New facets for the consecutive ones polytope

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Abstract

A 0/1 matrix has the Consecutive Ones Property if a permutation of its columns exists such that the ones appear consecutively in each row. In many applications, one has to find a matrix having the consecutive ones property and optimizing a linear objective function. For this problem, literature proposes, among other approaches, an Integer Linear Programming formulation, based on Tucker characterization and on some classes of facet defining inequalities introduced by Oswald and Reinelt. We propose a method based on asteroidal triple free-graphs to derive new and more general classes of facets, and we embed them in a branch-and-cut algorithm applied to random and real instances.

keywords: Consecutive Ones Problem, Facets, Branch-and-Cut

1 Introduction

A matrix $A \in \{0,1\}^{m \times n}$ has the consecutive ones property for rows (or, briefly, A is C1) if there exists a permutation of its columns such that the ones in each row appear consecutively. Checking if A is C1 (and finding a related column permutation) can be done in $O(n+m)$ time [1]. If the property does not hold, an interesting combinatorial optimization problem arises: how can A be *minimally* transformed so that it becomes C1? The problem is known as the Consecutive Ones Problem (C1P) and can be formally stated as follows: given $A \in \{0,1\}^{m \times n}$, determine the minimum number of entries to be switched from

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$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \longleftrightarrow$$

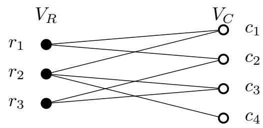

Figure 1: A 0/1 matrix and the associated bipartite graph.

0 to 1 or from 1 to 0 in order to obtain a new C1 matrix $X \in \{0,1\}^{m\times n}$. The problem is NP-Hard and, together with some specializations, has relevant applications in several fields, e.g., computational biology (physical mapping problems [2]), cutting stock, production planning or logistics (open stack problems [3]), electronic circuit design (gate matrix layout problems - GMLP [4]).

In this paper, we will focus on possible Integer Linear Programming (ILP) formulations and, in particular, on the representation of the constraint "X is C1", which is relevant in all the above mentioned applications.

A first ILP formulation can be obtained thanks to the Tucker's characterization of C1 matrices [9], based on the following definitions. Given $A \in \{0,1\}^{m\times n}$, we define the *associated bipartite graph* $G(A) = (V_R, V_C, E)$ where vertices correspond to rows (V_R) and columns (V_C), and edges to 1-entries of A (an example is shown in Figure 1). Three vertices of V_C are a *column asteroidal triple* of $G(A)$ if between any two of them there exists a path having no vertex adjacent to the third vertex (e.g., c_1, c_2 and c_3 in Figure 1).

Theorem 1 (Tucker [9]) *The following statements are equivalent:*

- *$A \in \{0,1\}^{m\times n}$ is C1;*
- *$G(A)$ contains no column asteroidal triples;*
- *$G(A)$ contains none of the graphs I_k, II_k, III_k (with integer $k \geq 1$), IV and V as induced subgraphs (see Figure 2);*
- *A and any of its row and/or column permutations contain none of the matrices $T_k^1 \in \{0,1\}^{(k+2)\times(k+2)}$, $T_k^2 \in \{0,1\}^{(k+3)\times(k+3)}$, $T_k^3 \in \{0,1\}^{(k+2)\times(k+3)}$ ($k \geq 1$), $T^4 \in \{0,1\}^{4\times 6}$ or $T^5 \in \{0,1\}^{4\times 5}$ as submatrices (see Figure 3).*

Let $\mathcal{T}_k^1$, $\mathcal{T}_k^2$, $\mathcal{T}_k^3$, $\mathcal{T}^4$ and $\mathcal{T}^5$ be obtained from, respectively, T_k^1, T_k^2, T_k^3, T^4 and T^5 by replacing all 0-entries by -1. If we consider X as a matrix of binary

