A data-independent distance to infeasibility for linear conic systems

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Abstract

We offer a unified treatment of distinct measures of well-posedness for homogeneous conic systems. To that end, we introduce a distance to infeasibility based entirely on geometric considerations of the elements defining the conic system. Our approach sheds new light into and connects several well-known condition measures for conic systems, including *Renegar's* distance to infeasibility, the *Grassmannian* condition measure, a measure of the *most interior* solution, as well as the *sigma* and *symmetry* measures.

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1 Introduction

The focus of this work is the geometric interpretation and coherent unified treatment of measures of well-posedness for homogeneous conic problems. We relate these different measures via a new geometric notion of a distance to infeasibility.

The development of condition measures in optimization was pioneered by Renegar [22, 24, 25] and has been further advanced by a number of scholars. Condition measures provide a fundamental tool to study various aspect of problems such as the behavior of solutions, robustness and sensitivity analysis [7, 18, 20, 23], and performance of algorithms [5, 14, 15, 19, 21, 25]. Renegar's condition number for conic programming is defined in the spirit of the classical matrix condition number of linear algebra, and is explicitly expressed in terms of the distance to infeasibility, that is, the smallest perturbation on the data defining a problem instance that renders the problem infeasible [24, 25]. By construction, Renegar's condition number is inherently data-dependent. A number of alternative approaches for condition measures are defined in terms of the intrinsic geometry of the problem and independently of its data representation. Condition measures of this kind include the *symmetry* measure studied by Belloni and Freund [3], the *sigma* measure used by Ye [27], and the *Grassmannian* measure introduced by Amelunxen and Bürgisser [1]. In addition, other condition measures such as the ones used by Goffin [16], Cheung and Cucker [9], Cheung et al. [11], and by Peña and Soheili [21] are defined in terms of *most interior solutions*. The perspective presented in this paper highlights common ideas and differences underlying most of the above condition measures, reveals some extensions, and establishes new relationships among them.

Condition measures are typically stated for feasibility problems in linear conic form. Feasibility problems of this form are pervasive in optimization. The constraints of linear, semidefinite, and more general conic programming problems are written explicitly as the intersection of a (structured) convex cone with a linear (or, more generally, affine) subspace. The fundamental signal recovery property in compressed sensing can be stated precisely as the infeasibility of a homogeneous conic system for a suitable choice of a cone and linear subspace as explained in [2, 8].

We focus on the feasibility problems that can be represented as the intersection of a closed convex cone with a linear subspace. Our *data-independent* distance to infeasibility is a measure of proximity between the orthogonal complement of this linear subspace and the dual cone. This distance depends only on the norm, cone, and linear subspace. Specific choices of norms lead

to interpretations of this distance as the Grassmannian measure [1] as well as a measure of the most interior solution [11]. Our approach also yields neat two-way bounds between the sigma measure [27] and symmetry measure [3,4] in terms of this geometric distance. Our work is inspired by [1], and is similar in spirit to an abstract setting of convex processes [6, Section 5.4] (also see [12]). For a more general take on condition numbers for unstructured optimization problems and for an overview of recent developments we refer the reader to [28].

The main sections of the paper are organized as follows. We begin by defining our data-independent distance to infeasibility in Section 2, where we also show that it coincides with the Grassmannian distance of [1] for the Euclidean norm. In Section 3 we discuss Renegar's distance to infeasibility and show in Theorem 1 that the ratio of the geometric distance to infeasibility and Renegar's distance is sandwiched between the reciprocal of the norm of the matrix and the norm of its set-valued inverse, hence extending [1, Theorem 1.4] to general norms. In Section 4 we show that the cone induced norm leads to the interpretation of the distance to infeasibility in terms of the most interior solution (Proposition 3). We also provide further interpretation as eigenvalue estimates for the cone of positive semidefinite matrices and for the nonnegative orthant.

In Section 5 we propose an extension of the sigma measure of Ye and establish bounds relating the sigma measure and the distance to infeasibility (Proposition 5). Section 6 relates our distance infeasibility and the sigma measure to the symmetry measure used by Belloni and Freund via neat symmetric bounds in Theorem 2 and Corollary 1. Finally, Section 7 describes extensions of our main developments via a more flexible choice of norms.

2 Data-independent distance to infeasibility

Let E be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$, endowed with a (possibly non-Euclidean) norm $\| \cdot \|$. Recall that the dual norm $\| \cdot \|^*$ is defined for $u \in E$ as

$$\|u\|^* := \max_{\|x\|=1} \langle u, x \rangle.$$

Notice that by construction, the following *Hölder's inequality* holds for all $u, x \in E$

$$| \langle u, x \rangle | \leq \|u\|^* \cdot \|x\|. \tag{1}$$

Let $K \subseteq E$ be a closed convex cone. Given a linear subspace $L \subseteq E$, consider the feasibility problem

$$\text{find } x \in L \cap K \setminus \{0\} \tag{2}$$

and its alternative

$$\text{find } u \in L^\perp \cap K^* \setminus \{0\}. \tag{3}$$

Here K^* denotes the *dual cone* of K, that is,

$$K^* := \{u \in E : \langle u, x \rangle \geq 0 \ \forall x \in K\},$$

and $L^\perp$ is the *orthogonal complement* of the linear subspace L,

$$L^\perp := \{u \in E : \langle u, x \rangle = 0 \ \forall x \in L\}.$$

In what follows we assume that $K \subseteq E$ is a closed convex cone that is also *regular*, that is, $\text{int}(K) \neq \emptyset$ and K contains no lines. In our analysis the cone K is fixed, and the linear subspace L is treated as the problem instance. This is a standard approach that stems from the real-world models, where the cone is a fixed object with well-known structure that encodes the model's structure (for instance, the nonnegative orthant, the cone of positive semidefinite matrices, copositive or hyperblicity cone), and the problem instance is encoded via the coefficients of a linear system that in our case corresponds to the linear subspace.

Observe that (2) and (3) are *alternative systems:* one of them has a strictly feasible solution if and only if the other one is infeasible. When neither problem is strictly feasible, they both are *ill-posed:* each problem becomes infeasible for arbitrarily small perturbations of the linear subspace.

