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Strong formulations for conic quadratic optimization with indicator variables

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Abstract We study the convex hull of the mixed-integer set given by a conic quadratic inequality and indicator variables. Conic quadratic terms are often used to encode uncertainties, while the indicator variables are used to model fixed costs or enforce sparsity in the solutions. We provide the convex hull description of the set under consideration when the continuous variables are unbounded. We propose valid nonlinear inequalities for the bounded case, and show that they describe the convex hull for the two-variable case. All the proposed inequalities are described in the original space of variables, but extended SOCP-representable formulations are also given. We present computational experiments demonstrating the strength of the proposed formulations.

Keywords Conic quadratic optimization · Submodularity · Convexification · Conic quadratic cuts

Mathematics Subject Classification (2010) 90C11 · 90C26 · 90C57

1 Introduction

Given $\sigma \geq 0$ and set $N = \{1, \ldots, n\}$, we consider the set defined by a conic quadratic constraint and indicator variables

$$X_\sigma = \left\{ (x,y,t) \in \{0,1\}^N \times \mathbb{R}_+^N \times \mathbb{R}_+ : \sqrt{\sigma^2 + \sum_{i \in N} (c_i y_i)^2} \leq t,\ y_i(1-x_i) = 0,\ \forall i \in N \right\}.$$

Set X_σ arises directly in robust least squares problems [27], best subset selection problems with shrinkage [45], chance-constrained problems with independent normal distributions [19], robust optimization with ellipsoidal uncertainty sets [16, 17]

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and distributionally robust optimization with mean-covariance information [28, 57]. Additionally, X_σ is also encountered implicitly in rotated cone form $y'y \leq ts$ where $s,t \in \mathbb{R}_+$ are variables, e.g., in joint location-inventory problems [11, 50], value-at-risk minimization with discrete uncertainty sets [3], and feature selection problems with respect to information criteria [35, 46].

Constructing strong convex relaxations of non-convex sets is a critical step towards devising efficient algorithms for optimization problems involving such sets. There is an increasing literature on constructing strong formulations for pure- and mixed-integer conic quadratic optimization, including Gomory cuts [23], MIR cuts [9], cut generating functions [52], lifting [10, 51], subadditive inequalities [49] and minimal inequalities [41]. Additionally, disjunctive programming has received a vast amount of attention in the context of mixed-integer nonlinear programming. The convex hull of a set defined as a disjunction of convex sets can be characterized in an extended formulation [22, 53]. However, such formulations create a copy of each variable for each disjunction; the resulting formulations do not scale for X_σ, defined as the disjunction of 2^n sets. Thus, researchers have focused on characterizing the convex hulls of sets involving conic quadratic constraints and simpler disjunctions [2, 13, 14, 15, 20, 40, 42, 47, 48, 56], in particular, disjunctions involving only two sets. The resulting valid inequalities are directly applicable to a broad class of pure-integer conic quadratic optimization problems, but do not exploit the specific structure of X_σ. Moreover, focusing only on two-term disjunctions may be insufficient to provide a good approximation of the convex hull of X_σ.

Although X_σ has received relatively few attention in the literature, the related quadratic set defined by the constraint $y'Qy \leq t$ with $Q \succeq 0$ and indicator variables has been studied in depth [18, 24, 39]. In particular, if Q is a diagonal matrix, then the perspective reformulation is a widely used technique for constructing strong convex relaxations for quadratic (and in general separable nonlinear) optimization with indicator variables [1, 21, 29, 31, 32, 33, 36, 38, 44, 54, 55]. For general positive-definite matrices Q, a standard technique is to extract a positive diagonal matrix D from Q such that $Q-D \succeq 0$, and reformulate each separable quadratic term of the form $D_{ii}y_i^2$ using its perspective function [25, 30, 58]. More recently, approaches which extract simple but non-diagonal matrices from Q have also been considered [4, 5, 12, 34, 37, 39]. However, perspective reformulation approaches are not applicable to X_σ as the nonlinear function $\sqrt{\sigma^2+\sum_{i\in N}(c_iy_i)^2}$ is non-separable due to the square root.

More recently, Atamtürk and Gómez [6] propose strong formulations for binary conic quadratic optimization with terms of the form $\sqrt{x'Qx} \leq t$, based on diagonal decompositions of the matrix Q. Specifically, a conic quadratic constraint $\sqrt{x'Qx} \leq t$ can be written in an extended formulation as $\sqrt{s^2+y^2} \leq t$, $\sqrt{x'Dx} \leq s$, $\sqrt{x'(Q-D)x} \leq y$. Valid inequalities of the form $\pi' x \leq s$ can then be added to the formulation, where π is an extreme point of the extended polymatroid associated with the submodular function $\sqrt{x'Dx}$ [8, 26, 43]. Strong relaxations of X_σ can similarly be exploited to derive strong formulations of the set

$$H = \left\{(x,y,t) \in \{0,1\}^N \times \mathbb{R}_+^N \times \mathbb{R}_+ : \sqrt{y'Qy} \leq t,\ y_i(1-x_i)=0,\ \forall i \in N\right\}.$$

However, the convex hull of X_σ is not well understood to date. To the best of our knowledge, set X_σ was first (explicitly) considered in [7]. The authors study the case where the continuous variables are bounded, i.e., $y_i \leq x_i$, and propose valid linear inequalities based on lifting the extended polymatroid inequalities for the pure-binary restriction of X_σ where $y_i = x_i$. Nonetheless, since the convex hull of X_σ is non-polyhedral, it cannot be described by a finite number of linear inequalities.

The main contributions of this paper are:

1. We give the complete description of the convex hull of X_σ.
2. We illustrate how the resulting valid inequalities can be used as building blocks to derive strong formulations for optimization problems involving subsets of X_σ. We focus in particular on the case with bounded continuous variables.

The rest of the paper is organized as follows. In Section 2 we study the set X_σ and prove the main result of the paper. In Section 3 we use the insights gained from X_σ to propose valid inequalities for the bounded case. In Section 4 we present computations illustrating the strength of the inequalities, and in Section 5 we conclude the paper.

Notation Given a set $X \subseteq \mathbb{R}^N$, let $\text{conv}(X)$ denote the closure of its convex hull. Given a vector $x \in \mathbb{R}^N$ and $S \subseteq N$, let x_S denote the subvector of x induced by S, let $x(S) = \sum_{i \in S} x_i$, and let $\Pi(S)$ be the set of all permutations of S.

2 Study of the set X_σ

In this section we study the set X_σ. We first point out that $\text{conv}(X_\sigma)$ is trivial when $\sigma = 0$.

Proposition 1 *$conv(X_0) = \left\{(x,y,t) \in [0,1]^N \times \mathbb{R}_+^N \times \mathbb{R}_+ : \sqrt{\sum_{i \in N} (c_i y_i)^2} \leq t\right\}$.*

Proof Let $\bar{X}_0 = \left\{(x,y,t) \in [0,1]^N \times \mathbb{R}_+^N \times \mathbb{R}_+ : \sqrt{\sum_{i \in N} (c_i y_i)^2} \leq t\right\}$. We prove that the optimization problems

$$(P_0) \quad \min_{(x,y,t) \in X_0} a'x - b'y + \Omega t, \text{ and}$$
$$(Q_0) \quad \min_{(x,y,t) \in \bar{X}_0} a'x - b'y + \Omega t,$$

where $a, b \in \mathbb{R}^N$ and $\Omega \in \mathbb{R}$, are either both unbounded, or share a same optimal solution. First suppose that there exists $(\bar{y}, \bar{t}) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $\sqrt{\sum_{i \in N} (c_i \bar{y}_i)^2} \leq \bar{t}$ and $-b'\bar{y} + \Omega \bar{t} < 0$. Then by setting $x = 1$, $y = \lambda \bar{y}$ and $t = \lambda \bar{t}$, we obtain feasible solutions for both problems with arbitrarily low values as $\lambda \to \infty$, thus both problems are unbounded. Now suppose that $-b'y + \Omega t \geq 0$ for all y, t such that $\sqrt{\sum_{i \in N} (c_i y_i)^2} \leq t$. Then there exists an optimal solution for (P_0) and (Q_0) with $y = t = 0$, and clearly in this case (P_0) and (Q_0) are equivalent. □

In the rest of this section, we assume $\sigma > 0$ to rule out the trivial case. First, in Section 2.1, we study optimization over X_σ. Then, in Section 2.2, we introduce a family of nonlinear valid inequalities for X_σ. Finally, in Section 2.3, we show that the proposed inequalities actually describe $\text{conv}(X_\sigma)$.

2.1 Optimization over X_σ

Consider the optimization problem

$$(P) \qquad \min_{(x,y,t)\in X_\sigma} a'x - b'y + \Omega t,$$

where $a,b \in \mathbb{R}^N$ and $\Omega \in \mathbb{R}$. Note that if $\Omega < 0$ then (P) is unbounded by letting $t \to \infty$, and if $\Omega = 0$ then (P) is a simple linear optimization. Thus we assume $\Omega > 0$ and, by scaling, we can set $\Omega = 1$. Since $t = \sqrt{\sigma^2 + \sum_{i\in N}(c_i y_i)^2}$ in any optimal solution, we write (P) as

$$\min_{(x,y)\in\{0,1\}^N\times\mathbb{R}^N_+} a'x - b'y + \sqrt{\sigma^2 + \sum_{i\in N}(c_i y_i)^2} \text{ subject to } y_i(1-x_i) = 0,\ \forall i \in N.$$

Moreover, we see that if $b_i \le 0$, then there exists an optimal solution where $y_i = 0$; and if $a_i \le 0$, then there exists an optimal solution where $x_i = 1$. Thus we assume that $a,b > 0$. Finally, observe that if $c_i = 0$, then the problem is unbounded by letting $y_i \to \infty$. Thus we also assume $c > 0$.

Atamtürk and Jeon [7] give a $O(n^2)$ algorithm for the case with bounded continuous variables that could be used for (P) using big-M constraints. We now show that (P) can be solved in $O(n\log n)$ time. For $i \in N$, define $\zeta_i := ({b_i}/{c_i})^2$. Moreover, for all $S \subseteq N$ such that $\zeta(S) \le 1$, define

$$\phi(S) := a(S) + \sigma\sqrt{1-\zeta(S)}.$$

Finally, define $g_S : \mathbb{R}^S \to \mathbb{R}$ as $g_S(y_S) = -b'y_S + \sqrt{\sigma^2 + \sum_{i\in S}(c_i y_i)^2}$ for all $S \subseteq N$. Proposition 2 shows how to solve (P), depending on the value of $\zeta(N)$. In particular, if $\zeta(N) \le 1$, then (P) is equivalent to the submodular minimization problem $\min_{S\subseteq N}\phi(S)$.

