

# Multi-Product Newsvendor Problem with Customer-driven Demand Substitution: A Stochastic Integer Program Perspective

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This paper studies a multi-product newsvendor problem with customer-driven demand substitution, where each product, once run out of stock, can be proportionally substituted by the others. This problem has been widely studied in the literature, however, due to nonconvexity and intractability, only limited analytical properties have been reported and no efficient approaches have been proposed. This paper first completely characterizes the optimal order policy when the demand is known and reformulates this nonconvex problem as a binary quadratic program. When the demand is random, we formulate the problem as a two-stage stochastic integer program, derive several necessary optimality conditions, prove the submodularity of the profit function, and also develop polynomial-time approximation algorithms and show their performance guarantees. We further propose a tight upper bound via nonanticipativity dual, which is proven to be very close to the optimal value and can yield a good-quality feasible solution under a mild condition. Our numerical investigation demonstrates effectiveness of the proposed algorithms. Moreover, several useful findings and managerial insights are revealed from a series of sensitivity analyses.

*Key words:* Newsvendor Problem, Demand Substitution, Stochastic Program, Submodularity, Approximation Algorithm

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## 1. Introduction

The U.S. e-commerce sales have been growing rapidly over the last decade. For example, the sales in the first quarter of 2017 (\$106.38 billion) was three times larger than that of 2009, and the total e-commerce revenue in the U.S. was \$453.46 billion in 2017, which is 13% of the total retail sales during that year in the U.S., as reported in Zaroban (2018). Many e-commerce supply chains typically involve products with many alternatives. As it has also been observed by others (see e.g., Ganesh et al. 2008), often, such products can substitute each other's demand proportionally, in particular, when some of them are out-of-stock. For instance, a customer might turn to iPhone 8 if iPhone X is unavailable. This

phenomenon is known as “*customer-driven demand substitution*”. Besides, the stochasticity and unpredictability of customers’ demand further complicate the production planning as well as the revenue management for the supply chain companies of these products. Therefore, in this paper, we propose to model and analyze these aspects, and also, develop efficient algorithms for their implementation in multi-product supply chain environments.

The substitution problem in a multi-product supply chain contains the following three unique aspects. First, the customers’ demand is, typically, highly uncertain. However, even if the demand were known, because of substitution, it would be challenging to distribute across different products. Second, due to substitution effect, the order quantities of some products can be reduced while those of the others might be increased, which causes the underlying supply chain to be significantly different from a single product supply chain that is typically addressed in the literature. Third, the left-over products at the end of the planning period have to be salvaged at a relatively low price, which requires a sophisticated predetermined optimal ordering policy. These aspects result in severe modeling and algorithmic challenges that require sophisticated methodologies to address them. In this paper, we plan to directly address these features.

### 1.1. Relevant Literature

The multi-product substitution in the inventory management problem has been extensively studied in the literature. In Rajaram and Tang (2001), Stavroulaki (2011), Choi (2012), the authors revealed that the substitution is very effective in hedging against risks from demand uncertainty and increasing the sales for retailers. In the work of Chopra and Meindl (2007), the authors studied and characterized the substitution as the use of one product to satisfy the demand of a different product within a specific product category. Zhang et al. (2014) suggested that the products with similar functionality, color, style, size, or price can be substituted with each other. In Kraiselburd et al. (2004), the authors studied different decision scenarios for a basic supply chain with one manufacturer and one retailer. There are many notions of substitution and interested readers can refer to Shin et al. (2015) for a comprehensive review. Other extensions of the inventory problem with substitution can be found in Bassok et al. (1999), Kraiselburd et al. (2004), Bish and Suwandechochai (2010), Nagarajan and Rajagopalan (2008), Honhon et al. (2010), Shumsky and Zhang (2009), Wu et al. (2014), and Yu et al. (2015), etc. In this paper, we will focus on customer-driven demand substitution, where the substitution is driven by customers rather than by companies and there is no substitution restriction among different products.

There are many studies on analyzing the multi-product newsvendor problem with *customer-driven demand substitutions* (e.g., Parlar and Goyal 1984, Cachon and Netessine 2006, Huang et al. 2011, Rajaram and Tang 2001, Netessine and Rudi 2003). An earlier work can be found in Parlar and Goyal (1984), where the authors proved the concavity of the objective function for a two-product newsvendor problem under a mild condition. This paper also provided necessary conditions to quantify the optimal solution. When the products are decentralized among different retailers, Cachon and Netessine (2006) formulated the problem as a decentralized game and investigated properties of their underlying model through game theory. In their model, there are two retailers who sell the same products, where the customers of an out-of-stock retailer can turn to the products of the other retailer. Later on, Huang et al. (2011) characterized the conditions for existence and uniqueness of Nash equilibrium of a decentralized model and provided an iterative algorithm to obtain it. In this paper, our focus is on modeling a centralized multi-product newsvendor problem. This model is highly nonconvex, and for which only very few solution algorithms have been attempted. In Rajaram and Tang (2001), the authors developed a service rate heuristic algorithm to solve a model developed only for two products. In addition, they proposed an upper bound by solving a Lagrangian relaxation problem in order to evaluate the performance of the proposed heuristic approach. Netessine and Rudi (2003) showed some analytical properties of the centralized multi-product newsvendor problem, where they demonstrated that the deterministic objective function can be quasiconcave or bimodal. However, their results are not sufficient to completely characterize the model properties or develop any efficient solution algorithms. In this paper, we develop several useful insights and efficient solution algorithms with performance guarantees for both deterministic and stochastic demand.

## 1.2. Summary of Main Contributions

The objective of this study, motivated by e-commerce, is to determine optimal order quantities of a multi-product supply chain under customer-driven demand substitution, which maximizes the expected profit including sales profit and salvage value. The main contributions of this work are summarized as below:

- (i) When the demand is deterministic, we show a complete characterization of the optimal order quantity for each product, i.e., the optimal order quantity of each product will be either 0 or equal to its effective demand. This characterization allows us to

reformulate the entire problem as a binary quadratic program (BQP), which admits a tight semidefinite program (SDP) relaxation bound.

- (ii) When the demand is stochastic, we first apply sampling average approximation (SAA) to approximate the model, i.e., we formulate the model as a two-stage stochastic program with finite support. We derive first order necessary conditions of the optimal order quantities, and based on these conditions, we give tight lower and upper bounds of optimal order quantities. We then prove that the profit function is continuous submodular in the order quantities, i.e., the marginal benefit of increasing one product's order quantity decreases as another product's order quantity increases.
- (iii) The model properties in Part (ii) further motivate us to derive efficient solution approaches. First of all, the observation that optimal order quantity of each product is bounded, allows us to derive two mixed-integer linear program (MILP) formulations. We show that one MILP formulation is stronger than the other, which can be solved to optimality by the off-the-shelf solvers if the number of products is not very large.
- (iv) We also investigate approximation algorithms for the stochastic model, which are still efficiently computable, in particular when MILP formulations fail. Our first approximation algorithm is based on relaxing nonanticipativity constraints of the two-stage stochastic program model, which enables the decomposition of the stochastic model into a series of deterministic ones.
- (v) Finally, we conduct numerical experiments to test the performance of proposed algorithms as well as to illustrate useful managerial insights. We show that the MILP models work well for small- or medium-size instances; however, for larger instances, the approximation algorithms consistently outperform the MILP models. In addition, we show that the substitution among multiple products can reduce the risks from demand uncertainty significantly as well as increase the expected profit.

The remainder of the paper is organized as follows. Section 2 introduces the problem setting and our model formulation. Section 3 presents the properties and main results of the model for the special case of known demand. In Sections 4 and 5, we study model properties for the case of stochastic demand, and develop two mixed integer programming formulations along with several efficient solution approaches. Section 6 presents results of our numerical investigation of proposed solution algorithms and also discusses managerial insights gained from this investigation. Finally, Section 7 concludes the paper.

*Notation:* The following notation is used throughout the paper. We use bold-letters (e.g.,  $\mathbf{x}, \mathbf{A}$ ) to denote vectors and matrices, and use corresponding non-bold letters to denote their components. We let  $\mathbf{e}$  be the vector of all ones, and let  $\mathbf{e}_i$  be the  $i$ th standard basis vector. Given an integer  $n$ , we let  $[n] := \{1, 2, \dots, n\}$ , and use  $\mathbb{R}_+^n := \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$  and  $\mathbb{R}_{++}^n := \{\mathbf{x} \in \mathbb{R}^n : x_i > 0, \forall i \in [n]\}$ . Given a real number  $t$ , we let  $(t)_+ := \max\{t, 0\}$ ,  $\lceil t \rceil$  be its round-up value and  $\lfloor t \rfloor$  be its round-down value. Given a finite set  $I$ , we let  $|I|$  denote its cardinality. We let  $\tilde{\boldsymbol{\xi}}$  denote a random vector with support  $\Xi$  and denote its realizations by  $\boldsymbol{\xi}$ . Given a vector  $\mathbf{x} \in \mathbb{R}^n$ , let  $\text{supp}(\mathbf{x})$  be its support, i.e.,  $\text{supp}(\mathbf{x}) := \{i \in [n] : x_i \neq 0\}$ , and let  $(\mathbf{x}|x_j \leftarrow a)$  be a vector obtained by replacing  $j$ th entry of  $\mathbf{x}$  with  $a$ . Additional notation will be introduced as needed.

## 2. Model Formulation

In this section, we present a model formulation for our Multi-product Newsvendor Problem with Customer-driven Demand Substitution (MPNP-CDS).

In the MPNP-CDS, we suppose that there are  $n$  products for sale which are indexed by  $[n] := \{1, \dots, n\}$ . These products are similar to each other, and each product  $i \in [n]$ , bears a random demand  $\tilde{D}_i$ . Similar to many newsvendor problems, we assume that each product  $i \in [n]$  has cost  $c_i$ , price  $p_i$ , and salvage value  $s_i$  at the end of planning horizon with  $p_i \geq c_i \geq s_i$ . Due to substitution effects, when a product  $j \in [n]$  runs out of stock, its demand can be often proportionally substituted by other products. Let  $\alpha_{ji}^1$  be the substitution rate of the unmet demand of product  $j \in [n]$  by another product  $i \in [n]$ , i.e., there are  $\alpha_{ji}$  units of product  $i$  that can substitute one unit of the unmet demand of product  $j$ . By convention, we let  $\alpha_{ii} = 0$ . In this model, the decision variable is the order quantity of each product  $i \in [n]$ , denoted as  $Q_i$ .

Note that for each product  $i \in [n]$ , its effective demand function, denoted as  $\tilde{D}_i^s(\mathbf{Q})$ , constitutes of two parts, i.e., primary demand  $\tilde{D}_i$  and substitutable demand  $\sum_{j \in [n]} \alpha_{ji}(\tilde{D}_j - Q_j)_+$ , i.e.,

$$\tilde{D}_i^s(\mathbf{Q}) = \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji}(\tilde{D}_j - Q_j)_+. \quad (1)$$

<sup>1</sup> Note that we allow the units of products to be different from each other, thus,  $\alpha_{ji}$  might be larger than 1. For example, one unit of product 1 is a dozen of apples, and one unit of product 2 is only 1 apple. Suppose that 50% customers of product 1 will choose to buy product 2 if product 1 is unavailable, then  $\alpha_{12} = 50\% \times 12 = 6$ .

In view of the notation introduced above, the MPNP-CDS can be formulated as follows:

$$v^* = \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) := \mathbb{E} \left[ \sum_{i \in [n]} \left( p_i \min(Q_i, \tilde{D}_i^s(\mathbf{Q})) - c_i Q_i + s_i (Q_i - \tilde{D}_i^s(\mathbf{Q}))_+ \right) \right] \right\}, \quad (2)$$

where  $(x)_+ = \max\{x, 0\}$  denotes the nonnegative part of number  $x$  and  $\Pi(\cdot)$  denotes the expected profit function. In the above Model (2), the objective is to maximize the expected profit, where the first term is the expected revenue from sales, the second term is the cost incurred, and the last term is the expected salvage value. Let  $\bar{P}_i = p_i - c_i$  and  $\bar{S}_i = p_i - s_i$  for each product  $i \in [n]$ . Also, because of the identity that  $\min(Q_i, \tilde{D}_i^s(\mathbf{Q})) = Q_i - (Q_i - \tilde{D}_i^s(\mathbf{Q}))_+$ , the above Model (2) is equivalent to

$$v^* = \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \mathbb{E} \left[ \sum_{i \in [n]} \bar{S}_i (Q_i - \tilde{D}_i^s(\mathbf{Q}))_+ \right] \right\}, \quad (3)$$

Note that if there is no substitution effect (i.e.,  $\alpha_{ji} = 0$  for each  $i, j \in [n]$ ), then Model (3) reduces to  $n$  classical newsvendor problems, one for each product. On the other hand, with the customer-driven demand substitution, the profit function  $\Pi(\mathbf{Q})$  is usually neither convex nor concave even if there are only two products ( $n = 2$ ) (cf., Netessine and Rudi 2003).

Finally, we remark that, for the MPNP-CDS, the substitution rate matrix  $\boldsymbol{\alpha}$  plays an important role, therefore various methods have been developed in the literature to estimate matrix  $\boldsymbol{\alpha}$ . For example, Vaagen et al. (2011) have suggested to treat the substitution rates as probabilities and assumed independent choices among different alternatives for a given product. Deflem and Van Nieuwenhuysse (2011) have presented three different approaches, i.e., multi-logit model, locational choice model, and exogenous demand model, to construct the substitution rate matrix. Smith and Agrawal (2000) have proposed to build the substitution rate matrix by using the proportional substitution matrix, given that the market share of each product and the loss probability for each product are known. Kök and Fisher (2007) have studied a random substitution model, where the substitution rates are estimated by regression.

### 3. Deterministic Demand

In this section, we study a special case of MPNP-CDS (denoted as MPNP-CDS(D)), for which the demand is known. We will first show a complete characterization of optimal

order quantities, reformulate the MPNP-CDS(D) as a discrete submodular maximization problem as well as a binary quadratic program (BQP), and also show the complexity of the MPNP-CDS(D). The formulation and model properties developed in this section also serve as a foundation for subsequent sections.

To begin with, we formally state our assumption on the deterministic demand.

**Assumption 1** *The demand for each product  $i \in [n]$  is known, i.e.,  $\tilde{D}_i = D_i$ , where  $D_i$  is a positive constant.*

Under this assumption, the MPNP-CDS (3) reduces to MPNP-CDS(D), which can be formulated as below:

$$v_D^* = \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{i \in [n]} \bar{S}_i (Q_i - D_i^s(\mathbf{Q}))_+ \right\}, \quad (4)$$

where the  $i$ th effective demand function is  $D_i^s(\mathbf{Q}) = D_i + \sum_{j \in [n]} \alpha_{ji} (D_j - Q_j)_+$  for each  $i \in [n]$ .

### 3.1. Characterization of Optimal Order Quantities and Model Reformulation of MPNP-CDS(D)

For notational convenience, we let  $\mathbf{Q}^*$  denote the optimal order quantities in (4). In this subsection, we first show a characterization of optimal order quantities  $\mathbf{Q}^*$  and thus, reformulate the MPNP-CDS(D) (4) as a submodular optimization problem.