The main object of this paper is the following data-independent *distance to infeasibility* of (2):

$$\nu(L) := \min_{\substack{u \in K^*, y \in L^\perp \\ \|u\|^* = 1}} \|u - y\|^*. \tag{4}$$

Observe that $\nu(L) \geq 0$ and $L \cap \text{int}(K) \neq \emptyset$ if and only if $\nu(L) > 0$. Furthermore, $\nu(L)$ is the distance between the space $L^\perp$ and the set $\{u \in K^* : \|u\|^* = 1\}$, or equivalently between $L^\perp$ and $\{u \in K^\circ : \|u\|^* = 1\}$ for $K^\circ = -K^*$, as illustrated in Figure 1. Since both (2) and (3) are defined via a cone and a linear subspace, there is a natural symmetric version of distance to infeasibility for (3) obtained by replacing K^*, $L^\perp$ and $\| \cdot \|^*$ in (4) with their primal counterparts.
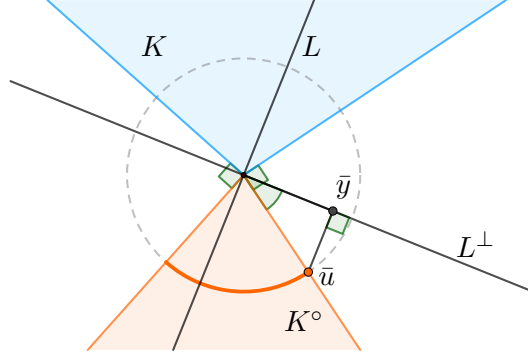
Figure 1: Illustration of $\nu(L)$ when $\nu(L) > 0$. Here $\bar{u}$ and $\bar{y}$ denote the points attaining the minimum in (4), so that $\nu(L) = \|\bar{u} - \bar{y}\|^*$.

When the norm $\|\cdot\|$ is Euclidean, that is, $\|v\| = \|v\|^* = \|v\|_2 = \sqrt{\langle v, v \rangle}$, the distance to infeasibility (4) coincides with the *Grassmann distance to ill-posedness* defined by Amelunxen and Bürgisser [1]. To see this, first observe that the Euclidean norm is naturally related to angles. Given $x, y \in E \setminus \{0\}$ let $\angle(x, y) := \arccos \frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2} \in [0, \pi]$. Given a linear subspace $L \subseteq E$ and a closed convex cone $C \subseteq E$, let

$$\angle(L, C) := \min\{\angle(x, v) : x \in L \setminus \{0\}\ v \in C \setminus \{0\}\} \in [0, \pi/2].$$

Proposition 1. *If $\|\cdot\| = \|\cdot\|_2$ then*

$$\nu(L) = \sin \angle(L^\perp, K^*).$$

Proof. Since $\angle(L^\perp, K^*) \in [0, \pi/2]$ we have

$$\sin \angle(L^\perp, K^*) = \min_{\substack{u \in K^*, y \in L^\perp \\ u, y \neq 0}} \sin \angle(y, u) = \min_{\substack{u \in K^*, y \in L^\perp \\ \|u\|_2 = 1}} \|u - y\|_2 = \nu(L).$$

$\square$

Proposition 1 and [1, Proposition 1.6] imply that when $\|\cdot\| = \|\cdot\|_2$ the distance to infeasibility $\nu(L)$ matches the Grassmann distance to ill-posedness of [1]. The flexibility in the choice of norm in E is an interesting feature in our construction of $\nu(L)$ as some norms are naturally more compatible with the cone. Suitable choice of norms generally yield sharper results in various kinds of analyses. In particular, in condition-based complexity estimates an appropriately selected norm typically leads to tighter bounds.

The articles [10, 21] touch upon this subject, and consistently in [7] a sup-norm is deemed a convenient choice for the perturbation analysis of linear programming problems. We discuss this matter in some depth via *induced norms* in Section 4.

We conclude this section with a useful characterization of $\nu(L)$.

Proposition 2. *If L is a linear subspace of E and $L \cap \mathrm{int}(K) \neq \emptyset$ then the distance to infeasibility* (4) *can be equivalently characterized as*

$$\nu(L) = \min_{\substack{u \in K^* \\ \|u\|^* = 1}} \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle u, x \rangle .$$

Proof. By properties of norms and convex duality for all $u \in E$ we have

$$\min_{y \in L^\perp} \|u - y\|^* = \min_{y \in L^\perp} \max_{\substack{x \in E \\ \|x\| \leq 1}} \langle u - y, x \rangle = \max_{\substack{x \in E \\ \|x\| \leq 1}} \min_{y \in L^\perp} \langle u - y, x \rangle = \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle u, x \rangle .$$

Therefore $\nu(L) = \min_{\substack{u \in K^*, y \in L^\perp \\ \|u\|^* = 1}} \|u - y\|^* = \min_{\substack{u \in K^* \\ \|u\|^* = 1}} \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle u, x \rangle .$ □

3 Renegar's distance to infeasibility

We next relate the condition measure $\nu(\cdot)$ with the classical Renegar's distance to infeasibility. A key conceptual difference between Renegar's approach and the approach used above is that Renegar [24, 25] considers conic feasibility problems where the linear spaces L and $L^\perp$ are explicitly defined as the image and the kernel of the adjoint of some linear mapping.

For a linear mapping $A : F \to E$ between two normed real vector spaces F and E consider the conic systems (2) and (3) defined by taking $L = \mathrm{Im}(A)$. These two conic systems can respectively be written as

$$Ax \in K \setminus \{0\} \tag{5}$$

and

$$A^* w = 0, \ w \in K^* \setminus \{0\}. \tag{6}$$

Here $A^* : E \to F$ denotes the *adjoint* operator of A, that is, the linear mapping satisfying $\langle y, Aw \rangle = \langle A^* y, w \rangle$ for all $y \in E, w \in F$.

Let $\mathcal{L}(F, E)$ denote the set of linear mappings from F to E. Endow $\mathcal{L}(F, E)$ with the operator norm, that is,

$$\|A\| := \max_{\substack{w \in F \\ |w| \leq 1}} \|Aw\|,$$

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where $|\cdot|$ is the norm in F.