Proposition 2 *Problem (P) can be solved as follows:*

1. *If $\zeta(N) > 1$, then (P) is unbounded, i.e., there exist feasible solutions with arbitrarily low objective values.*
2. *If $\zeta(N) < 1$, then $\min_{(x,y,t)\in X_\sigma} a'x - b'y + t = \min_{S\subseteq N}\phi(S)$. Moreover, if $S^* \in \arg\min_{S\subseteq N}\phi(S)$, then there exists an optimal solution to (P) where $y_i^* = \sigma\frac{\zeta_i}{b_i\sqrt{1-\zeta(S^*)}}$ and $x_i^* = 1$ if $i \in S^*$, and $y_i^* = x_i^* = 0$ otherwise.*
3. *If $\zeta(N) = 1$, then $\inf_{(x,y,t)\in X_\sigma} a'x - b'y + t = \min_{S\subseteq N}\phi(S)$. Moreover, if $S^* \in \arg\min_{S\subseteq N}\phi(S)$ with $S^* \subsetneq N$, then there exists an optimal solution as in the case $\zeta(N) < 1$. If $\arg\min_{S\subseteq N}\phi(S) = \{N\}$, then there does not exist an optimal solution.*

Proof 1. First suppose $\zeta(N) > 1$. Then, for $\lambda > 0$, let $y_i(\lambda) = \lambda\frac{b_i}{c_i^2}$ for $i \in N$, and observe that

$$\lim_{\lambda\to\infty} g_N(y(\lambda)) = \lim_{\lambda\to\infty} -\zeta(N)\lambda + \sqrt{\sigma^2 + \zeta(N)\lambda^2} = -\infty.$$

Thus, setting $x_i = 1$ for all $i \in N$ and $y = y(\lambda)$, we find feasible solutions with arbitrarily low objective values.

2. Now suppose $\zeta(N) < 1$, let $S \subseteq N$, and consider the convex optimization problem $\min_{y \in \mathbb{R}^S_+} g_S(y_S)$. From the KKT conditions we find that, for $k \in S$,

$$b_k - \frac{c_k^2 y_k}{\sqrt{\sigma^2 + \sum_{i \in S} (c_i y_i)^2}} = -\alpha_k,\ \alpha_k \geq 0 \text{ and } \alpha_k y_k = 0. \tag{1}$$

Observe that if $y_k = 0$, then conditions (1) cannot be satisfied since $b_k > 0$ and $\alpha_k \geq 0$. Thus we see that $y_k > 0$ and $\alpha_k = 0$ for $k \in S$. In particular, we have

$$y_k = \frac{b_k}{c_k^2} \sqrt{\sigma^2 + \sum_{i \in S} (c_i y_i)^2} \tag{2}$$

$$\implies \sum_{i \in S} (c_i y_i)^2 = \zeta(S) \left(\sigma^2 + \sum_{i \in S} (c_i y_i)^2 \right) \tag{3}$$

$$\Leftrightarrow \sum_{i \in S} (c_i y_i)^2 = \frac{\zeta(S)}{1 - \zeta(S)} \sigma^2, \tag{4}$$

where (4) follows since $\zeta(S) < 1$. In particular, from (2) and (4), we find

$$y_k^* = \frac{b_k}{c_k^2 \sqrt{1 - \zeta(S)}} \sigma. \tag{5}$$

Now observe that, for $S \subseteq N$ and for all $i \in S$, if $x_i = 1$ and y_i is given by (5), and $x_i = y_i = 0$ for $i \in N \setminus S$, we find that

$$\begin{aligned} a'x - b'y + \sqrt{\sigma^2 + \sum_{i \in N} (c_i y_i)^2} &= a(S) - \frac{\sigma \zeta(S)}{\sqrt{1 - \zeta(S)}} + \frac{\sigma}{\sqrt{1 - \zeta(S)}} \\ &= a(S) + \sigma \sqrt{1 - \zeta(S)} = \phi(S). \end{aligned}$$

In particular, we find that (P) is equivalent to solving $\min_{S \subseteq N} \phi(S)$.

3. Finally, suppose $\zeta(N) = 1$. Observe that if $S \subsetneq N$, we have $\zeta(S) < 1$ and optimal solutions to $\min_{y \in \mathbb{R}^S_+} g_S(y_S)$ are given by (5), as in the case $\zeta(N) < 1$.

   Now consider the optimization problem $\inf_{y \in \mathbb{R}^N_+} g_N(y)$. Observe that the KKT conditions (3) cannot be satisfied, since $\zeta(S) = \zeta(N) = 1$ and $\sigma > 0$, thus there does not exist any KKT point (or optimal solution). However, note that setting $y_i(\lambda) = \frac{b_i}{c_i^2} \lambda$ for all $i \in N$, we find that $\lim_{\lambda \to \infty} g_N(y(\lambda)) = 0$. Thus, by setting $x_i = 1$ for all $i \in N$ and $y = y(\lambda)$ as $\lambda \to \infty$, we find feasible solutions with objective value arbitrarily close to $a(N)$.

   Therefore, we see that $\inf_{(x,y,t) \in X_\sigma} a'x - b'y + t = \min_{S \subseteq N} \phi(S)$, and that the existence of an optimal solution depends on whether there exists an optimal solution with $S \subsetneq N$ or not. □

Note that the set function ϕ is submodular, and optimization with function ϕ was studied in [8]. In particular, index the variables such that $\frac{a_1}{\zeta_1} \leq \frac{a_2}{\zeta_2} \leq \ldots \leq \frac{a_n}{\zeta_n}$ (breaking ties arbitrarily), and define $S_0 = \emptyset$ and $S_i = \{1, \ldots, i\}$ for $i \in N$.

Proposition 3 (Atamtürk and Narayanan [8]) *The set of all optimal solutions of* $\min_{S \subseteq N} \phi(S)$ *is some collection* $\mathscr{S}$ *of nested sets* $S_{i_1} \subsetneq S_{i_2} \subsetneq \ldots \subsetneq S_{i_k}$ *for some* $1 \le k \le n$.

Corollary 1 *Optimization problem* (P) *can be solved in* $O(n \log n)$ *time.*

Proof Checking whether $\zeta(N) > 1$ can be done in linear time. If $\zeta(N) \le 1$, then solving (P) requires sorting the variables in terms of $\frac{a_i}{\zeta_i}$, which can be done with $O(n \log n)$ comparisons. □

2.2 Valid inequalities for X_σ

In this section we propose a new class of valid inequalities for X_σ.

Definition 1 Given a permutation $v = ((1),(2),\ldots,(n)) \in \Pi(N)$, define $x_{(0)} = 1$, define $N_i = \{(i),(i+1),\ldots,(n)\}$ for $i = 0,\ldots,n$ and define $f^v_{\sigma,i} : \{0,1\}^{N_{i-1}} \times \mathbb{R}^{N_i}_+ \to \mathbb{R}_+$ recursively as

$$f^v_{\sigma,n+1}(x_{(n)}) := \sigma x_{(n)}$$

$$f^v_{\sigma,i}(x_{N_{i-1}}, y_{N_i}) := \sigma(x_{(i-1)} - x_{(i)}) + \sqrt{f^v_{\sigma,i+1}(x_{N_i}, y_{N_{i+1}})^2 + (c_{(i)} y_{(i)})^2}, \quad i = 1,\ldots,n.$$

□

For simplicity we write $f^v_{\sigma,i}(x,y)$ instead of $f^v_{\sigma,i}(x_{N_{i-1}}, y_{N_i})$, and $f^v_{\sigma,n+1}(x,y)$ instead of $f^v_{\sigma,n+1}(x_{(n)})$, in the rest of the paper; the variables of which $f^v_{\sigma,i}$ is a function of are implied by the index i and the permutation v. Moreover, we also omit the index i when equal to 1 from our notation, i.e.,

$$f^v_\sigma(x,y) := f^v_{\sigma,1}(x,y).$$

Example 1 Let $N = \{1,2,3\}$, and let $v = (1,2,3)$ be the permutation corresponding to the natural order of the variables. Then

$$f^v_\sigma(x,y) = \sigma(1-x_1) + \sqrt{\left(\sigma(x_1-x_2) + \sqrt{\left(\sigma(x_2-x_3) + \sqrt{(\sigma x_3)^2 + (c_3 y_3)^2}\right)^2 + (c_2 y_2)^2}\right)^2 + (c_1 y_1)^2}.$$

□

Consider the nonlinear inequality

$$f^v_\sigma(x,y) \le t. \tag{6}$$

Inequality (6) can be used with most off-the-shelf conic quadratic solvers in an extended formulation: introduce non-negative continuous variables s_i and t_i, $i = 2,\ldots,n+1$, and write

$$\sigma(1 - x_{(1)}) + t_2 \le t, \quad s_2^2 + (c_{(1)} y_{(1)})^2 \le t_2^2 \tag{7a}$$

$$\sigma(x_{(i-1)} - x_{(i)}) + t_{i+1} \le s_i, \quad s_{i+1}^2 + (c_{(i)} y_{(i)})^2 \le t_{i+1}^2, \qquad i = 2,\ldots,n \tag{7b}$$

$$\sigma x_{(n)} \le s_{n+1}. \tag{7c}$$

In particular, (6) is SOCP-representable and convex. We now prove that inequalities (6) are valid for X_σ.

Proposition 4 *For all $(x,y) \in X_\sigma$ and all $i = 1,\dots,n+1$,*

$$f^v_{\sigma,i}(x,y) \le \sqrt{\sigma^2 x_{(i-1)} + \sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2}.$$

Proof We prove the result by induction, starting from the base case $i = n+1$.

Base case Trivial.

Induction step We assume the results holds for $f^v_{\sigma,i+1},\dots,f^v_{\sigma,n+1}$, and we show it holds for $f^v_{\sigma,i}$.

• First suppose $x_{(i-1)} = x_{(i)}$. We find

$$f^v_{\sigma,i}(x,y) = \sqrt{f^v_{\sigma,i+1}(x,y)^2 + \left(c_{(i)} y_{(i)}\right)^2} \le \sqrt{\sigma^2 x_{(i)} + \sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2},$$

where the inequality follows from the induction hypothesis. Since $x_{(i)} = x_{(i-1)}$, the result follows.

• Now suppose $x_{(i-1)} = 1$ and $x_{(i)} = 0$. Note that if $x_{(j)} = y_{(j)} = 0$ for all $j \ge i$, then we find

$$f^v_{\sigma,i}(x,y) = \sigma = \sqrt{\sigma^2 x_{(i-1)} + \sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2}.$$

Otherwise, let $k \le n$ be the minimum index $k > i$ such that $x_{(k)} = 1$. We find

$$\begin{aligned} f^v_{\sigma,i}(x,y) = \sigma x_{(i-1)} + f^v_{\sigma,k}(x,y) &= \sigma x_{(i-1)} - \sigma x_{(k)} + \sqrt{f^v_{\sigma,k+1}(x,y)^2 + \left(c_{(k)} y_{(k)}\right)^2} \\ &\le \sigma x_{(i-1)} - \sigma x_{(k)} + \sqrt{\sigma^2 x_{(k)} + \sum_{j=k}^{n} \left(c_{(j)} y_{(j)}\right)^2} \\ &= \sqrt{\sigma^2 x_{(i-1)} + \sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2}, \end{aligned}$$

where the inequality follows from the induction hypothesis, and the last equality follows from $x_{(i-1)} = x_{(k)}$.

