

Refining the partition for multifold conic optimization problems

Héctor Ramírez C.* Vera Roshchina†

March 30, 2018

Abstract

In this paper we give a unified treatment to different definitions of complementarity partition for a primal-dual pair of linear conic optimization problem.

1 Introduction

The complementarity partitions for linear programming and linear complementarity problems are well understood and are utilised heavily in the analysis of optimisation problems as well as in numerical methods. In addition to the strong duality, linear programming problems exhibit strict complementarity: in an appropriately formulated pair of primal and dual problems each primal variable has a dual counterpart, and a fully complementary pair, such that exactly one variable in the pair is positive and the other one is zero. The situation is much more complex for general linear conic problems. Even in the absence of the duality gap it may happen that both primal and dual variables are zero, or as it happens in the case of a multifold system, both lie on the boundary of component cones. Moreover, there are several different definitions of complementarity that do not coincide and lead to different characterisations.

In some special cases, especially when a homogeneous feasibility problem is considered, complementarity partition can be identified via an algorithm. The polyhedral (linear programming) case was considered in [14, 16], and it was shown in [15] that it is possible to identify the partition of [2] via a variant of an interior point method for second-order conic programming. The complementarity partition is related to generalised condition measures for multifold conic systems, see [5].

Recall that in the linear case the complementarity partition has a simple and well-studied structure. Consider the primal-dual pair of linear programming problems

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \geq b, \end{array} \quad (\text{LP}) \qquad \begin{array}{ll} \max_{y \in \mathbb{R}^m} & b^\top y \\ \text{s.t.} & A^\top y = c, \\ & y \geq 0. \end{array} \quad (\text{LD})$$

Here $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The classical Goldman-Tucker theorem [6] (also see [1, Proposition 17.16] for modern treatment) states that when either one of the problems has a

*Departamento de Ingeniería Matemática & Centro de Modelamiento Matemático (UMI 2807, CNRS), Universidad de Chile, Beauchef 851, Casilla 170-3, Santiago 3, Chile. E-mail: hramirez@dim.uchile.cl

†School of Science, RMIT University and CIAO, Federation University Australia, Victoria, Australia. E-mail: vera.roshchina@rmit.edu.au

finite optimal value, there exists a unique partition (B, N) of the index set $J = \{1, \dots, m\}$ that is of maximal complementarity. In other words, there is a pair (\bar{x}, \bar{y}) of optimal solutions to (LP) and (LD) such that for $\bar{s} = A\bar{x} - b$ and \bar{y} we have $\bar{s}_B > 0$, $\bar{y}_N > 0$, and moreover for any optimal solution $(s = Ax - b, y)$ always $y_B = 0$, $s_N = 0$.

The optimal partition also features in the study of monotone linear complementarity problems. Let Q and R be $n \times n$ matrices, and let $h \in \mathbb{R}^n$. The linear complementarity problem

$$\begin{cases} xs = 0, \\ Qx + Rs = h, \\ x \geq 0, \quad s \geq 0, \end{cases}$$

is called monotone if $Qu + Rv = 0$ yields $u^\top v \geq 0$. For monotone linear complementarity problems the optimal partition is of the form (B, N, T) , where T corresponds to the subset of indices for which both s and x components are zero for every solution.

The linear complementarity partition is reminiscent of the situation encountered in multifold conic systems. When the system of linear inequalities is replaced by a product of general closed convex cones (as in the case of second-order cone programming for instance), the partition of indices becomes more complex since the component variables may end up on the boundary of the cones.

In [2] a four-set partition was introduced for general multifold conic systems, based on regularity conditions involving normal cones. In [10] a refined six-set partition was suggested for homogeneous feasibility problems and a geometric characterization of such partition was obtained. Our goal is to generalize the latter partition to multifold conic optimization problems and compare the two partitions.

We focus on the case when strong duality holds for the primal-dual pair of conic optimization problems. In case this condition does not hold, primal problem can be rewritten by means of the facial reduction approach. This leads to an equivalent primal problem for which Slater condition holds as well as the strong duality provided an ad-hoc dual problem is considered. We refer the reader to recent works on facial reduction [9, 4, 17] and Ramana Duals [12, 11] for modern approaches to inducing the Slater condition and the strong duality in ill-posed conic problems.

We introduce the two types of partitions in Section 2, show the relations between them in Lemma 5, treat the special case of second-order cones in Lemma 7 and provide some illustrative examples. In Section 3 we relate the geometric relations of [10] to the four-set partition of [2] in Lemma 10. We finish with a study of the relations between partitions of reformulated (lifted) problems in Section 4, strengthening some results of [2].

Throughout the paper we work in a Hilbert finite dimensional (Euclidean) setting. By $x \cdot y$ we denote the standard inner product of two vectors $x, y \in \mathbb{R}^n$, $x \cdot y := \sum_{i=1}^n x_i y_i$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and use standard Euclidean norm $\|x\| = \sqrt{x \cdot x}$.