In the next theorem, we show a characteristic of the optimal order quantities  $\mathbf{Q}^*$  of MPNP-CDS(D) (4). To derive this result, we first compare  $Q_i^*$  with  $D_i$  and  $D_i^s(\mathbf{Q}^*)$ , analyze the optimality condition of  $\mathbf{Q}^*$ , and finally reformulate MPNP-CDS(D) (4) as a combinatorial optimization problem.

**Theorem 1** *Let  $\mathbf{Q}^*$  be optimal order quantities for the MPNP-CDS(D) (4). Then,*

(i)

$$Q_j^* = \begin{cases} 0, & \text{if } f(\Gamma^* \setminus \{j\}) \leq f(\Gamma^*) \\ D_j^s(\mathbf{Q}^*) = D_j + \sum_{i \in \Gamma^*} \alpha_{ij} D_i, & \text{otherwise.} \end{cases} \quad (5)$$

for each  $j \in [n]$ ; and

(ii)

$$v_D^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus \Gamma} \bar{P}_i D_i \right\} := f(\Gamma^*), \quad (6)$$

where  $[n] \setminus \text{supp}(\mathbf{Q}^*) = \Gamma^*$ , i.e.,  $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$ .

*Proof:* See the Online Supplement.  $\square$

By Theorem 1, we can conclude that, when demand is known, the optimal order quantity of product  $i \in [n]$ , denoted as  $Q_i^*$ , is either zero or equal to its effective demand  $D_i^s(\mathbf{Q}^*)$ . The result in Theorem 1 is essential to prove the complexity of MPNP-CDS and derive tight approximation algorithms in the subsequent sections.

Although Theorem 1 provides the characterization of optimal order quantities, we prove in the next subsection that it is in general NP-hard to determine them. Therefore, we provide a special condition under which the optimal order quantity of a product is equal to its demand, i.e., substitution does not take effect. This particular condition is given as below.

**Corollary 1** *Suppose (1)  $\bar{P}_i = \bar{P}_j, \forall i, j \in [n]$  and (2) for each product  $j \in [n]$ ,  $\sum_{i \in [n]} \alpha_{ji} \leq 1$ . Then  $Q_j^* = D_j$  for all  $j \in [n]$ .*

*Proof:* Note that in this case, the optimal subset  $\Gamma^* = \emptyset$  according to Theorem 1. Therefore,  $Q_j^* = D_j^s(\mathbf{Q}^*) = D_j$  for all  $j \in [n]$ .  $\square$

Next, we show that the set function  $f(\Gamma)$  in (6) is submodular, i.e., it has diminished marginal benefit when the size of set  $\Gamma$  grows. Below, we briefly introduce the definition of submodularity and interested readers are referred to Edmonds (1970), Lovász (1983), Shioura et al. (2016) for more details.

**Definition 1 (Discrete Submodularity)** *Given a finite set  $\Theta$ , let  $2^\Theta$  denote its power set. Then a set function  $g : 2^\Theta \rightarrow \mathbb{R}$  is “submodular” if and only if it satisfies the following condition:*

- for every  $X, Y \subseteq \Theta$  with  $X \subseteq Y$  and every  $x \in \Theta \setminus Y$ , we must have  $g(X \cup \{x\}) - g(X) \geq g(Y \cup \{x\}) - g(Y)$ .

By directly checking the definition, we can show that

**Proposition 1** *The set function  $f(\Gamma)$ , defined in (6), is submodular.*



*Proof:* See the Online Supplement.  $\square$

The submodular property in Proposition 1 implies that the increment in expected profit becomes smaller as the subset  $\Gamma$  (denoting a subset of products with zero order quantities) grows. In the next subsection, we derive a BQP reformulation of (6), which has a tight semidefinite program (SDP) relaxation bound.

### 3.2. Complexity of the MPNP-CDS and Alternative BQP Formulation

In this subsection, we first show that the MPNP-CDS(D) is strongly NP-hard. Since Model (6) is a special case of the MPNP-CDS (i.e., with known demand), therefore, solving the MPNP-CDS in general is also strongly NP-hard. Then, in light of Model (6), we present an alternative BQP for MPNP-CDS(D) and its SDP relaxation. As we will show in subsequent sections, this result is also useful in designing a tight bound for Model (3) with stochastic demand. To prove the complexity result, we show that the well-known strongly NP-hard problem - weighted max-cut problem is polynomially reducible to Model (6) of MPNP-CDS(D).

**Theorem 2** *The MPNP-CDS(D) is strongly NP-hard, so is the MPNP-CDS (3).*

*Proof:* See the Online Supplement.  $\square$

To reformulate Model (6) as an equivalent BQP, we follow the idea from Goemans and Williamson (1995) on reformulating a Maximum Directed Cut Problem (MAX DICUT) as a BQP. To do so, we first introduce a binary variable  $y_i \in \{-1, 1\}$  for each product  $i \in [n]$ . We also introduce an additional variable  $y_{n+1} \in \{-1, 1\}$  to differentiate between sets  $\Gamma$  and  $[n] \setminus \Gamma$ , that is, set  $\Gamma = \{j \in [n] : y_j = y_{n+1}\}$ . For notational convenience, let us denote  $w_{ij} = \alpha_{ji} \bar{P}_i D_j$ ,  $w_{i(n+1)} = \bar{P}_i D_i$ ,  $w_{ii} = w_{(n+1)i} = 0$  for all  $i, j \in [n]$ .

With the notation above, we are ready to show that

**Proposition 2** *Model (6) is equivalent to*

$$v_D^* = \max_{\mathbf{y}} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - y_i y_{n+1} + y_j y_{n+1} - y_i y_j) : y_i \in \{-1, 1\}, \forall i \in [n+1] \right\}. \quad (7)$$

*Proof:* See the Online Supplement.  $\square$

In Model (7), let us define  $\mathbf{Y} = \mathbf{y}\mathbf{y}^T$ . Then, Model (7) is equivalent to:

$$v_D^* = \max_{\mathbf{Y}} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - Y_{i(n+1)} + Y_{j(n+1)} - Y_{ij}) : Y_{jj} = 1, \forall j \in [n+1], \mathbf{Y} \succeq 0, \text{rank}(\mathbf{Y}) = 1 \right\} \quad (8)$$

If we relax the constraint  $\text{rank}(\mathbf{Y}) = 1$  in Model (8),  $v_D^*$  is upper bounded by the optimal value of the following SDP:

$$\bar{v}_D = \max_{\mathbf{Y}} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - Y_{i(n+1)} + Y_{j(n+1)} - Y_{ij}) : Y_{jj} = 1, \forall j \in [n+1], \mathbf{Y} \succeq 0 \right\}. \quad (9)$$

It follows from Goemans and Williamson (1995) that BQP Model (7) can be viewed a special case of MAX DICUT, which implies the following bound comparison result.

**Corollary 2**  $0.79607\bar{v}_D \leq v_D^* \leq \bar{v}_D$ .

Corollary 2 provides the strength of SDP relaxation Model (9), and this result will be useful for the analysis of Lagrangian relaxation approach in section 5.2.

#### 4. Stochastic Demand: Model Properties and MILP Reformulations

In this section, we study the general MPNP-CDS (3) when the demand is stochastic. We derive first-order necessary conditions, and show that the profit function is continuous submodular and that the optimal order quantities are bounded. Consequently, we formulate the MPNP-CDS (3) as a two-stage stochastic MILP and further improve this formulation by exploring the model properties.

To begin with, we make the following assumption.

**Assumption 2** *The random demand  $\tilde{\mathbf{D}}$  has a finite support  $\{\mathbf{D}^k\}_{k \in [N]}$ , where each  $k \in [N]$  is referred to as a scenario. For each scenario  $k \in [N]$ ,  $m_k$  denotes its associated probability mass, i.e.,  $\mathbb{P}\{\tilde{\mathbf{D}} = \mathbf{D}^k\} = m_k$ .*

Under this assumption, the expectation in Model (3) is equivalent to a finite summation, thus MPNP-CDS (3) can be reformulated as the following scenario-based model

$$\begin{aligned} v^* &= \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \mathbb{E} \left[ \sum_{i \in [n]} \bar{S}_i (Q_i - \tilde{D}_i^s(\mathbf{Q}))_+ \right] \right\} \\ &= \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i (Q_i - D_i^{sk}(\mathbf{Q}))_+ \right] \right\}, \end{aligned} \quad (10)$$

where for each product  $i \in [n]$  and scenario  $k \in [N]$ , its effective demand function is

$$D_i^{sk}(\mathbf{Q}) = D_i^k + \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+.$$

We remark that if the random demand  $\tilde{D}$  is not finitely supported, then one might generate  $N$  i.i.d samples,  $\{\mathbf{D}^k\}_{k \in [N]}$ . By the sampling average approximation (SAA) method (cf., Shapiro et al. 2009), Model (3) can be approximated to an arbitrary accuracy by the scenario Model (10) if  $N$  is large enough (polynomial both in  $n$  and accuracy).

#### 4.1. Model Properties

In this subsection, we derive the first-order necessary conditions for the scenario Model (10) of MPNP-CDS and show that the profit function is continuous submodular.

Note that the objective function  $\Pi(\mathbf{Q})$  is a nonsmooth function. Therefore, the main proof idea in this subsection is based upon the perturbation method, i.e., suppose the vector of the optimal order quantities  $\mathbf{Q}^*$  is known, then we analyze the inequality  $\Pi(\mathbf{Q}^* + \boldsymbol{\epsilon}) \leq \Pi(\mathbf{Q}^*)$  for a sufficiently small vector  $\boldsymbol{\epsilon} \in \mathbb{R}^n$ . Our first result specifies the range of  $\mathbf{Q}^*$ , which further implies that  $\mathbf{Q}^*$  is a bounded vector. A similar result has been developed in Netessine and Rudi (2003) with continuous demand, however, the discrete demand has different necessary conditions and also requires a very different proof.

**Theorem 3** *Let  $\mathbf{Q}^*$  be the vector of optimal quantities of Model (10). Then,*

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\mathbf{Q}^*)\right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* < \tilde{D}_i\right) \geq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n], \quad (11a)$$

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\mathbf{Q}^*)\right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n]. \quad (11b)$$

*Proof:* See the Online Supplement. □

We remark that: (i) in (11a), if there is no substitution effect (i.e.,  $\alpha_{ij} = 0$  for all  $i, j \in [n]$ ), then the second term is zero since  $\tilde{D}^s(\mathbf{Q}^*) = \tilde{D}$ . Thus, (11a) reduces to the necessary optimal condition for the classical newsvendor problem with discrete demand; (ii) in (11), as  $\mathbf{Q}^*$  exists in the effective demand function  $\tilde{D}^s(\mathbf{Q}^*)$ , it can be very difficult to obtain a closed-form expression of the optimal order quantity for each product.

Next, we use the result of Theorem 3 to derive upper and lower bounds on the optimal order quantities. The main idea in proving this result is to relax the inequalities in (11) until arriving at the desired results.

**Proposition 3** Let  $\mathbf{Q}^*$  be the vector of optimal quantities of Model (10). Then,  $\mathbf{Q}^*$  is upper and lower bounded by  $\overline{\mathbf{Q}}$  and  $\underline{\mathbf{Q}}$ , respectively, i.e., for each product  $i \in [n]$ ,  $\overline{Q}_i \geq Q_i^* \geq \underline{Q}_i$  with

$$\underline{Q}_i = \begin{cases} F_{\tilde{D}_i}^{-1} \left( \frac{\bar{P}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}{\bar{S}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j} \right), & \text{if } \bar{S}_i > \sum_{j \in [n]} \alpha_{ij} \bar{S}_j, \\ 0, & \text{otherwise} \end{cases}, \quad (12a)$$

$$\overline{Q}_i = \bar{F}_{\tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j}^{-1} \left( \frac{\bar{P}_i}{\bar{S}_i} \right), \quad (12b)$$

where  $F_{\tilde{X}}^{-1}, \bar{F}_{\tilde{X}}^{-1}$  denote the lower and upper inverse distribution function of random variable  $\tilde{X}$ , respectively, i.e.,  $F_{\tilde{X}}^{-1}(t) = \inf \left\{ \kappa : \mathbb{P} \left( \tilde{X} \leq \kappa \right) \geq t \right\}$  and  $\bar{F}_{\tilde{X}}^{-1}(t) = \inf \left\{ \kappa : \mathbb{P} \left( \tilde{X} < \kappa \right) \geq t \right\}$ .

*Proof:* See the Online Supplement.  $\square$

Note that to compute the lower bound in (12a), we can simply sort  $\{D_i^k\}_{k \in [N]}$  in an ascending order such that  $D_i^{(1)} \leq \dots \leq D_i^{(N)}$ , where  $\{(1), \dots, (N)\}$  is a permutation of  $[N]$ , and choose the smallest index  $k_{\min}$  such that  $\sum_{i \in [k_{\min}]} m(i) \geq \frac{\bar{P}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}{\bar{S}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}$ , then  $\underline{Q}_i = D_i^{(k_{\min})}$ . Similarly, we can compute the upper bound  $\overline{Q}_i$  in (12b) efficiently.

Next, we show that profit function  $\Pi(\mathbf{Q})$  defined in (10) is a continuous submodular function, which is defined as below.

**Definition 2 (Continuous Submodularity)** Given a closed set  $\Theta$ , let  $g : \Theta \mapsto \mathbb{R}$  be a continuous function. Then  $g(\cdot)$  is continuous submodular if and only if for every  $\mathbf{x}, \mathbf{y} \in \Theta$ , we must have  $g(\mathbf{x}) + g(\mathbf{y}) \geq g(\max\{\mathbf{x}, \mathbf{y}\}) + g(\min\{\mathbf{x}, \mathbf{y}\})$ .

Other equivalent definitions of continuous submodularity can be found in Bach (2015), Bian et al. (2016).

Next we will show that  $\Pi(\mathbf{Q})$  is continuous submodular. Note that in Netessine and Rudi (2003), the authors proved that the profit function is continuous submodular in demand when  $N = 1$ , i.e., demand is deterministic.

**Proposition 4** The profit function  $\Pi(\mathbf{Q})$  defined in (10) is continuous submodular.

*Proof:* See the Online Supplement.  $\square$

The continuous submodular property in Proposition 4 implies that if the  $i$ th product's order quantity becomes larger, then an increment in the expected profit becomes smaller

when another product  $j$ 's order quantity increases, i.e., the marginal benefit diminishes if we increase order quantities of both the products  $i, j$ . We will show in the next section that by exploring the submodularity of profit function  $\Pi(\mathbf{Q})$ , there exists an efficient double greedy algorithm that can solve Model (10) to near optimality with an approximation ratio  $1/3$ .

#### 4.2. MILP Formulations

Note that Model (10) is in general nonconvex and nonsmooth. In this subsection, we introduce two different MILP formulations by linearizing nonconvex functions in the form  $g(x, y) = \max\{x, y\}$  with additional binary variables. We also show that the second model, which is slightly less intuitive, is stronger than the first model.

We first reformulate Model (10) as a two-stage stochastic program as follows:

$$v^* = \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i + \sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k) \right\}, \quad (13a)$$

where the second-stage function

$$H(\mathbf{Q}, \mathbf{D}^k) = - \sum_{i \in [n]} \bar{S}_i \left( Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+ \right)_+, \quad (13b)$$

for each  $k \in [N]$ . Note that the nonconvexity of function  $H(\mathbf{Q}, \mathbf{D}^k)$  arises from unmet demand terms  $\{(D_j^k - Q_j)_+\}_{j \in [n]}$ . Our main idea is to linearize them by introducing additional variables.