• Finally, suppose $x_{(i-1)} = 0$ and $x_{(i)} = 1$. We find

$$\begin{aligned} f^v_{\sigma,i}(x,y) = -\sigma + \sqrt{f^v_{\sigma,i+1}(x,y)^2 + \left(c_{(i)} y_{(i)}\right)^2} &\le -\sigma + \sqrt{\sigma^2 + \sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2} \\ &\le \sqrt{\sum_{j=i}^{n} \left(c_{(j)} y_{(j)}\right)^2}, \end{aligned}$$

where the first inequality follows from the induction hypothesis, and the second inequality follows from concavity of the square root function. $\square$

Corollary 2 *Inequality* (6) *is valid for X_σ.*

Proof For any $(x,y,t) \in X_\sigma$ we have

$$f_\sigma^v(x,y) \leq \sqrt{\sigma^2 + \sum_{i\in N} (c_i y_i)^2} \leq t,$$

where the first inequality follows from Proposition 4 and $x_{(0)} = 1$. □

Before proving that inequalities (6) are in fact sufficient to describe $\text{conv}(X_\sigma)$ (Section 2.3), we first give additional definitions and properties concerning the functions $f_{\sigma,i}^v$ that provide useful insights and will be used through the paper.

Remark 1 If $\sigma = 0$, then $f_{\sigma,i}^v(x,y) = \sqrt{\sum_{j=i}^n (c_{(j)} y_{(j)})^2}$ and $f_\sigma^v(x,y) = \sqrt{\sum_{j\in N}(c_j y_j)^2}$. □

Definition 2 Given a permutation $v = ((1),(2),\ldots,(n)) \in \Pi(N)$, define $\bar{f}_{\sigma,i}^v : \{0,1\}^{N_i} \times \mathbb{R}_+^{N_i} \to \mathbb{R}_+$ as

$$\bar{f}_{\sigma,n+1}^v(x,y) := \sigma x_{(n)}$$
$$\bar{f}_{\sigma,i}^v(x,y) := -\sigma x_{(i)} + \sqrt{f_{\sigma,i+1}^v(x_{N_i}, y_{N_{i+1}})^2 + (c_{(i)} y_{(i)})^2}, \quad i = 1,\ldots,n.$$

□

We also use the simpler notation $\bar{f}_\sigma^v(x,y)$ instead of $\bar{f}_{\sigma,1}^v(x,y)$.

Remark 2 Observe that functions $f_{\sigma,i}^v$ and $\bar{f}_{\sigma,i}^v$ are related by the identities $f_\sigma^v(x,y) = \sigma + \bar{f}_\sigma^v(x,y)$, and $f_{\sigma,i}^v(x,y) = \sigma x_{(i-1)} + \bar{f}_{\sigma,i}^v(x,y)$ for $i = 2,\ldots,n$. Additionally, observe that $\bar{f}_{\sigma,i}^v$ is positively homogeneous of degree 1, i.e., for $\lambda \geq 0$, $\bar{f}_{\sigma,i}^v(\lambda x, \lambda y) = \lambda \bar{f}_{\sigma,i}^v(x,y)$. □

Remark 3 (Perspective reformulation) Note that $\sqrt{\sigma^2 + (c_1 y_1)^2} = \sigma + (\sqrt{\sigma^2 + (c_1 y_1)^2} - \sigma)$. Using the perspective reformulation [29, 31, 36] for the term in parenthesis, we obtain the valid inequality

$$f_\sigma(x_1,y_1) = \sigma + x_1 \bar{f}_\sigma(1, y_1/x_1) = \sigma + \left(\sqrt{(x_1\sigma)^2 + (c_1 y_1)^2} - \sigma x_1\right) \leq t.$$

Thus, inequalities (6) coincide with the perspective reformulation for the one-variable case. □

Remark 4 (Partial derivatives of f_σ^v) The partial derivatives of function f_σ^v are

$$\frac{\partial f_\sigma^v}{\partial y_{(k)}}(x,y) = \left(\prod_{i=1}^{k-1} \frac{f_{\sigma,i+1}^v(x,y)}{\sqrt{f_{\sigma,i+1}^v(x,y)^2 + \left(c_{(i)} y_{(i)}\right)^2}}\right) \cdot \frac{c_{(k)}^2 y_{(k)}}{\sqrt{f_{\sigma,k+1}^v(x,y)^2 + \left(c_{(k)} y_{(k)}\right)^2}}$$
$$\frac{\partial f_\sigma^v}{\partial x_{(k)}}(x,y) = \left(\prod_{i=1}^{k-1} \frac{f_{\sigma,i+1}^v(x,y)}{\sqrt{f_{\sigma,i+1}^v(x,y)^2 + \left(c_{(i)} y_{(i)}\right)^2}}\right) \cdot \sigma \left(\frac{f_{\sigma,k+1}^v(x,y)}{\sqrt{f_{\sigma,k+1}^v(x,y)^2 + \left(c_{(k)} y_{(k)}\right)^2}} - 1\right).$$

□

Note that if there exist an index ℓ such that $x_{(i)} = y_{(i)} = 0$ for all $i > \ell$, then $\sqrt{f^v_{\sigma,i+1}(x,y)^2 + \left(c_{(i)}y_{(i)}\right)^2} = 0$, and the partial derivatives above are undefined for $k > \ell$ since they involve division by 0.

Finally, as Remark 5 below points out, the functions $f^v_{\sigma,i}(x,y)$ and its partial derivatives can be simplified for certain permutations and values of (x,y).

Remark 5 Let $v = ((1),\ldots,(\ell),\ldots,(n))$ and $\bar{x} \in \{0,1\}^n$ such that $\bar{x}_{(i)} = 1$ for $i \le \ell$ and $\bar{x}_{(i)} = 0$ for $i > \ell$. Then

$$f^v_{\sigma,k}(\bar{x},y) = \begin{cases} 0 & \text{if } k = n+1 \\ \sqrt{\sum_{i=k}^n (c_{(i)}y_{(i)})^2} & \text{if } \ell+2 \le k \le n \\ \sigma + \sqrt{\sum_{i=\ell+1}^n (c_{(i)}y_{(i)})^2} & \text{if } k = \ell+1 \\ \sqrt{\sum_{i=k}^{\ell} (c_{(i)}y_{(i)})^2 + \left(\sigma + \sqrt{\sum_{i=\ell+1}^n (c_{(i)}y_{(i)})^2}\right)^2} & \text{if } k \le \ell. \end{cases}$$

Moreover, for $k = 1,\ldots,\ell$, we find

$$\frac{\partial f^v_\sigma}{\partial y_{(k)}}(\bar{x},y) = \frac{c^2_{(k)}y_{(k)}}{\sqrt{\sigma^2 + \sum_{i=1}^{\ell}(c_{(i)}y_{(i)})^2}}.$$

If, additionally, y is given as in Proposition 2, i.e., $y_{(k)} = \frac{\sigma\zeta_{(k)}}{b_{(k)}\sqrt{1-\sum_{i=1}^{\ell}\zeta_{(i)}}}$ for $k = 1,\ldots,\ell$ and $y_{(i)} = 0$ otherwise, then for $k = 1,\ldots,\ell$ we find

$$\frac{\partial f^v_\sigma}{\partial x_{(k)}}(\bar{x},y) = \sigma\sqrt{1 - \sum_{i=1}^{k}\zeta_{(i)}} - \sigma\sqrt{1 - \sum_{i=1}^{k-1}\zeta_{(i)}},$$

corresponding to marginal contributions of the submodular set function $\phi(S) - a(S)$. For $k = \ell+1,\ldots,n$, the partial derivatives are undefined. □

2.3 Convex hull of X_σ

This section is devoted to proving the main result of the paper, i.e.,

Theorem 1 (Convex hull of X_σ) *The convex hull of X_σ is described by inequalities* (6) *and bound constraints, i.e.,*

$$conv(X_\sigma) = \left\{(x,y,t) \in [0,1]^N \times \mathbb{R}^N_+ \times \mathbb{R} : f^v_\sigma(x,y) \le t,\ \forall v \in \Pi(N)\right\},$$

and is SOCP-representable.

Note that SOCP-representability follows directly from representation (7). We prove the rest of Theorem 1 by showing that (P) and the optimization problem

$$
(P_C) \qquad \begin{aligned} &\min a'x - b'y + t \\ &\text{s.t. } f_\sigma^v(x,y) \leq t, \quad \forall v \in \Pi(N) \\ &\quad x \in [0,1]^N, y \in \mathbb{R}_+^N, t \in \mathbb{R} \end{aligned}
$$

are equivalent. We can assume that $a_i, b_i, c_i > 0$ for all $i \in N$, as otherwise $x_i = 1$, $y_i = 0$ or $y_i \to \infty$ in an optimal solution of both (P) and (P_C). Suppose that $(\hat{x}, \hat{y}, \hat{t})$ is a (fractional) feasible solution of (P_C), and assume without loss of generality that the variables are indexed such that $\hat{x}_1 \geq \ldots \geq \hat{x}_n$. Let $v = (1, 2, \ldots, n)$ be the permutation corresponding to the natural order of the variables, and consider Algorithm 1 below, which shows how to construct an integral feasible solution (x^*, y^*, t^*) of (P_C) such that $a'x^* - b'y^* + t^* \leq a'\hat{x} - b'\hat{y} + \hat{t}$, or detects that both (P) and (P_C) are unbounded. We use the notation that for $k = 1, \ldots, n$,

$$[k] = \{1, \ldots, k\}.$$

Algorithm 1

Input: $(\hat{x}, \hat{y}, \hat{t})$ feasible point of P_C with $\hat{x}_1 \geq \ldots \geq \hat{x}_n \geq 0$.
Output: Integral solution (x^*, y^*, t^*)

Initialize $(x^*, y^*) \leftarrow (\hat{x}, \hat{y})$
$k \leftarrow 1$ ▷ iteration counter
repeat
  if $\sum_{i=k}^n (a_i x_i^* - b_i y_i^*) + \sqrt{1-\zeta([k-1])}\,\bar{f}_{\sigma,k}^v(x^*, y^*) < 0$ **and** $x_k^* > 0$ **then**
    $\lambda \leftarrow 1/x_k^*$
    $x_i^* \leftarrow \lambda x_i^*, y_i^* \leftarrow \lambda y_i^*, \quad \forall i \geq k$ ▷ $x_k^* = 1$
    $y_i^* \leftarrow b_i/c_i^2 \sqrt{\frac{1}{1-\zeta([k-1])}} \left(\sigma + \bar{f}_{\sigma,k}^v(x^*, y^*)\right), \quad \forall i < k$
    $k \leftarrow k+1$ ▷ Increase iteration counter
  else if $\sum_{i=k}^n (a_i x_i^* - b_i y_i^*) + \sqrt{1-\zeta([k-1])}\,\bar{f}_{\sigma,k}^v(x^*, y^*) < 0$ **and** $x_k^* = 0$ **then**
    return $-\infty$ ▷ Problems (P) and (P_C) are unbounded
  else
    $x_i^* \leftarrow 0, y_i^* \leftarrow 0, \quad \forall i \geq k$
    $y_i^* \leftarrow b_i/c_i^2 \sqrt{\frac{1}{1-\zeta([k-1])}}\,\sigma, \quad \forall i < k$
    $k \leftarrow n$ ▷ Terminate
  end if
until $k = n$
$t^* \leftarrow f_\sigma^v(x^*, y^*)$
return (x^*, y^*, t^*)

We first prove the invariant of the algorithm.