2 Partitions for multifold conic optimization problems

Consider a general linear conic optimization problem with constraints in product form, i.e.,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c \cdot x \\ \text{s.t.} \quad & A^j x - b^j \in K_j \quad \forall j \in J, \end{aligned} \tag{P}$$

where K_j is a closed convex cone in \mathbb{R}^{q_j} , $q_i \in \mathbb{N}$ for every $j \in J$, and $J = \{1, \dots, r\}$ is a finite index set. We set $K := K_1 \times \dots \times K_r$, and define $A = (A^1; \dots; A^r)$ as the matrix whose rows

are those of A^1 to A^r , and $b := \text{vec}(b^1, \dots, b^r)$ so that (P) is equivalent to

$$\min_{x \in \mathbb{R}^n} \{c \cdot x; Ax - b \in K\}.$$

The dual problem is

$$\begin{aligned} \max_{y^1, \dots, y^r} \quad & \sum_{j=1}^r b^j \cdot y^j \\ \text{s.t.} \quad & \sum_{j=1}^r (A^j)^\top y^j = c, \\ & y^j \in K_j^+ \quad \forall j \in J, \end{aligned} \tag{D}$$

where the (positive) polar of a set $C \subset \mathbb{R}^m$ is defined as $C^+ := \{y \in \mathbb{R}^m; y \cdot z \geq 0, \forall z \in C\}$. Generally speaking, such primal-dual pairs of linear conic problems may have a nonzero duality gap (see [7, Section 11.6] for a detailed discussion and [13, Section 3.2] for examples). The Slater condition together with the bounded objective for either (P) or (D) guarantees zero duality gap. If the primal and dual optimal values are equal, a pair (x, y) of optimal solutions to primal and dual problems is characterized by the complementarity system

$$\begin{aligned} A^j x - b^j &\in K_j, \\ y^j &\in K_j^+, \\ y^j \cdot (A^j x - b^j) &= 0 \quad \forall j \in J, \\ A^\top y &= c. \end{aligned} \tag{C}$$

We denote by $S(C)$ the set of solutions for relations (C). Observe that $S(C)$ is nonempty if and only if the pair (P)–(D) has zero duality gap. This condition will be assumed throughout the manuscript.

We say that *strict primal (resp. dual) feasibility* holds for $j \in J$ if there exists $x \in F(\text{P})$ such that $A^j x - b^j \in \text{int } K_j$ (resp. $y \in F(\text{D})$ such that $y^j \in \text{int } K_j^+$). Here by $F(\text{P})$ and $F(\text{D})$ we denote the set of feasible solutions to the problems (P) and (D) respectively.

One says (e.g., [3, Def. 4.74]) that the *strict complementarity hypothesis* holds for problem (P) if there exists a pair (x, y) solution of the optimality system (C), such that $-y \in \text{ri } N_K(Ax - b)$, where N_K is the normal cone of convex analysis (see [8, Section A.5.2]). Since K is a closed convex cone, we have for $s \in K$ that

$$N_K(s) = \begin{cases} (-K^+) \cap s^\perp, & s \in \partial K, \\ \emptyset, & s \notin K, \\ \{0\}, & s \in \text{int } K, \end{cases} \tag{1}$$

where s^\perp denotes the orthogonal complement to the linear span of s .

For problems with constraints in the product form as in (P), we introduce the notion of *componentwise strict complementarity*, which means that for each component j there exists a pair $(x, y) \in S(C)$, such that $-y^j \in \text{ri } N_{K_j}(A^j x - b^j)$. It is shown in Corollary 4 that the two notions are equivalent.

In [2] the notion of optimal partition, well known for linear programming and monotone linear complementarity problems (see e.g. [1, Section 18.2.4]), was extended to the abstract framework. Let

$$\begin{aligned} B &= \{j \mid \exists (x, y) \in S(C) \text{ s.t. } A^j x - b^j \in \text{int } K_j\} \\ N &= \{j \mid \exists (x, y) \in S(C) \text{ s.t. } y^j \in \text{int } K_j^+\} \\ R^0 &= \{j, \mid \exists (x, y) \in S(C) \text{ s.t. } -y^j \in \text{ri } N_{K_j}(A^j x - b^j)\}. \end{aligned}$$

Define $R := R^0 \setminus (B \cup N)$, $T := J \setminus (R^0 \cup B \cup N)$. The next statement follows from [2, Lemma 3].

Lemma 1. *If $S(C)$ is not empty, the partition (B, N, R, T) is a disjoint partition of the index set J .*

Definition 2. *Any pair $(x, y) \in S(C)$ satisfying the relations below is said to be of maximal complementarity:*

$$\begin{cases} \text{(i)} & A^j x - b^j \in \text{int } K_j, \quad \forall j \in B, \\ \text{(ii)} & y^j \in \text{int } K_j^+, \quad \forall j \in N, \\ \text{(iii)} & -y^j \in \text{ri } N_{K_j}(A^j x - b^j), \quad \forall j \in R. \end{cases} \quad (2)$$

For each $j \in B \cup N \cup R$ let $(x(j), y(j)) \in S(C)$ be such that

$$\begin{cases} A^j x^j(j) - b^j \in \text{int } K_j & \text{if } j \in B, \\ y^j(j) \in \text{int } K_j & \text{if } j \in N, \\ -y^j(j) \in \text{ri } N_{K_j}(A^j x^j(j) - b^j) & \text{if } j \in R. \end{cases}$$

We define

$$\hat{x} := (|B| + |R|)^{-1} \sum_{j \in B \cup R} x(j); \quad \hat{y} := (|N| + |R|)^{-1} \sum_{j \in N \cup R} y(j). \quad (3)$$

The next result can be found in [2, Lemma 7].