Before introducing our two MILP models, we first start with the following two properties of Model (13). From Proposition 3, we know that the optimal order quantities  $\mathbf{Q}^*$  can be bounded from above, which is formally stated as follows:

**Property 1** In (13), let  $Q_i \leq M_i$  for each  $i \in [n]$ .

Note that one might simply choose  $\mathbf{M} := \bar{\mathbf{Q}}$  in Proposition 3 as a valid upper bound, or derive another tighter one.

For the second property, as there is only a finite number of realizations  $\{\mathbf{D}^k\}_{k \in [N]}$  of the random demand  $\tilde{\mathbf{D}}$ , therefore, for each product, we can sort the realizations of its demand in an ascending order, i.e.,

**Property 2** for each  $i \in [n]$ , let  $D_i^{(1)} \leq \dots \leq D_i^{(N)}$ , where  $\{(1), \dots, (N)\}$  is a permutation of  $[N]$ .

Please note that the demand realizations of different products can be sorted differently, i.e., different products might not share the same permutations to sort their demand realizations.

**MILP Model 1.** In this model, we introduce nonnegative variables  $y_i^k$  and  $u_i^k$  to represent salvaged units of product  $\left(Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji}(D_j^k - Q_j)_+\right)_+$  and unmet demand  $(D_i^k - Q_i)_+$  for each product  $i \in [n]$ , respectively. In addition, we introduce a binary variable  $z_i^k = 1$  if  $Q_i \geq D_i^k$  (i.e.,  $u_i^k = 0$ ) and  $z_i^k = 0$  if  $Q_i < D_i^k$  (i.e.,  $u_i^k = D_i^k - Q_i$ ). In view of the notation above, the second-stage function  $H(\mathbf{Q}, \mathbf{D}^k)$  can be equivalently represented by the following MILP:

$$H(\mathbf{Q}, \mathbf{D}^k) = \max_{\mathbf{u}^k, \mathbf{y}^k, \mathbf{z}^k} - \sum_{i \in [n]} \bar{S}_i y_i^k \quad (14a)$$

$$\text{s.t. } y_i^k \geq Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji} u_j^k, \forall i \in [n], \quad (14b)$$

$$D_i^k - Q_i + M_j z_i^k \geq u_i^k \geq D_i^k - Q_i - M_j z_i^k, \forall i \in [n], \quad (14c)$$

$$0 \leq u_i^k \leq D_i^k (1 - z_i^k), \forall i \in [n], \quad (14d)$$

$$y_i^k \geq 0, \forall i \in [n], \quad (14e)$$

$$z_i^k \in \{0, 1\}, \forall i \in [n]. \quad (14f)$$

Note that for each product  $i \in [n]$ , the objective function (14a), constraints (14b) and non-negativity constraints (14e) together enforce that  $y_i^k = Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji} u_j^k$ . Constraints (14c), (14d), and (14f) along with Property 1 imply that

$$u_i^k = (D_i^k - Q_i)_+ = \begin{cases} 0, & \text{if } Q_i \geq D_i^k \\ D_i^k - Q_i, & \text{otherwise} \end{cases}.$$

We conclude the validity of MILP Model 1 in the following proposition.

**Proposition 5** The second-stage function  $H(\mathbf{Q}, \mathbf{D}^k)$  is equivalent to the MILP formulation (14) for each scenario  $k \in [N]$ , i.e., and MILP Model 1 is equivalent to

$$v^* = \max_{\mathbf{Q}, \mathbf{z}, \mathbf{u}, \mathbf{y}} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} \sum_{i \in [n]} \bar{S}_i m_k y_i^k : (14b) - (14f) \right\}.$$

**MILP Model 2.** In the formulation (14), note that, the expressions  $\{(D_i^k - Q_i)_+\}_{i \in [n]}$  share the same monotonicity as  $\{D_i^k\}_{i \in [n]}$ . This property motivates us to derive a stronger formulation to represent expected second-stage functions, i.e.,  $\sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k)$ .

In MILP Model 2, we use the same variables  $\{\mathbf{u}^k\}_{k \in [N]}, \{\mathbf{y}^k\}$  as in MILP Model 1, i.e., nonnegative variables  $y_i^k$  and  $u_i^k$  represent salvaged units of product  $(Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+)_+$  and unmet demand  $(D_i^k - Q_i)_+$  for each product  $i \in [n]$ , respectively. From Property 2, we know that the demand realizations for each product are sorted as

$$D_i^{(1)} \leq \dots \leq D_i^{(N)}.$$

Let  $\widehat{D}_i^{(k)} = \min\{D_i^{(k)}, M_i\}$  for each  $k \in [N]$  since  $M_i$  might be smaller than some of  $\{D_i^{(k)}\}_{k \in [N]}$ . We note that, for each product  $i \in [n]$ , its order quantity  $Q_i$  must belong to one of the following  $N + 1$  intervals:

$$\left[\widehat{D}_i^{(0)}, \widehat{D}_i^{(1)}\right], \left[\widehat{D}_i^{(1)}, \widehat{D}_i^{(2)}\right], \dots, \left[\widehat{D}_i^{(N)}, \widehat{D}_i^{(N+1)}\right].$$

where  $\widehat{D}_i^{(0)} = 0, \widehat{D}_i^{(N+1)} = M_i$ . Therefore, we introduce one binary variable for each interval to indicate whether  $Q_i$  is in this interval or not, i.e., we let  $\chi_i^{(k)} = 1$  if  $Q_i \in [\widehat{D}_i^{(k-1)}, \widehat{D}_i^{(k)}]$ , and 0, otherwise. Also, we let  $\sum_{k \in [N+1]} \chi_i^{(k)} = 1$  to enforce that  $Q_i$  indeed belong to only one interval (we break the boundary points arbitrarily).

In order to formulate the model as a mathematical program, for each product  $i \in [n]$  and  $k \in [N + 1]$ , we introduce another variable  $w_i^{(k)}$  to be equal to  $Q_i$  if  $Q_i \in [\widehat{D}_i^{(k-1)}, \widehat{D}_i^{(k)}]$ ; and 0, otherwise. That is,  $\widehat{D}_i^{(k-1)} \chi_i^{(k)} \leq w_i^{(k)} \leq \widehat{D}_i^{(k)} \chi_i^{(k)}$ , and  $\sum_{k \in [N+1]} w_i^{(k)} = Q_i$ .

Now, we can represent  $u_i^{(k)}$  with variables  $\{\chi_i^{(\tau)}\}_{\tau \in [N]}$  and  $\{w_i^{(\tau)}\}_{\tau \in [N]}$  for each product  $i \in [n]$  and  $k \in [N]$  as follows:

$$u_i^{(k)} = D_i^{(k)} \sum_{\tau \in [k]} \chi_i^{(\tau)} - \sum_{\tau \in [k]} w_i^{(\tau)}$$

which is equal to 0 if  $Q_i > D_i^{(k)}$ , and otherwise, it is equal to  $D_i^{(k)} - Q_i$ .

In view of the above development, we can represent  $\sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k)$  as the following mathematical program:

$$\sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k) = \max_{\mathbf{u}, \mathbf{w}, \boldsymbol{\chi}, \mathbf{y}} - \sum_{i \in [n]} \sum_{k \in [N]} \bar{S}_i m_i^{(k)} y_i^{(k)} \quad (15a)$$

$$\text{s.t. } y_i^{(k)} \geq Q_i - D_i^{(k)} - \sum_{j \in [n]} \alpha_{ji} u_j^{(k)}, \forall i \in [n], \forall k \in [N], \quad (15b)$$

$$\widehat{D}_i^{(k-1)} \chi_i^{(k)} \leq w_i^{(k)} \leq \widehat{D}_i^{(k)} \chi_i^{(k)}, \forall i \in [n], \forall k \in [N], \quad (15c)$$

$$u_i^{(k)} = D_i^{(k)} \sum_{\tau \in [k]} \chi_i^{(\tau)} - \sum_{\tau \in [k]} w_i^{(\tau)}, \forall i \in [n], \forall k \in [N], \quad (15d)$$

$$\sum_{k \in [N+1]} w_i^{(k)} = Q_i, \forall i \in [n], \forall k \in [N], \quad (15e)$$

$$\sum_{k \in [N+1]} \chi_i^{(k)} = 1, \forall j \in [n], \forall k \in [N], \quad (15f)$$

$$y_i^{(k)} \geq 0, u_i^{(k)} \geq 0, \forall i \in [n], w_i^{(k)} \geq 0, \forall i \in [n], \forall k \in [N], \quad (15g)$$

$$\chi_i^{(k)} \in \{0, 1\}, \forall i \in [n], \forall k \in [N], \quad (15h)$$

where for notational convenience, we let  $\{m_i^{(k)}\}_{k \in [N]}$  be the permutation of  $\{m_k\}_{k \in [N]}$  in the same order as  $\{D_i^{(k)}\}_{k \in [N]}$ .

From the discussion above, we conclude that Model (15) is a valid representation of  $\sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k)$ . Therefore, we can formally state the validity of MILP Model 2 as follows.

**Proposition 6** *The expected second-stage function  $\sum_{k \in [N]} m_k H(\mathbf{Q}, \mathbf{D}^k)$  is equivalent to the MILP formulation (15), and MILP Model 2 is equivalent to*

$$v^* = \max_{\mathbf{Q}, \boldsymbol{\chi}, \mathbf{u}, \mathbf{w}, \mathbf{y}} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{i \in [n]} \sum_{k \in [N]} \bar{S}_i m_i^{(k)} y_i^{(k)} : (15b) - (15h) \right\}.$$

Next, we show that the MILP Model 2 (from Proposition 6) is stronger than the MILP Model 1 (from Proposition 5). To this end, the main idea is to show that  $\bar{v}_M^2$  is no smaller than  $\bar{v}_M^1$ , where  $\bar{v}_M^1, \bar{v}_M^2$  denote the continuous relaxation values of these models, respectively, i.e.,

$$\bar{v}_M^1 = \max_{\mathbf{Q}, \mathbf{z}, \mathbf{u}, \mathbf{y}} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} \sum_{i \in [n]} \bar{S}_i m_k y_i^k : (14b) - (14e), \mathbf{Q} \in \mathbb{R}_+^n, \mathbf{z} \in [0, 1]^{n \times N} \right\}, \quad (16a)$$

$$\bar{v}_M^2 = \max_{\mathbf{Q}, \boldsymbol{\chi}, \mathbf{u}, \mathbf{w}, \mathbf{y}} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{i \in [n]} \sum_{k \in [N]} \bar{S}_i m_i^{(k)} y_i^{(k)} : (15b) - (15g), \mathbf{Q} \in \mathbb{R}_+^n, \boldsymbol{\chi} \in [0, 1]^{n \times (N+1)} \right\}. \quad (16b)$$



**Theorem 4** *The MILP Model 2 is stronger than MILP Model 1, i.e., their continuous relaxation values satisfy  $\bar{v}_M^1 \leq \bar{v}_M^2$ , where  $\bar{v}_M^1, \bar{v}_M^2$  are defined in (16a), (16b), respectively.*

*Proof:* See the Online Supplement. □

## 5. Stochastic Demand: Approximation Algorithms

Although the MILP Model 2 is proven to be stronger, there are  $O(nN)$  binary variables in the both formulations. These additional binary variables may cause difficulties in solving these models, especially for large-sized instances, where the number of products or number of scenarios is large. Therefore, in this section, we present efficient approximation algorithms to solve Model (10) with provable performance guarantees. We will adopt the same notation and assumption as the previous section.

### 5.1. Double Greedy Algorithm

From Proposition 4, we know that the profit function (10) is continuous submodular. Recent development in continuous submodular optimization has shown that a double greedy algorithm can solve a nonnegative continuous submodular function maximization with bounded feasible region efficiently, leading to an approximation ratio of  $1/3$  (cf., Bian et al. 2016), i.e., suppose  $\mathbf{Q}$  denotes the output of double greedy algorithm, then  $\Pi(\mathbf{Q}) \geq v^*/3$ .

The detailed implementation of double greedy can be found in Algorithm 1. The algorithm requires  $n$  iterations and proceeds as follows. Let  $\underline{\mathbf{Q}}, \overline{\mathbf{Q}}$  be the lower and upper bounds of optimal order quantities  $\mathbf{Q}$  according to Proposition 3. We first initiate two vectors  $\mathbf{x}^0, \mathbf{y}^0$  to be  $\underline{\mathbf{Q}}, \overline{\mathbf{Q}}$ , respectively. During  $i$ th iteration, we solve the following two univariate optimization problems

$$\max_{Q_a \in [\underline{Q}_i, \overline{Q}_i]} \Pi(\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow Q_a) \quad (17a)$$

$$\max_{Q_b \in [\underline{Q}_i, \overline{Q}_i]} \Pi(\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow Q_b) \quad (17b)$$

where for a vector  $\mathbf{x}$ ,  $(\mathbf{x} | x_j \leftarrow a)$  denotes a copy of vector  $\mathbf{x}$  except the  $j$ th coordinate is replaced by  $a$ . Let  $\widehat{Q}_a, \widehat{Q}_b$  be the optimal solutions to optimization Models (17a) and (17b), respectively. Next we check the improvements of the new solutions,  $\delta_a = \Pi(\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow \widehat{Q}_a) - \Pi(\mathbf{x}^{i-1})$  and  $\delta_b = \Pi(\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow \widehat{Q}_b) - \Pi(\mathbf{y}^{i-1})$ . If  $\delta_a \geq \delta_b$ , then

let  $\mathbf{x}^i, \mathbf{y}^i$  be  $(\mathbf{x}^{i-1}|x_i^{i-1} \leftarrow \widehat{Q}_a), (\mathbf{y}^{i-1}|y_i^{i-1} \leftarrow \widehat{Q}_a)$ , respectively; otherwise, let  $\mathbf{x}^i, \mathbf{y}^i$  be  $(\mathbf{x}^{i-1}|x_i^{i-1} \leftarrow \widehat{Q}_b), (\mathbf{y}^{i-1}|y_i^{i-1} \leftarrow \widehat{Q}_b)$ .

Next we show that both the univariate optimization models in (17) are efficiently solvable.

**Proposition 7** *Suppose that  $\mathbf{Q} \in \mathbb{R}_+^n$  is known. Then,*

(i) *the following optimization model is efficiently solvable,*

$$\max_{q \in [\underline{Q}_i, \overline{Q}_i]} \Pi(\mathbf{Q}|Q_i \leftarrow q) \quad (18)$$

for each  $i \in [n]$ ; and

(ii) *an optimal solution to Model (18) belongs to set  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ , where*

$$\mathcal{R}_1 = \left\{ D_i^k : D_i^k \in [\underline{Q}_i, \overline{Q}_i], \forall k \in [N+1] \right\}, \quad (19a)$$

$$\mathcal{R}_2 = \left\{ D_i^{sk} : D_i^{sk} \in [\underline{Q}_i, \overline{Q}_i], \forall k \in [N] \right\}, \quad (19b)$$

$$\mathcal{R}_3 = \left\{ D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} : D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} \in [\underline{Q}_i, D_i^k], \forall j \in [n], k \in [N] \right\}. \quad (19c)$$

*Proof:* See the Online Supplement. □

By Proposition 7, we also note that to solve Model (18), we might simply check the objective value for each  $q \in \mathcal{R}$  and the one with the largest objective value must be an optimal solution. There are at most  $2N + nN$  points in set  $\mathcal{R}$ , thus, Model (18) is efficiently solvable.