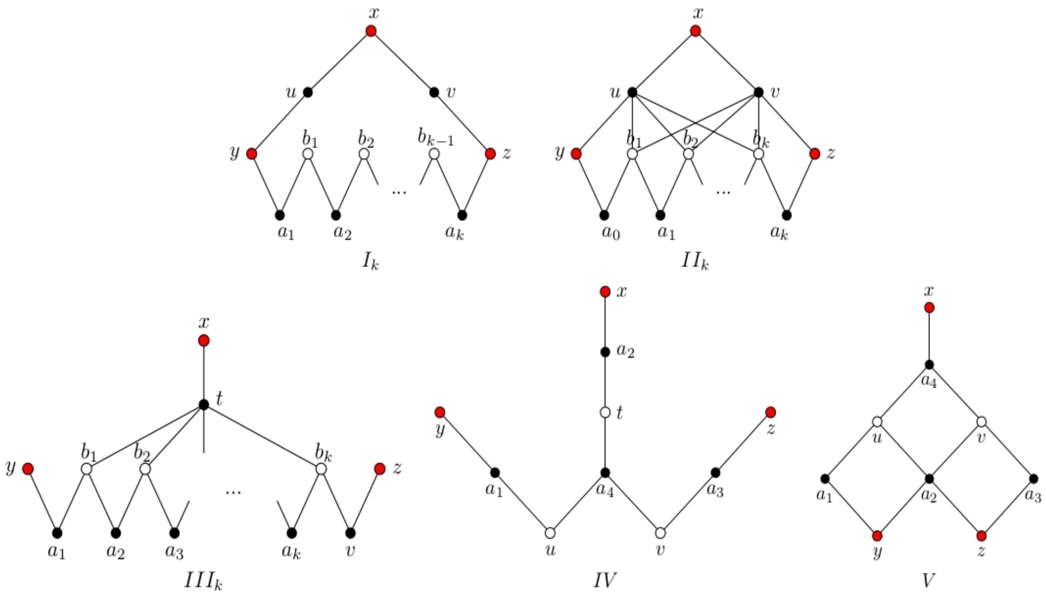

Figure 2: Tucker forbidden subgraphs ($k \geq 1$).

$$T_k^1 = \begin{pmatrix} 1 & 1 & & & \mathbf{0} \\ & 1 & 1 & & \\ \mathbf{0} & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad T_k^2 = \begin{pmatrix} 1 & 1 & & \mathbf{0} & & 0 \\ & 1 & 1 & & & 0 \\ \mathbf{0} & & \ddots & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix} \quad T_k^3 = \begin{pmatrix} 1 & 1 & & \mathbf{0} & & 0 \\ & 1 & 1 & & & 0 \\ \mathbf{0} & & \ddots & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad T^5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 3: Tucker forbidden submatrices ($k \geq 1$).

variables x_{ij}, with $i = 1 \ldots m$ and $j = 1 \ldots n$, the constraint "X is C1" can be formulated with the following linear inequalities [7]:

$$
\begin{aligned}
&\mathcal{T}_k^1 \circ X_{IJ} \leq 2k+3 && \forall\, I \in \rho^{k+2}, J \in \gamma^{k+2},\ k \geq 1 && (1)\\
&\mathcal{T}_k^2 \circ X_{IJ} \leq 4k+5 && \forall\, I \in \rho^{k+3}, J \in \gamma^{k+3},\ k \geq 1 && (2)\\
&\mathcal{T}_k^3 \circ X_{IJ} \leq 3k+2 && \forall\, I \in \rho^{k+2}, J \in \gamma^{k+3},\ k \geq 1 && (3)\\
&\mathcal{T}^4 \circ X_{IJ} \leq 8 && \forall\, I \in \rho^{4}, J \in \gamma^{6} && (4)\\
&\mathcal{T}^5 \circ X_{IJ} \leq 10 && \forall\, I \in \rho^{4}, J \in \gamma^{5} && (5)\\
&x_{ij} \in \{0,1\} && \forall\, i = 1 \ldots m,\ j = 1 \ldots n && (6)
\end{aligned}
$$

where, given two $p \times q$ matrices B and C, $B \circ C = \sum_{i=1}^{p} \sum_{j=1}^{q} b_{ij}\, c_{ij}$, X_{IJ} is the submatrix of X defined by the ordered subsets I and J of its rows and columns, and ρ^t (resp. γ^t) is the set of all permutations of all subsets of t rows (resp. columns). The formulation is valid, since if X or any of its row/column permutation contains a Tucker submatrix, then (at least) one inequality is violated.

Let $x = (x_{11} \ldots x_{1n} \ldots x_{m1} \ldots x_{mn})$ be a vector associated with a matrix $X \in \{0,1\}^{m \times n}$. We define the Consecutive Ones Polytope as follows [7]:

$$P_{C1}^{m,n} = \text{conv}\{x \in \{0,1\}^{m \cdot n} | X \in \{0,1\}^{m \times n} \text{ is C1}\}.$$