Lemma 3. *The pair (\hat{x}, \hat{y}) defined in (3) is of maximal complementarity. Moreover, any pair $(x, y) \in \text{ri } S(P) \times \text{ri } S(D)$ is of maximal complementarity.*

An immediate consequence of Lemma 3 is that componentwise strict complementarity yields strict complementarity, therefore the two notions are equivalent.

Corollary 4. *Componentwise strict complementarity of (P) is equivalent to strict complementarity.*

Proof. Observe that strict complementarity yields componentwise strict complementarity, as we have $\text{ri } N_K(Ax - b) = \text{ri } N_{K_1}(A^1 x - b^1) \times \text{ri } N_{K_2}(A^2 x - b^2) \times \cdots \times \text{ri } N_{K_r}(A^r x - b^r)$. Any pair of strictly complementary solutions is hence componentwise strictly complementary.

On the other hand, Lemma 3 provides a construction for maximal complementary pair that proves the reverse inclusion. \square

In a separate development [10] a different partition for a homogeneous feasibility problem was suggested. In what follows we extend this definition to optimization problems, and relate it to the previously defined four-sets partition. Define the index subsets B^0 and N^0 as follows:

$$\begin{aligned} B^0 &= \{j \mid \forall (x, y) \in S(C) \quad y^j = 0\}, \\ N^0 &= \{j \mid \forall (x, y) \in S(C) \quad A^j x - b^j = 0\}. \end{aligned}$$

Define the following four sets:

$$B' = B^0 \setminus (N^0 \cup B), \quad N' = N^0 \setminus (B^0 \cup N), \quad O = B^0 \cap N^0, \quad C = J \setminus (N^0 \cup B^0).$$

It follows immediately from the complementarity conditions of (C) that $B \subset B^0$ and $N \subset N^0$, and that $B' \cap N' = \emptyset$. Therefore, it is not difficult to observe that the sets B, N, B', N', O, C form a disjoint partition of the index set J . To avoid confusion, we will refer to this partition as the six-partition, whereas the four-set partition introduced before will be referred to as the four-partition.

Lemma 5. *The following relations between the four- and six-partitions hold.*

$$T \supset B' \cup N' \cup O = (N^0 \cup B^0) \setminus (N \cup B); \quad R \subset C = J \setminus (B^0 \cup N^0); \quad (4)$$

Proof. It is enough to show that $R \subset C$. Assume it is not so. Then there exists an index $j^* \in R \setminus C$. Since $C = J \setminus (B^0 \cup N^0)$, $j^* \in B^0 \cup N^0$.

Since $j^* \in R$, there exists a solution $(x(j^*), y(j^*)) \in S(C)$ with

$$-y^{j^*}(j^*) \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}). \quad (5)$$

First assume that $j^* \in B^0 \cap R$. This yields $y_{j^*}(j^*) = 0$, and hence

$$0 \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}).$$

This is only possible when either $N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}) = \{0\}$ or $N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*})$ contains lines. In the first case $A^{j^*}x(j^*) - b^{j^*} \in \text{int } K_{j^*}$, and $j^* \in B$, and that contradicts the definition of R . The second case is impossible by our assumption that all cones K_j , $j \in J$ have a nonempty interior.

Now assume that $j^* \in N^0 \cap R$. We have $A^{j^*}x(j^*) - b^{j^*} = 0$, hence, (5) yields

$$-y^{j^*}(j^*) \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}) = \text{ri } N_{K_{j^*}}(0) = \text{ri}(-K_{j^*}^+) = -\text{int } K_{j^*}^+,$$

which yields $j^* \in N$, and contradicts the fact that j^* is in R . We therefore conclude that

$$R \subset C = J \setminus (N^0 \cup B^0). \quad (6)$$

□

We demonstrate the relations of Lemma 5 using a Venn diagram in Fig. 1.

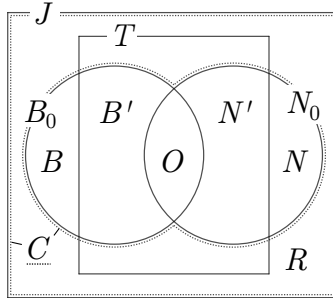


Figure 1: The relations between index sets (see Lemma 5).

The next example shows that the sets R and C can be different.

Example 6. *Consider the following linear semidefinite problem:*

$$\min_{x=(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4} x_1; \quad \begin{bmatrix} -x_1 & -x_3 & -x_2 \\ -x_3 & -x_2 & -x_1 \\ -x_2 & -x_4 & 1 - x_3 \end{bmatrix} \in S_+^3,$$

where S_+^n stands for set of symmetric semidefinite positive matrices of dimension n . It is easy to see that the only solution of this problem is $x_1 = x_2 = x_3 = x_4 = 0$.

On the other hand, its dual is as follows

$$\max_{Y=(Y_{ij}) \in S_+^3} -Y_{33}; \quad Y_{11} = 1, \quad -2Y_{13} - Y_{22} = 0, \quad -2Y_{12} - Y_{33} = 0, \quad Y_{23} = 0,$$

which unique solution is given by

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, it is clear that strict complementarity does not hold for this unique block (so, $R = \emptyset$). However, this clearly belongs to C .

We next show that in the case of second-order cone programming the complementarity partition has special properties. Recall that a Lorentz cone L_n is a closed convex cone in \mathbb{R}^{n+1} (for some $n \geq 1$) defined as follows,

$$L_n = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \mid x_0 \geq \|\bar{x}\|\}.$$

For consistency, we also let $L_0 := \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Second-order conic programming problems are of type (P) and (D), where each cone K_j , for $j \in J$, is a Lorentz cone. Notice that linear programming case can be cast a special case of second-order conic programming when $K_j = L_0$ for all $j \in J$.