From the discussions above, we can conclude that

**Corollary 3** *Algorithm 1 is a polynomial-time approximation algorithm with an approximation ratio of  $1/3$ , i.e., suppose  $\mathbf{Q}$  denotes the output of Algorithm 1, then  $\Pi(\mathbf{Q}) \geq v^*/3$ .*

## 5.2. Lagrangian Relaxation Approach

In this subsection, we derive an efficiently computable upper bound of Model (10) based on the nonanticipativity Lagrangian dual of stochastic program (cf., Shapiro et al. 2009) as well as the results of the deterministic MPNP-CDS (i.e., MPNP-CDS(D)) in Section 3. We show that this upper bound can be equivalently computed via a SDP, which is more effective than computing the Lagrangian dual. We prove that this upper bound is only

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**Algorithm 1** Double greedy algorithm to solve Model (10) (Bian et al. 2016)

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1:  $\mathbf{x}^0 \leftarrow \mathbf{Q}, \mathbf{y}^0 \leftarrow \bar{\mathbf{Q}}$ 
2: for  $i = 1$  to  $n$  do
3:   Find  $\hat{Q}_a \in \arg \max_{Q_a \in [\underline{Q}_i, \bar{Q}_i]} \Pi(\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow Q_a), \hat{Q}_b \in \arg \max_{Q_b \in [\underline{Q}_i, \bar{Q}_i]} \Pi(\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow Q_b)$ 
4:   Let  $\delta_a \leftarrow \Pi(\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow \hat{Q}_a) - \Pi(\mathbf{x}^{i-1})$  and  $\delta_b \leftarrow \Pi(\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow \hat{Q}_b) - \Pi(\mathbf{y}^{i-1})$ 
5:   if  $\delta_a \geq \delta_b$  then
6:      $\mathbf{x}^i \leftarrow (\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow \hat{Q}_a), \mathbf{y}^i \leftarrow (\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow \hat{Q}_a);$ 
7:   else
8:      $\mathbf{y}^i \leftarrow (\mathbf{y}^{i-1} | y_i^{i-1} \leftarrow \hat{Q}_b), \mathbf{x}^i \leftarrow (\mathbf{x}^{i-1} | x_i^{i-1} \leftarrow \hat{Q}_b)$ 
9:   end if
10: end for
11: Output  $\mathbf{Q} = \mathbf{x}^n$  (or  $\mathbf{y}^n$ );

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a constant-factor away from the true optimal value  $v^*$  given that the random demand is mainly concentrated on the mean. Finally, we derive a heuristic algorithm simply by letting order quantities  $\mathbf{Q}$  equal to one vector of this linear program, which has a similar economic interpretation as order quantities.

The nonanticipativity Lagrangian dual method has been demonstrated as one of the effective approaches to solve the two-stage stochastic (integer) program (cf., Rockafellar and Wets 1976, Birge 1997, Birge and Louveaux 1997, Schultz 2003, Ahmed et al. 2017). It decomposes a large-scale stochastic program model into scenario-based subproblems. As a special case of a two-stage stochastic program, we apply this approach to Model (10). First of all, we create  $N$  copies of vector  $\mathbf{Q}$ , one for each scenario, denoted as  $\{\mathbf{Q}^k\}_{k \in [N]}$ , and enforce them to be equal, i.e.,

$$\mathbf{Q}^k = \mathbf{Q}, \forall k \in [N], \quad (20)$$

where constraints (20) are known as “*nonanticipativity constraints*”.

Then, Model (10) can be equivalently reformulated as follows:

$$v^* = \max_{\mathbf{Q}^k \in \mathbb{R}_+^n, \forall k \in [N], \mathbf{Q}} \left\{ \sum_{k \in [N]} m_k \sum_{i \in [n]} \bar{P}_i Q_i^k - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i (Q_i^k - D_i^{sk}(\mathbf{Q}))_+ \right] : (20) \right\}. \quad (21)$$

The Lagrangian dual problem is to relax the nonanticipativity constraints with Lagrangian multipliers  $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}^k\}_{k \in [N]} \in \mathbb{R}^{n \times N}$ , which can be written as

$$v^{LD} = \inf_{\boldsymbol{\lambda}} \mathcal{L}(\boldsymbol{\lambda}) \quad (22a)$$

where

$$\mathcal{L}(\boldsymbol{\lambda}) := \max_{\mathbf{Q}^k \in \mathbb{R}_+^n, \forall k \in [N], \mathbf{Q}} \left\{ \sum_{k \in [N]} m_k \sum_{i \in [n]} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i (Q_i^k - D_i^{sk}(\mathbf{Q}))_+ \right] - \sum_{k \in [N]} \sum_{i \in [n]} \lambda_i^k Q_i \right\}. \quad (22b)$$

We note that: (i) since  $\mathbf{Q}$  is a free vector in (22b), we must have  $\sum_{k \in [N]} \lambda_i^k = 0$ , otherwise,  $\mathcal{L}(\boldsymbol{\lambda}) = +\infty$ ; and (ii) since  $Q_i^k$  can be positive infinity, we also have  $\bar{P}_i + \frac{\lambda_i^k}{m_k} \leq \bar{S}_i$  for all  $i \in [n]$  and  $k \in [N]$ , otherwise, if we suppose that there exists a pair  $(i, k)$  such that  $\bar{P}_i + \frac{\lambda_i^k}{m_k} > \bar{S}_i$ , then we can have  $\left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \bar{S}_i (Q_i^k - D_i^{sk}(\mathbf{Q}))_+ \rightarrow \infty$  if  $Q_i^k \rightarrow +\infty$ , i.e.,  $\mathcal{L}(\boldsymbol{\lambda}) \rightarrow +\infty$ ; and (iii) we also notice that for any given Lagrangian multipliers  $\boldsymbol{\lambda}$ , the maximization of the dual problem (22b) can be decomposed into  $N$  subproblems, one for each scenario.

Therefore, the Lagrangian dual problem can be rewritten as

$$v^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Q}^k \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \sum_{i \in [n]} \bar{S}_i (Q_i^k - D_i^{sk}(\mathbf{Q}))_+ \right\}, \quad (23)$$

where

$$\Omega = \left\{ \boldsymbol{\lambda} : \sum_{\tau \in [N]} \lambda_i^\tau = 0, \bar{P}_i + \frac{\lambda_i^k}{m_k} \leq \bar{S}_i, \forall k \in [N], \forall i \in [n] \right\}.$$

For each  $k \in [N]$ , the inner maximization in (23) is a special case of the deterministic MPNP-CDS (4) with  $\bar{P}_i \leftarrow \bar{P}_i + \frac{\lambda_i^k}{m_k}$  and  $D_i \leftarrow D_i^k$  for all  $i \in [n]$ . Therefore, by Proposition 2, the inner maximization in (23) is equivalent to Model (8), so Lagrangian dual (23) is equivalent to

$$v^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\}, \quad (24)$$

where  $C = \{\mathbf{Y} : Y_{jj} = 1, \forall j \in [n+1], \mathbf{Y} \succeq 0, \text{rank}(\mathbf{Y}) = 1\}$ ,  $w_{ij}^k = \alpha_{ji} D_j^k$ ,  $w_{i(n+1)}^k = D_i^k$ ,  $w_{j,i}^k = w_{(n+1)i}^k = 0$  for all  $i, j \in [n]$ . If we relax the rank-one constraint, then we obtain an upper bound of  $v^{LD}$ , denoted as  $v_R^{LD}$ , as follows:

$$v_R^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C_R} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\}, \quad (25)$$

where  $C_R = \{\mathbf{Y} : Y_{jj} = 1, \forall j \in [n+1], \mathbf{Y} \succeq 0\}$ .

The following theorem summarizes the above model comparison results, i.e.,  $v^* \leq v^{LD} \leq v_R^{LD}$  and shows an equivalent SDP to obtain  $v_R^{LD}$ .

**Theorem 5** *Let  $v^*, v^{LD}, v_R^{LD}$  denote the optimal values obtained for Models (10), (22b), and (25), respectively. Then,*

(i)  $v^* \leq v^{LD} \leq v_R^{LD}$ ; and

(ii)

$$v_R^{LD} = \max_{\boldsymbol{\pi}, \boldsymbol{\beta}} \sum_{i \in [n]} \bar{P}_i \beta_i - \sum_{k \in [N]} m_k \sum_{i \in [n]} \bar{S}_i \pi_i^k, \quad (26a)$$

$$\text{s.t. } \beta_i - \pi_i^k = \sum_{j=1}^{n+1} \frac{1}{4} w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k), \forall i \in [n], \forall k \in [N], \quad (26b)$$

$$\pi_i^k \geq 0, \forall i \in [n], \forall k \in [N], \quad (26c)$$

$$\beta_i \geq 0, \forall i \in [n] \quad (26d)$$

$$Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N] \quad (26e)$$

$$\mathbf{Y}^k \succeq 0, \forall k \in [N] \quad (26f)$$

*Proof:* See the Online Supplement. □

Suppose that  $(\boldsymbol{\pi}^*, \boldsymbol{\beta}^*)$  is an optimal solution to Model (26). From constraints (26d), it is clear that  $\boldsymbol{\beta}^* \in \mathbb{R}_+^n$ . We also note that from the proof of Theorem 5, variable  $\beta_i^*$  in (26) is the shadow price of Lagrangian dual constraint  $\sum_{\tau \in [N]} \lambda_i^\tau = 0$  for each product  $i \in [n]$ , while from (22b) and the subsequent derivations, we can also see that, its order quantity  $Q_i$  can be also viewed as the shadow price of Lagrangian dual constraint  $\sum_{\tau \in [N]} \lambda_i^\tau = 0$  for each  $i \in [n]$ . This motivates us to construct a lower bound of Model (10) simply by letting  $\mathbf{Q} = \boldsymbol{\beta}^*$ .

**Remark 1** *Let  $(\boldsymbol{\pi}^*, \boldsymbol{\beta}^*)$  be an optimal solution to Model (26). Then,  $\boldsymbol{\beta}^*$  is feasible to Model (10), and  $\Pi(\boldsymbol{\beta}^*)$  is a lower bound to the optimal value  $v^*$ .*

Theorem 5 provides an efficiently computable upper bound  $v_R^{LD}$  for MPNP-CDS (10). Next, we investigate the quality of the proposed bound compared with the optimal value  $v^*$  of Model (10). Before introducing our main result, we first assume that the random demand  $\tilde{\mathbf{D}}$  mainly concentrates on its mean or a particular constant vector  $\mathbf{D}$ .

**Assumption 3** Given an  $n$ -dimensional nonnegative vector  $\mathbf{D} \in \mathbb{R}_+^n$  and two constants  $\underline{\delta} \in [0, 1), \bar{\delta} \in [0, +\infty)$ , let the random demand  $\tilde{\mathbf{D}}$  satisfy

$$\mathbb{P} \left\{ (1 - \underline{\delta})D_i \leq \tilde{D}_i \leq (1 + \bar{\delta})D_i \right\} = 1.$$

i.e.,  $(1 - \underline{\delta})D_i \leq D_i^k \leq (1 + \bar{\delta})D_i$  for all  $k \in [N]$ .

Under this assumption, we are able to show that  $v_R^{LD}$  and  $v^{LD}$  are only constant-factor away from the true optimal  $v^*$ .

**Theorem 6** Let  $v^*, v^{LD}, v_R^{LD}$  denote the optimal value of Models (10), (22b), and (25), respectively. Then,

(i)  $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$ ; and

(ii) if Assumption 3 holds, then

$$v_R^{LD} \leq \frac{v^{LD}}{0.79607} \leq \frac{(1 + \bar{\delta})}{0.79607(1 - \underline{\delta})} v^*$$

*Proof:* See the Online Supplement. □

## 6. Numerical Investigation

We conduct two sets of numerical experiments on Model (10). For the first set of numerical experiments, we test the performances of solving Model (10) by MILP Model 1 (from Proposition 5); MILP Model 2 (from Proposition 6); double greedy Algorithm 1, relaxed Lagrangian dual bound  $v_R^{LD}$  in (26) and its related heuristic algorithm from Remark 1. For the second set of numerical experiments, we investigate model properties and conduct sensitivity analyses that reveal useful managerial insights. The numerical instances in this section can be found at <https://drive.google.com/drive/folders/1RhwozZCb7IS6p5my0trquyUrneXlJtU?usp=sharing>.

### 6.1. Performances of Different Approaches

To study the computational effectiveness of the two MILP formulations and the approximation algorithms, 20 numerical instances of varying sizes were generated. We consider two values of  $n$ , namely,  $n = 10$  and  $n = 20$  products. For each product, its demand was assumed to be uniformly distributed between 5 and 100, its unit price to range from 85 to 95, unit cost to vary from 40 to 50, and the salvage value between 22 and 30. All the products are sold or purchased in the same unit of measurement, and thus their substitution

rates are between 0 and 1.<sup>2</sup> These substitution rates were chosen uniformly between 0 and 1, and then were normalized to satisfy  $\sum_{j \in [n]} \alpha_{ij} = 0.8$  and  $\alpha_{ii} = 0$  for each  $i \in [n]$ . To study the model performances across different scenarios, we generated 10 different types of samples with sample size,  $N = 100, 200, \dots, 1000$ . The results for each pair value of  $(N, n)$  are the average values by performing 5 replications. All the algorithms were coded in Python 2.7 with calls to Gurobi 7.5 on a personal computer with 2.3 GHz Intel Core i5 processor and 8G of memory. The CPU time limit of Gurobi was set to be 3600 seconds.

Table 1 and Table 2 display the computational results of MILP Model 1 in Proposition 5 and MILP Model 2 in Proposition 6 with  $n = 10$  and  $n = 20$ , respectively. In particular, LB and UB denote best lower and upper bounds, RB represents the root bound (i.e., continuous relaxation value), while Gap is the optimality gap, computed as  $(UB-LB)/LB$ . Note that, when  $n = 10$ , MILP Model 2 can be solved to optimality or near-optimality, while MILP Model 1 takes a longer time to solve or it ends with larger optimality gaps. When the number of products increases to 20, both models cannot be solved to optimality; however, MILP Model 2 yields much smaller optimality gaps. The root bounds of MILP Model 2 are also much smaller than those for MILP Model 1, which is consistent with the model comparison results presented in Theorem 5. Therefore, MILP Model 2 outperforms MILP Model 1. We also notice that the computational time and optimality gaps for both the models increase as the number of scenarios  $N$  grows. Thus, a good choice of  $N$  will be crucial for these models, in particular for MILP Model 2.

The results of the approximation algorithms presented in Section 4 are shown in Table 3 and Table 4. In particular, we let LB denote the best objective value obtained using the procedure specified in Remark 1 or Algorithm 1, Gap is computed as  $(v_R^{LD}-LB)/LB$ , where  $v_R^{LD}$  denotes relaxed Lagrangian dual bound.

