad \text{for all } (t, d) \in B_\varepsilon(\bar{t}, \bar{d}) .$$

**Proof.** We make use of the result in Theorem 6.9. To do so, we first notice that under our assumptions (cf., (3.9)) the Hessian of the objective function  $N(x, t)$  is

continuous and positive definite on  $\mathbb{R}_+^n \times T$ . In particular for  $\bar{x}, \bar{t}$  with the smallest eigenvalue  $\bar{\lambda}_0 > 0$  of  $\nabla_x^2 N(\bar{x}, \bar{t})$  the second order condition holds (cf., (6.56)):

$$\xi^T \nabla_x^2 N(\bar{x}, \bar{t}) \xi \geq \bar{\lambda}_0 \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^n .$$

For our program the condition  $A_{ex}$  is automatically fulfilled. Indeed for  $\bar{d} > 0$  it is not difficult to see that the feasible set  $X_{\bar{d}}$  is not a singleton and for any  $d \geq 0$  the sets  $X_d$  are nonempty. So we can apply Theorem 6.9, saying that there exist some  $\varepsilon, c_0 > 0$ , such that with  $r = Cd$  (see, Lemma 8.3(b)) (use again  $\|t\| + \|d\| \leq \|(t, d)\| \leq \sqrt{2}(\|t\| + \|d\|)$ )

$$\begin{aligned} \|x(t, d) - \bar{x}\| &\leq c_0 \|(t, Cd) - (\bar{t}, C\bar{d})\| \\ &\leq c_0 (\|t - \bar{t}\| + \|Cd - C\bar{d}\|) \\ &\leq c_0 \max\{1, \|C\|\} (\|t - \bar{t}\| + \|d - \bar{d}\|) \\ &\leq c_0 \max\{1, \|C\|\} \sqrt{2} \|(t, d) - (\bar{t}, \bar{d})\| . \end{aligned}$$

Note that the continuity of the minimizer function  $x(t, d)$  on  $\text{dom } S$  follows by the continuity statement in Theorem 8.6(a).  $\square$

### Toll prize, first best prize

An important special case of (8.19) is given by the costs

$$c_e(x_e, t) = c_e(x_e) + t_e, \quad e \in E ,$$

where the parameter values  $t_e$  can be interpreted as a toll for using edge  $e$  and the demand  $d$  is fixed. With regard to the system optimum  $x^s$  (cf., Section 3) an interesting question arrizes.

**Question.** Can the government choose a toll vector  $t = (t_e, e \in E) \geq 0$  such that the user equilibrium wrt. costs  $c_e(x_e) + t_e$  coincides with the system optimum  $x^s$  wrt. the original costs  $c_e(x_e)$ ?

With such a toll  $t$  the selfish user would be "forced" into a system optimum.

The answer is, yes!

**COROLLARY 8.2.** *Let  $c_e(x_e)$  satisfy the assumptions of Theorem 3.3 and let  $x^s$  be the system optimum for given demand  $d$ . If we choose tolls*

$$(8.22) \quad t_e^s = \frac{\partial}{\partial x_e} S(x^s) - c_e(x_e^s) = x_e^s c_e'(x_e^s) ,$$

*then the WE  $x(t^s) \in X_d$  wrt. the costs  $c_e(x_e) + t_e^s$  coincides with  $x^s$ .*

**Proof.** To prove the statement we only have to compare the KKT conditions for the minimizer  $\bar{x}$  of  $P_{WE}^x$  wrt.  $c_e(x_e) + t_e^s$ ,

$$(8.23) \quad \nabla c(\bar{x}) + t^s + B^T \mu = 0, \quad \mu^T (Cd - B\bar{x}) = 0 ,$$

( $\mu$  is the Lagrange multiplier wrt.  $Bx \leq Cd$ , see also Lemma 8.3(a)) with the KKT relations for the minimizer  $x^s$  of  $P_s^x$  (cf., (3.11)):

$$\nabla S(x^s) + B^T \mu = 0, \quad \mu^T (Cd - Bx^s) = 0.$$

The choice  $t_e^s$  in (8.22) yields

$$\nabla S(x^s) = c(x^s) + t^s,$$

so that  $\bar{x} = x^s$  satisfies the KKT conditions (8.23) and by Remark 3.1 the flow  $x^s$  is a solution of  $P_{WE}^x$  wrt. the costs  $c_e(x_e) + t_e^s$ . □