Lemma 7. *If for every $j \in J$ the cone K_j is a Lorentz cone, then*

$$R = C = J \setminus (B^0 \cup N^0). \quad (7)$$

Proof. Let $j \in C$. Then there exists a solution $(x(j), y(j)) \in S(C)$ with both $y_j(j)$ and $s = A^j x(j) - b^j$ nonzero. Since $y_j(j) \perp s$, we have $y_j(j) \in (K_j^+) \cap s^\perp = -N_{K_j}(s)$. Observe that for a second-order cone the normal cone $N_{K_j}(s)$ is one-dimensional provided $s \neq 0$, hence, $\text{ri } N_{K_j}(s) = N_{K_j}(s) \setminus \{0\} \ni y_j(j)$, and therefore $j \in R$. We have then shown that $C \subset R$. The reverse inclusion follows from Lemma 5 \square

We next show a simple example of mixed polyhedral and SOCP cones in which it can happen that $R \neq C$.

Example 8. *We consider a feasibility problem, so in this case $b = 0$ and $c = 0$. We let $K = L_3 \times \mathbb{R}_+^1 \times \mathbb{R}_+^3$, and*

$$A^\top = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solving the problem analytically, we obtain all possible solutions as the parametric family

$$x = (0, 0, \alpha)^\top, \quad Ax = (\alpha, 0, 0, \alpha, 0, 0, 0, \alpha)^\top, \quad y = (\beta, 0, 0, -\beta, \gamma, 0, \gamma, 0)^\top,$$

where $\alpha, \beta, \gamma \geq 0$. We deduce that $1, 3 \in C$, $2 \in N$. Now observe that

$$N_{K_3}(A_3 x) = N_{\mathbb{R}_+^3}(0, 0, \alpha) = -\mathbb{R}_+^2 \times \{0\},$$

and hence

$$\text{ri } N_{K_3}(A_3 x) = -\text{int } \mathbb{R}_+^2 \times \{0\}.$$

The only possible solution is $y_3 = (0, \gamma, 0)$. Clearly, $-y_3 \notin \text{ri } N_{K_3}(A_3 x)$, hence, $3 \notin R$.

Observe that in this problem we did not consider the ‘maximal’ possible decomposition, i.e. the equivalent formulation when

$$K = L_3 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1.$$

In this case $1 \in C$, $2, 4 \in N$, $3 \in O$, $5 \in B$, and $R = \{1\}$, since the first cone is a SOC.

3 Homogeneous feasibility problems

Homogeneous feasibility problems are of special interest in duality theory. We obtain such problems as a special case of (P)–(D) when $b = 0$, $c = 0$. In this case the primal problem consists of finding an x such that

$$A^j x \in K_j \quad \forall j \in J, \quad (\text{FP})$$

and the dual problem consists of finding y satisfying

$$\sum_{j=1}^r (A^j)^\top y^j = 0, \quad y^j \in K_j^+ \quad \forall j \in J. \quad (\text{FD})$$

Observe that in the case of the homogeneous feasibility problems we always have strong duality. The relevant complementarity conditions can be obtained from (C),

$$\begin{aligned} A^j x &\in K_j, \\ y^j &\in K_j^+, \\ y^j \cdot A^j x &= 0 \quad \forall j \in J, \\ A^\top y &= 0. \end{aligned} \quad (\text{FC})$$

Denote by $\text{Lin } C$ the lineality space of a convex set C . In [10, Theorem 1] a dual characterization of the homogeneous six-partition was obtained.

Theorem 9. *The sets B , N , B^0 and N^0 can be described as follows.*

$$\begin{aligned} B &= \{j \in J \mid A_j^\top (K_j^+ \setminus \{0\}) \cap \text{Lin}(\overline{A^\top K^+}) = \emptyset\}, \\ N &= \{j \in J \mid \text{ri } A_j^\top K_j^+ \cap \text{Lin}(A^\top K^+) \neq \emptyset\}, \\ B^0 &= \{j \in J \mid A_j^\top (K_j^+ \setminus \{0\}) \cap \text{Lin}(A^\top K^+) = \emptyset\}, \\ N^0 &= \{j \in J \mid \text{ri } A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+}) \neq \emptyset\}. \end{aligned}$$

It appears that in the case of feasibility problem the set R^0 has a similar dual characterization.

Lemma 10. *In the case of feasibility problem*

$$R^0 \subset \tilde{R} := \{j \in J \mid A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+}) = A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+})\}.$$