Note that when there are only  $n = 10$  products, all of these approximation algorithms can find good-quality feasible solutions. The relaxed Lagrangian dual bound  $v_R^{LD}$  is also quite close to the true optimal value. When  $n = 20$ , Lagrangian relaxation can find good-quality feasible solutions, while the solutions obtained by the double greedy algorithm are slightly worse. On the other hand, the running times of double greedy algorithm grow

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<sup>2</sup>This assumption is due to the demonstration convenience. If there are different units of measurement among the products, then we can stick to one particular product's unit of measurement and convert all the other products' values including their unit costs, unit prices, salvage values, and demands into this unit of measurement.

significantly as the number of scenarios  $N$  increases, while the relaxed Lagrangian dual bound,  $v_R^{LD}$ , can be obtained within 4 seconds even when  $N = 1000$ . In comparison with the results of the MILP models in Table 1 and Table 2, the relaxed Lagrangian dual bound can be tighter than the best upper bound generated by the MILP Model 2, especially when  $n = 20, N \geq 800$ . These observations suggest that for large-scale instances, the relaxed Lagrangian dual bound and the feasible solution constructed by the method specified in Remark 1 are preferable.

**Table 1** Computational results of MILP Model 1, and MILP Model 2 with  $n = 10$

$N$	MILP Model 1					MILP Model 2				
	time	LB	UB	RB	Gap	time	LB	UB	RB	Gap
100	26	20111	20113	28354	0.01%	29	20111	20113	21798	0.01%
200	153	19244	19246	27513	0.01%	253	19244	19246	20965	0.01%
300	774	19437	19439	27741	0.01%	593	19437	19439	21162	0.01%
400	3600	19017	19022	27288	0.03%	1122	19017	19019	20704	0.01%
500	3600	19384	19389	27702	0.02%	2788	19384	19387	21126	0.02%
600	3600	19547	19568	27866	0.11%	3600	19547	19551	21276	0.02%
700	3600	19429	19448	27728	0.10%	3600	19429	19435	21133	0.03%
800	3600	19318	19350	27591	0.16%	3600	19318	19323	21013	0.02%
900	3600	19434	19515	27744	0.42%	3600	19434	19437	21167	0.02%
1000	3600	19545	19631	27852	0.44%	3600	19545	19548	21266	0.02%

**Table 2** Computational results of MILP Model 1, and MILP Model 2 with  $n = 20$

$N$	MILP Model 1					MILP Model 2				
	time	LB	UB	RB	Gap	time	LB	UB	RB	Gap
100	3600	39879	42703	80394	7.08%	3600	39907	40656	48553	1.88%
200	3600	39342	45561	79994	15.81%	3600	39326	40489	47970	2.96%
300	3600	38970	45563	79704	16.92%	3600	38911	40089	47575	3.03%
400	3600	39448	46603	80250	18.14%	3600	39442	40550	48181	2.81%
500	3600	39339	46808	80124	18.99%	3600	38994	40680	48132	4.32%
600	3600	39014	46738	79778	19.90%	3600	38596	40715	47738	5.49%
700	3600	39593	47365	80546	19.63%	3600	39277	42285	48489	7.66%
800	3600	38088	47565	80181	24.88%	3600	38552	43510	48219	12.86%
900	3600	39110	46951	80003	20.05%	3600	38861	43998	47914	13.22%
1000	3600	37473	47156	79910	25.84%	3600	38478	45470	47810	18.17%

We further compare the theoretical approximation ratio with that obtained computationally, termed “*computational ratio*”, of double greedy algorithm and Lagrangian relaxation approach. The computational ratios are illustrated in Figures 1(a) and 1(b). Note that, for double greedy algorithm, the computational approximation ratios (i.e., best lower bound divided by  $v_R^{LD}$ ) for all the instances are larger than the theoretical ratio  $1/3$ . For the Lagrangian relaxation approach, according to Theorem 6, the theoretical approximation ratio is  $1/19 * 0.79607 \approx 0.042$ , however, the computational ratios (lower bound divided by



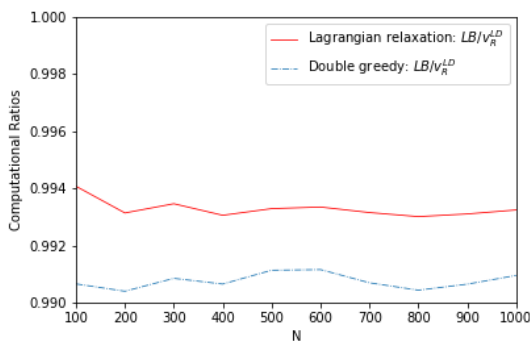
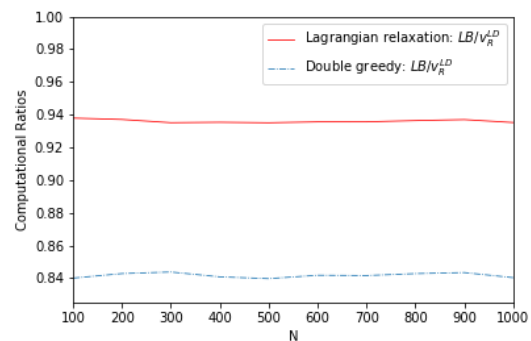
**Table 3** Computational results of approximation algorithms when  $n = 10$ 

$N$	Lagrangian Relaxation				Double Greedy		
	time	LB	$v_R^{LD}$	Gap	time	LB	Gap
100	0.19	20105	20263	0.79%	6	19538	3.71%
200	0.33	19240	19409	0.88%	20	18768	3.42%
300	0.47	19432	19598	0.85%	47	18935	3.50%
400	0.62	19011	19182	0.90%	81	18559	3.35%
500	0.85	19378	19545	0.86%	127	18924	3.28%
600	0.90	19541	19709	0.86%	190	19070	3.35%
700	1.08	19422	19596	0.89%	259	18951	3.40%
800	1.15	19311	19484	0.89%	348	18845	3.39%
900	1.31	19430	19601	0.88%	430	18964	3.36%
1000	1.42	19538	19709	0.87%	520	19051	3.45%

**Table 4** Computational results of approximation algorithms when  $n = 20$ 

$N$	Lagrangian Relaxation				Double Greedy		
	time	LB	$v_R^{LD}$	Gap	time	LB	Gap
100	0.62	39809	41801	5.00%	22	35572	17.51%
200	1.09	39341	41307	5.00%	83	35283	17.07%
300	1.34	39029	41028	5.12%	193	35098	16.90%
400	2.14	39459	41445	5.03%	336	35335	17.29%
500	2.40	39387	41409	5.14%	506	35278	17.38%
600	2.67	39082	41077	5.11%	738	35069	17.13%
700	3.24	39718	41713	5.02%	1040	35603	17.16%
800	3.57	39467	41463	5.06%	1403	35423	17.05%
900	3.79	39221	41179	4.99%	1708	35207	16.96%
1000	4.36	39157	41159	5.11%	2071	35085	17.31%

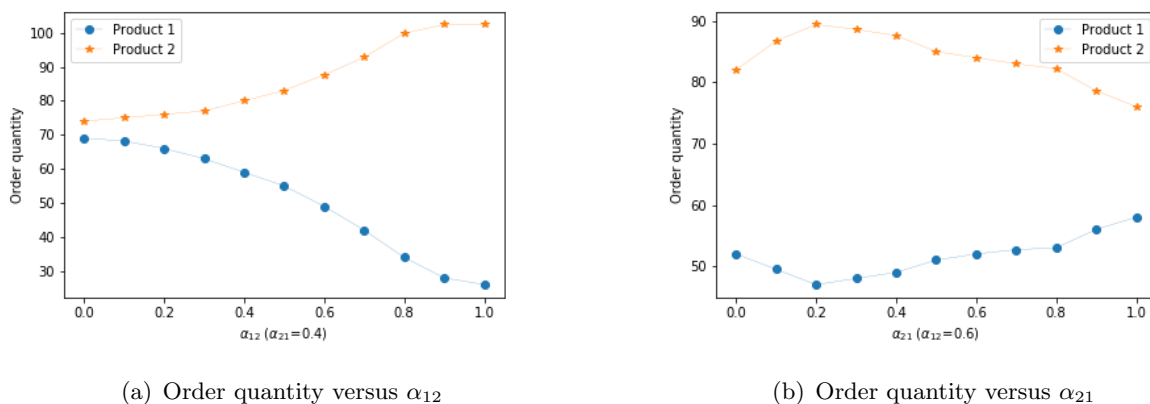
$v_R^{LD}$ ) for all the instances are much larger than the theoretical ratio. We also notice that in both figures, the computational ratios of the Lagrangian relaxation approach are even better than that of the double greedy algorithm. This suggests a potential to improve the approximation result in Theorem 6.

(a) Computational ratios when  $n = 10$ (b) Computational ratios when  $n = 20$ **Figure 1** Computational ratios of Lagrangian relaxation approach and double greedy algorithm

## 6.2. Model Properties and Sensitivity Analyses

In this subsection, we illustrate model properties, and also conduct sensitivity analyses for the MPNP-CDS by using the case of two products, i.e.,  $n = 2$ . For all the different instances below, the optimal solutions and values are obtained by solving the MILP Model 2 to optimality.

**Effects of Substitution Rates.** To study this effect, the demand was assumed to follow a uniform distribution between 5 and 100, and we generated  $N = 1000$  i.i.d. samples. Also, we let  $p_1 = 91, p_2 = 92, c_1 = 45, c_2 = 41, s_1 = 23, s_2 = 28$ . Figure 2 illustrates how order quantities vary with different substitution rates, where in Figure 2(a), we set  $\alpha_{21} = 0.4$  and let  $\alpha_{12}$  vary from 0 to 1, while in Figure 2(b), we set  $\alpha_{12} = 0.6$  and let  $\alpha_{21}$  vary from 0 to 1.

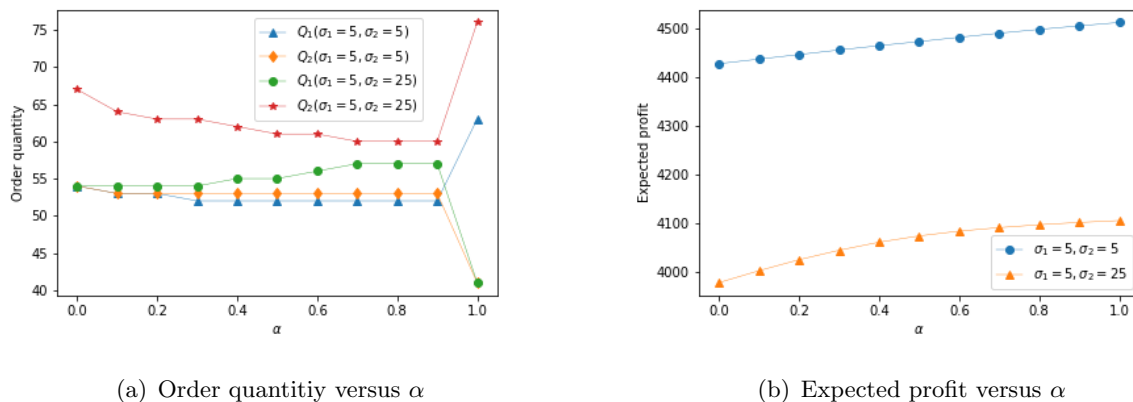


**Figure 2** Optimal order quantities versus substitution rates

In Figure 2(a), we observe that as the value of  $\alpha_{12}$  increases, the optimal order quantity of Product 2 increases as well, while the optimal order quantity of Product 1 decreases. In Figure 2(b), as  $\alpha_{21}$  increases, the optimal order quantity of Product 1 decreases in the beginning and then increases. On the contrary, the optimal order quantity of Product 2 increases at first and then decreases. Both figures imply that the optimal order quantity does not necessarily monotonically increase or decrease with increment in substitution rate. These observations are consistent with the result in Theorem 3. Therefore, we recommend that the retailer should increase the orders of products which have higher substitution rates and order less on the other products which will be substituted.

**Effects of Demand Variation.** To study the effect of demand variation, we assumed that the demand for each product to be uniformly distributed with mean  $\mu = 50$ , and the standard deviation is  $\sigma_1 = 5$  for product 1. The standard deviation of product 2 (i.e.,  $\sigma_2$ ) varied from 5 to 25. For each configuration  $(\sigma_1, \sigma_2)$  of random demand, we generated  $N = 500$  samples and solved their corresponding MILP Model 2. For the rest of parameters, we let  $p_i = 92, s_i = 24, c_i = 45$  for each product  $i = 1, 2$ , and also, let  $\alpha_{12} = \alpha_{21} = \alpha$ , which varies from 0 to 1.

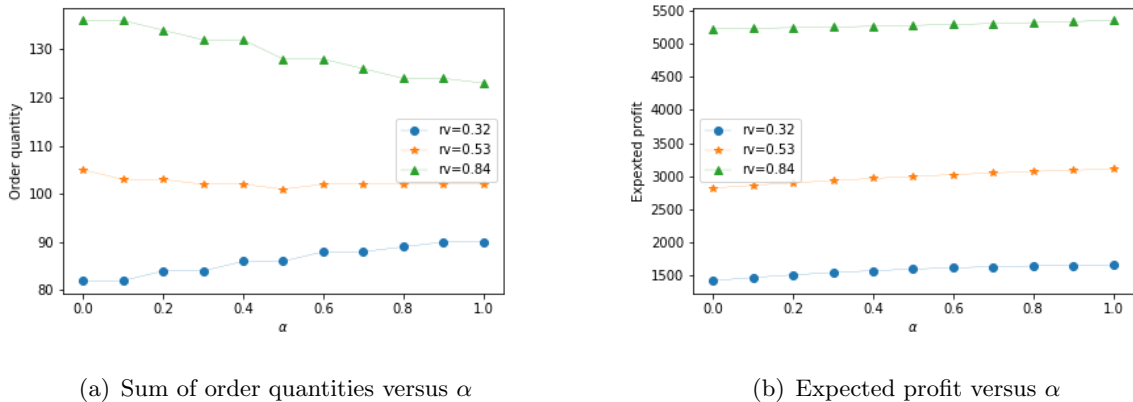
Figure 3 illustrates the results obtained. In Figure 3(a), we see that, when the product 2's standard deviation increases, both products' order quantities will increase. This suggests that the retailer should increase the orders of both products to hedge against the risks from demand variation. When the substitution rate is close to 1, we observe that the difference between these two products' order quantities becomes large. This might be because the customers are insensitive to either product and thus, the retailer probably can remove one product to reduce the products' assortment on the shelf. In Figure 3(b), we observe that as the demand standard deviation increases, the expected profit decreases. This is mainly because a higher standard deviation implies a higher risk of understock or overstock. On the contrary, increasing substitution rate  $\alpha$  can mitigate the risks caused by demand variability, thus contributes to a higher expected profit, which demonstrates the important role of substitution rates. As a result, we recommend that the retailer should focus on a subset of products, among which there are substantial substitution effects.



**Figure 3** Optimal order quantities and the optimal expected profit versus substitution rate under different standard deviations

**Effects of Ratio Value.** To study this effect, we assume the demand for each product to be uniformly distributed between 5 and 100. We generate  $N = 1000$  samples. For the rest of parameters, we let  $p_i = 92, s_i = 24$  while  $c_i = c$ , which takes three values, namely, 35, 45 and 70, for each product  $i = 1, 2$ , and we also let  $\alpha_{12} = \alpha_{21} = \alpha$ , which varies from 0 to 1. Therefore, we define the “ratio value” as  $rv = \frac{\bar{P}}{\bar{S}}$ , where  $\bar{P}_i = \bar{P}$  and  $\bar{S}_i = \bar{S}$  for each product  $i = 1, 2$ . Clearly,  $rv$  takes three values-0.84, 0.53, 0.32.