### Monotonicity results for Wardrop equilibria

Let us recall the equilibrated path costs  $\gamma_w$  (see Remark 3.2) of a WE  $(\bar{x}, \bar{f})$  (a solution of  $P_{WE}$ ) for given demands  $d_w, w \in W$ :

$$\gamma_w = c_p(\bar{x}) \quad \text{for all } p \in P_w \text{ with } f_p > 0.$$

We expect that the equilibrated path costs  $\gamma_w$  increase if the demand  $d_w$  increases. In the same way, if for fixed  $d$  we increase the costs  $c_e$  of an edge  $e \in E$ , we expect the edge flow  $x_e$  of a WE  $\bar{x}$  to go down. The next theorem reveals such monotonicity relations.

Let us also recall that under AT a flow  $(x, f) \in F_d$  is a WE if and only if the KKT conditions

$$(8.24) \quad \Lambda^T \gamma = \Delta^T c(x) - \mu, \quad \mu^T f = 0$$

hold with Lagrange multipliers  $\mu \geq 0$  and  $\gamma$  (the equilibrated path cost vector).

#### **THEOREM 8.7.** [Monotonicity relations for static Wardrop equilibria]

Let the costs  $c_e(x_e, t)$ ,  $e \in E$ , depend on the parameter  $t$  such that for any fixed  $t$  the functions  $c_e(x_e, t)$  satisfy the conditions AT in the variables  $x_e$  (see Section 3).

(a) [Hall's result [15], change in demand  $d$ ] For fixed  $t$ , let  $(x, f)$  with corresponding multipliers  $\gamma, \mu$  be a WE wrt. demand  $d$  and let  $(x', f')$  with corresponding  $\gamma', \mu'$  be a WE wrt.  $d'$ . Then we have

$$(\gamma' - \gamma)^T (d' - d) \geq \mu'^T f + \mu^T f' \geq 0.$$

(b) [change in costs] For fixed demand  $d$  let  $x$ , resp.,  $x'$  be WE corresponding to the costs  $c_e(x_e, t)$ , resp.,  $c_e(x_e, t')$ . Then it holds,

$$\begin{aligned} (c(x, t') - c(x, t))^T (x' - x) &\leq 0 \quad \text{and} \\ (c(x', t') - c(x', t))^T (x' - x) &\leq 0. \end{aligned}$$

**Proof.** (a) By AT the functions  $c_e(x_e, t)$  (for  $t$  fixed) are increasing in  $x_e$ , i.e.,  $(c_e(x'_e, t) - c_e(x_e, t))(x'_e - x_e) \geq 0$ , and thus

$$(c(x', t) - c(x, t))^T (x' - x) \geq 0.$$

Using  $\Delta f = x$ ,  $\Delta f' = x'$  we find

$$(c(x', t) - c(x, t))^T \Delta(f' - f) = (\Delta^T c(x', t) - \Delta^T c(x, t))^T (f' - f) \geq 0 .$$

In view of the KKT condition  $\Delta^T c(x, t) = \Lambda^T \gamma + \mu$  etc. (see (8.24)) it follows

$$(\Lambda^T \gamma' - \Lambda^T \gamma)^T (f' - f) + (\mu' - \mu)^T (f' - f) \geq 0 ,$$

or by  $\Lambda f = d$ ,  $\Lambda f' = d'$ ,

$$(\gamma' - \gamma)^T (d' - d) \geq -(\mu' - \mu)^T (f' - f) .$$

The complementarity conditions  $\mu^T f = \mu'^T f' = 0$  finally yield

$$-(\mu' - \mu)^T (f' - f) = \mu^T f' + \mu'^T f ,$$

and hence using  $\mu', \mu, f', f \geq 0$ ,

$$(\gamma' - \gamma)^T (d' - d) \geq \mu^T f' + \mu'^T f \geq 0 .$$

(b) Monotonicity of  $c_e(x_e, t')$  in  $x_e$  (for fixed  $t'$ ) and the optimality conditions in Theorem 3.1(c) for  $x, x'$  lead to the inequalities

$$\begin{aligned} (c(x', t') - c(x, t'))^T (x' - x) &\geq 0 \\ c(x, t)^T (x' - x) &\geq 0 \\ c(x', t')^T (x - x') &\geq 0 . \end{aligned}$$