Proof. Fix an index $j \in J$ and consider the linear subspaces

$$L = \{y \mid A_j^\top y \in \text{Lin}(A^\top K^+)\}, \quad \bar{L} = \{y \mid A_j^\top y \in \text{Lin}(\overline{A^\top K^+})\},$$

and let

$$F = L \cap K_j^+, \quad \bar{F} = \bar{L} \cap K_j^+.$$

Pick an arbitrary solution x and any $v \in \bar{F}$. Since $v \in \bar{F}$, we have $A_j^\top v \in \text{Lin}(\overline{A^\top K^+})$, hence, $-A_j^\top v \in \text{Lin}(\overline{A^\top K^+}) \subset \overline{A^\top K^+}$. Then there exist sequences $\{v_k^+\}$, $\{v_k^-\}$ such that $v_k^-, v_k^+ \in K^+$, $A^\top v_k^-, A^\top v_k^+ \in A^\top K^+$, $A^\top v_k^\pm \rightarrow \pm A_j^\top v$. Since $A^\top v_k^-, A^\top v_k^+ \in A^\top K^+$, and since x is a solution (i.e. $Ax \in K$), we have

$$0 \leq (Ax)^\top \cdot v_k^- = x \cdot A^\top v_k^- = x \cdot v_k^-,$$

and analogously $0 \leq x \cdot v_k^+$ for all n . Hence, passing to the limit, we have $A_j x \cdot v = 0$. Therefore, $\bar{F} \subset (A_j x)^\perp$. By definition, $\bar{F} \subset K_j^+$, hence,

$$\bar{F} \subset K_j^+ \cap (A_j x)^\perp.$$

Now assume that $j \in R^0 \setminus \tilde{R}$. Since $j \in R^0$, there exists a pair of feasible solutions (x, y) such that

$$y_j \in -\text{ri } N_{K_j}(A_j x) = K_j^+ \cap (A_j x)^\perp. \quad (8)$$

Let $\bar{y} \in \mathbb{R}^n$ be such that $\bar{y}_i = 0$ for all $i \neq j$, and $\bar{y}_j = y_j$. Observe that $\bar{y} \in K^+$, hence, $A_j^\top y_j = A^\top \bar{y} \in A^\top K^+$, and since $A^\top y = 0$, we have $-A_j^\top y_j \in A^\top K$, hence, $A_j^\top y_j \in \text{Lin}(AK)$, and therefore $y_j \in F$.

Since $y_j \in F \subsetneq \bar{F} \subset K_j^+ \cap (A_j x)^\perp$, $F = \bar{F} \cap L$, and all three sets F , \bar{F} and $K_j^+ \cap (A_j x)^\perp$ are cones, their affine hulls are linear subspaces, moreover, we have the following representations

$$\text{Aff } F = L_1, \quad \text{Aff } \bar{F} = L_2, \quad \text{Aff } (K_j^+ \cap (A_j x)^\perp) = L_3,$$

where L_1, L_2, L_3 are linear subspaces, $L_1 \subsetneq L_2 \subset L_3$. Observe that $L_1 \subsetneq L_2$ follows from the fact that $F = \bar{F} \cap L$, where L is a linear subspace and $F \neq \bar{F}$. Also note that since $F = L \cap \bar{F}$, and $\bar{F} = \bar{L} \cap (K^+ \cap (A_j x)^\perp)$, we have $F = \bar{F} \cap L_1$, $\bar{F} = L_2 \cap (K^+ \cap (A_j x)^\perp)$.

From (8) we deduce that there exists $\varepsilon > 0$ such that $(y_j + B_\varepsilon) \cap L_3 \subset K_j^+ \cap (A_j x)^\perp$. Observe that since $\text{Aff } F = L_1 \oplus (L_2 \cap L_1^\perp)$, there exists a unit vector u in $L_1^\perp \cap L_2$ such that $y_j + \varepsilon u \in \bar{F} \subset L_2 \subset L_3$, therefore $y_j - \varepsilon u \in L_3$, and since $\pm \varepsilon u \in B_\varepsilon$, we have $y_j \pm \varepsilon u \in L_2 \cap (K_j^+ \cap (A_j x)^\perp)$, hence, $y_j \pm \varepsilon u \in \bar{F}$, as $\bar{F} = \bar{L}_2 \cap (K^+ \cap (A_j x)^\perp)$. Therefore, $A_j^\top (y_j \pm \varepsilon u) \in A^\top K^+$. Since $A_j^\top y_j \in \text{Lin}(AK)$, we have $-A_j^\top y_j \in A^\top K^+$, and therefore we deduce that $-A_j^\top y_j \pm \varepsilon u \in AK$, and hence $A_j^\top (y_j \pm u) \in \text{Lin}(AK)$, which means that $y_j \pm \varepsilon u \notin F$, which contradicts the construction of u . Therefore, our assumption that there exist $j \in R^0 \setminus \tilde{R}$ is wrong, and hence $R^0 \subset \tilde{R}$. □

4 Equivalent optimization problems

We now introduce another problem related to (P). Let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_r$ be another finite family of closed convex cones in \mathbb{R}^{r_j} , $j \in J$, and M^j be $r_j \times q_j$ matrices such that

$$s^j \in \mathcal{K}_j \text{ iff } M^j s^j \in \mathcal{K}_j, \quad j = 1, \dots, J. \quad (9)$$

The latter is inspired by the relation between Lorentz cones and the cone of the semidefinite positive matrices. Indeed, denote these cones by L_m and \mathcal{S}_+^{m+1} , respectively, when their dimensions are $m + 1$. For $s = (s_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^m$ it is easy to check that

$$s \in L_m \text{ iff } \text{Arw}(s) := \begin{pmatrix} s_0 & \bar{s}^\top \\ \bar{s} & I_m \end{pmatrix} \in \mathcal{S}_+^{m+1}. \quad (10)$$

Linear function $\text{Arw}(\cdot)$ is the so-called *arrow function* and it has several interesting properties that will be exploited in the next sections.