Proposition 5 *At the beginning of iteration $k < n$ of Algorithm 1,*

1. $x_1^* = \ldots = x_{k-1}^* = 1$,
2. $1 \geq x_k^* \geq \ldots \geq x_n^* \geq 0$,
3. $a'x^* - b'y^* + f_\sigma^v(x^*, y^*) \leq a'\hat{x} - b'\hat{y} + f_\sigma^v(\hat{x}, \hat{y})$.

Proof Observe that after iteration i, either x_i^* is fixed to one (and never modified again) or the algorithm terminates. Additionally, since all non-fixed x variables are multiplied by the same constant, the relative order does not change. Thus (1) are (2) are easily verified. To prove (3), we show that the value $a'x^* - b'y^* + f_\sigma^v(x^*, y^*)$ does not increase after performing the updates in Algorithm 1. Let $(\bar{x}, \bar{y})$ be the value of (x^*, y^*) at the beginning of iteration k. Observe that, since $\bar{x}_1 = \ldots = \bar{x}_{k-1} = 1$,

$$\begin{aligned} f_\sigma^v(\bar{x}, \bar{y}) &= \sqrt{\sum_{i=1}^{k-1} (c_i\bar{y}_i)^2 + \left(\sigma + \sqrt{(c_k\bar{y}_k)^2 + f_{\sigma,k+1}^v(\bar{x}, \bar{y})} - \sigma\bar{x}_k\right)^2} \\ &= \sqrt{\sum_{i=1}^{k-1} (c_i\bar{y}_i)^2 + \left(\sigma + \bar{f}_{\sigma,k}^v(\bar{x}, \bar{y})\right)^2}. \end{aligned}$$

We now show how to construct a solution where either $x_k = 1$ or $x_k = \ldots = x_n = 0$ and $y_k = \ldots = y_n = 0$ without increasing the value $a'\bar{x} - b'\bar{y} + f_\sigma^v(\bar{x}, \bar{y})$. Consider the optimization problem

$$\begin{aligned} (P_C^k) \qquad & \min g_k(x,y) := a'x - b'y + \sqrt{\sum_{i=1}^{k-1} (c_iy_i)^2 + \left(\sigma + \bar{f}_{\sigma,k}^v(x,y)\right)^2} \\ & \text{s.t. } x_1 = \ldots = x_{k-1} = 1 \\ & \quad x \in [0,1]^N,\ y \in \mathbb{R}_+^N. \end{aligned}$$

Observe that for any feasible solution (x,y) of (P_C^k) the corresponding objective value $g_k(x,y)$ is precisely $a'x - b'y + f_\sigma^v(x,y)$, and that $(\bar{x}, \bar{y})$ is feasible for (P_C^k). Moreover, note that for any fixed value of (x_i, y_i), $i = k, \ldots, n$, the optimal values of y_i, $i < k$, can be found by setting $\frac{\partial g_k}{\partial y_i}(x,y) = 0$. In particular, we find that for $i < k$,

$$\begin{aligned} & \frac{\partial g_k}{\partial y_i}(x,y) = 0 \\ \Leftrightarrow\ & -b_i + \frac{c_i^2 y_i}{\sqrt{\sum_{j=1}^{k-1} (c_jy_j)^2 + \left(\sigma + \bar{f}_{\sigma,k}^v(x,y)\right)^2}} = 0 \\ \Longrightarrow\ & y_i = \frac{b_i}{c_i^2}\sqrt{\sum_{j=1}^{k-1} (c_jy_j)^2 + \left(\sigma + \bar{f}_{\sigma,k}^v(x,y)\right)^2} \qquad (8) \\ \Longrightarrow\ & \sum_{j=1}^{k-1} (c_jy_j)^2 = \zeta([k-1]) \left(\sum_{j=1}^{k-1} (c_jy_j)^2 + \left(\sigma + \bar{f}_{\sigma,k}^v(x,y)\right)^2\right) \\ \Longrightarrow\ & \sum_{j=1}^{k-1} (c_jy_j)^2 = \frac{\zeta([k-1])}{1 - \zeta([k-1])} \left(\sigma + \bar{f}_{\sigma,k}^v(x,y)\right)^2. \qquad (9) \end{aligned}$$

Moreover, by replacing $\sum_{j=1}^{k-1}(c_j y_j)^2$ in (8) using (9), we find the values

$$y_i = \frac{b_i}{c_i^2}\sqrt{\frac{1}{1-\zeta([k-1])}}\left(\sigma + \bar{f}^v_{\sigma,k}(x,y)\right). \tag{10}$$

If we substitute $y_{[k-1]}$ in (P_C^k) with their optimal values given by (10), then we find the equivalent optimization problem

$$(P_C^{k'}) \qquad \min g'_k(x,y) := a([k-1]) + \sum_{i\in N_k}(a_i x_i - b_i y_i) + \sqrt{1-\zeta([k-1])}\left(\sigma + \bar{f}^v_{\sigma,k}(x,y)\right)$$
$$\text{s.t. } x \in [0,1]^{N_k},\ y \in \mathbb{R}_+^{N_k},$$

where $N_k = \{k,\ldots,n\}$. Observe that from a given feasible solution of $(P_C^{k'})$ we can recover a feasible solution of (P_C^k) with same objective value simply by setting $y_{[k-1]}$ using equality (10). Moreover, by construction, $g'_k(\bar{x}_{N_k}, \bar{y}_{N_k}) \le g_k(\bar{x},\bar{y})$, since the values $\bar{y}_{[k-1]}$ were changed to the values that minimize g_k. Additionally, we find that for $\lambda \ge 0$,

$$g'_k(\lambda\bar{x}_{N_k}, \lambda\bar{y}_{N_k}) = \sqrt{1-\zeta([k-1])}\sigma + a([k-1]) + \lambda\left(a'\bar{x}_{N_k} - b'\bar{y}_{N_k} + \sqrt{1-\zeta([k-1])}\bar{f}^v_{\sigma,k}(\bar{x},\bar{y})\right).$$

Thus, if $a'\bar{x}_{N_k} - b'\bar{y}_{N_k} + \sqrt{1-\zeta([k-1])}\bar{f}^v_{\sigma,k}(\bar{x},\bar{y}) \ge 0$, we can improve the objective value by setting $\lambda = 0$, i.e., setting $x_k = \ldots = x_n = y_k = \ldots = y_n = 0$. If $a'\bar{x}_{N_k} - b'\bar{y}_{N_k} + \sqrt{1-\zeta([k-1])}\bar{f}^v_{\sigma,k}(\bar{x},\bar{y}) < 0$ and $\bar{x}_k > 0$, then we can improve the objective value by setting $\lambda = 1/\bar{x}_k$.

Finally, if $a'\bar{x}_{N_k} - b'\bar{y}_{N_k} + \bar{f}^v_{\sigma,k}(\bar{x},\bar{y}) < 0$ and $\bar{x}_k = \ldots = \bar{x}_n = 0$, then letting $\lambda \to \infty$ we find feasible solutions of $(P_C^{k'})$ with arbitrarily low objectives values – observe that these solutions are not feasible for (P) as they do not satisfy the complementary constraints $y_i(1-x_i)$ for $i \ge k$. However, it follows from Remark 5 that for any $y \in \mathbb{R}_+^{N_k}$,

$$\bar{f}^v_{\sigma,k}(\bar{x},y) = \bar{f}^v_{\sigma,k}(0_{N_k},y) = \sqrt{\sum_{i=k}^n (c_i y_i)^2} \ge \sqrt{\sigma^2 + \sum_{i=k}^n (c_i y_i)^2} - \sigma = \bar{f}^v_{\sigma,k}(e_{N_k},y),$$

where e is a vector of ones. Therefore, setting $x_k = \ldots = x_n = 1$, $y_i = \lambda\bar{y}_i$ for $i = k,\ldots,n$ and y_i given by (10) for $i = 1,\ldots,k-1$, we find feasible solutions of $(P_C^{k'})$, (P) and (P_C) with arbitrarily low objective values.

The three cases above and the corresponding updates are precisely the updates done in Algorithm 1. Therefore, at each iteration the value of the objective function of (P_C^k) does not increase, i.e., $g_k(x^*,y^*) \ge g_{k+1}(x^*,y^*)$. □

Proof (Proof of Theorem 1) Given any optimal solution $(\hat{x},\hat{y},\hat{t})$ of (P_C) with $\hat{x}_1 \ge \ldots \ge \hat{x}_n$, we have that $a'\hat{x} - b'\hat{y} + \hat{t} \ge a'\hat{x} - b'\hat{y} + f^v_\sigma(\hat{x},\hat{y})$ from feasibility. Additionally, we can use Algorithm 1 to find a solution (x^*,y^*,t^*) integral in x such that $a'x^* - b'y^* + t^* \le a'\hat{x} - b'\hat{y} + f^v_\sigma(\hat{x},\hat{y})$ by Proposition 5. Finally, (x^*,y^*,t^*) is feasible for (P_C) since $t^* = f^v_\sigma(x^*,y^*) = \sqrt{\sigma^2 + \sum_{i\in S^*}(c_i y_i)^2}$ where $S^* = \{i \in N : x_i^* = 1\}$ (Remark 5), and this point is not cut off by inequalities (6) by validity. Therefore, if there exists an optimal solution to (P_C), then there exists an optimal integral solution, and problems (P) and (P_C) are equivalent. □

We close this section by presenting a compact notation for the inequalities describing $\text{conv}(X_\sigma)$, which will be used in Section 3. Define the convex function

$$f_\sigma(x,y) = \max_{v \in \Pi(N)} f_\sigma^v(x,y); \tag{11}$$

function $f_\sigma(x,y)$ is convex as it is the maximum of a finite number of convex functions. Moreover,

$$\text{conv}(X_\sigma) = \left\{(x,y,t) \in [0,1]^N \times \mathbb{R}_+^N \times \mathbb{R} : f_\sigma(x,y) \le t\right\}.$$

3 Valid inequalities for the bounded case

In this section we discuss valid inequalities the set with bounded continuous variables given by

$$Z_\sigma = \{(x,y,t) \in X_\sigma : y_i \le x_i,\ \forall i \in N\}.$$

The upper bound of 1 for the bounded continuous variables is without loss of generality, as the variables can be scaled otherwise. The inequalities we propose are based on the results for the unbounded relaxation X_σ given in Section 2. First, we illustrate the derivation of the inequalities in Example 2.