Let $A \in \mathcal{L}(F,E)$ be such that (5) is feasible. The *distance to infeasibility* of (5) is defined as

$$\text{dist}(A,\mathcal{I}) := \inf\left\{\|A - \tilde{A}\| : \tilde{A}x \in K \setminus \{0\} \text{ is infeasible}\right\}$$
$$= \inf\left\{\|A - \tilde{A}\| : \tilde{A}^*w = 0 \text{ for some } w \in K^* \setminus \{0\}\right\}.$$

Observe that (5) is strictly feasible if and only if $\text{dist}(A,\mathcal{I}) > 0$.

Given $A \in \mathcal{L}(F,E)$, let $A^{-1} : \text{Im}(A) \rightrightarrows F$ be the set-valued mapping defined via $x \mapsto \{w \in F : Aw = x\}$ and

$$\|A^{-1}\| := \max_{\substack{x \in \text{Im}(A) \\ \|x\| \leq 1}} \min_{w \in A^{-1}(x)} |w|.$$

The following result is inspired by and extends [1, Theorem 1.4]. More precisely, [1, Theorem 1.4] coincides with Theorem 1 for the special case $\|\cdot\| = \|\cdot\|_2$.

Theorem 1. *Let $A \in \mathcal{L}(F,E)$ be such that* (5) *is strictly feasible and let* $L := \text{Im}(A)$. *Then*

$$\frac{1}{\|A\|} \leq \frac{\nu(L)}{\text{dist}(A,\mathcal{I})} \leq \|A^{-1}\|.$$

Proof. First, we prove $\text{dist}(A,\mathcal{I}) \leq \nu(L)\|A\|$. To that end, let $\bar{u} \in K^*$ be such that $\|\bar{u}\|^* = 1$ and $\nu(L) = \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle \bar{u}, x \rangle$ as in Proposition 2. Then

$$|A^*\bar{u}|^* = \max_{\substack{w \in F \\ |w| \leq 1}} \langle \bar{u}, Aw \rangle \leq \nu(L)\|A\|. \tag{7}$$

Let $\bar{v} \in E$ be such that $\|\bar{v}\| = 1$ and $\langle \bar{u}, \bar{v} \rangle = \|\bar{u}\|^* = 1$. Now construct $\Delta A : F \to E$ as follows

$$\Delta A(w) := -\langle A^*\bar{u}, w \rangle \bar{v}.$$

Observe that $\|\Delta A\| = |A^*\bar{u}|^* \cdot \|\bar{v}\| \leq \nu(L)\|A\|$ (by (7)) and $\Delta A^* : E \to F$ is defined by

$$\Delta A^*(y) = -\langle y, \bar{v} \rangle A^*\bar{u}.$$

In particular $(A+\Delta A)^*\bar{u} = A^*\bar{u} - \langle \bar{u}, \bar{v} \rangle A^*\bar{u} = 0$ and $\bar{u} \in K^* \setminus \{0\}$. Therefore

$$\text{dist}(A,\mathcal{I}) \leq \|\Delta A\| \leq \nu(L)\|A\|.$$

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Next, we prove $\nu(L) \leq \text{dist}(A,\mathcal{I})\|A^{-1}\|$. To that end, suppose $\tilde{A} \in \mathcal{L}(F,E)$ is such that $\ker(\tilde{A}^*) \cap K^* \setminus \{0\} \neq \emptyset$. Let $\bar{u} \in K^*$ be such that $\|\bar{u}\|^* = 1$ and $\tilde{A}^*(\bar{u}) = 0$. From the construction of $\|A^{-1}\|$, it follows that for all $x \in L = \text{Im}(A)$ there exists $w \in A^{-1}(x)$ such that $\|w\| \leq \|A^{-1}\| \cdot \|x\|$. Since $\bar{u} \in K^*$ and $\|\bar{u}\|^* = 1$, Proposition 2 implies that

$$\nu(L) \leq \max_{\substack{x \in \text{Im}(A) \\ \|x\| \leq 1}} \langle \bar{u}, x \rangle \leq \max_{\substack{w \in F \\ |w| \leq \|A^{-1}\|}} \langle \bar{u}, Aw \rangle = \|A^{-1}\| \cdot |A^*\bar{u}|^*.$$

Next, observe that $|A^*\bar{u}|^* = |(\tilde{A} - A)^*\bar{u}|^* \leq \|\tilde{A} - A\|$ because $\|\bar{u}\|^* = 1$ and $\tilde{A}^*\bar{u} = 0$. Thus $\nu(L) \leq \|A^{-1}\| \cdot \|\tilde{A} - A\|$. Since this holds for all $\tilde{A} \in \mathcal{L}(F,E)$ such that $\ker(\tilde{A}^*) \cap K^* \setminus \{0\} \neq \emptyset$ it follows that

$$\nu(L) \leq \|A^{-1}\| \, \text{dist}(A,\mathcal{I}).$$

$\square$

4 Induced norm and induced eigenvalue mappings

In addition to our assumption that $K \subseteq E$ is a regular closed convex cone, throughout the sequel we assume that $e \in \text{int}(K)$ is fixed. We next describe a norm $\|\cdot\|_e$ in E and a mapping $\lambda_e : E \to \mathbb{R}$ induced by the pair (K,e). These norm and mapping yield a natural alternative interpretation of $\nu(L)$ as a measure of the *most interior* solution to the feasibility problem $x \in L \cap \text{int}(K)$ when this problem is feasible.