Let $M = (M^1; \cdots; M^r)$ be the matrix whose rows are those of M^j . Then, (P) is equivalent to the linear conic problem

$$\text{Min}_{x \in \mathbb{R}^n} c \cdot x; \quad M^j(A^j x - b^j) \in \mathcal{K}_j, \quad j \in J, \quad (\text{MP})$$

whose dual is

$$\text{Max}_{z \in \mathcal{K}^+} \sum_{j=1}^r b^j \cdot (M^j)^\top z^j; \quad \sum_{j=1}^r (A^j)^\top (M^j)^\top z^j = c; \quad z^j \in \mathcal{K}_j^+, j \in J. \quad (\text{MD})$$

If the primal and dual values are equal, an optimal pair (x, y) of the primal and dual problems is characterized by the optimality system

$$\begin{cases} M^j(A^j x - b^j) \in \mathcal{K}_j, \quad z^j \in \mathcal{K}_j^+, \quad z^j \cdot M^j(A^j x - b^j) = 0, \quad j \in J; \\ \sum_{j=1}^r (A^j)^\top (M^j)^\top z^j = c. \end{cases} \quad (\text{MC})$$

We first recall some known results from [2] in the following lemma.

Lemma 11. *The following relations hold:*

- (i) $S(\text{P}) = S(\text{MP})$, $M^\top \mathcal{K}^+ \subset K^+$, and $M^\top S(\text{MD}) \subset S(\text{D})$.
- (ii) If $M^\top \mathcal{K}^+$ is closed, then $M^\top \mathcal{K}^+ = K^+$ and $M^\top S(\text{MD}) = S(\text{D})$.
- (iii) Closedness of $M^\top \mathcal{K}^+$ holds if M^\top is coercive on \mathcal{K}^+ , i.e., if there exists $\gamma > 0$ such that $\|M^\top z\| \geq \gamma \|z\|$ for all $z \in \mathcal{K}^+$. In that case, $S(\text{MD})$ is bounded iff $S(\text{D})$ is bounded.
- (iv) Assume in addition that M is one to one. Then, $M^\top \text{int } \mathcal{K}^+ = \text{int } K^+$ and $M^\top \text{int } S(\text{MD}) = \text{int } S(\text{D})$. Moreover, for all $s \in K$, $M^\top \text{ri}(\mathcal{K}^+ \cap (Ms)^\perp) \subset \text{ri}(K^+ \cap s^\perp)$.

Proof. See Lemmas 8 and 9 of [2]. □

Let (B_P, N_P, R_P, T_P) denote the optimal partition of (P), and adopt a similar convention for (MP). The next result strengthens Lemma 10 of [2].

Lemma 12. *Assume that $M^\top \mathcal{K}^+$ is closed, that M is one to one, and that*

$$\text{For all } s^j \in K_j, \quad M^j s^j \in \partial \mathcal{K}_j \text{ iff } s^j \in \partial K_j. \quad (11)$$

Then, the four-partition of problems (P) and (MP) coincide, that is

$$\begin{cases} B_P = B_{MP}, \quad N_P = N_{MP}, \\ R_P = R_{MP}, \quad T_P = T_{MP}. \end{cases} \quad (12)$$

In particular, the strict complementarity hypothesis holds for (P) iff it holds for (MP).

Proof. Due to Lemma 10 of [2] it only remains to prove that $R_P \subset R_{MP}$. Let $j \in R_P$. This implies the existence of $(x, y) \in S(\text{C})$ such that $s^j := A^j x^j - b^j \in \partial K_j \setminus \{0\}$ and $y^j(j) \in \partial \mathcal{K}_j^+ \setminus \{0\}$. It follows from Lemma 11, Parts (i) and (ii), that $x \in S(\text{MP})$ and that there exist $z \in S(\text{MD})$ such that $y = M^\top z$, respectively. In particular, $y^j = (M^j)^\top z^j$ and

$$0 = s^j \cdot y^j = s^j \cdot (M^j)^\top z^j = M^j s^j \cdot z^j. \quad (13)$$

Moreover, from (11) and the fact that M is one to one, we deduce that $M^j s^j$ is a nonzero element of $\partial \mathcal{K}_j$. This in particular implies that $z^j \notin \text{int}(\mathcal{K}_j^+)$ (because otherwise (13) yields to $M^j s^j = 0$). Finally, since $(M^j)^\top z^j = y^j \neq 0$ it directly follows that $z^j \neq 0$. Summarizing, we have proved that $(x, z) \in S(\text{MC})$, and $M^j s^j$ and z^j are nonzero elements of $\partial \mathcal{K}_j$ and $\partial \mathcal{K}_j^+$, respectively. This means that $j \in R_{MP}$, which concludes our proof. □

Let us study now the relation with the six-partition.

Lemma 13. *Assume that $M^\top \mathcal{K}^+$ is closed. Then*

$$B_P^0 \supset B_{MP}^0.$$

If, in addition, for all $z \in \mathcal{K}^+$ such that $M^\top z = 0$ yields $z = 0$, then

$$B_P^0 = B_{MP}^0.$$

Proof. The relation $B_P^0 \supset B_{MP}^0$ follows directly from Lemma 11(ii). If for every $(x, z) \in S(MC)$ we have $z_j = 0$, then $y_j = M_j^\top(z_j) = 0$.