Figure 4 illustrates how the sum of optimal order quantities and the expected profit change with different substitution rates and ratio values. Note that both the order quantities and expected profit increase as ratio value increases, which is because the higher ratio value is, the more profitable products are. In Figure 4(a), the sum of total order quantities can increase or decrease with an increment in substitution rate, which depends on their current ratio values. On the contrary, as depicted in Figure 4(b), the expected profit increases with an increment in substitution rate. Thus, a higher ratio value will help increase the retailer’s expected profit, but does not necessarily imply that the retailer should always order more products.



**Figure 4** Sum of optimal order quantities and expected profit versus substitution rates under different ratio values.

## 7. Concluding Remarks

In this paper, we study a multi-product newsvendor problem with customer-driven demand substitution (MPNP-CDS) for both the cases of deterministic and stochastic demands. When demand is known, we show that each product is either not ordered or ordered up

to the effective demand. This fact allows us to reformulate the MPNP-CDS as an equivalent binary quadratic program, and to prove that the MPNP-CDS is NP-hard. When the demand is stochastic, we derive first-order necessary conditions for the MPNP-CDS, and show that the profit function is continuous submodular. This fact enables us to develop two different mixed integer linear program (MILP) models and compare their strengths. Inspired by the model properties, we develop several approximation algorithms and prove their performance guarantees. Our numerical investigation on the performance of the proposed solution algorithms shows that the stronger MILP model works well for small- or medium- sized problem instances, while the approximation algorithms consistently provide high-quality solutions. We further conducted sensitivity analyses to reveal how the model performs when the values of parameters change. Our findings suggested that the retailer should increase the orders of the products which have higher substitution rates and also should order more products if their demand variation increases. Additionally, a higher ratio value can help increase the expected profit. A potential idea for future research is to investigate distributionally robust MPNP-CDS, where the probability distribution of the random demand is not fully specified but instead, some empirical data is available. Also, it would be interesting to develop exact and efficient algorithms to solve large-scale instance of MPNP-CDS instances when demand is stochastic.

## Acknowledgment

The authors would like to thank Professor Mohit Singh (Georgia Tech) for providing helpful comments on improving the earlier version of the paper. We also appreciate the valuable comments from associate editor and two anonymous referees. The authors would like to thank Olivares-Nadal (2021) for pointing out a typo in Part (i) of Theorem 1.

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# Online Supplement for Paper: Multi-Product Newsvendor Problem with Customer-driven Demand Substitution: A Stochastic Integer Program Perspective

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## A. Proofs

### A.1 Proof of Theorem 1

**Theorem 1** Let  $\mathbf{Q}^*$  be optimal order quantities for the MPNP-CDS(D) (4). Then,

(i)

$$Q_j^* = \begin{cases} 0, & \text{if } f(\Gamma^* \setminus \{j\}) \leq f(\Gamma^*) \\ D_j^s(\mathbf{Q}^*) = D_j + \sum_{i \in \Gamma^*} \alpha_{ij} D_i, & \text{otherwise.} \end{cases} \quad (5)$$

for each  $j \in [n]$ ; and

(ii)

$$v_D^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus \Gamma} \bar{P}_i D_i \right\} := f(\Gamma^*), \quad (6)$$

where  $[n] \setminus \text{supp}(\mathbf{Q}^*) = \Gamma^*$ , i.e.,  $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$ .

*Proof:* We prove the result by using the following three arguments.

(1) Let  $x_i = Q_i - D_i$  denotes the unsold units of  $i$ th product for each  $i \in [n]$ . Then, Model

(4) is equivalent to

$$v_D^* = \max_{\mathbf{x} \geq -\mathbf{D}} \left\{ g(\mathbf{x}) := \sum_{i \in [n]} \bar{P}_i x_i - \sum_{i \in [n]} \bar{S}_i \left( x_i - \sum_{j \in [n]} \alpha_{ji} (-x_j)_+ \right)_+ + \sum_{i \in [n]} \bar{P}_i D_i \right\}, \quad (27)$$

To simplify function  $g(\mathbf{x})$ , we classify  $n$  products into the following three sets according to the value of  $\mathbf{x}$ , i.e.,

$$I_+ = \{i : x_i \geq 0\}, I_- = \{i : x_i \leq 0\}, I_{++} = \left\{ i \in I_+ : x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0 \right\}.$$

Consequently, we can remove  $(\cdot)_+$  from (27), and we have

$$g(\mathbf{x}) = \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i x_i + \sum_{i \in I_{++}} (\bar{P}_i - \bar{S}_i) x_i + \sum_{j \in I_-} \left( \bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that in the above function, for each  $i \in I_+ \setminus I_{++}$ , the coefficient of  $x_i$  is positive as  $\bar{P}_i = p_i - c_i > 0$ , and by definition,  $x_i \leq -\sum_{j \in I_-} \alpha_{ji} x_j$ . Therefore, by letting  $x_i = -\sum_{j \in I_-} \alpha_{ji} x_j$  for each  $i \in I_+ \setminus I_{++}$ , function  $g(\mathbf{x})$  is upper bounded by

$$\begin{aligned} g(\mathbf{x}) &\leq \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i \left( -\sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{i \in I_{++}} (\bar{P}_i - \bar{S}_i) x_i + \sum_{j \in I_-} \left( \bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i \\ &= \sum_{i \in I_{++}} (\bar{P}_i - \bar{S}_i) \left( x_i + \sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{j \in I_-} \left( \bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i. \end{aligned}$$

For each  $i \in I_{++}$ , we have  $\bar{P}_i - \bar{S}_i = s_i - c_i < 0$  by definition, and  $x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0$  by definition of set  $I_{++}$ . Thus, by letting  $I_{++} = \emptyset$ , function  $g(\mathbf{x})$  is further upper bounded by

$$g(\mathbf{x}) \leq \sum_{j \in I_-} \left( \bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that for each  $j \in I_-$ , we note that  $x_j \in [-D_j, 0]$ . Hence, for each  $j \in I_-$ , let  $x_j = 0$  if  $\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \geq 0$ , and  $-D_j$ , otherwise. Then  $g(\mathbf{x})$  is further upper bounded by

$$g(\mathbf{x}) \leq \sum_{j \in I_-} \left( \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i,$$

where the equality is achieved when  $I_{++} = \emptyset$  and for each  $j \in [n]$ ,

$$x_j = \begin{cases} 0, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \leq 0, j \in I_- \\ -D_j, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j > 0, j \in I_- \\ -\sum_{i \in I_-} \alpha_{ij} x_i, & \text{otherwise} \end{cases} \quad (28)$$

Note that  $I_+ = [n] \setminus I_-$ . Therefore, Model (27) is further equivalent to the following combinatorial optimization problem

$$v_D^* = \max_{I_- \subseteq [n]} \left\{ \hat{g}(I_-) := \sum_{j \in I_-} \left( \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i \right\}. \quad (29)$$

(2) Next, we prove the following property of Model (29).

**Claim 1** In the Model (29), for any subset  $I_- \subseteq [n]$ , let  $J_0 = \left\{ j \in I_- : \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i \leq \bar{P}_j \right\}$ , then  $\hat{g}(I_-) \leq \hat{g}(I_- \setminus J_0)$ .

*Proof:* Let us define  $\hat{I}_- = I_- \setminus J_0$ . By definitions of sets  $I_-$ ,  $J_0$  and  $\hat{I}_-$ , we have

$$\begin{aligned} \hat{g}(I_-) &= \sum_{j \in I_-} \left( \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i \\ &= \sum_{j \in I_- \setminus J_0} \left( \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i \\ &= \sum_{j \in I_- \setminus J_0} \left( \sum_{i \in [n] \setminus (I_- \setminus J_0)} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i - \sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \\ &= \hat{g}(\hat{I}_-) - \sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \\ &\leq \hat{g}(\hat{I}_-) \end{aligned}$$

where the inequality holds due to  $\sum_{j \in I_- \setminus J_0} \sum_{i \in J_0} \alpha_{ji} \bar{P}_i D_j \geq 0$ .

◇

By Claim 1 and equation (28), we note that there exists an optimal solution to Model (27)  $\mathbf{x}^*$  with subset  $I_-^* := \left\{ j : \sum_{i \in [n] \setminus I_-^*} \alpha_{ji} \bar{P}_i > \bar{P}_j \right\}$  such that

$$\mathbf{x}_j^* = \begin{cases} -D_j, & \text{if } j \in I_-^* \\ -\sum_{i \in I_-} \alpha_{ij} x_i^*, & \text{otherwise.} \end{cases}$$

Let  $\mathbf{Q}^* = \mathbf{x}^* + \mathbf{D}$ , and  $\Gamma^* = I_-^*$ . Clearly,  $\mathbf{Q}^*$  satisfies (5) and is an optimal solution to Model (4) and  $v_D^* = f(\Gamma^*)$ .

(3) Finally, by Claim 1 and letting  $\Gamma = I_-$ , Model (29) reduces to

$$v_D^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \left( \sum_{i \in [n] \setminus \Gamma} \alpha_{ji} \bar{P}_i - \bar{P}_j \right) D_j + \sum_{i \in [n]} \bar{P}_i D_i \right\},$$

which is equivalent to (6).

□

## A.2 Proof of Proposition 1

**Proposition 1** *The set function  $f(\Gamma)$ , defined in (6), is submodular.*

*Proof:* Let  $A \subseteq B \subseteq [I], k \in [n] \setminus B$ . By Definition 1, we only need to show that

$$f(A \cup \{k\}) - f(A) \geq f(B \cup \{k\}) - f(B).$$

Note that

$$\begin{aligned} f(B \cup \{k\}) - f(B) &= \sum_{j \in B \cup \{k\}} \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ji} \bar{P}_i D_j - \sum_{j \in B} \sum_{i \in [n] \setminus B} \alpha_{ji} \bar{P}_i D_j - \bar{P}_k D_k \\ &= \sum_{j \in B} \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ji} \bar{P}_i D_j - \sum_{j \in B} \sum_{i \in [n] \setminus B} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k \\ &= -\sum_{j \in B} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k \end{aligned}$$

Similarly,

$$f(A \cup \{k\}) - f(A) = -\sum_{j \in A} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (A \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k.$$

Hence,

$$\begin{aligned} &(f(B \cup \{k\}) - f(B)) - (f(A \cup \{k\}) - f(A)) \\ &= -\sum_{j \in B} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \left( -\sum_{j \in A} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (A \cup \{k\})} \alpha_{ki} \bar{P}_i D_k \right) \\ &= -\sum_{j \in B \setminus A} \alpha_{jk} \bar{P}_k D_j - \sum_{i \in B \setminus A} \alpha_{ki} \bar{P}_i D_k \leq 0 \end{aligned}$$

where the inequality follows because  $\sum_{j \in B \setminus A} \alpha_{jk} \bar{P}_k D_j \geq 0$  and  $\sum_{i \in B \setminus A} \alpha_{ki} \bar{P}_i D_k \geq 0$ . Thus,  $f(\Gamma)$  is submodular.  $\square$

## A.3 Proof of Theorem 2

**Theorem 2** *The MPNP-CDS( $D$ ) is strongly NP-hard, so is the MPNP-CDS (3).*

*Proof:* We prove this result by showing that the weighted max-cut problem (WMCP) is a special case of the MPNP-CDS( $D$ ).

**(Weighted Max-Cut Problem)** Given an undirected graph  $G = (V, E)$  with  $|V| = n$ , and a nonnegative integer weight  $w_{ij}$  associated with each edge  $(i, j) \in E$  in the graph, (and  $w_{ij} = 0$  if there is no edge between nodes  $i, j$ .) find a subset  $\Lambda \subseteq V$  which maximizes the total weights of edges between subsets  $\Lambda$  and  $[n] \setminus \Lambda$ .

Clearly, this problem can be formulated as:

$$v_w = \max_{\Lambda \subseteq [n]} \left\{ \sum_{j \in \Lambda} \sum_{i \in [n] \setminus \Lambda} w_{ji} \right\}. \quad (30)$$

Without loss of generality, we assume that there is at least one edge  $(i, j) \in E$  such that  $w_{ij} > 0$ , otherwise, the weighted max-cut problem is trivial.

Consider a special instance of MPNP-CDS(D), where  $\alpha_{ji} = \alpha_{ij} = (n+1)w_{ji}$  and  $\bar{P}_i = D_i = 1$  for all  $i, j \in [n]$ . Under this setting, Model (6) reduces to

$$v_{DW} = \max_{\Lambda \subseteq [n]} \left\{ (n+1) \sum_{j \in \Lambda} \sum_{i \in [n] \setminus \Lambda} w_{ji} + n - |\Lambda| \right\}. \quad (31)$$

Let  $\lfloor x \rfloor$  denote the floor function of number  $x$ . It remains to show that

**Claim 2**  $\lfloor \frac{v_{DW}}{n+1} \rfloor = v_w$ .

*Proof:* We separate the proof into two steps.

$v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor$  Let  $\Lambda^*$  be an optimal solution to (30). Clearly,  $\Lambda^*$  is feasible to (31), thus

$$(n+1)v_w \leq (n+1) \sum_{j \in \Lambda^*} \sum_{i \in [n] \setminus \Lambda^*} w_{ji} + n - |\Lambda^*| \leq v_{DW}.$$

Due to our assumption that all the weights are integral, we have  $v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor$ . Next we show that

$v_w \geq \lfloor \frac{v_{DW}}{n+1} \rfloor$  Suppose that  $v_w < \lfloor \frac{v_{DW}}{n+1} \rfloor$ , i.e.,  $v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor - 1$ , which implies that

$$(n+1)v_w \leq v_{DW} - (n+1).$$

Let  $\hat{\Lambda}$  be an optimal solution to (31). We have

$$(n+1)v_w \leq v_{DW} - (n+1) = (n+1) \sum_{j \in \hat{\Lambda}} \sum_{i \in [n] \setminus \hat{\Lambda}} w_{ji} + n - |\hat{\Lambda}| - (n+1)$$

which implies that

$$\sum_{j \in \hat{\Lambda}} \sum_{i \in [n] \setminus \hat{\Lambda}} w_{ji} \geq v_w + \frac{1 + |\hat{\Lambda}|}{n+1} > v_w$$

a contradiction that  $v_w$  is the optimal value to (30). ◇

Hence, it follows that we can solve the MPNP-CDS(D) efficiently, only if we can solve the weighted max-cut problem (30) efficiently. However, the weighted max-cut problem is strongly NP-hard. Therefore, the MPNP-CDS is also NP-hard, and consequently, so is the MPNP-CDS. □

#### A.4 Proof of Proposition 2

**Proposition 2** Model (6) is equivalent to

$$v_D^* = \max_{\mathbf{y}} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - y_i y_{n+1} + y_j y_{n+1} - y_i y_j) : y_i \in \{-1, 1\}, \forall i \in [n+1] \right\}. \quad (7)$$

*Proof:* Let  $\widehat{v}_D^*$  be the optimal value of Model (7). We need to show  $\widehat{v}_D^* = v_D^*$ .