Define the norm $\|\cdot\|_e$ in E induced by (K,e) as follows (see [10])

$$\|x\|_e := \min\{\alpha \geq 0 : x + \alpha e \in K, \ -x + \alpha e \in K\}.$$

For the special case of the nonnegative orthant $\mathbb{R}^n_+$ this norm has a natural interpretation: it is easy to check that for $e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\mathsf{T}$ we obtain $\|\cdot\|_e = \|\cdot\|_\infty$. The geometric interpretation is shown in Figure 2. Define the *eigenvalue* mapping $\lambda_e : E \to \mathbb{R}$ induced by (K,e) as follows

$$\lambda_e(x) := \max\{t \in \mathbb{R} : x - te \in K\}.$$

Observe that $x \in K \Leftrightarrow \lambda_e(x) \geq 0$ and $x \in \text{int}(K) \Leftrightarrow \lambda_e(x) > 0$. Furthermore, observe that when $x \in K$

$$\lambda_e(x) = \max\{r \geq 0 : \|v\|_e \leq r \Rightarrow x + v \in K\}.$$

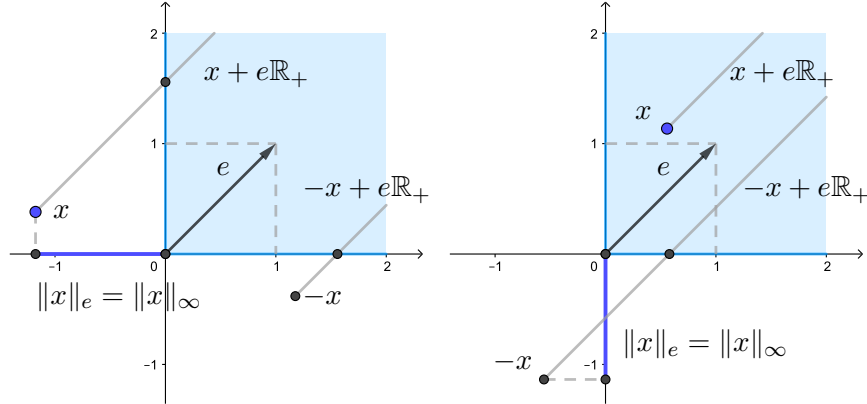Thus for $x \in K$, $\lambda_e(x)$ is a measure of how interior x is in the cone K.

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Figure 2: Induced norm for the nonnegative orthant.

It is easy to see that $\|u\|_e^* = \langle u, e \rangle$ for $u \in K^*$. In analogy to the standard simplex, let

$$\Delta(K^*) := \{u \in K^* : \|u\|_e^* = 1\} = \{u \in K^* : \langle u, e \rangle = 1\}.$$

It is also easy to see that the eigenvalue mapping λ_e has the following alternative expression

$$\lambda_e(x) = \min_{u \in \Delta(K^*)} \langle u, x \rangle.$$

The next result readily follows from Proposition 2 and convex duality.

Proposition 3. *If $\| \cdot \| = \| \cdot \|_e$, then for any linear subspace $L \subseteq E$*

$$\nu(L) = \min_{u \in \Delta(K^*)} \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle x, u \rangle = \max_{\substack{x \in L \\ \|x\| \leq 1}} \min_{u \in \Delta(K^*)} \langle x, u \rangle = \max_{\substack{x \in L \\ \|x\| \leq 1}} \lambda_e(x).$$

In particular, when $L \cap \operatorname{int}(K) \neq \emptyset$ the quantity $\nu(L)$ can be seen as a measure of the most interior point in $L \cap \operatorname{int}(K)$.

We next illustrate Proposition 3 in two important cases. The first case is $E = \mathbb{R}^n$ with the usual dot inner product, $K = \mathbb{R}^n_+$ and $e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n_+$. In this case $\| \cdot \|_e = \| \cdot \|_\infty$, $\| \cdot \|_e^* = \| \cdot \|_1$, $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ and $\Delta(\mathbb{R}^n_+)$ is the standard simplex $\Delta_{n-1} := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$. Thus $\lambda_e(x) = \min_{i=1,\ldots,n} x_i$ and for $\| \cdot \| = \| \cdot \|_e$ we have

$$\nu(L) = \max_{\substack{x \in L \\ \|x\| \leq 1}} \min_{j=1,\ldots,n} x_j. \tag{8}$$

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The second special case is $E = \mathbb{S}^n$ with the trace inner product, $K = \mathbb{S}^n_+$ and $e = I \in \mathbb{S}^n_+$. In this case $\| \cdot \|_e$ and $\| \cdot \|_e^*$ are respectively the operator norm and the nuclear norm in $\mathbb{S}^n$. More precisely

$$\|X\|_e = \max_{i=1,\ldots,n} |\lambda_i(X)|, \ \|X\|_e^* = \sum_{i=1}^n |\lambda_i(X)|,$$

where $\lambda_i(X)$, $i = 1, \ldots, n$ are the usual eigenvalues of X. Furthermore, $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$ and $\Delta(\mathbb{S}^n_+)$ is the *spectraplex* $\{X \in \mathbb{S}^n_+ : \sum_{i=1}^n \lambda_i(X) = 1\}$. Thus $\lambda_e(x) = \min_{j=1,\ldots,n} \lambda_j(X)$. In addition, in a nice analogy to (8), for $\| \cdot \| = \| \cdot \|_e$ we have

$$\nu(L) = \max_{\substack{X \in L \\ \|X\| \leq 1}} \min_{j=1,\ldots,n} \lambda_j(X). \tag{9}$$

5   Sigma measure

The induced eigenvalue function discussed in Section 4 can be defined more broadly. Given $v \in K \setminus \{0\}$ define $\lambda_v : E \to [-\infty, \infty)$ as follows

$$\lambda_v(x) := \max\{t : x - tv \in K\}.$$

Define the *sigma* condition measure of a linear subspace $L \subseteq E$ as follows

$$\sigma(L) := \min_{\substack{v \in K \\ \|v\|=1}} \max_{\substack{x \in L \\ \|x\| \leq 1}} \lambda_v(x). \tag{10}$$

The quantity $\sigma(L)$ can be interpreted as a measure of the depth of $L \cap K$ within K along all directions $v \in K$. Proposition 3 and Proposition 5(c) below show that $\sigma(L)$ coincides with the measure $\nu(L)$ of the most interior point in $L \cap K$ when $\| \cdot \| = \| \cdot \|_e$.

The construction (10) of $\sigma(L)$ can be seen as a generalization of the sigma measure introduced by Ye [27]. Observe that $L \cap \text{int}(K) \neq \emptyset$ if and only if $\sigma(L) > 0$. Furthermore, in this case Proposition 5 below shows that the quantities $\sigma(L)$ and $\nu(L)$ are closely related. To that end, we rely on the following analogue of Proposition 2.