Now assume that $j \in B_{(P)}^0$. Then every solution $(x, z) \in S(MC)$ satisfies $M_j^\top z_j = 0$. By assumption this yields $z_j = 0$. \square

Lemma 14. *We have*

$$N_P^0 \subset N_{MP}^0.$$

If, in addition, M is one-to-one, then

$$N_P^0 = N_{MP}^0.$$

Proof. From Lemma 11 (i) we have $MS(P) = S(MP)$, hence, the first relation $N_P^0 \subset N_{MP}^0$ is obvious. With the assumption that M is one-to-one, $M_j(A_j x - b_j) = 0$ yields $A_j x - b_j = 0$, hence, the equality $N_P^0 = N_{MP}^0$. \square

Lemma 15. *Under the assumptions of Lemma 12,*

$$\begin{cases} B_P = B_{MP}, & N_P = N_{MP}, \\ N_P^0 = N_{MP}^0, & B_P^0 \supset B_{MP}^0. \end{cases} \quad (14)$$

If, in addition, for all $z \in \mathcal{K}^+$ such that $M^\top z = 0$ yields $z = 0$, then $B_P^0 = B_{MP}^0$.

Proof. Follows directly from Lemmas 13 and 14. \square

The latter in particular implies that, if all the assumptions of Lemma 15 hold true, then the six-partition of problems (P) and (MP) coincide.

We finally provide alternative hypotheses to those in Lemma 12 to prove that the four-partition of problems (P) and (MP) coincide.

Corollary 16. *Suppose that the assumptions of Lemma 12 are fulfilled. Suppose also that for all $z \in \mathcal{K}^+$ such that $M^\top z = 0$ yields $z = 0$, $R_P = C_P$ and $R_{MP} = C_{MP}$. Then*

$$R_P = R_{MP}, \quad T_P = T_{MP}$$

Proof. Follows directly from Lemma 15 and definition of the index set C . \square

We next show that the assumptions of Lemma 15 are crucial for the equality between partitions.

Example 17. *Consider a pair of feasibility problems*

$$Ax \in K, \quad (P) \quad A^\top y = 0, y \in K^+, \quad (D)$$

where

$$K = K^+ = L_2 \times \mathbb{R}_+^1,$$

and L_2 is the three-dimensional Lorentz cone. Let

$$A^\top = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We can solve this feasibility problem directly and obtain the parametric family of solutions

$$x^* = (0, \alpha)^\top, \alpha \geq 0, \text{ with } Ax^* = (\alpha, 0, \alpha, 0)^\top; \quad y^* = (\beta, 0, -\beta, 0)^\top, \beta \geq 0.$$

It is not difficult to see that for the first index the solutions lie on the boundary of the cones, hence, $1 \in C = R$. For the second index, both primal and dual components are zero, hence, $2 \in O$.

Now we transform our problem into a higher-dimensional one. We let

$$M_1 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the transformed problem, we let $\mathcal{K}_1 = K_1 = L_2$, $\mathcal{K}_2 = L_1$. Observe that for every $y \in \mathbb{R}$ we have

$$M_2 y = (y, y)^\top,$$

and hence $M_2 y \in \mathcal{K}_2 = L_1$ iff $y \in K_2 = \mathbb{R}_+^1$. Further,

$$\mathcal{A}_1 = M_1 A_1 = A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}_2 = M_2 A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $\mathcal{A} = [\mathcal{A}_1; \mathcal{A}_2]$, then we can write the transformed feasibility problem as

$$\mathcal{A}u \in \mathcal{K}, \quad (P') \quad \mathcal{A}^\top v = 0, v \in \mathcal{K}^+. \quad (D')$$

Solving the problem directly, we obtain the family of optimal solutions

$$u = (\lambda, 0), \lambda \geq 0 \text{ with } \mathcal{A}u = (\lambda, 0, \lambda, 0, 0); \quad v = (\mu, 0, -\mu, \gamma, -\gamma), \mu, \gamma \geq 0.$$

Therefore, we conclude that $1_M \in C$, $2_M \in N^0 \setminus (B^0 \cup N) = N'$, and hence the partition has changed.

The situation above occurs is because M_2 does not satisfy that $M_2^\top z = 0$ yields $z = 0$ for all $z \in \mathcal{K}^+ = L_1$. Indeed, it is enough to take $z = (1, -1)^\top$ to check this hypothesis fails.

4.1 Application to SOCP-SDP conversion

The aim of this section is to apply the results of the above section to the (LSOCP) and (LSDP). For this, let us consider the following sets $K_j := L_{m_j+1}$, $\mathcal{K}_j := \mathcal{S}_+^{m_j+1}$, and $M^j s^j = \text{Arw}^j s^j$, where the arrow function $\text{Arw}^j : \mathbb{R}^{m_j+1} \rightarrow \mathcal{S}^{m_j+1}$ was defined in (10), and \mathcal{S}^{m+1} denotes the set of square symmetric matrices of dimension $m+1$.

In what follows consider a generic nonnegative integer value m and omit index j from the arrow function. Note that we can write

$$\text{Arw}(s) = (s_0 - \|\bar{s}\|)I_{m+1} + \begin{pmatrix} \|\bar{s}\| & \bar{s}^\top \\ \bar{s} & \|\bar{s}\|I_m \end{pmatrix}. \quad (15)$$

This shows that for $s \in L_{m+1} \setminus \{0\}$, $\text{Arw}(s)$ is of rank m iff $s \in \partial L_{m+1}$, and of rank $m+1$ otherwise. In particular, $\text{Arw} \partial L_{m+1} \subset \partial \mathcal{S}_+^{m+1}$, and $\text{Arw} \text{int} L_{m+1} \subset \text{int} \mathcal{S}_+^{m+1}$. Therefore (11) holds. Also Arw is clearly one-to-one.