Example 2 Consider set Z_σ with $n=3$. Observe that if $(x,y,t) \in Z_\sigma$, then

$$\begin{aligned}
t \ge& \sqrt{\sigma^2 + (c_1y_1)^2 + (c_2y_2)^2 + (c_3y_3)^2}\\
=&\sigma + \left(\sqrt{\sigma^2 + (c_1y_1)^2} - \sigma\right) + \left(\sqrt{\sigma^2 + (c_1y_1)^2 + (c_2y_2)^2 + (c_3y_3)^2} - \sqrt{\sigma^2 + (c_1y_1)^2}\right)\\
\ge&\sigma + \left(\sqrt{\sigma^2 + (c_1y_1)^2} - \sigma\right) + \left(\sqrt{\sigma^2 + c_1^2 + (c_2y_2)^2 + (c_3y_3)^2} - \sqrt{\sigma^2 + c_1^2}\right) &(12)\\
\ge&\sigma + \left(\sqrt{(\sigma x_1)^2 + (c_1y_1)^2} - \sigma x_1\right)\\
&+\left(\sqrt{\left((x_2 - x_3)\sqrt{\sigma^2 + c_1^2} + \sqrt{(\sigma^2 + c_1^2)x_3^2 + (c_3y_3)^2}\right)^2 + (c_2y_2)^2} - x_2\sqrt{\sigma^2 + c_1^2}\right). &(13)
\end{aligned}$$

Inequality (12) follows from concavity of the square root function and from $(c_1y_1)^2 \le c_1^2$ (since $y_1 \le 1$); inequality (13) holds since

$$\bar{f}_\sigma(x_1,y_1) = \sqrt{(\sigma x_1)^2 + (c_1y_1)^2} - \sigma x_1 \le \sqrt{\sigma^2 + (c_1y_1)^2} - \sigma$$

by validity of inequalities (6), and

$$\begin{aligned}
\bar{f}^{(2,3)}_{\sqrt{\sigma^2+c_1^2}}(x_2,x_3,y_2,y_3) =& \sqrt{\left((x_2 - x_3)\sqrt{\sigma^2 + c_1^2} + \sqrt{(\sigma^2 + c_1^2)x_3^2 + (c_3y_3)^2}\right)^2 + (c_2y_2)^2} - x_2\sqrt{\sigma^2 + c_1^2}\\
\le& \sqrt{\sigma^2 + c_1^2 + (c_2y_2)^2 + (c_3y_3)^2} - \sqrt{\sigma^2 + c_1^2},
\end{aligned}$$

also by validity of (6). Thus, we find that inequality

$$\sigma + \bar{f}_\sigma(x_1,y_1) + \bar{f}^{(2,3)}_{\sqrt{\sigma^2+c_1^2}}(x_2,x_3,y_2,y_3) \leq t$$

is valid. □

In general, let $v = ((1),(2),\ldots,(n))$ be a permutation of N, let $m \leq n$ and let $k \in \mathbb{Z}_+^{m+1}$ such that $1 = k_1 < k_2 < \ldots < k_m < k_{m+1} = n+1$, let $\sigma_j = \sqrt{\sigma^2 + \sum_{i=1}^{k_j-1} c_{(i)}^2}$, let $N_{ij} = \{(i),(i+1),\ldots,(j-1)\}$ for $j > i$, and consider the inequality

$$\sigma + \sum_{j=1}^{m} \bar{f}_{\sigma_j}(x_{N_{k_j,k_{j+1}}}, y_{N_{k_j,k_{j+1}}}) \leq t, \tag{14}$$

where $\bar{f}_{\sigma_j}$ are the functions given in Definition 2.

Proposition 6 *Inequality* (14) *is valid for* Z_σ.

Proof Let $(x,y,t) \in Z_\sigma$. We have that

$$\begin{aligned}\sqrt{\sigma^2 + \sum_{i\in N}(c_i y_i)^2} &= \sigma + \sum_{j=1}^{m}\left(\sqrt{\sigma^2 + \sum_{i=1}^{k_{j+1}-1}(c_{(i)}y_{(i)})^2} - \sqrt{\sigma^2 + \sum_{i=1}^{k_j-1}(c_{(i)}y_{(i)})^2}\right) \\ &\geq \sigma + \sum_{j=1}^{m}\left(\sqrt{\sigma_j^2 + \sum_{i=k_j}^{k_{j+1}-1}(c_{(i)}y_{(i)})^2} - \sigma_j\right),\end{aligned}$$

where the last inequality follows from $\sqrt{\sigma^2 + \sum_{i=1}^{k_j-1}(c_{(i)}y_{(i)})^2} \leq \sigma_j$ for any feasible point in Z_σ and concavity of the square root function. Since $(x,y,t) \in Z_\sigma$ implies that $(x,y,t) \in X_\sigma$, we find from validity of the inequalities discussed in Section 2 that

$$\sqrt{\sigma_j^2 + \sum_{i=k_j}^{k_{j+1}-1}(c_{(i)}y_{(i)})^2} - \sigma_j \geq f_{\sigma_j}(x_{N_{k_j,k_{j+1}}}, y_{N_{k_j,k_{j+1}}}) - \sigma_j = \bar{f}_{\sigma_j}(x_{N_{k_j,k_{j+1}}}, y_{N_{k_j,k_{j+1}}}),$$

and we obtain inequality (14). □

We now investigate the strength of inequalities (14), and discuss their separation.

3.1 Connections with previous work

We show that inequalities (14) dominate the first class of linear inequalities proposed in [7]. Observe that if $k_j = j$ for all $j = 1,\ldots,m+1$ and $m = n$, then inequalities (14) reduce to the simpler form (Remark 3)

$$\sigma + \sum_{i=1}^{n}\left(\sqrt{(\sigma_i x_{(i)})^2 + (c_{(i)}y_{(i)})^2} - \sigma_i x_{(i)}\right) \leq t. \tag{15}$$

Linear cuts can be derived from (15) using gradients. In particular, a linear outer approximation around the point $(\bar{x}, \bar{y})$ is given by

$$\sigma + \sum_{i=1}^{n} \left(\left(\frac{\sigma_i^2 \bar{x}_{(i)}}{\sqrt{(\sigma_i \bar{x}_{(i)})^2 + (c_{(i)} \bar{y}_{(i)})^2}} - \sigma_i \right) x_i + \frac{c_{(i)}^2 \bar{y}_{(i)}}{\sqrt{(\sigma_i \bar{x}_{(i)})^2 + (c_{(i)} \bar{y}_{(i)})^2}} y_i \right) \leq t. \tag{16}$$

If, additionally, $\bar{x} = \bar{y}$, then (16) simplifies to

$$\sigma + \pi' x \leq t + \alpha(x - y), \tag{17}$$

where $\pi_{(i)} = \sqrt{\sigma_i^2 + c_{(i)}^2} - \sigma_i$ and $\alpha_{(i)} = c_{(i)}^2 / \sqrt{\sigma_i^2 + c_{(i)}^2}$. Observe that π is an extreme point of the extended polymatroid associated with the submodular set function $\varphi(S) = \sqrt{\sigma^2 + \sum_{i \in S} c_i^2}$, and (17) corresponds to the first class of linear inequalities proposed in [7].

3.2 The two-variable case

We now show that inequalities (14) and bound constraints describe $\operatorname{conv}(Z_\sigma)$ when $\sigma = 0$ and $|N| = 2$. For this case, there are three valid inequalities (14):

1. If $v = (1,2)$, $m = 2$ with $k_1 = 1$, $k_2 = 2$ and $k_3 = 3$, then $\sigma_1 = 0$, $\sigma_2 = c_1$ and (14) is of the form (15) and reduces to $c_1 y_1 + \sqrt{(c_1 x_2)^2 + (c_2 y_2)^2} - c_1 x_2 \leq t$.
2. If $v = (2,1)$, $m = 2$ with $k_1 = 1$, $k_2 = 2$ and $k_3 = 3$, then $\sigma_1 = 0$, $\sigma_2 = c_2$ and (14) is of the form (15) and reduces to $c_2 y_2 + \sqrt{(c_2 x_1)^2 + (c_1 y_1)^2} - c_2 x_1 \leq t$.
3. If $m = 1$ with $k_1 = 1$ and $k_2 = 3$, then for any permutation v we have $\sigma_1 = 0$ and (14) reduces to the original inequality $\sqrt{(c_1 y_1)^2 + (c_2 y_2)^2} \leq t$.

Consider the optimization problem

$$\begin{aligned}
(P_B) \qquad & \min\ a_1 x_1 + a_2 x_2 - b_1 y_1 - b_2 y_2 + t && \text{(18a)}\\
& \text{s.t. } \sqrt{(c_1 y_1)^2 + (c_2 y_2)^2} \leq t && \text{(18b)}\\
& c_1 y_1 + \sqrt{(c_1 x_2)^2 + (c_2 y_2)^2} - c_1 x_2 \leq t && \text{(18c)}\\
& c_2 y_2 + \sqrt{(c_2 x_1)^2 + (c_1 y_1)^2} - c_2 x_1 \leq t && \text{(18d)}\\
& 0 \leq y_1 \leq x_1 \leq 1,\ 0 \leq y_2 \leq x_2 \leq 1,\ t \geq 0. && \text{(18e)}
\end{aligned}$$

Proposition 7 *Problem (P_B) has an optimal solution integral in x.*

Proof Let (x^*, y^*, t^*) be an optimal solution of (P_B). If $0 < x^* < 1$, then $(\lambda x^*, \lambda y^*, \lambda t^*)$ is also feasible for $0 \leq \lambda \leq 1/\max\{x_1^*, x_2^*\}$, and a solution with the same objective value can be found at one of the bounds. If $\lambda = 0$, then we have found that $x = y = t = 0$ is optimal (and integral). Otherwise, we may assume without loss of generality that