Oswald and Reinelt [7] show that inequalities (1-5) are not tight for $P_{C1}^{m,n}$ and propose the following alternative formulation:

$$
\begin{aligned}
&\mathcal{F}_k^1 \circ X_{IJ} \leq 2k+3 && \forall\, I \in \rho^{k+2}, J \in \gamma^{k+2},\ k \geq 1 && (7)\\
&\mathcal{F}_k^2 \circ X_{IJ} \leq 2k+3 && \forall\, I \in \rho^{k+2}, J \in \gamma^{k+3},\ k \geq 1 && (8)\\
&\mathcal{F}^3 \circ X_{IJ} \leq 8 && \forall\, I \in \rho^{4}, J \in \gamma^{6} && (9)\\
&\mathcal{F}^4 \circ X_{IJ} \leq 8 && \forall\, I \in \rho^{4}, J \in \gamma^{5} && (10)\\
&x_{ij} \in \{0,1\} && \forall\, i = 1 \ldots m,\ j = 1 \ldots n && (11)
\end{aligned}
$$

where $\mathcal{F}_k^1$, $\mathcal{F}_k^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$ are the, respectively, $(k+2)\times(k+2)$, $(k+2)\times(k+3)$, 4×6 and 4×5 matrices reported in Figure 4, with entries in $\{-1, 0, 1\}$ (we write "$-$" and "$+$" instead of, respectively, -1 and $+1$). Notice that the coordinates of the -1 in the last row and the first -1 in the first column of $\mathcal{F}_k^1$ are, respectively $(k+2, s)$ and $(s, k+2)$, with $2 \leq s \leq k+1$. Moreover, $\mathcal{F}^4$ is taken from [4], since the (1,4)-entry reported in [7, 8] is "+", yielding non valid inequalities.

Theorem 2 (Oswald and Reinelt [7]) $P_{C1}^{m,n}$ *has the following properties:*

$$\mathcal{F}_k^1 = \begin{pmatrix} + & + & & & & & & & - \\ 0 & + & + & & & & & & - \\ \vdots & & \ddots & \ddots & & & \mathbf{0} & & \vdots \\ 0 & & & + & + & & & & - \\ - & & & & + & + & & & 0 \\ \vdots & \mathbf{0} & & & & \ddots & \ddots & & \vdots \\ - & & & & & & + & + & 0 \\ - & & & & & & & + & + \\ + & 0 & \cdots & 0 & - & 0 & \cdots & 0 & + \end{pmatrix} \qquad \mathcal{F}_k^2 = \begin{pmatrix} + & + & & \mathbf{0} & & - & - \\ & + & + & & & - & - \\ \mathbf{0} & & \ddots & \ddots & & \vdots & \vdots \\ & & & + & + & - & - \\ - & 0 & \cdots & & + & + & \\ + & 0 & \cdots & 0 & - & - & + \end{pmatrix}$$

$$\mathcal{F}^3 = \begin{pmatrix} + & + & - & 0 & - & 0 \\ - & 0 & + & + & - & 0 \\ - & 0 & - & 0 & + & + \\ - & + & - & + & - & + \end{pmatrix} \qquad \mathcal{F}^4 = \begin{pmatrix} + & + & 0 & - & - \\ + & 0 & 0 & + & - \\ - & 0 & + & + & - \\ + & - & - & + & + \end{pmatrix}$$

Figure 4: Matrices $\mathcal{F}_k^1$, $\mathcal{F}_k^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$ ($k \geq 1$).

- $P_{C1}^{m,n}$ *is full-dimensional;*
- *if an inequality defines a facet of* $P_{C1}^{m,n}$*, then it also defines a facet of* $P_{C1}^{m',n'}$*, for any* $m' \geq m$ *and* $n' \geq n$*;*
- *inequalities* $0 \leq x_{ij} \leq 1$ *define facets of* $P_{C1}^{m,n}$*;*
- *inequalities (7-10) define facets of* $P_{C1}^{m,n}$*.*

Formulation (7-10) is valid for $P_{C1}^{m,n}$, since constraints (7) forbid Tucker submatrices T_k^1 and T_k^2, and (8), (9) and (10) forbid T_k^3, T^4 and T^5, respectively. The formulation includes an exponential number of facet defining inequalities: a separation procedure and a branch-and-cut approach is proposed in [8], and tested on random instances of C1P. In [7], it is also noticed that the four classes of facets does not provide a complete description of $P_{C1}^{m,n}$ (unless m and n are very small).