Let us decompose any matrix $Y \in \mathcal{S}^{m+1}$ as follows

$$Y = \begin{pmatrix} Y_{00} & \bar{Y}_0^\top \\ \bar{Y}_0 & \bar{Y} \end{pmatrix}, \quad (16)$$

where $Y_{00} \in \mathbb{R}$, $\bar{Y}_0 \in \mathbb{R}^m$ and $\bar{Y} \in \mathcal{S}^m$. We note that for any $s \in \mathbb{R}^{m+1}$ we get

$$\text{Arw}(s) \cdot Y = s_0 \text{Tr}(Y) + 2\bar{s} \cdot \bar{Y}_0. \quad (17)$$

It follows that $\text{Arw}^\top : \mathcal{S}^{m+1} \rightarrow \mathbb{R}^{m+1}$ is nothing but

$$\text{Arw}^\top Y := \begin{pmatrix} \text{Tr}(Y) \\ 2\bar{Y}_0 \end{pmatrix}. \quad (18)$$

Consequently

$$M^\top(Y^1, \dots, Y^r) = \text{vec} \left(\begin{pmatrix} \text{Tr}(Y^1) \\ 2\bar{Y}_0^1 \end{pmatrix}, \dots, \begin{pmatrix} \text{Tr}(Y^r) \\ 2\bar{Y}_0^r \end{pmatrix} \right). \quad (19)$$

Then, $M^\top \mathcal{K}$ is clearly closed. Moreover, since $\text{Arw}^\top Y = 0$ implies $\text{Tr}(Y) = 0$, when $Y \in \mathcal{S}_+^{m+1}$ the latter yields $Y = 0$. Hence, all the assumptions of Lemma 12, and of Lemma 15, are fulfilled. We thus obtain as a consequence of those statements the following result.

Proposition 18. *Problems (LSOCP) and (LSDP) have the same four and six-partition.*

5 Acknowledgements

The research was partially supported by Australian Research Council grant DE150100240. The second author was supported by FONDECYT regular project 1160567 and Basal Program CMM-AFB 170001, both from CONICYT (Chile).

References

- [1] J. Frédéric Bonnans, J. Charles Gilbert, Claude Lemaréchal, and Claudia A. Sagastizábal. *Numerical optimization*. Universitext. Springer-Verlag, Berlin, 2003. Theoretical and practical aspects, Translated and revised from the 1997 French original.
- [2] J. Frédéric Bonnans and Héctor Ramírez C. Perturbation analysis of second-order cone programming problems. *Math. Program.*, 104(2-3, Ser. B):205–227, 2005.
- [3] J. Frédéric Bonnans and Alexander Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, New York, 2000.
- [4] Masakazu Muramatsu Bruno F. Lourenco and Takashi Tsuchiya. Facial reduction and partial polyhedrality. *arXiv:1512.02549*.
- [5] Dennis Cheung, Felipe Cucker, and Javier Peña. A condition number for multifold conic systems. *SIAM J. Optim.*, 19(1):261–280, 2008.

- [6] A. J. Goldman and A. W. Tucker. Theory of linear programming. In *Linear inequalities and related systems*, Annals of Mathematics Studies, no. 38, pages 53–97. Princeton University Press, Princeton, N.J., 1956.
- [7] Osman Güler. *Foundations of optimization*, volume 258 of *Graduate Texts in Mathematics*. Springer, New York, 2010.
- [8] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2001. Abridged version of it Convex analysis and minimization algorithms. I [Springer, Berlin, 1993; MR1261420 (95m:90001)] and it II [ibid.; MR1295240 (95m:90002)].
- [9] Gábor Pataki. Strong duality in conic linear programming: facial reduction and extended duals. In *Computational and analytical mathematics*, volume 50 of *Springer Proc. Math. Stat.*, pages 613–634. Springer, New York, 2013.
- [10] Javier Peña and Vera Roshchina. A complementarity partition theorem for multifold conic systems. *Math. Program.*, 142(1-2, Ser. A):579–589, 2013.
- [11] Motakuri V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Programming*, 77(2, Ser. B):129–162, 1997. Semidefinite programming.
- [12] Motakuri V. Ramana, Levent Tunçel, and Henry Wolkowicz. Strong duality for semidefinite programming. *SIAM J. Optim.*, 7(3):641–662, 1997.
- [13] James Renegar. *A mathematical view of interior-point methods in convex optimization*. MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2001.
- [14] Negar Soheili and Javier Peña. A condition-based algorithm for solving polyhedral feasibility problems. *J. Complexity*, 30(6):673–682, 2014.
- [15] Tamás Terlaky and Zhouhong Wang. On the identification of the optimal partition of second order cone optimization problems. *SIAM J. Optim.*, 24(1):385–414, 2014.
- [16] Stephen A. Vavasis and Yinyu Ye. Condition numbers for polyhedra with real number data. *Oper. Res. Lett.*, 17(5):209–214, 1995.
- [17] Hayato Waki and Masakazu Muramatsu. Facial reduction algorithms for conic optimization problems. *J. Optim. Theory Appl.*, 158(1):188–215, 2013.