$x_1^* = 1$. Since $c_2y_2 + \sqrt{c_2^2 + (c_1y_1)^2} - c_2 \leq \sqrt{(c_1y_1)^2 + (c_2y_2)^2}$, we find that constraint (18d) is implied by (18b) when $x_1^* = 1$, and

$$(x_2^*, y_1^*, y_2^*) \in \arg\min a_2x_2 - b_1y_1 - b_2y_2 + t \tag{19a}$$
$$\text{s.t. } \sqrt{(c_1y_1)^2 + (c_2y_2)^2} \leq t \tag{19b}$$
$$c_1y_1 + \sqrt{(c_1x_2)^2 + (c_2y_2)^2} - c_1x_2 \leq t \tag{19c}$$
$$0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq x_2 \leq 1,\ t \geq 0. \tag{19d}$$

Observe that if $0 < y_1^* < 1$ and $0 < x_2^* < 1$, we can use a similar scaling argument to find an optimal solution where either $x_2 = y_1 = 0$, $x_2 = 1$, or $y_1 = 1$. For the first two cases the proof is complete; therefore we assume without loss of generality that $y_1^* = 1$. Since $c_1 + \sqrt{(c_1x_2)^2 + (c_2y_2)^2} - c_1x_2 \geq c_1 + \sqrt{(c_1)^2 + (c_2y_2)^2} - c_1$, we find that constraint (19b) is implied by (19c) when $y_1^* = 1$, and

$$(x_2^*, y_2^*) \in \underset{0 \leq y_2 \leq x_2 \leq 1}{\arg\min}\ a_2x_2 - b_2y_2 + \sqrt{(c_1x_2)^2 + (c_2y_2)^2} - c_1x_2.$$

It follows from a third scaling argument that there exists an optimal solution with $x_2^* \in \{0, 1\}$. □

Corollary 3 *Inequalities* (14) *and bound constraints* $0 \leq y \leq x \leq 1$ *describe conv*(Z_0) *when* $|N| = 2$.

Note that Corollary 3 can be rephrased as "inequalities (15), the original inequality (18b) and bound constraints describe $\text{conv}(Z_0)$ when $|N| = 2$." Nonetheless, as Example 3 below shows, the more general inequalities (14) are indeed stronger than inequalities (15) for $|N| \geq 3$.

Example 3 Consider the optimization problem

$$w^* = \min a_1x_1 + a_2x_2 + a_3x_3 - b_1y_1 - b_2y_2 - b_3y_3 + \Omega t \tag{20a}$$
$$\text{s.t. } \sqrt{(c_1y_1)^2 + (c_2y_2)^2 + (c_3y_3)^2} \leq t \tag{20b}$$
$$0 \leq y_1 \leq x_1,\ 0 \leq y_2 \leq x_2,\ 0 \leq y_3 \leq x_3,\ t \geq 0 \tag{20c}$$
$$x_1, x_2, x_3 \in \{0, 1\}, \tag{20d}$$

where the coefficients a, b, c are given in Table 1 and $\Omega = 2.3479$. The solution corresponding to setting $x_i = y_i = 1$ for $i = 1, \ldots, 3$ and $t = 37.06$ has objective value $w^* = -0.001$ and is optimal for (20). Now consider the solutions of the following convex relaxations of (20), computed using CPLEX solver:

R1 The natural convex relaxation, obtained by relaxing the integrality constraints on x, has optimal solution $x = (1.00, 0.48, 0.39)$, $y = (1.00, 0.48, 0.39)$ and $t = 21.98$, with objective value $w_1 = -6.002$.

R2 If inequality (15) for permutation $(1,2,3)$ is added, i.e.,

$$c_1y_1+\sqrt{(c_1x_2)^2+(c_2y_2)^2}-c_1x_2+\sqrt{(c_1^2+c_2^2)x_3^2+(c_3y_3)^2}-x_3\sqrt{c_1^2+c_2^2}\leq t, \tag{21}$$

then the optimal solution of the resulting relaxation is $x=(1.00,1.00,0.84)$, $y=(1.00,0.78,0.84)$ and $t=31.11$, with objective value $w_2=-0.485$.

R3 If inequalities (15) for all six permutations are added, then the optimal solution of the resulting relaxation is $x=(0.95,1.00,0.77)$, $y=(0.96,0.78,0.77)$ and $t=30.07$, with objective value $w_3=-0.459$.

R4 If in addition to adding inequality (21), inequality (14) for permutation $(1,2,3)$ and partition $k_1=1$, $k_2=2$ and $k_3=4$ is added, i.e.,

$$c_1y_1+\sqrt{\left(c_1x_2-c_1x_3+\sqrt{(c_1x_3)^2+(c_3y_3)^2}\right)^2+(c_2y_2)^2}-c_1x_2\leq t, \tag{22}$$

then the optimal solution of the resulting relaxation is $x=(0.94,1.00,0.86)$, $y=(0.94,0.97,0.86)$ and $t=34.60$, with objective value $w_4=-0.029$.

R5 If in addition to adding inequalities (21) and (22), inequality (14) for permutation $(2,1,3)$ and partition $k_1=1$, $k_2=2$ and $k_3=4$ is added, i.e.,

$$c_2y_2+\sqrt{\left(c_2x_1-c_2x_3+\sqrt{(c_2x_3)^2+(c_3y_3)^2}\right)^2+(c_1y_1)^2}-c_2x_1\leq t,$$

then the optimal solution of the resulting relaxation is $x=(1.00,1.00,1.00)$, $y=(1.00,1.00,1.00)$ and $t=37.06$, with objective value $w_5=-0.001$. This solution is optimal (up to the precision given by the solver).

Therefore, in this example, inequalities (15) are still effective, with a single valid inequality closing most of the gap (R2). Nevertheless, the remaining five inequalities (15) are ineffective at reducing the remaining gap (R3). In contrast, with the addition of two inequalities (14), an optimal integer solution can be recovered (R4-R5). □

Table 1 Values for coefficients a,b and c in Example 3.

Coeff.	Index		
	1	2	3
a	477.0160	10.7861	687.5810
b	509.9840	48.3004	704.1120
c	15.8881	26.9137	19.9159

3.3 Separation

We now discuss how to, given a fractional point $(\bar{x},\bar{y},\bar{t})$ and a permutation $v\in\Pi(N)$, find the partition of N defined by the indices $k_1,\ldots,k_m$ resulting in a most violated inequality (14).

Construct a directed acyclic graph G as follows: let $N \cup \{n+1\}$ be the set of vertices; let $(i,j) \in A$ for $i < j$ with cost $\bar{f}^{v}_{\sigma_i}(\bar{x}_{N_{ij}}, \bar{y}_{N_{ij}})$. Observe that the cost of path $(k_1, k_2, \ldots, k_{m+1})$ with $k_1 = 1$ and $k_{m+1} = n+1$ is precisely the left hand side of (14) evaluated at $(\bar{x}, \bar{y})$ (minus σ). Thus, a most violated inequality (14) can be found by finding a longest path from 1 to $n+1$ in G, which can be done with $O(n^2)$ evaluations of $\bar{f}^{v}_{\sigma_i}(\bar{x}_{N_{ij}}, \bar{y}_{N_{ij}})$ and $O(n^2)$ comparisons.

4 Computations

In this section we report computational experiments which demonstrate the strength of the proposed *nonlinear* inequalities. All experiments are conducted using CPLEX 12.7 solver with a single thread. We set a time limit of one hour, and CPLEX default settings are used.

4.1 Computations with Z_σ

First, we test the valid inequalities in optimization problems over Z_σ. The main purpose is to test how well the inequalities proposed in Section 3 are able to approximate $\text{conv}(Z_\sigma)$. In addition, the computations in this section also illustrate the difficulties of implementing the valid inequalities as cuts using current off-the-shelf branch-and-bound solvers.

4.1.1 Instances

We test the proposed formulations in instances of the form

$$\begin{aligned} \min\ & a'x - b'y + \Omega t \\ \text{s.t.}\ & \sum_{i \in N} (c_i y_i)^2 \le t^2 \\ & 0 \le y \le x \\ & x \in \{0,1\}^N, y \in \mathbb{R}^N_+,\ t \in \mathbb{R}. \end{aligned}$$

The vectors a, b, c are generated randomly, using the same parameters as the ones used in [7]: coefficients a_i are drawn from integer uniform $[5, 20]$, coefficients c_i^2 are drawn from integer uniform $[0.9n, 1.2n]$, and b_i is set to $a_i + h_i$, where h_i is drawn from integer uniform $[1, 4]$. In order to make the problems as hard as possible for the solver, we set[1] $\Omega = 0.999 \cdot \frac{-a(N)+b(N)}{\sqrt{\sum_{i \in N} c_i^2}}$, which causes a difficult trade-off between the linear and nonlinear parts of the objective.

[1] If $\Omega = {-a(N)+b(N)}/{\sqrt{\sum_{i \in N} c_i^2}}$, then the objective values corresponding to the solutions $x = y = 0$ and $x = y = 1$ have objective values of 0; in most cases, such solutions are optimal, resulting in initial/root gaps of infinity. Thus, we multiply Ω by 0.999 to ensure optimal solutions with objective value different from 0.

4.1.2 Methods

We compare the strength of the formulations obtained by adding cuts (14), (15) and (17). We test three approaches to add cuts:

`Nonlinear` The inequalities are added as nonlinear inequalities as described in (7). Since adding nonlinear cuts is not possible using CPLEX callbacks, the cuts are added by repeatedly solving the convex relaxations and manually adding the nonlinear inequality to the formulation. Using this method, only the convex relaxations are computed but the discrete problems are not solved.

`Gradient force` The nonlinear inequalities are underestimated using the gradients at fractional points (see Remark 4), and added as linear cuts using CPLEX user cut callbacks. The cuts are added using the CPLEX parameter "Force", i.e., all violated cuts found are added and kept in the formulation. The setting "Force" is not the default choice in CPLEX and often results in worse branch-and-bound performance, as too many cuts may be added and the size of the formulations may increase substantially.

`Gradient filter` The nonlinear inequalities are underestimated using the gradients at fractional points (see Remark 4), and added as linear cuts using CPLEX user cut callbacks. The cuts are added using the CPLEX parameter "Filter", i.e., CPLEX may decide to not add the cut, or to remove it from the formulation at a latter stage. The setting "Filter" is the default choice in CPLEX, and the size of the formulations is kept in check by not adding or removing cuts in the branch-and-bound process.

In all cases, inequalities are added only at the root node, and only inequalities with a violation of more than 10^{-4} are added, e.g., given a fractional point $(\bar{x},\bar{y},\bar{t})$, inequality (14) is added only if $\sigma + \sum_{j=1}^{m} \bar{f}_{\sigma_j}(\bar{x}_{N_{k_j,k_{j+1}}}, \bar{y}_{N_{k_j,k_{j+1}}}) - \bar{t} > 10^{-4}$. To separate the inequalities, we use the same method as [7] to find a permutation: given a fractional point $(\bar{x},\bar{y},\bar{t})$, we add inequalities (14), (15) and (17) based on the permutation $v = ((1),(2),\dots,(n))$ such that $\bar{x}_{(1)} \geq \bar{x}_{(2)} \geq \dots \geq \bar{x}_{(n)}$; for inequalities (14), we chose a partition using the longest path method discussed in Section 3.3.