Proposition 4. *Let $L \subseteq E$ be a linear subspace. Then*

$$\sigma(L) = \min_{\substack{v \in K \\ \|v\|=1}} \max_{\substack{x \in L \\ \|x\| \leq 1}} \lambda_v(x) = \min_{\substack{v \in K, y \in L^\perp, u \in K^* \\ \|v\|=1, \langle u,v \rangle = 1}} \|u - y\|^*. \tag{11}$$

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Proof. Assume $v \in K$ is fixed. The construction of λ_v implies that

$$
\begin{aligned}
\max_{\substack{x \in L \\ \|x\| \leq 1}} \lambda_v(x) &= \max_{\substack{x \in L, t \in \mathbb{R} \\ \|x\| \leq 1, x - tv \in K}} t \\
&= \max_{\substack{x \in L, t \in \mathbb{R} \\ \|x\| \leq 1}} \min_{u \in K^*} \left(t + \langle u, x - tv \rangle \right) \\
&= \min_{u \in K^*} \max_{\substack{x \in L, t \in \mathbb{R} \\ \|x\| \leq 1}} \left(t + \langle u, x - tv \rangle \right) \qquad (12) \\
&= \min_{\substack{u \in K^* \\ \langle u, v \rangle = 1}} \max_{\substack{x \in L \\ \|x\| \leq 1}} \langle u, x \rangle \\
&= \min_{\substack{u \in K^*, y \in L^\perp \\ \langle u, v \rangle = 1}} \|u - y\|^*,
\end{aligned}
$$

where on the second line we used the von Neumann minimax theorem [26] (also see [17, Theorem 11.1.]), and the last step follows from the identity $\max_{x \in L, \|x\| \leq 1} \langle u, x \rangle = \min_{y \in L^\perp} \|u - y\|^*$ established in the proof of Proposition 2. We thus get (11) by taking minimum in (12) over the set $\{v \in K : \|v\| = 1\}$. $\quad\square$

Proposition 5. *Let $L \subseteq E$ be a linear subspace such that $L \cap \mathrm{int}(K) \neq \emptyset$.*

(a) *For any norm $\| \cdot \|$ in E the following holds*

$$
1 \leq \min_{\substack{v \in K, u \in K^* \\ \|v\| = 1, \langle u, v \rangle = 1}} \|u\|^* \leq \frac{\sigma(L)}{\nu(L)} \leq \frac{1}{\min_{\substack{u \in K^* \\ \|u\|^* = 1}} \max_{\substack{v \in K \\ \|v\| = 1}} \langle u, v \rangle}.
$$

(b) *If $\| \cdot \| = \| \cdot \|_2$ then*

$$
1 \leq \frac{\sigma(L)}{\nu(L)} \leq \frac{1}{\cos(\Theta(K^*, K))}.
$$

where

$$
\Theta(K^*, K) := \max_{u \in K^* \setminus \{0\}} \min_{v \in K \setminus \{0\}} \angle(u, v).
$$

In particular, if $K^* \subseteq K$ then $\nu(L) = \sigma(L)$.

(c) *If $\| \cdot \| = \| \cdot \|_e$ then*

$$
\sigma(L) = \nu(L).
$$

Proof. **(a)** The first inequality is an immediate consequence of Hölder's inequality (1). Next, from Proposition 4 it follows that $\sigma(L) = \|\bar{u} - \bar{y}\|^*$

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for some $\bar{v} \in K, \bar{y} \in L^\perp, \bar{u} \in K^*$ with $\|\bar{v}\| = 1, \langle \bar{u}, \bar{v} \rangle = 1$. Thus from the construction of $\nu(L)$ we get

$$\nu(L) \leq \frac{\|\bar{u} - \bar{y}\|^*}{\|\bar{u}\|^*} \leq \frac{\sigma(L)}{\min\limits_{\substack{v \in K, u \in K^* \\ \|v\|=1, \langle u, v \rangle = 1}} \|u\|^*}$$

and hence the second inequality follows.

For the third inequality assume $\nu(L) = \|\hat{u} - \hat{y}\|^*$ for some $\hat{u} \in K^*, \hat{y} \in L^\perp$ with $\|\hat{u}\|^* = 1$. Then by Proposition 4 we get

$$\sigma(L) = \min_{\substack{v \in K, y \in L^\perp, u \in K^* \\ \|v\|=1, \langle u, v \rangle = 1}} \|u - y\|^* \leq \inf_{\substack{v \in K, y \in L^\perp, \\ \|v\|=1, \langle \hat{u}, v \rangle \neq 0}} \|\frac{\hat{u}}{\langle \hat{u}, v \rangle} - y\|^*$$

$$= \inf_{\substack{v \in K, y \in L^\perp, \\ \|v\|=1, \langle u, v \rangle = 1}} \frac{\|\hat{u} - y\|^*}{\langle \hat{u}, v \rangle} = \frac{\min\limits_{y \in L^\perp} \|\hat{u} - y\|^*}{\max\limits_{\substack{v \in K \\ \|v\|=1}} \langle \hat{u}, v \rangle}$$

$$\leq \frac{\|\hat{u} - \hat{y}\|^*}{\max\limits_{\substack{v \in K \\ \|v\|=1}} \langle \hat{u}, v \rangle}.$$

Hence

$$\sigma(L) \leq \frac{\|\hat{u} - \hat{y}\|^*}{\max\limits_{\substack{v \in K \\ \|v\|=1}} \langle \hat{u}, v \rangle} \leq \frac{\nu(L)}{\min\limits_{\substack{u \in K^* \\ \|u\|^*=1}} \max\limits_{\substack{v \in K \\ \|v\|=1}} \langle u, v \rangle}$$

and the third inequality follows.

(b) The first inequality follows from part (a). For the second inequality observe that since $\cos(\cdot)$ is decreasing in $[0, \pi]$

$$\cos(\Theta(K^*, K)) = \min_{u \in K^* \setminus \{0\}} \max_{v \in K \setminus \{0\}} \cos(\angle(u, v))$$

$$= \min_{u \in K^* \setminus \{0\}} \max_{v \in K \setminus \{0\}} \frac{\langle u, v \rangle}{\|u\|_2 \cdot \|v\|_2}$$

$$= \min_{\substack{u \in K^* \\ \|u\|_2=1}} \max_{\substack{v \in K \\ \|v\|_2=1}} \langle u, v \rangle.$$

The second inequality then follows from part (a) as well.