( $\widehat{v}_D^* \leq v_D^*$ ) Given an optimal solution  $\mathbf{y}^*$  of Model (7), we define a set  $\widehat{\Gamma} = \{j \in [n] : y_j^* = y_{n+1}^*\}$ . Clearly,  $\widehat{\Gamma}$  is a feasible solution of Model (6) and

$$\begin{aligned} \widehat{v}_D^* &= \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*), \\ &= \sum_{j \in [n]} \sum_{i \in [n]} \frac{1}{4} w_{ij} (1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*) + \sum_{i \in [n]} \frac{1}{2} w_{i(n+1)} (1 - y_i^* y_{n+1}^*), \\ &= \sum_{j \in \widehat{\Gamma}} \sum_{i \in [n]} \frac{1}{4} w_{ij} (2 - y_i^* y_{n+1}^* - y_i^* y_j^*) + \sum_{j \in [n] \setminus \widehat{\Gamma}} \sum_{i \in [n]} \frac{1}{4} w_{ij} (-y_i^* y_{n+1}^* - y_i^* y_j^*) + \\ &\quad \sum_{i \in [n] \setminus \widehat{\Gamma}} w_{i(n+1)}, \\ &= \sum_{j \in \widehat{\Gamma}} \sum_{i \in [n]} \frac{1}{4} w_{ij} (2 - 2y_i^* y_{n+1}^*) + \sum_{i \in [n] \setminus \widehat{\Gamma}} w_{i(n+1)}, \\ &= \sum_{j \in \widehat{\Gamma}} \sum_{i \in [n] \setminus \widehat{\Gamma}} w_{ij} + \sum_{i \in [n] \setminus \widehat{\Gamma}} w_{i(n+1)}, \\ &= \sum_{j \in \widehat{\Gamma}} \sum_{i \in [n] \setminus \widehat{\Gamma}} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus \widehat{\Gamma}} \bar{P}_i D_i, \end{aligned} \quad (32)$$

where the third equality is because  $y_j^* y_{n+1}^* = 1$  for all  $j \in \widehat{\Gamma}$ , and  $-1$ , otherwise; the fourth equality is due to the definition of set  $\widehat{\Gamma}$ , we have  $y_i^* y_{n+1}^* + y_i^* y_j^* = 2y_i^* y_{n+1}^*$  if  $j \in \widehat{\Gamma}$ , and  $0$ , otherwise; the fifth equality is due to that  $y_i^* y_{n+1}^* = 1$  for all  $i \in \widehat{\Gamma}$ ,  $-1$ , otherwise; and the last equality is due to the definition of  $w_{ij}$ . Thus,  $\widehat{v}_D^* \leq v_D^*$ .

( $\widehat{v}_D^* \geq v_D^*$ ) Given an optimal solution  $\Gamma^*$  of Model (6), let us construct vector  $\widehat{\mathbf{y}} \in \{-1, 1\}^{n+1}$  as follows: If  $j \in \Gamma^*$ , then  $\widehat{y}_j = \widehat{y}_{n+1} = 1$ , otherwise,  $\widehat{y}_j = -1 \neq \widehat{y}_{n+1}$ . Clearly,  $\widehat{\mathbf{y}}$  is feasible to Model (7) and following the same derivation as (32), we have

$$v_D^* = \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} (1 - \widehat{y}_i \widehat{y}_{n+1} + \widehat{y}_j \widehat{y}_{n+1} - \widehat{y}_i \widehat{y}_j).$$

Thus,  $\widehat{v}_D^* \geq v_D^*$ .

□

### A.5 Proof of Theorem 3

**Theorem 3** Let  $\mathbf{Q}^*$  be the vector of optimal quantities of Model (10). Then,

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\mathbf{Q}^*)\right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* < \tilde{D}_i\right) \geq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n], \quad (11a)$$

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\mathbf{Q}^*)\right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ji} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n]. \quad (11b)$$

*Proof:* For notational convenience, given a vector  $\mathbf{Q}$ , let  $(\mathbf{Q}|Q_i \leftarrow q)$  denote a new vector that is the same as  $\mathbf{Q}$  except that the  $i$ th entry is  $q$ . Note that  $\mathbf{Q}^*$  is optimal to (10), i.e.,

$$\mathbf{Q}^* \in \arg \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \mathbb{E} \left[ \sum_{i \in [n]} \bar{S}_i \left( Q_i - \tilde{D}_i^s(\mathbf{Q}) \right)_+ \right] \right\}.$$

We note that in the above optimization model, since  $0 < \bar{P}_i < \bar{S}_i$  for each  $i \in [n]$  and demand  $\tilde{D}$  is nonnegative, thus,  $\Pi(\mathbf{Q}) < \Pi((\mathbf{Q})_+)$  if  $\mathbf{Q} \notin \mathbb{R}_+^n$  is not a nonnegative vector. Therefore, we can relax the domain of  $\mathbf{Q}$  to be  $\mathbb{R}^n$  as below:

$$\mathbf{Q}^* \in \arg \max_{\mathbf{Q}} \left\{ \Pi(\mathbf{Q}) = \sum_{i \in [n]} \bar{P}_i Q_i - \mathbb{E} \left[ \sum_{i \in [n]} \bar{S}_i \left( Q_i - \tilde{D}_i^s(\mathbf{Q}) \right)_+ \right] \right\}.$$

By the optimality of  $\mathbf{Q}^*$ , for each  $i \in [n]$ ,  $Q_i^*$  is also optimal to the following mathematical program

$$Q_i^* \in \operatorname{argmax} \left\{ G(Q_i) := \bar{P}_i Q_i - \bar{S}_i \mathbb{E} \left( Q_i - \tilde{D}_i^s(\mathbf{Q}^* | Q_i^* \leftarrow Q_i) \right)_+ - \sum_{j \in [n], j \neq i} \mathbb{E} \left( \bar{S}_j \left( Q_j^* - \tilde{D}_j^s(\mathbf{Q}^* | Q_i^* \leftarrow Q_i) \right)_+ \right) \right\}.$$

Let  $Q_i^1 := Q_i^* + \varepsilon$ ,  $Q_i^2 := Q_i^* - \varepsilon$ , where  $\varepsilon > 0$  is a sufficiently small positive constant.

Simple calculation yields

$$G(Q_i^1) - G(Q_i^*) = \bar{P}_i \varepsilon - \mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\mathbf{Q}^*)\right) \bar{S}_i \varepsilon - \sum_{j \in [n], j \neq i} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* < \tilde{D}_i\right) \bar{S}_j \alpha_{ji} \varepsilon \leq 0,$$

$$G(Q_i^2) - G(Q_i^*) = -\bar{P}_i \varepsilon + \mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\mathbf{Q}^*)\right) \bar{S}_i \varepsilon + \sum_{j \in [n], j \neq i} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* \leq \tilde{D}_i\right) \bar{S}_j \alpha_{ji} \varepsilon \leq 0,$$

where  $G(Q_i^1) - G(Q_i^*) \leq 0$ , and  $G(Q_i^2) - G(Q_i^*) \leq 0$  are due to the optimality of  $Q_i^*$ . Thus, we arrive at (11).  $\square$

### A.6 Proof of Proposition 3

**Proposition 3** Let  $\mathbf{Q}^*$  be the vector of optimal quantities of Model (10). Then,  $\mathbf{Q}^*$  is upper and lower bounded by  $\bar{\mathbf{Q}}$  and  $\underline{\mathbf{Q}}$ , respectively, i.e., for each product  $i \in [n]$ ,  $\bar{Q}_i \geq Q_i^* \geq \underline{Q}_i$  with

$$\underline{Q}_i = \begin{cases} F_{\tilde{D}_i}^{-1} \left( \frac{\bar{P}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}{\bar{S}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j} \right), & \text{if } \bar{S}_i > \sum_{j \in [n]} \alpha_{ij} \bar{S}_j, \\ 0, & \text{otherwise} \end{cases}, \quad (12a)$$

$$\bar{Q}_i = \bar{F}_{\tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j}^{-1} \left( \frac{\bar{P}_i}{\bar{S}_i} \right), \quad (12b)$$

where  $F_{\tilde{X}}^{-1}, \bar{F}_{\tilde{X}}^{-1}$  denote the lower and upper inverse distribution function of random variable  $\tilde{X}$ , respectively, i.e.,  $F_{\tilde{X}}^{-1}(t) = \inf \left\{ \kappa : \mathbb{P} \left( \tilde{X} \leq \kappa \right) \geq t \right\}$  and  $\bar{F}_{\tilde{X}}^{-1}(t) = \inf \left\{ \kappa : \mathbb{P} \left( \tilde{X} < \kappa \right) \geq t \right\}$ .

*Proof:* Note that by definition, for each product  $i \in [n]$ , we have

$$\tilde{D}_i \leq \tilde{D}_i^s(\mathbf{Q}^*) = \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \left( \tilde{D}_j - Q_j^* \right)_+ \leq \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j,$$

where the second inequality holds because  $\tilde{D}_j, Q_j^*$  are nonnegative for all  $j \in [n]$ . Therefore,

$$\begin{aligned} \mathbb{P} \left( \tilde{D}_i \leq Q_i^* \right) &\geq \mathbb{P} \left( \tilde{D}_i^s(\mathbf{Q}^*) \leq Q_i^* \right) \geq \mathbb{P} \left( \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j \leq Q_i^* \right), \\ \mathbb{P} \left( \tilde{D}_i < Q_i^* \right) &\geq \mathbb{P} \left( \tilde{D}_i^s(\mathbf{Q}^*) < Q_i^* \right) \geq \mathbb{P} \left( \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j < Q_i^* \right), \end{aligned} \quad (33)$$

Now, we separate the rest of the proof into two parts.

(1) Clearly, by (10),  $\mathbf{Q}^* \in \mathbb{R}_+^n$ . According to (11a) in Theorem 3, we have

$$\begin{aligned} \mathbb{P} \left( Q_i^* \geq \tilde{D}_i \right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ij} \mathbb{P} \left( Q_i^* < \tilde{D}_i \right) &\geq \mathbb{P} \left( Q_i^* \geq \tilde{D}_i^s(\mathbf{Q}^*) \right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ij} \mathbb{P} \left( Q_j \geq \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* < \tilde{D}_i \right) \\ &\geq \frac{\bar{P}_i}{\bar{S}_i} \end{aligned}$$

where the first inequality follows because (33) and  $\mathbb{P} \left( Q_j \geq \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* < \tilde{D}_i \right) \leq \mathbb{P} \left( Q_i^* < \tilde{D}_i \right)$ . Since  $\mathbb{P} \left( Q_i^* < \tilde{D}_i \right) = 1 - \mathbb{P} \left( Q_i^* \geq \tilde{D}_i \right)$  and multiply  $\bar{S}_i$  on both sides of  $\mathbb{P} \left( Q_i^* \geq \tilde{D}_i \right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ij} \mathbb{P} \left( Q_i^* < \tilde{D}_i \right) \geq \frac{\bar{P}_i}{\bar{S}_i}$ , we obtain

$$\mathbb{P} \left( Q_i^* \geq \tilde{D}_i \right) \geq \frac{\bar{P}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}{\bar{S}_i - \sum_{j \in [n]} \alpha_{ij} \bar{S}_j}$$



given that  $\bar{S}_i > \sum_{j \in [n]} \alpha_{ij} \bar{S}_j$ , i.e., we arrive at (12a).

(2) According to (11b) in Theorem 3, we have

$$\mathbb{P} \left( Q_i^* > \tilde{D}_i + \sum_{j \in [n]} \alpha_{ji} \tilde{D}_j \right) \leq \mathbb{P} \left( Q_i^* > \tilde{D}_i^s(\mathbf{Q}^*) \right) + \sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ij} \mathbb{P} \left( Q_j > \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* \leq \tilde{D}_i \right) \leq \frac{\bar{P}_i}{\bar{S}_i}$$

where the first inequality is due to (33) and  $\sum_{j \in [n]} \frac{\bar{S}_j}{\bar{S}_i} \alpha_{ij} \mathbb{P} \left( Q_j > \tilde{D}_j^s(\mathbf{Q}^*), Q_i^* \leq \tilde{D}_i \right) \geq 0$ .

Thus, we arrive at (12b). □

### A.7 Proof of Proposition 4

**Proposition 4** *The profit function  $\Pi(\mathbf{Q})$  defined in (10) is continuous submodular.*

*Proof:*

$$(10) = \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i (Q_i - D_i^{sk}(\mathbf{Q}))_+ \right] \quad (34a)$$

$$= \sum_{i \in [n]} \bar{P}_i Q_i + \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i \min(D_i^{sk}(\mathbf{Q}) - Q_i, 0) \right] \quad (34b)$$

In (34b),  $D_i^{sk}(\mathbf{Q}) = D_i^k + \sum_{i \neq j} \alpha_{ij} (D_j^k - Q_j)_+ = D_i^k + \sum_{i \neq j} \alpha_{ij} \min(D_j^k - Q_j, 0) = D_i^k - \sum_{i \neq j} \alpha_{ij} Q_j + \sum_{i \neq j} \alpha_{ij} \min(D_i^k, Q_j)$ . As proved in Topkis (2011),  $D_i^{sk}(\mathbf{Q})$  is submodular and supermodular on  $\mathbf{D}$ . Thus,  $D_i^s(\mathbf{Q})$  is submodular and supermodular on  $\mathbf{Q}$ , since  $\mathbf{Q}$  and  $\mathbf{D}$  are symmetric in the function  $\min(D_i, Q_i)$ , and also the summation of linear function are still submodular or supermodular. So  $Q_i - D_i^{sk}(\mathbf{Q})$  is submodular on  $\mathbf{Q}$ . Since  $\min\{t, 0\}$  is non-decreasing and concave on  $t$ , according to Topkis (1978),  $\min\{f(\mathbf{Q}), 0\}$  is submodular if  $f(\mathbf{Q})$  is submodular. Therefore,  $\min(D_i^{sk}(\mathbf{Q}) - Q_i, 0)$  is submodular on  $\mathbf{Q}$ . The first term  $\sum_{i \in [n]} \bar{P}_i Q_i$  in (34b) is linear function and  $m_k, \bar{P}_i \geq 0$ , for all  $k \in [N]$  and  $i \in [n]$ . Thus, (10) is submodular on  $\mathbf{Q}$ . □

### A.8 Proof of Theorem 4

**Theorem 4** *The MILP Model 2 is stronger than MILP Model 1, i.e., their continuous relaxation values satisfy  $\bar{v}_M^1 \leq \bar{v}_M^2$ , where  $\bar{v}_M^1, \bar{v}_M^2$  are defined in (16a), (16b), respectively.*

*Proof:* Let  $(\mathbf{Q}^*, \boldsymbol{\chi}^*, \mathbf{u}^*, \mathbf{w}^*, \mathbf{y}^*)$  be an optimal solution to relaxed Model (16b). For each  $i \in [n], k \in [N]$ , define

$$z_i^{(k)*} = 1 - \sum_{\tau \in [k]} \chi_i^{(\tau)*}.$$

Clearly,  $\mathbf{z}^* \in [0, 1]^{n \times N}$ . We need to show that  $(\mathbf{Q}^*, \mathbf{z}^*, \mathbf{u}^*, \mathbf{y}^*)$  is feasible to relaxed Model (16a). Note that  $(\mathbf{Q}^*, \mathbf{z}^*, \mathbf{u}^*, \mathbf{y}^*)$  satisfies constraints (14b) and (14e).