4.1.3 Results

Table 2 shows the results for the `nonlinear` method. It shows, for different sizes n and the different classes of cuts, the initial gap (`igap`), the root improvement (`rimp`), the time used for adding cuts (`rtime`) and the number of cuts added (`#cuts`). The initial gap is computed as $\texttt{igap} = \frac{\text{obj}_{\text{opt}} - \text{relax}}{|\text{obj}_{\text{opt}}|} \times 100$, where "$\text{obj}_{\text{opt}}$" denotes the optimal objective value and "relax" is the objective value of the natural continuous relaxation. The root improvement is computed as $\texttt{rimp} = \frac{\text{root}_{\text{relax}} - \text{relax}}{\text{obj}_{\text{obt}} - \text{relax}} \times 100$, where "$\text{root}_{\text{relax}}$" is the objective value of the relaxation obtained after adding cuts. Each row represents the average over five instances generated with identical parameters.

We see that the linear cuts (17) achieve a similar root improvement as the simplest nonlinear extension (15). The root improvement achieved by (15) and (17) decreases as the size of the problems increases. On the other hand, the formulation resulting

Table 2 Results for the `nonlinear` method.

n	igap	cplex		cplex+(17)			cplex+(15)			cplex+(14)		
		rimp	rtime	rimp	rtime	#cuts	rimp	rtime	#cuts	rimp	rtime	#cuts
50	519.8	0.0	0.0	56.3	0.1	8	64.9	0.3	11	99.9	1.6	17
100	111.2	0.0	0.0	46.0	0.1	9	49.7	0.6	10	100.0	4.4	18
200	24.9	0.0	0.0	37.7	0.3	8	39.0	1.6	10	100.0	26.0	21
300	10.1	0.0	0.0	33.2	0.4	7	32.2	1.7	8	99.9	33.9	17
500	3.0	0.0	0.1	27.3	0.5	6	26.7	1.9	6	99.8	24.2	12

from the most general version of the nonlinear inequalities given by (14) is substantially stronger than the rest: the root improvement is close to 100% in all instances, regardless of the size of the problem. Additionally, the root improvement is achieved using a relatively small number of cuts, and the number of nonlinear inequalities added does not seem to depend on the size of the problem. The root times corresponding to (14) are larger than for the other formulations (as expected), since more cuts are added and each cut requires adding $2n$ additional variables and constraints.

Table 3 presents the results for the `gradient force` on the same instances. It shows, for different sizes n and the different classes of cuts, the root improvement (`rimp`), the time used to solve the problems using branch-and-bound (`time`), the number of branch-and-bound nodes used (`nodes`) and the number of cuts added as reported by CPLEX (`#cuts`). We see that approximating the highly nonlinear cuts (14) using gradients results in a very poor performance. For instances with $n = 50$, finding a good-outer approximation of the nonlinear functions requires on average 1,203 linear cuts (compared with 17 nonlinear cuts), and it requires substantially more time to find such an approximation (182 seconds, versus 1.6 when using nonlinear cuts). For instances with $n = 100$, after adding 2,000 linear cuts (and spending one hour doing so), the quality of the outer approximation is still far from ideal (62.3 root improvement, compared with 100.0 when using nonlinear cuts). For instances with $n \geq 200$, a large number of cuts is added but the root improvement is marginal. We conjecture that constructing the linear outer approximation requires adding many parallel cuts, causing the poor performance. For the method `gradient filter`, presented in Table 4, CPLEX actually decides against adding cuts (14) –or adds a few cuts and removes them latter in the branch-and-bound tree–, and the performance is similar to default CPLEX.

Table 3 Results for the `gradient force` method.

n	cplex			cplex+(17)				cplex+(15)				cplex+(14)			
	rimp	time	nodes	rimp	time	nodes	#cuts	rimp	time	nodes	#cuts	rimp	time	nodes	#cuts
50	0.0	0.2	22	61.3	4.3	9	96	61.0	2.0	11	43	99.8	182.5	1	1,203
100	0.0	0.4	12	50.0	13.6	10	124	51.7	16.1	8	144	62.3	3,600.0	-	2,079
200	0.0	1.3	111	32.1	74.6	19	198	32.2	24.0	20	138	9.6	3,600.0	-	1,577
300	0.0	2.4	82	46.9	399.6	183	323	45.2	415.7	194	334	0.0	3,600.0	-	1,220
500	0.0	3.5	214	7.8	2,019.3	369	700	9.3	2,895.0	796	138	0.1	3,600.0	-	907

Table 4 Results for the `gradient filter` method.

n	cplex			cplex+(17)				cplex+(15)				cplex+(14)			
	rimp	time	nodes	rimp	time	nodes	#cuts	rimp	time	nodes	#cuts	rimp	time	nodes	#cuts
50	0.0	0.2	22	60.3	1.7	14	14	51.1	1.3	8	15	5.5	0.5	3	2
100	0.0	0.4	12	47.6	2.3	10	15	47.1	3.6	10	14	0.0	0.5	12	0
200	0.0	1.3	111	30.8	19.6	15	26	27.7	2.6	17	16	0.0	1.3	111	0
300	0.0	2.4	82	45.2	1,011.8	98	104	41.6	574.6	157	91	0.0	2.7	81	0
500	0.0	3.5	214	0.0	3.5	214	0	0.0	3.6	214	0	0.0	4.0	214	0

Contrary to expectations, the root improvement attained by adding cuts (17) differs between the `nonlinear` method (Table 2) and the `gradient force` method (Table 3), even though the cuts added are linear in both cases. Note that when the cuts are added via a callback in a branch-and-bound algorithm, the solver may automatically perform some additional strengthening, and we would expect the root improvements in that case (Table 3) to be better as a consequence. This is indeed the case for $n = 50$, $n = 100$ and $n = 300$; however, the root improvements for the `nonlinear` method are better for $n = 200$ and $n = 500$.

We conjecture that this discrepancy is due to the different algorithms used to solve the convex subproblems. For the `nonlinear` method, at every iteration a nonlinear conic quadratic relaxation is solved *to optimality* using an interior point method, and the resulting solution is used to generate a cut. In contrast, for the `gradient force` (and `gradient filter`) method, CPLEX only solves a linear outer approximation at the root node (and perhaps this linear relaxation is not even solved to optimality) and the resulting solution is used to generate a cut. Thus, if the linear outer approximation is weak, the root improvement of the `gradient force` methods is directly affected by this weak relaxation, and the points used to generate the cuts (17) could be ill-chosen as well (resulting in little strengthening). Moreover, the author has observed in the past (e.g., see Table 3 in [12]) that the quality of the linear outer approximations degrades substantially as the size of the problem increases – which may explain the poor root improvements of the `gradient force` and `gradient filter` method for $n = 500$.

4.2 Computations with value-at-risk minimization problems

We now report computational experiments with value-at-risk minimization problems of the form

$$w^* = \min \ -b'y + \Omega\sqrt{y'Qy} \tag{23a}$$
$$\text{s.t.} \sum_{i \in N} x_i \le r \tag{23b}$$
$$y \le x \tag{23c}$$
$$x \in \{0,1\}^N, \ y \in \mathbb{R}^N_+. \tag{23d}$$

4.2.1 Instances

The instances are generated as follows. The matrix Q is generated according to a factor model, i.e., $Q = WFW' + C$ where $F \in \mathbb{R}^{r\times r}$ is the factor covariance matrix, $W \in \mathbb{R}^{n\times r}$ is the exposure matrix and $C \in \mathbb{R}^{n\times n}$ is the diagonal matrix with the specific covariances. The matrix Q and vector b are generated similarly to [6][2]: $F = GG'$, with $G \in \mathbb{R}^{r\times r}$ and G_{ij} is drawn from continuous uniform $[-1.1]$, W_{ij} is drawn from continuous uniform $[0,1]$ with probability 0.2 and $W_{ij} = 0$ otherwise, C_{ii} is drawn from continuous uniform $[0,\delta\bar{q}]$ where $\delta \geq 0$ is a diagonal dominance parameter and $\bar{q} = (1/n)\sum_{i\in N}(WFW')_{ii}$, and b_i is drawn from continuous uniform $[0.85\sqrt{Q_{ii}}, 1.15\sqrt{Q_{ii}}]$.

4.2.2 Methods

In order to add cuts based on Z_σ, we use the idea introduced in [6] and rewrite problem (23) as

$$\begin{aligned}
\min\ & -b'y + \Omega\sqrt{y'(WFW')y + t^2}\\
\text{s.t.}\ & \sqrt{\sum_{i\in N}(c_i y_i)^2} \leq t\\
& \text{(23b), (23c), (23d),}
\end{aligned}$$

where $c_i = \sqrt{C_{ii}}$. Cuts can then be added based on the constraint $\sqrt{\sum_{i\in N}(c_i y_i)^2} \leq t$.

Given the computational results obtained in Section 4.1, we see that the nonlinear inequalities may be difficult to implement with CPLEX branch-and-bound algorithm using callbacks. Thus we focus on evaluation the strength of the convex relaxations obtained by adding the different classes of cuts presented in paper, as per the `nonlinear` method given in Section 4.1.2.

In order to compute root improvements, we use a simple rounding heuristic to obtain feasible solutions for (23). Given any feasible solution $(\bar{x},\bar{y})$ for a convex relaxation of (23) such that $\bar{x}_1 \geq \ldots \geq \bar{x}_r \geq \ldots \geq \bar{x}_n$ (without loss of generality), a feasible solution of (23) can be recovered simply by setting $x_{r+1} = \ldots = x_n = y_{r+1} = \ldots = y_n = 0$. Therefore, each time a convex relaxation of (23) is solved and a cut is added, we also recover a feasible solution, and let obj_{heur} be the objective value of the best solution obtained in this way. Then we can compute upper bounds on the initial gap as `igap` $\leq \frac{\text{obj}_{\text{opt}}-\text{relax}}{|\text{obj}_{\text{heur}}|} \times 100$ and lower bounds on the root improvement `rimp` as `rimp` $\geq \frac{\text{root}_{\text{relax}}-\text{relax}}{\text{obj}_{\text{heur}}-\text{relax}} \times 100$.

[2] We also tested the cuts with data generated according to [7]. However in such cases the initial gaps are very small – less than 5%– and the simple linear cuts (17) result in close to 100% root gap improvement, thus the stronger inequalities yield a marginal improvement at best. In contrast, the instances generated according to [6] are more challenging, with initial gaps up to 80%.