Adding, directly yields

$$(c(x, t) - c(x, t'))^T (x' - x) \geq 0$$

as claimed. The second relation can be proven similarly. □

The monotonicity condition

$$(8.25) \quad \sum_{w \in W} (\gamma'_w - \gamma_w)(d'_w - d_w) \geq \mu'^T f + \mu^T f \geq 0$$

of Theorem 8.7(a) in particular means the following:

- if one demand  $d_{w_0}$  is increased and the others remain unchanged, i.e.,  $d'_{w_0} \geq d_{w_0}$  and  $d'_w = d_w$  for all  $w \neq w_0$ , then the equilibrated cost  $\gamma_{w_0}$  increases, i.e.,  $\gamma'_{w_0} \geq \gamma_{w_0}$ ,
- if all demands increase, i.e.,  $d'_w \geq d_w$  for all  $w \in W$ , then at least one of the equilibrated costs  $\gamma_w$  must increase.

An analogous interpretation can be given for the monotonicity condition in Theorem 8.7(b).

Similar monotonicity formulas can be derived for the elastic Wardrop equilibrium,



see Section 3. Recall that here, a point  $(x, f, d)$  feasible for  $P_{eWE}$  is an elastic WE if and only if with corresponding Lagrangean multipliers  $\gamma$  and  $\mu, \sigma \geq 0$  we have

$$(8.26) \quad \begin{aligned} \Lambda^T \gamma &= \Delta^T c(x) - \mu, & \mu^T f &= 0 \\ \gamma &= g(d) + \sigma, & \sigma^T d &= 0 \end{aligned} .$$

**THEOREM 8.8.** *[monotonicity relations for elastic Wardrop equilibria]*

(a) *[Hall's result, change in elastic demand]* For fixed costs  $c_e(x_e), e \in E$ , let the functions  $d_w(\gamma_w, v)$  depend on a parameter  $v \in V \subset \mathbb{R}^p$ , such that for any fixed  $v$  the functions  $d_w(\gamma_w, v)$  (and thus also  $g_w(d_w, v) = d_w^{-1}(\gamma_w, v)$ ) satisfy the conditions (3.12) wrt.  $\gamma_w$ .

Assume now that  $(x, f, d)$  with corresponding multipliers  $\gamma, \mu, \sigma$  is an eWE wrt.  $g_w(d_w, v)$  and that  $(x', f', d')$  with corresponding  $\gamma', \mu', \sigma'$  is an eWE wrt.  $g_w(d_w, v')$ . Then we have

$$\begin{aligned} (g(d, v') - g(d, v))^T (d' - d) &\geq \mu'^T f + \mu^T f' + \sigma'^T d + \sigma^T d' \geq 0, \\ (g(d', v') - g(d', v))^T (d' - d) &\geq \mu'^T f + \mu^T f' + \sigma'^T d + \sigma^T d' \geq 0. \end{aligned}$$

(b) *[change in costs]* For fixed elastic demand functions  $d_w(\gamma_w), w \in W$ , (and thus fixed  $g_w(d_w)$ ) let the costs  $c_e(x_e, t)$  depend on the parameter  $t \in T \subset \mathbb{R}^p$ , such that for any fixed  $t$  the functions  $c_e(x_e, t)$  satisfy AT wrt.  $x_e$ .