According to (15d), for each  $i \in [n]$  and  $k \in [N]$ , we have

$$\begin{aligned} u_i^{(k)*} + Q_i^* - D_i^{(k)} &= D_i^{(k)} \left[ \sum_{\tau \in [k]} \chi_i^{(\tau)*} - 1 \right] + Q_i^* - \sum_{\tau \in [k]} w_i^{(\tau)*} \\ &\geq D_i^{(k)} \left[ \sum_{\tau \in [k]} \chi_i^{(\tau)*} - 1 \right] \geq -M_i z_i^{(k)*} \end{aligned}$$

where the first inequality is due to  $Q_i^* \geq \sum_{\tau \in [k]} w_i^{(\tau)*}$  and the second inequality is due to  $z_i^{(k)*} = 1 - \sum_{\tau \in [k]} \chi_i^{(\tau)*} = 0$  if  $D_i^{(k)} > M_i$ , and  $z_i^{(k)*} = 1 - \sum_{\tau \in [k]} \chi_i^{(\tau)*} \in [0, 1]$ , otherwise. On the other hand,

$$\begin{aligned} u_i^{(k)*} + Q_i^* - D_i^{(k)} &= D_i^{(k)} \left[ \sum_{\tau \in [k]} \chi_i^{(\tau)*} - 1 \right] + Q_i^* - \sum_{\tau \in [k]} w_i^{(\tau)*} = D_i^{(k)} \left[ \sum_{\tau \in [k]} \chi_i^{(\tau)*} - 1 \right] + \sum_{\tau \in [N+1] \setminus [k]} w_i^{(\tau)*} \\ &\leq M_i \sum_{\tau \in [N+1] \setminus [k]} \chi_i^{(\tau)*} := M_i z_i^{(k)*} \end{aligned}$$

where the second equality follows because  $Q_i^* = \sum_{\tau \in [N+1]} w_i^{(\tau)*}$ , the first inequality is due to  $D_i^{(k)} \left[ \sum_{\tau \in [k]} \chi_i^{(\tau)*} - 1 \right] \leq 0$  and  $w_i^{(\tau)*} \leq \widehat{D}_i^{(\tau)} \chi_i^{(\tau)*} \leq M_i \chi_i^{(\tau)*}$  for each  $\tau \in [N+1] \setminus [k]$ . Therefore,  $(\mathbf{Q}^*, \mathbf{z}^*, \mathbf{u}^*, \mathbf{y}^*)$  satisfies constraints (14c).

Finally, we note that  $u_i^{(k)*} \geq 0$  for each  $i \in [n]$  and  $k \in [N]$ . In addition, by (15d), we have

$$u_i^{(k)*} = D_i^{(k)} \sum_{\tau \in [k]} \chi_i^{(\tau)*} - \sum_{\tau \in [k]} w_i^{(\tau)*} \leq D_i^{(k)} \sum_{\tau \in [k]} \chi_i^{(\tau)*} := D_i^{(k)} (1 - z_i^{(k)*})$$

where the inequality because  $\sum_{\tau \in [k]} w_i^{(\tau)*} \geq 0$ . Thus,  $(\mathbf{Q}^*, \mathbf{z}^*, \mathbf{u}^*, \mathbf{y}^*)$  satisfies constraints (14d).  $\square$

## A.9 Proof of Proposition 7

**Proposition 7** *Suppose that  $\mathbf{Q} \in \mathbb{R}_+^n$  is known. Then,*

(i) *the following optimization model is efficiently solvable,*

$$\max_{q \in [\underline{Q}_i, \overline{Q}_i]} \Pi(\mathbf{Q} | Q_i \leftarrow q) \quad (18)$$

for each  $i \in [n]$ ; and

(ii) an optimal solution to Model (18) belongs to set  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ , where

$$\mathcal{R}_1 = \left\{ D_i^k : D_i^k \in [\underline{Q}_i, \overline{Q}_i], \forall k \in [N+1] \right\}, \quad (19a)$$

$$\mathcal{R}_2 = \left\{ D_i^{sk} : D_i^{sk} \in [\underline{Q}_i, \overline{Q}_i], \forall k \in [N] \right\}, \quad (19b)$$

$$\mathcal{R}_3 = \left\{ D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} : D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} \in [\underline{Q}_i, D_i^k], \forall j \in [n], k \in [N] \right\}. \quad (19c)$$

*Proof:* First of all, we can simplify Model (18) to an equivalent form by eliminating all of the constant terms, i.e., the following optimization problem has the same optimal solutions as Model (18):

$$\max_{q \in [\underline{Q}_i, \overline{Q}_i]} \bar{P}_i q - \sum_{k \in [N]} m_k \bar{S}_i (q - D_i^{sk}(\mathbf{Q} | Q_i \leftarrow q))_+ - \sum_{k \in [N]} m_k \sum_{j \in [n], j \neq i} \bar{S}_j (Q_j - D_j^{sk}(\mathbf{Q} | Q_i \leftarrow q))_+,$$

which is further equivalent to

$$\max_{q \in [\underline{Q}_i, \overline{Q}_i]} \bar{P}_i q - \sum_{k \in [N]} m_k \bar{S}_i (q - D_i^{sk}(\mathbf{Q}))_+ - \sum_{k \in [N]} m_k \sum_{j \in [n], j \neq i} \bar{S}_j (Q_j - D_j^{sk}(\mathbf{Q} | Q_i \leftarrow q))_+, \quad (35)$$

since  $\alpha_{ii} = 0$  and  $D_i^{sk}(\mathbf{Q} | Q_i \leftarrow q) = D_i^{sk}(\mathbf{Q}) = D_i^k + \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+$  is a constant.

Notice that

$$D_j^{sk}(\mathbf{Q} | Q_i \leftarrow q) = D_j^k + \sum_{\tau \in [n], \tau \neq i} \alpha_{\tau j} (D_\tau^k - Q_\tau)_+ + \alpha_{ij} (D_i^k - q)_+ := D_{j,-i}^{sk}(\mathbf{Q}) + \alpha_{ij} (D_i^k - q)_+$$

where  $D_{j,-i}^{sk}(\mathbf{Q}) = D_j^k + \sum_{\tau \in [n], \tau \neq i} \alpha_{\tau j} (D_\tau^k - Q_\tau)_+$ .

From Property 2, we know that the demand of product  $i$  is sorted as

$$D_i^{(1)} \leq \dots \leq D_i^{(N)}.$$

Now let  $\widehat{D}_i^{(k)} = \max \left\{ \min \left\{ D_i^{(k)}, \overline{Q}_i \right\}, \underline{Q}_i \right\}$ . Hence, the optimal order quantity  $q^*$  of Model (35) must belong to one of the following  $N+1$  intervals:

$$\left[ \widehat{D}_i^{(0)}, \widehat{D}_i^{(1)} \right], \left[ \widehat{D}_i^{(1)}, \widehat{D}_i^{(2)} \right], \dots, \left[ \widehat{D}_i^{(N)}, \widehat{D}_i^{(N+1)} \right].$$

where  $\widehat{D}_i^{(0)} = \underline{Q}_i$ ,  $\widehat{D}_i^{(N+1)} = \overline{Q}_i$ . Let us set  $I_r = \left\{ \tau : D_i^\tau \geq \widehat{D}_i^{(r)} \right\}$  for each  $r \in [N]$ . By removing constant terms, Model (35) further reduces to

$$\max_{r \in [N+1]} \max_{q \in [\widehat{D}_i^{(r-1)}, \widehat{D}_i^{(r)}]} \bar{P}_i q - \sum_{k \in [N]} m_k \bar{S}_i (q - D_i^{sk}(\mathbf{Q}))_+ - \sum_{k \in I_r} m_k \sum_{\substack{j \in [n] \\ j \neq i}} \bar{S}_j (\alpha_{ij} q - D_{j,-i}^{sk}(\mathbf{Q}) - \alpha_{ij} D_i^k + Q_j)_+.$$

Note that in the above optimization model, the inner optimization is to maximize a piecewise linear concave function with optimal value achieved by one of its extreme points, which are included in the set of all the breaking points of the piecewise linear concave function. Therefore, one of the optimal solution to the above maximization model is contained in a set  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ , where  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$  are defined in (19). There are at most  $2N + nN$  points in set  $\mathcal{R}$ , thus, Model (18) is efficiently solvable.  $\square$

#### A.10 Proof of Theorem 5

**Theorem 5** *Let  $v^*, v^{LD}, v_R^{LD}$  denote the optimal values obtained for Models (10), (22b), and (25), respectively. Then,*

(i)  $v^* \leq v^{LD} \leq v_R^{LD}$ ; and

(ii)

$$v_R^{LD} = \max_{\pi, \beta} \sum_{i \in [n]} \bar{P}_i \beta_i - \sum_{k \in [N]} m_k \sum_{i \in [n]} \bar{S}_i \pi_i^k, \quad (26a)$$

$$s.t. \quad \beta_i - \pi_i^k = \sum_{j=1}^{n+1} \frac{1}{4} w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k), \forall i \in [n], \forall k \in [N], \quad (26b)$$

$$\pi_i^k \geq 0, \forall i \in [n], \forall k \in [N], \quad (26c)$$

$$\beta_i \geq 0, \forall i \in [n] \quad (26d)$$

$$Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N] \quad (26e)$$

$$\mathbf{Y}^k \succeq 0, \forall k \in [N] \quad (26f)$$

(i) Clearly, by the discussion above, we have  $v^* \leq v^{LD} \leq v_R^{LD}$ .

(ii) Notice that in Model (25), the inner maximization problem are separable, thus, we can swap summation and max operators as below,

$$v_R^{LD} = \inf_{\lambda \in \Omega} \max_{\mathbf{Y} \in C_R^N} \sum_{k \in [N]} m_k \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\},$$

where  $C_R^N$  denotes  $n$ -fold Cartesian product of set  $C_R$  and  $\mathbf{Y} = \{\mathbf{Y}^k\}_{k \in [N]}$ . Note that  $C_R$  is a bounded convex set and  $\Omega$  is a nonempty polyhedral set, The above function is bilinear in  $\lambda$  and  $\mathbf{Y}$ . According to the well-known Sion's minimax theorem (cf., Sion 1958), we can switch the inf and max operators, i.e., Model (25) is equivalent to

$$v_R^{LD} = \max_{\mathbf{Y} \in C_R^N} \inf_{\lambda \in \Omega} \sum_{k \in [N]} m_k \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\}.$$

By the strong duality of linear program, we can reformulate the inner infimum in the above formulation as an equivalent maximization problem with dual variables  $\beta = \{\beta_i\}_{i \in [n]}$ ,  $\pi = \{\pi_i^k\}_{i \in [n], k \in [N]}$  corresponding to the constraints in  $\Omega$ , i.e., Model (25) is equivalent to

$$\begin{aligned}
v_R^{LD} &= \max_{\pi, \beta, \mathbf{Y}} \sum_{i \in [n]} \bar{P}_i \beta_i - \sum_{k \in [N]} \sum_{i \in [n]} \bar{S}_i \pi_i^k, \\
\text{s.t. } \beta_i m_k - \pi_i^k &= \sum_{j \in [n+1]} \frac{1}{4} m_k w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k), \forall i \in [n], \forall k \in [N], \\
\pi_i^k &\geq 0, \forall i \in [n], \forall k \in [N], \\
\beta_i &\geq 0, \forall i \in [n] \\
Y_{jj}^k &= 1, \forall j \in [n+1], \forall k \in [N] \\
Y^k &\geq 0, \forall k \in [N]
\end{aligned}$$

Redefining  $\pi_i^k := \frac{\pi_i^k}{m_k}$  for each  $i \in [n], k \in [N]$ , we arrive at Model (26).  $\square$

### A.11 Proof of Theorem 6

**Theorem 6** Let  $v^*, v^{LD}, v_R^{LD}$  denote the optimal value of Models (10), (22b), and (25), respectively. Then,

(i)  $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$ ; and

(ii) if Assumption 3 holds, then

$$v_R^{LD} \leq \frac{v^{LD}}{0.79607} \leq \frac{(1 + \bar{\delta})}{0.79607(1 - \bar{\delta})} v^*$$

*Proof:*

(i) We first prove  $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$ . From (25), we have

$$\begin{aligned}
v_R^{LD} &= \inf_{\lambda \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C_R} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\} \\
&\leq \frac{1}{0.79607} \inf_{\lambda \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\} \\
&= \frac{1}{0.79607} v^{LD}
\end{aligned}$$

where the inequality follows by the result of Corollary 2.

(ii) It remains to show that  $v^{LD} \leq \frac{1+\bar{\delta}}{1-\underline{\delta}} v^*$  under Assumption 3. By (24), we have

$$\begin{aligned}
v^{LD} &= \inf_{\lambda \in \Omega} \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left( \bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\} \\
&\leq \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\} \\
&\leq (1 + \bar{\delta}) \sum_{k \in [N]} m_k \max_{\mathbf{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij}^k (1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k) \right\} \\
&= (1 + \bar{\delta}) \sum_{k \in [N]} m_k v_D^* \tag{37} \\
&= (1 + \bar{\delta}) v_D^*, \tag{38}
\end{aligned}$$

where the first inequality follows because we let  $\lambda = \mathbf{0}$ , the second inequality holds because  $D_i^k \leq (1 + \bar{\delta}) D_i$  for all  $k \in [N]$ , the second equality follows by the definition of  $v_D^*$  in (8), and the third equality is due to  $\sum_{k \in [N]} m_k = 1$ .

On the other hand, note that for any fixed  $\mathbf{Q} \in \mathbb{R}_+^n$ ,  $D_i^{sk}(\mathbf{Q}) = D_i^k + \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+$  is nondecreasing in  $\mathbf{D}^k$ . Since  $(1 - \underline{\delta}) D_i \leq D_i^k$  for all  $k \in [N]$ , we have

$$D_i^{sk}(\mathbf{Q}) \geq (1 - \underline{\delta}) D_i^s \left( \frac{\mathbf{Q}}{1 - \underline{\delta}} \right) := (1 - \underline{\delta}) \left[ D_i + \sum_{j \in [n]} \alpha_{ji} \left( D_j - \frac{Q_j}{1 - \underline{\delta}} \right)_+ \right].$$

Thus, by (10), we have

$$\begin{aligned}
v^* &= \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i (Q_i - D_i^{sk}(\mathbf{Q}))_+ \right] \right\} \\
&\geq \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i \left( Q_i - (1 - \underline{\delta}) D_i^s \left( \frac{\mathbf{Q}}{1 - \underline{\delta}} \right) \right)_+ \right] \right\} \\
&= (1 - \underline{\delta}) \max_{\mathbf{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i \frac{Q_i}{1 - \underline{\delta}} - \sum_{k \in [N]} m_k \left[ \sum_{i \in [n]} \bar{S}_i \left( \frac{Q_i}{1 - \underline{\delta}} - D_i^s \left( \frac{\mathbf{Q}}{1 - \underline{\delta}} \right) \right)_+ \right] \right\} \\
&= (1 - \underline{\delta}) v^*(\mathbf{D}) \tag{39}
\end{aligned}$$

where the first inequality follows because  $D_i^{sk}(\mathbf{Q}) \geq (1 - \underline{\delta}) D_i^s \left( \frac{\mathbf{Q}}{1 - \underline{\delta}} \right)$  for each  $k \in [N]$ , and the third equality obtained by letting  $Q_i := \frac{Q_i}{1 - \underline{\delta}}$  for each  $i \in [n]$ .

Combining (38) and (39), we have

$$v^* \geq \frac{1 - \delta}{1 + \delta} v^{LD}.$$

□