4.2.3 Results

Tables 5 and 6 show the results for $n = 200$ and $n = 500$, respectively. They show, for different cardinalities r, different nonlinear weight parameters Ω and δ, and the different classes of cuts, the initial gap (`igap`), the root improvement (`rimp`), the time used for adding cuts (`rtime`) and the number of cuts added (`#cuts`). Each row represents the average over five instances generated with identical parameters.

Table 5 Results in value-at-risk minimization problems with $n = 200$.

r	Ω	δ	`igap`	`cplex`		`cplex+`(17)			`cplex+`(15)			`cplex+`(14)		
				`rimp`	`rtime`	`rimp`	`rtime`	`#cuts`	`rimp`	`rtime`	`#cuts`	`rimp`	`rtime`	`#cuts`
		1	0.1	0.0	<0.1	64.8	0.1	2	64.8	0.1	2	64.8	0.1	2
		3	0.4	0.0	<0.1	91.1	0.1	3	91.1	0.2	3	91.1	0.3	3
	1	5	0.8	0.0	<0.1	96.4	0.2	4	96.3	0.3	4	96.4	0.4	4
		10	2.3	0.0	<0.1	91.7	0.5	12	91.9	1.5	11	91.6	3.3	11
		Average		**0.0**	**<0.0**	**86.0**	**0.2**	**5**	**86.0**	**0.5**	**5**	**86.0**	**1.0**	**5**
		1	1.0	0.0	<0.1	76.0	0.1	3	76.0	0.2	3	76.0	0.3	3
		3	4.4	0.0	<0.1	76.8	0.6	14	77.0	2.6	15	76.9	6.5	15
40	2	5	10.1	0.0	<0.1	52.4	0.7	15	52.5	2.9	16	52.5	6.9	16
		10	24.7	0.0	<0.1	27.6	0.3	7	34.3	3.6	18	34.8	11.5	19
		Average		**0.0**	**<0.0**	**58.2**	**0.4**	**10**	**59.9**	**2.3**	**13**	**60.0**	**6.3**	**13**
		1	3.1	0.0	<0.1	75.0	0.3	8	75.0	0.9	8	75.1	1.3	7
		3	22.7	0.0	<0.1	30.1	0.4	8	36.1	3.1	16	39.5	10.9	18
	3	5	46.6	0.0	<0.1	18.6	0.1	4	25.2	1.6	11	32.3	12.9	18
		10	84.8	0.0	<0.1	12.4	0.1	2	20.5	0.3	4	36.4	29.1	23
		Average		**0.0**	**<0.0**	**34.0**	**0.2**	**5**	**39.2**	**1.5**	**10**	**45.8**	**13.6**	**17**
		1	<0.1	0.0	<0.1	58.5	0.1	2	58.6	0.1	2	58.6	0.1	2
		3	0.1	0.0	<0.1	88.1	0.1	2	88.1	0.1	2	88.1	0.1	2
	1	5	0.2	0.0	<0.1	95.9	0.1	2	95.9	0.1	2	95.9	0.1	2
		10	0.5	0.0	<0.1	98.4	0.1	3	98.4	0.2	3	98.5	0.3	3
		Average		**0.0**	**<0.0**	**85.2**	**0.1**	**2**	**85.3**	**0.1**	**2**	**85.3**	**0.2**	**2**
		1	0.2	0.0	<0.1	65.0	0.1	2	64.9	0.1	2	65.0	0.1	2
		3	0.7	0.0	<0.1	92.9	0.1	4	93.0	0.3	4	93.0	0.4	4
80	2	5	1.7	0.0	<0.1	96.6	0.3	8	96.5	0.8	7	96.4	1.2	6
		10	6.1	0.0	<0.1	70.8	0.4	9	82.4	4.0	17	84.6	12.7	18
		Average		**0.0**	**<0.0**	**81.3**	**0.2**	**6**	**84.2**	**1.3**	**7**	**84.7**	**3.6**	**7**
		1	0.6	0.0	<0.1	59.8	0.1	2	67.4	0.1	2	67.6	0.2	2
		3	2.8	0.0	<0.1	65.3	0.2	4	80.7	0.6	6	92.6	1.0	6
	3	5	7.5	0.0	<0.1	48.5	0.1	3	63.4	0.7	6	94.2	7.0	13
		10	32.4	0.0	<0.1	18.3	0.1	2	24.6	0.3	4	63.9	52.3	29
		Average		**0.0**	**<0.0**	**48.0**	**0.1**	**3**	**59.0**	**0.4**	**5**	**79.5**	**15.1**	**12**

We see that the initial gap is larger in instances with tight cardinality constraints (i.e., small values of r), or instances with larger weights assigned to the nonlinear terms (i.e., large values of Ω and δ). Moreover, we see that, in general, all classes of

Table 6 Results in value-at-risk minimization problems with $n = 500$.

r	Ω	δ	igap	cplex		cplex+(17)			cplex+(15)			cplex+(14)		
				rimp	rtime	rimp	rtime	#cuts	rimp	rtime	#cuts	rimp	rtime	#cuts
		1	<0.1	0.0	<0.1	71.3	0.3	2	71.3	0.4	2	71.3	0.7	2
		3	0.1	0.0	<0.1	90.4	0.6	3	90.4	0.7	3	90.4	1.5	3
	1	5	0.3	0.0	<0.1	94.9	0.9	4	95.0	1.0	4	94.9	2.1	4
		10	0.8	0.0	<0.1	95.6	2.3	9	95.4	3.6	9	96.1	10.7	9
		Average		**0.0**	**<0.0**	**88.1**	**1.0**	**4**	**88.0**	**1.4**	**4**	**88.2**	**3.8**	**4**
		1	0.2	0.0	<0.1	69.3	0.6	3	69.3	0.7	3	69.3	1.4	3
		3	0.9	0.0	<0.1	92.7	1.5	6	92.9	2.3	6	92.9	5.1	6
100	2	5	2.1	0.0	<0.1	90.4	4.2	16	90.9	8.8	14	91.5	31.7	14
		10	6.8	0.0	<0.1	54.3	5.9	21	54.3	20.8	23	54.8	89.3	23
		Average		**0.0**	**<0.0**	**76.7**	**3.1**	**12**	**76.9**	**8.2**	**12**	**77.1**	**30.5**	**12**
		1	0.7	0.0	<0.1	77.4	0.7	3	77.4	0.8	3	77.4	1.3	3
		3	3.3	0.0	<0.1	89.0	2.7	10	93.7	8.6	14	94.0	19.5	11
	3	5	9.1	0.0	<0.1	49.2	1.7	7	65.6	20.6	22	67.3	82.1	20
		10	29.7	0.0	<0.1	20.9	0.9	4	32.1	5.3	11	40.7	97.7	20
		Average		**0.0**	**<0.0**	**59.1**	**1.5**	**6**	**67.2**	**8.8**	**13**	**69.9**	**50.2**	**14**
		1	<0.1	0.0	<0.1	54.8	0.2	1	55.0	0.2	1	55.0	0.4	1
		3	<0.1	0.0	<0.1	80.1	0.2	1	80.2	0.2	1	80.2	0.4	1
	1	5	<0.1	0.0	<0.1	87.1	0.3	2	87.1	0.4	2	87.1	0.7	2
		10	0.1	0.0	<0.1	97.8	0.7	3	96.5	0.8	3	96.5	1.5	3
		Average		**0.0**	**<0.0**	**79.9**	**0.4**	**2**	**79.7**	**0.4**	**2**	**79.7**	**0.8**	**2**
		1	<0.1	0.0	<0.1	68.1	0.4	2	68.2	0.4	2	68.2	0.8	2
		3	0.1	0.0	<0.1	89.9	0.6	3	89.9	0.8	3	89.9	1.6	3
200	2	5	0.3	0.0	<0.1	95.6	0.9	4	95.6	1.3	4	95.7	2.3	4
		10	1.1	0.0	<0.1	98.4	2.2	8	98.5	4.2	9	98.5	7.3	7
		Average		**0.0**	**<0.0**	**88.0**	**1.0**	**4**	**88.0**	**1.7**	**4**	**88.1**	**3.0**	**4**
		1	0.1	0.0	<0.1	66.2	0.4	2	66.2	0.5	2	66.2	1.0	2
		3	0.4	0.0	<0.1	86.6	1.0	4	88.1	1.2	4	88.9	1.8	3
	3	5	1.1	0.0	<0.1	69.6	0.8	4	85.7	2.1	5	95.5	2.4	4
		10	4.9	0.0	<0.1	40.7	0.9	4	67.5	5.8	10	98.1	12.3	8
		Average		**0.0**	**<0.0**	**65.8**	**0.8**	**3**	**76.9**	**2.4**	**5**	**87.2**	**4.3**	**4**

cuts perform comparatively better in instances with large values of r – as expected, as the cuts do not use/exploit the cardinality constraint.

On the one hand, in instances with $\Omega = 1$, the initial gaps are very small and the simpler linear cuts (17) are consistently able to achieve large root gap improvements, often over 90%; note that similar results were reported in [7]. In these instances the more sophisticated cuts (15) and (14) barely have any impact in terms of the strength of the relaxation, and increase the time required to solve the relaxations. On the other hand, in instances with large values of Ω and δ, the effectiveness of the linear cuts (17) sharply decreases, e.g., the average root improvement is only 18.3% in instances with $n = 200$, $r = 80$, $\Omega = 3$ and $\delta = 10$. In such instances, the more general inequalities (14) result in a much better root improvement, e.g., 63.9% in instances with $n = 200$, $r = 80$, $\Omega = 3$ and $\delta = 10$. The computational time to compute the relaxations corresponding to (14) increase with respect to the simpler relaxations, but are adequate for many practical applications, requiring in all cases less than 100 seconds.

4.3 Summary

The linear cuts (17) are able to achieve substantial root improvements in many of the instances considered, and may be the best choice for using with current branch-and-bound solvers due to their ease of implementation as linear cuts using callbacks. However, these linear cuts are ineffective in the more challenging instances with large weights for the nonlinear components and large initial gaps. In these instances, the nonlinear cuts (14) substantially improve over (17) in terms of quality of the resulting convex relaxation. Unfortunately, due to their nonlinearity, inequalities (14) are difficult to effectively implement using current off-the-shelf branch-and-bound software. The development of new codes for conic quadratic optimization with indicators that can effectively leverage the stronger relaxations is a promising direction for future research.

5 Conclusions

We present new nonlinear valid inequalities for conic quadratic optimization with indicator variables. The inequalities are ideal, i.e., describe the convex hull of the set considered when the continuous variables are unbounded, and result in significantly stronger formulations for the bounded case. The inequalities are SOCP-representable, but are highly nonlinear and are not amenable to use with current branch-and-bound solvers – which rely on linear outer approximations of the nonlinear terms. Nonetheless, our computational experiments indicate the new nonlinear inequalities have the potential to lead to improvements in challenging conic quadratic optimization problems with indicator variables, provided that the resulting convex relaxations are adequately handled in a branch-and-bound algorithm.

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