The last result motivates this paper, whose scope is a method to derive new facet-defining inequalities for $P_{C1}^{m,n}$, in order to strengthen the formulation of C1P. The method we propose in Section 2 moves from the relation between C1 matrices and column asteroidal triple-free bipartite graphs, as stated by Theorem 1. In fact, in order an inequality to be valid for $P_{C1}^{m,n}$, it has to be violated only by matrices whose associated bipartite graph contains at least one column asteroidal triple: for instance, inequalities (1-5) are valid because each of them is violated only by matrices containing the related Tucker submatrix (or any permutation), which is associated to a Tucker subgraph containing the asteroidal triple x, y and z (see Figure 2). We thus start from a bipartite graph including an asteroidal triple and derive a class of inequalities that are violated only by matrices whose associated bipartite graph contains the same triple, which attests for validity. Starting from Tucker subgraphs and their generalizations, we obtain new classes of inequalities which we prove to be facets. They generalize (7-9) and provide alternative inequalities excluding T^5, thus further strengthening the formulation for $P_{C1}^{m,n}$. Section 3 is devoted to some preliminary computational experiment. First, it discusses separation procedures for the new facets. They are embedded in preliminary implementations of branch-and-cut and row generation procedures. Tests on random C1P instances and real GMLP instances show the improved efficiency of the new formulation. Section 4 provides some final remarks and possible line of research towards the definition of further facets of $P_{C1}^{m,n}$ based on the equivalence between C1 matrices and column asteroidal triple-free graphs.

2 New classes of facet defining inequalities

Let $G = (V_R, V_C, E)$ be a bipartite graph containing a column asteroidal triple (x, y, z). Let P_x (resp. N_x) be the set of edges (resp. vertices) in the path between y and z with no vertex adjacent to x. Define P_y, N_y, P_z and N_z in a similar way and assume $E = P_x \cup P_y \cup P_z$. Let $\mathcal{G}$ be a $(|V_R| \times |V_C|)$-matrix where row i is associated with vertex $r_i \in V_C$ and column j is associated with vertex $c_j \in V_C$. Let $\mathcal{G}(i, j)$ denote the (i, j)-entry of $\mathcal{G}$ and set

- $\mathcal{G}(i, j) = 1$, if edge $\{r_i, c_j\} \in E$;
- $\mathcal{G}(i, j) = -1$, if $c_j \in \{x, y, z\}$ and $r_i \in V_R \cap N_{c_j}$;
- $\mathcal{G}(i, j) = 0$, otherwise.

Theorem 3 *The following inequalities are valid for $P_{C1}^{m,n}$, $m \geq |V_R|$, $n \geq |V_C|$:*

$$\mathcal{G} \circ X_{IJ} \leq |E| - 1 \qquad \forall\, I \in \rho^{|V_R|}, J \in \gamma^{|V_C|}. \tag{12}$$

Proof Let $\bar{X} \in \{0, 1\}^{m \times n}$ violate any of inequalities (12) and let I and J be the corresponding row and column subsets. For the matrix $A = \bar{X}_{IJ}$ we have, necessarily, (i) $a_{ij} = 1$ if $\mathcal{G}(i, j) = 1$, and (ii) $a_{ij} = 0$ if $\mathcal{G}(i, j) = -1$. The remaining entries may be either 0 or 1. Let $G(A) = (V_R, V_C, E')$ be the bipartite graph associated to A. By (i), $E \subseteq E'$, meaning that paths P_x, P_y and P_z exist in $G(A)$. Further, by (ii), none of the edges that would make

$$\mathcal{G}_V^1 \begin{pmatrix} + & + & - & 0 & 0 \\ 0 & + & + & 0 & - \\ 0 & - & + & + & 0 \\ + & - & - & + & + \end{pmatrix} \quad \mathcal{G}_V^2 \begin{pmatrix} + & + & - & 0 & - \\ + & 0 & + & 0 & - \\ 0 & - & + & + & 0 \\ + & - & - & + & + \end{pmatrix} \quad \mathcal{G}_V^3 \begin{pmatrix} + & + & - & 0 & 0 \\ 0 & + & 0 & + & - \\ 0 & - & + & + & - \\ + & - & - & + & + \end{pmatrix} \quad \mathcal{G}_V^4 \begin{pmatrix} + & + & - & 0 & - \\ + & 0 & 0 & + & - \\ 0 & - & + & + & - \\ + & - & - & + & + \end{pmatrix}$$

Figure 5: Matrices $\mathcal{G}_V^1$, $\mathcal{G}_V^2$, $\mathcal{G}_V^3$ and $\mathcal{G}_V^4$.

path P_c adjacent to c ($c \in \{x, y, z\}$) belongs to E'. Hence, (x, y, z) is still a column asteroidal triple of $G(A)$. By Theorem 1, A is not C1. Summarizing, if an inequality of type (12) is violated, the correponding matrix is not C1, that is, the inequality is valid. □