If in addition $K^* \subseteq K$ then $\Theta(K^*, K) = 0$ and consequently $\frac{\sigma(L)}{\nu(L)} = 1$.

(c) Since $\| \cdot \| = \| \cdot \|_e$, we have $\|e\| = 1$ and $\|u\|^* = \langle u, e \rangle$ for all $u \in K^*$. Thus $\min\limits_{\substack{u \in K^* \\ \|u\|^* = 1}} \max\limits_{\substack{v \in K \\ \|v\| = 1}} \langle u, v \rangle \geq \min\limits_{\substack{u \in K^* \\ \|u\|^* = 1}} \langle u, e \rangle = 1$. Therefore from part (a) it follows that $\frac{\sigma(L)}{\nu(L)} = 1$.

$\square$

The following example shows that the upper bound in Proposition 5(b) is tight.

Example 1. *Let $E = \mathbb{R}^2$ be endowed with the dot inner product and let $K :=$ $\{(x_1, x_2) \in E : \sin(\phi)x_2 \geq \cos(\phi)|x_1|\}$ where $\phi \in (0, \pi/2)$, $L = \{(x_1, x_2) \in E : x_1 = 0\}$, and $\| \cdot \| = \| \cdot \|_2$. Then $K^* = \{(x_1, x_2) \in E : \cos(\phi)x_2 \geq \sin(\phi)|x_1|\}$ and $\nu(L) = \sin(\phi)$. If $\phi \in (0, \pi/4)$ then $\sigma(L) = 1/(2\cos(\phi))$ and $\Theta(K, K^*) = \pi/2 - 2\phi$. Hence for $\phi \in (0, \pi/4)$*

$$\frac{\sigma(L)}{\nu(L)} = \frac{1}{2\sin(\phi)\cos(\phi)} = \frac{1}{\sin(2\phi)} = \frac{1}{\cos(\pi/2 - 2\phi)} = \frac{1}{\cos(\Theta(K, K^*))}.$$

On the other hand, if $\phi \in [\pi/4, \pi/2)$ then $\sigma(L) = \sin(\phi) = \nu(L)$, and $\Theta(K, K^*) = 0$.

6 Symmetry measure

Next, we will consider the symmetry measure of L, which has been used as a measure of conditioning [3, 4]. This measure is defined as follows. Given a set S in a vector space such that $0 \in S$, define

$$\mathrm{Sym}(0, S) := \max\{t \geq 0 : w \in S \Rightarrow -tw \in S\}. \tag{13}$$

Observe that $\mathrm{Sym}(0, S) \in [0, 1]$ with $\mathrm{Sym}(0, S) = 1$ precisely when S is perfectly symmetric around 0.

Let $A : E \to F$ be a linear mapping such that $L = \ker(A)$. Define the *symmetry* measure of L relative to K as follows.

$$\mathrm{Sym}(L) := \mathrm{Sym}(0, A(\{x \in K : \|x\| \leq 1\})). \tag{14}$$

It is easy to see that $\mathrm{Sym}(L)$ depends only on L, K and not on the choice of A. More precisely, $\mathrm{Sym}(0, A(\{x \in K : \|x\| \leq 1\})) = \mathrm{Sym}(0, A'(\{x \in K : \|x\| \leq 1\}))$ if $\ker(A) = \ker(A') = L$. Indeed, the quantity $\mathrm{Sym}(L)$ can be alternatively defined directly in terms of L and K with no reference to any linear mapping A as the next proposition states.

Proposition 6. *let $L \subseteq E$ be a linear subspace. Then*

$$\mathrm{Sym}(L) = \min_{\substack{v \in K \\ \|v\|=1}} \max_{\substack{x \in K \\ \|x\| \leq 1}} \{t \geq 0 : x + tv \in L\}.$$

Proof. Let $A : E \to F$ be such that $L = \ker(A)$. From (13) and (14) it follows that for $S := \{Ax : x \in K, \|x\| \leq 1\}$

$$\mathrm{Sym}(L) = \min_{\substack{v \in K \\ \|v\| \leq 1}} \max\{t \geq 0 : -tAv \in S\}$$

$$= \min_{\substack{v \in K \\ \|v\| \leq 1}} \max_{\substack{x \in K \\ \|x\| \leq 1}} \{t \geq 0 : -tAv = Ax\}$$

$$= \min_{\substack{v \in K \\ \|v\| \leq 1}} \max_{\substack{x \in K \\ \|x\| \leq 1}} \{t \geq 0 : x + tv \in L\}.$$

$\square$

Observe that $L \cap \mathrm{int}(K) \neq \emptyset$ if and only if $\mathrm{Sym}(L) > 0$. It is also easy to see that $\mathrm{Sym}(L) \in [0, 1]$ for any linear subspace L. The following result relating the symmetry and sigma measures is a general version of [13, Proposition 22].

Theorem 2. *Let $L \subseteq E$ be a linear subspace such that $L \cap \mathrm{int}\, K \neq \emptyset$. Then*

$$\frac{\mathrm{Sym}(L)}{1 + \mathrm{Sym}(L)} \leq \sigma(L) \leq \frac{\mathrm{Sym}(L)}{1 - \mathrm{Sym}(L)},$$

with the convention that the right-most expression above is $+\infty$ if $\mathrm{Sym}(L) = 1$. If there exists $e \in \mathrm{int}(K^*)$ such that $\|z\| = \langle e, z \rangle$ for all $z \in K$ then

$$\frac{\mathrm{Sym}(L)}{1 + \mathrm{Sym}(L)} = \sigma(L).$$

Proof. To ease notation, let $s := \mathrm{Sym}(L)$ and $\sigma := \sigma(L)$. First we show that $\sigma \geq \frac{s}{1+s}$. To that end, suppose $v \in K, \|v\| = 1$ is fixed. By Proposition 6 there exists $z \in K, \|z\| \leq 1$ such that $z + sv \in L$. Observe that $z + sv \neq 0$ because $z, v \in K$ are non-zero and $s \geq 0$. Thus $x := \frac{1}{\|z+sv\|}(z + sv) \in L, \|x\| = 1$ and

$$\lambda_v(x) \geq \frac{s}{\|z + sv\|} \geq \frac{s}{\|z\| + s\|v\|} \geq \frac{s}{1 + s}.$$

Since this holds for any $v \in K, \|v\| = 1$, it follows that $\sigma \geq \frac{s}{1+s}$.