Now let  $(x, f, d)$ , resp.,  $(x', f', d')$  be elastic WE corresponding to the costs  $c_e(x_e, t)$ , resp.,  $c_e(x_e, t')$ . Then it holds,

$$\begin{aligned} (c(x, t') - c(x, t))^T (x' - x) &\leq 0 \quad \text{and} \\ (c(x', t') - c(x', t))^T (x' - x) &\leq 0. \end{aligned}$$

**Proof.** (a) As in the unelastic case in Theorem 8.7(a) we obtain

$$(\gamma' - \gamma)^T (d' - d) \geq \mu'^T f + \mu^T f' \geq 0.$$

The KKT conditions  $\gamma = g(d, v) + \sigma$ ,  $\gamma' = g(d', v') + \sigma'$  imply

$$(g(d', v') - g(d, v))^T (d' - d) + (\sigma' - \sigma)^T (d' - d) \geq \mu'^T f + \mu^T f' \geq 0.$$

The complementarity conditions  $\sigma^T d = \sigma'^T d' = 0$  lead to

$$(g(d', v') - g(d, v))^T (d' - d) \geq \mu'^T f + \mu^T f' + \sigma'^T d' + \sigma^T d \geq 0.$$

By adding to the left-hand side the relation  $(g(d, v') - g(d', v'))^T (d' - d) \geq 0$  (due to the fact that  $g_w(d_w, v)$  for fixed  $v$  are decreasing in  $d_w$ ) we obtain

$$(g(d, v') - g(d, v))^T (d' - d) \geq \mu'^T f + \mu^T f' + \sigma^T d' + \sigma'^T d \geq 0.$$

The second relation follows similarly.

(b) Since for fixed  $t'$  the costs  $c_e(x_e, t'), e \in E$ , are nondecreasing in  $x_e$  and the

functions  $g_w(d_w)$ ,  $w \in W$ , are nonincreasing we find

$$\begin{aligned} (c(x', t') - c(x, t'))^T (x' - x) &\geq 0 \\ -(g(d') - g(d))^T (d' - d) &\geq 0 . \end{aligned}$$

Optimality conditions for  $(x, d)$  and  $(x', d')$  (cf., Theorem 3.4) yield

$$\begin{aligned} c(x, t)^T (x' - x) - g(d)^T (d' - d) &\geq 0 \\ c(x', t')^T (x - x') - g(d')^T (d - d') &\geq 0 . \end{aligned}$$

By adding all 4 inequalities we obtain

$$(c(x, t) - c(x, t'))^T (x' - x) \geq 0 .$$

Multiplication by  $-1$  proves the first relation. The second can be shown in the same way.

□

## CHAPTER 9

### Appendix

To keep the booklet largely selfcontained, in this appendix we list some facts from calculus which are used regularly.

#### Mean value formula, Taylor's formula.

We make use of the mean value formula (or first order Taylor formula) in different forms. For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f \in C^1$ , a well-known form is,

$$f(y) - f(x) = \nabla f(x)(y - x) + o(\|y - x\|) ,$$

or with some  $\tau = \tau(x, y)$ ,  $0 < \tau < 1$ ,

$$f(y) - f(x) = \nabla f(x + \tau(y - x))(y - x) .$$

From the Fundamental Theorem of Calculus we obtain the following integral form of the mean value.

**LEMMA 9.1.** *Let be given  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $D$  open,  $f \in C^1(D, \mathbb{R}^m)$  and  $y, x \in D$  such that the whole line segment  $\{x + \tau(y - x) \mid \tau \in [0, 1]\}$  is contained in  $D$ . Then we have*

$$f(y) - f(x) = \int_0^1 \nabla f(x + \tau(y - x))(y - x) d\tau$$

and thus

$$(9.1) \quad \|f(y) - f(x)\| \leq \max_{0 \leq \tau \leq 1} \|\nabla f(x + \tau(y - x))\| \|y - x\| .$$

**Proof.** Let  $f(x) = (f_i(x), i = 1, \dots, m)^T$  and define  $g_i : [0, 1] \rightarrow \mathbb{R}$  by

$$g_i(\tau) = f_i(x + \tau(y - x)) .$$

Then the Fundamental Theorem of Calculus yields

$$g_i(1) - g_i(0) = \int_0^1 g_i'(\tau) d\tau$$

and then by chain rule,

$$\begin{aligned} f_i(y) - f_i(x) &= g_i(1) - g_i(0) = \int_0^1 g_i'(\tau) d\tau \\ &= \int_0^1 \nabla f_i(x + \tau(y - x))(y - x) d\tau . \end{aligned}$$