Next we show that $\sigma \leq \frac{s}{1-s}$. Assume $s < 1$ as otherwise there is nothing to show. Let $v \in K, \|v\| = 1$ be such that

$$\max_{\substack{x \in K \\ \|x\| \leq 1}} \{t \geq 0 : x + tv \in L\} < 1. \tag{15}$$

At least one such v exists because $s = \mathrm{Sym}(L) < 1$.

It follows from the construction of $\sigma(L)$ that there exists $x \in L, \|x\| = 1$ such that $\lambda_v(x) \geq \sigma > 0$. In particular, $x - \sigma v \in K$. Furthermore, $x - \sigma v \neq 0$ as otherwise $v = \frac{1}{\sigma} x \in L$ and $x + v \in L$ which would contradict (15). Thus $z := \frac{x - \sigma v}{\|x - \sigma v\|} \in K, \|z\| = 1$ and $z + \frac{\sigma}{\|x - \sigma v\|} v \in L$ with $\frac{\sigma}{\|x - \sigma v\|} \geq \frac{\sigma}{1+\sigma}$. Since this holds for any $v \in K, \|v\| = 1$ satisfying (15), it follows that $s \geq \frac{\sigma}{1+\sigma}$ or equivalently $\sigma \leq \frac{s}{1-s}$.

Next consider the special case when there exists $e \in \mathrm{int}(K^*)$ such that $\|z\| = \langle e, z \rangle$ for all $z \in K$. In this case, $\|x - \sigma v\| = \langle e, x - \sigma v \rangle = \langle e, x \rangle - \langle e, \sigma v \rangle = \|x\| - \sigma \|v\| = 1 - \sigma$ in the previous paragraph and so the second inequality can be sharpened to $s \geq \frac{\sigma}{1-\sigma}$ or equivalently $\sigma \leq \frac{s}{1+s}$. $\qquad \square$

We also have the following relationship between the distance to infeasibility and the symmetry measure.

Corollary 1. *Let $L \subseteq E$ be a linear subspace such that $L \cap \mathrm{int}(K) \neq \emptyset$. Then*

$$\min_{\substack{u \in K^* \\ \|u\|^* = 1}} \max_{\substack{v \in K \\ \|v\| = 1}} \langle u, v \rangle \cdot \frac{\mathrm{Sym}(L)}{1 + \mathrm{Sym}(L)} \leq \nu(L) \leq \frac{\mathrm{Sym}(L)}{1 - \mathrm{Sym}(L)}.$$

In particular, if $\| \cdot \| = \| \cdot \|_2$ then

$$\cos(\Theta(K^*, K)) \cdot \frac{\mathrm{Sym}(L)}{1 + \mathrm{Sym}(L)} \leq \nu(L) \leq \frac{\mathrm{Sym}(L)}{1 - \mathrm{Sym}(L)}.$$

Proof. This is an immediate consequence of Proposition 5 and Theorem 2.
$\qquad \square$

7 Extended versions of $\nu(L)$ and $\sigma(L)$

The construction of the distance to infeasibility $\nu(L)$ can be extended by de-coupling the normalizing constraint of $u \in K^*$ from the norm defining its distance to $L^\perp$. More precisely, suppose $\| \cdot \|$ is an additional norm in the space E and consider the following extension of $\nu(L)$

$$\mathcal{V}(L) := \min_{\substack{u \in K^*, y \in L^\perp \\ \|u\|^* = 1}} \|y - u\|^*.$$

Proceeding as in Proposition 2, it is easy to see that $\mathcal{V}(L) = \min\limits_{\substack{u \in K^* \\ \|u\|^* = 1}} \max\limits_{\substack{x \in L \\ \|x\| \le 1}} \langle u, x \rangle$.

Thus only the restriction of $\|\cdot\|$ to L matters for $\mathcal{V}(L)$. We next consider a special case when this additional flexibility is particularly interesting. Suppose $L = \text{Im}(A)$ for some linear map $A : F \to E$ and define the norm $\|\cdot\|$ in L as follows

$$\|x\| := \min\limits_{w \in A^{-1}(x)} |w|, \tag{16}$$

where $|\cdot|$ denotes the norm in F. The proof of Theorem 1 readily shows that in this case $\mathcal{V}(L) = \frac{\text{dist}(A, \mathcal{I})}{\|A\|}$. In other words, $\mathcal{V}(L)$ coincides with Renegar's relative distance to infeasibility when the norm $\|\cdot\|$ in L is defined as in (16).

The additional flexibility of $\mathcal{V}(L)$ readily yields the following extension of Proposition 3: If $\|\cdot\| = \|\cdot\|_e$ for some $e \in \text{int}(K)$ then for any linear subspace $L \subseteq E$ and any additional norm $\|\cdot\|$ in L

$$\mathcal{V}(L) = \max\limits_{\substack{x \in L \\ \|x\| \le 1}} \lambda_e(x).$$

The construction of $\sigma(L)$ can be extended in a similar fashion by decoupling the normalizing constraints of $v \in K$ and $x \in L$. More precisely, let $\|\cdot\|$ be an additional norm in L and consider the following extension of $\sigma(L)$:

$$\Sigma(L) := \min\limits_{\substack{v \in K \\ \|v\| = 1}} \max\limits_{\substack{x \in L \\ \|x\| \le 1}} \lambda_v(x).$$

The additional flexibility of $\Sigma(L)$ readily yields the extension of Proposition 5 to the more general case where $\nu(L)$ and $\sigma(L)$ are replaced with $\mathcal{V}(L)$ and $\Sigma(L)$ respectively for any additional norm $\|\cdot\|$ in L.

Next, consider the following variant of $\nu(L)$ that places the normalizing constraint on $y \in L^\perp$ instead of $u \in K^*$:

$$\bar{\nu}(L) := \min\limits_{\substack{u \in K^*, y \in L^\perp \\ \|y\|^* = 1}} \|y - u\|^*.$$

It is easy to see that $\bar{\nu}(L) = \nu(L) = \sin\angle(L^\perp, K^*)$ when $\|\cdot\| = \|\cdot\|_2$. However, $\bar{\nu}(L)$ and $\nu(L)$ are not necessarily the same for other norms.

Like $\nu(L)$, its variant $\bar{\nu}(L)$ is closely related to Renegar's distance to infeasibility as stated in Proposition 7 below, which is a natural counterpart of Theorem 1. Suppose $A : E \to F$ is a linear mapping and consider the conic systems (2) and (3) defined by taking $L = \ker(A)$, that is,

$$Ax = 0, \; x \in K \setminus \{0\}, \tag{17}$$

and

$$A^*w \in K^* \setminus \{0\}. \tag{18}$$

In analogy to $\mathrm{dist}(A, \mathcal{I})$, define $\overline{\mathrm{dist}}(A, \mathcal{I})$ as follows

$$\overline{\mathrm{dist}}(A, \mathcal{I}) := \inf\left\{\|A - \tilde{A}\| : \tilde{A}x = 0, x \in K \setminus \{0\} \text{ is infeasible}\right\}$$
$$= \inf\left\{\|A - \tilde{A}\| : \tilde{A}^*w \in K^* \text{ for some } w \in F \setminus \{0\}\right\}.$$

A straightforward modification of the proof of Theorem 1 yields Proposition 7. We note that this proposition requires that A be surjective. This is necessary because $\overline{\mathrm{dist}}(A, \mathcal{I}) = 0$ whenever A is not surjective whereas $\|A\|, \|A^{-1}\|$, and $\overline{\nu}(L)$ may all be positive and finite. The surjectivity of A can be evidently dropped if the definition of $\overline{\mathrm{dist}}(A, \mathcal{I})$ is amended by requiring $\mathrm{Im}(\tilde{A}) = \mathrm{Im}(A)$.

Proposition 7. *Let $A \in \mathcal{L}(E, F)$ be a surjective linear mapping such that (17) is strictly feasible and let $L := \ker(A)$. Then*

$$\frac{1}{\|A\|} \leq \frac{\overline{\nu}(L)}{\overline{\mathrm{dist}}(A, \mathcal{I})} \leq \|A^{-1}\|.$$

Proof. First, we prove $\overline{\mathrm{dist}}(A, \mathcal{I}) \leq \overline{\nu}(L)\|A\|$. To that end, let $\bar{y} \in L^\perp$ and $\bar{u} \in K^*$ be such that $\|\bar{y}\|^* = 1$ and $\overline{\nu}(L) = \|\bar{y} - \bar{u}\|^*$. Since $\bar{y} \in L^\perp = \mathrm{Im}(A^*)$ and $\|\bar{y}\|^* = 1$, it follows that $\bar{y} = A^*\bar{v}$ for some $\bar{v} \in F$ with $|\bar{v}|^* \geq 1/\|A\|$. Let $\bar{z} \in F$ be such that $|\bar{z}| = 1$ and $\langle\bar{v}, \bar{z}\rangle = |\bar{v}|^* = 1$. Now construct $\Delta A : E \to F$ as follows

$$\Delta A(x) := \frac{\langle\bar{u} - \bar{y}, x\rangle}{|\bar{v}|^*}\bar{z}.$$

Observe that $\|\Delta A\| = \|\bar{y} - \bar{u}\|^*/|\bar{v}|^* \leq \nu(L)\|A\|$, and $\Delta A^* : F \to E$ is defined by

$$\Delta A^*(w) = \frac{\langle w, \bar{z}\rangle}{|\bar{v}|^*}(\bar{u} - \bar{y}).$$

In particular $(A + \Delta A)^*\bar{v} = A^*\bar{v} + (\bar{u} - \bar{y}) = \bar{u} \in K^*$ and $\bar{v} \in F \setminus \{0\}$. Therefore

$$\overline{\mathrm{dist}}(A, \mathcal{I}) \leq \|\Delta A\| \leq \nu(L)\|A\|.$$

Next, we prove $\overline{\nu}(L) \leq \|A^{-1}\| \cdot \overline{\mathrm{dist}}(A, \mathcal{I})$. To that end, suppose $\tilde{A} \in \mathcal{L}(E, F)$ is such that $\tilde{A}^*\bar{w} \in K^*$ for some $\bar{w} \in F \setminus \{0\}$. Since A is surjective, A^* is

one-to-one and thus $A^*\bar{w} \neq 0$. Without loss of generality we may assume that $\|A^*\bar{w}\|^* = 1$ and so $|\bar{w}|^* \leq \|A^{-1}\|$. It thus follows that

$$\bar{\nu}(L) \leq \min_{u \in K^*} \|A^*\bar{w} - u\| \leq \|A^*\bar{w} - \tilde{A}^*\bar{w}\|^* \leq \|A^{-1}\| \cdot \|\tilde{A} - A\|.$$

Since this holds for all $\tilde{A} \in \mathcal{L}(E, F)$ such that $\tilde{A}^*w \in K^*$ for some $w \in F \setminus \{0\}$, it follows that

$$\bar{\nu}(L) \leq \|A^{-1}\| \cdot \overline{\text{dist}}(A, \mathcal{I}).$$

$\square$

Finally, consider the extension of $\bar{\nu}(L)$ obtained by de-coupling the normalizing constraint of $y \in L^\perp$ from the norm defining its distance to K^*. Suppose $\|\|\cdot\|\|$ is an additional norm in the space $L^\perp$ and consider the following extension of $\bar{\nu}(L)$:

$$\overline{\mathcal{V}}(L) := \min_{\substack{u \in K^*, y \in L^\perp \\ \|\|y\|\|^* = 1}} \|y - u\|^*.$$

To illustrate the additional flexibility of $\overline{\mathcal{V}}(L)$ consider the special case when $L = \ker(A)$ for some surjective linear mapping $A : E \to F$ and define the norm $\|\|\cdot\|\|$ in $L^\perp$ as follows

$$\|\|x\|\| := |Ax|, \tag{19}$$

where $|\cdot|$ denotes the norm in F. The proof of Proposition 7 shows that $\overline{\mathcal{V}}(L) = \frac{\overline{\text{dist}}(A, \mathcal{I})}{\|A\|}$ for this choice of norm.

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