By definition of the Jacobian this leads to,

$$f(y) - f(x) = \int_0^1 \nabla f(x + \tau(y - x))(y - x) d\tau$$

and

$$\begin{aligned} \|f(y) - f(x)\| &\leq \int_0^1 \|\nabla f(x + \tau(y - x))\| \|y - x\| d\tau \\ &\leq \max_{0 \leq \tau \leq 1} \|\nabla f(x + \tau(y - x))\| \|y - x\|. \end{aligned}$$

□

We also state the second order Taylor formula for  $C^2$ -functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  near a point  $\bar{x} \in \mathbb{R}^n$  in the form

$$(9.2) \quad f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) + o(\|x - \bar{x}\|^2)$$

or with some  $\tau = \tau(x, \bar{x})$ ,  $0 < \tau < 1$ ,

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x} + \tau(x - \bar{x}))(x - \bar{x}).$$

### Lipschitz continuity and Lipschitz constant.

**DEFINITION 9.1.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called (locally) Lipschitz continuous at  $\bar{x} \in \mathbb{R}^n$  if there exists a neighborhood  $B_\delta(\bar{x})$ ,  $\delta > 0$ , and a Lipschitz constant  $L \geq 0$ , such that

$$\|f(x) - f(\bar{x})\| \leq L\|x - \bar{x}\| \quad \forall x \in B_\delta(\bar{x}).$$

We say that  $f : S \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is (globally) Lipschitz continuous on  $S$  if there exists  $L_1 \geq 0$  such that

$$\|f(x') - f(x)\| \leq L_1\|x' - x\| \quad \forall x', x \in S.$$

$L_1$  is called Lipschitz constant of  $f$  on  $S$ .

A differentiable function is always Lipschitz continuous.

**LEMMA 9.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a  $C^1$ -function. Let  $D \subset \mathbb{R}^n$  be convex and compact. Then with the Lipschitz constant  $L = \max_{x \in D} \|\nabla f(x)\|$  the function  $f$  satisfies a Lipschitz condition on  $D$ :

$$\|f(y) - f(x)\| \leq L\|y - x\|, \quad \forall y, x \in D.$$

**Proof.** Take any pair  $y, x \in D$ . By convexity of  $D$  the whole line segment  $\{\tau x + (1 - \tau)y \mid \tau \in [0, 1]\}$  is contained in  $D$  and the formula (9.1) yields,

$$\begin{aligned} \|f(y) - f(x)\| &\leq \max_{0 \leq \tau \leq 1} \|\nabla f(x + \tau(y - x))\| \|y - x\| \\ &\leq \max_{x \in D} \|\nabla f(x)\| \|y - x\|. \end{aligned}$$

□

### The Weierstrass Theorem: existence of a minimizer

The example

$$\min e^{-x} \quad \text{s.t.} \quad x \geq 0$$

shows that a minimization problem needs not possess a minimizer even when the min-value is bounded from below. We however have

**THEOREM 9.1.** *[Weierstrass Theorem]*

*If the function  $f : D \rightarrow \mathbb{R}$  is continuous on  $D \subset \mathbb{R}^n$  and the domain  $D \neq \emptyset$  is bounded and closed (compact). Then the problem*

$$P : \quad \min f(x) \quad \text{s.t.} \quad x \in D$$

*has a minimizer (at least one).*

*The same holds if  $D$  is closed,  $f$  is continuous on  $D$  and for some  $x_0 \in D$  the lower level set*

$$\ell(f, x_0) = \{x \in D \mid f(x) \leq f(x_0)\}$$

*is compact.*

**Proof.** The first statement is the famous Weierstrass theorem (see, e.g., [29, 4.16 Theorem]). The second statement follows from the observation that by the Weierstrass theorem a minimizer of  $f$  on the compact set  $\ell(f, x_0) \neq \emptyset$  exists and that this minimizer is also a minimizer of  $P$ .

□



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