Theorem 3 allows us to obtain one class of valid inequalities for each graph containing a column asteroidal triple, each of these triples, and each set of paths showing that the triple is asteroidal. For example, let us consider the Tucker subgraph V containing the asteroidal triple (x, y, z). We set
$P_y = \{\{x, a_4\}, \{a_4, v\}, \{v, a_3\}, \{a_3, z\}\}$,
$P_z = \{\{x, a_4\}, \{a_4, u\}, \{v, a_1\}, \{a_1, y\}\}$, and
$P_x = \{\{y, a_2\}, \{a_2, z\}\}$ (choice 1), or $P_x = \{\{y, a_1\}, \{a_1, u\}, \{u, a_2\}, \{a_2, z\}\}$ (choice 2), or $P_x = \{\{y, a_2\}, \{a_2, v\}, \{v, a_3\}, \{a_3, z\}\}$ (choice 3), or
$P_x = \{\{y, a_1\}, \{a_1, u\}, \{u, a_2\}, \{a_2, v\}, \{v, a_3\}, \{a_3, z\}\}$ (choice 4). We thus obtain the four matrices $\mathcal{G}_V^1$, $\mathcal{G}_V^2$, $\mathcal{G}_V^3$ and $\mathcal{G}_V^4$ shown in Figure 5 (the rows correspond to vertices a_1, a_2, a_3 and a_4, the columns to vertices u, y, z, v and x, in the order they appear).

Theorem 4 *Inequalities* $\mathcal{G}_V^t \circ X_{IJ} \leq 8, I \in \rho^4, J \in \gamma^5$, *where* $t = 1 \ldots 4$, *define facets of* $P_{C1}^{m,n}$, $m \geq 4$, $n \geq 5$.

Proof The inequalities are valid by Theorem 3. Using PORTA we verify that they are facet defining for $P_{C1}^{4,5}$. The general result follows from Theorem 2. □

We now apply Theorem 3 to the graphs shown with solid lines in Figures 6 and 7, which can be seen as a generalization of subgraphs I_k, III_k, IV and V. Different graphs can be obtained for different integer values $s, d, l, u \geq 1$ ($l \geq 0$ for V_{usdl}). Each graph contains the column asteroidal triple (x, y, z) and the definition of sets P_x, P_y and P_z is unique (taking into account that l may be 0 in V_{usdl}). For example, P_x in I_{sdl} is defined by the path $(y, a_1, b_1, \ldots, a_{s-1}, b_{s-1}, a_s, z)$. We thus build the matrices $\mathcal{G}_{s,d,l}^1$, $\mathcal{G}_{s,d,l}^2$, $\mathcal{G}_{s,d,l}^3$ and $\mathcal{G}_{u,s,d,l}^4$ according to graphs I_{sdl}, III_{sdl}, IV_{sdl} and V_{usdl}, respectively. In particular, we will set to +1 the entries corresponding to solid lines in Figures 6 and 7, to −1 the entries corresponding to dashed lines (edges that would make a path adjacent to the related vertex), and to 0 the remaining entries.

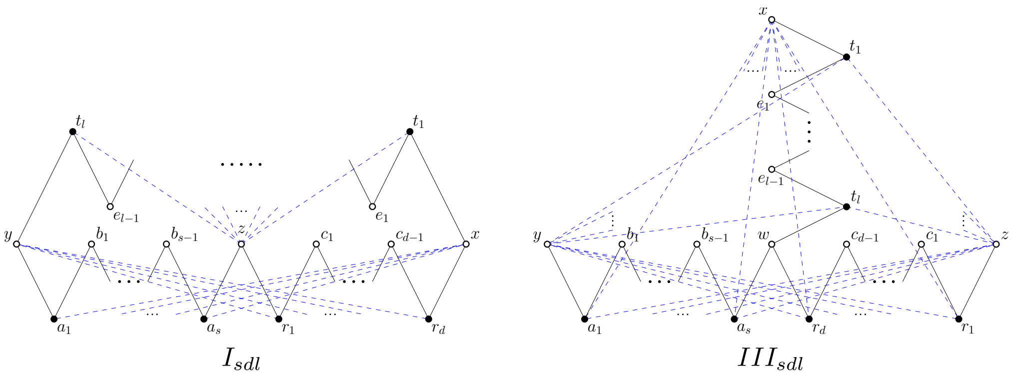

Figure 6: General graphs with a column asteroidal triple $(s, d, l, u \geq 1)$.

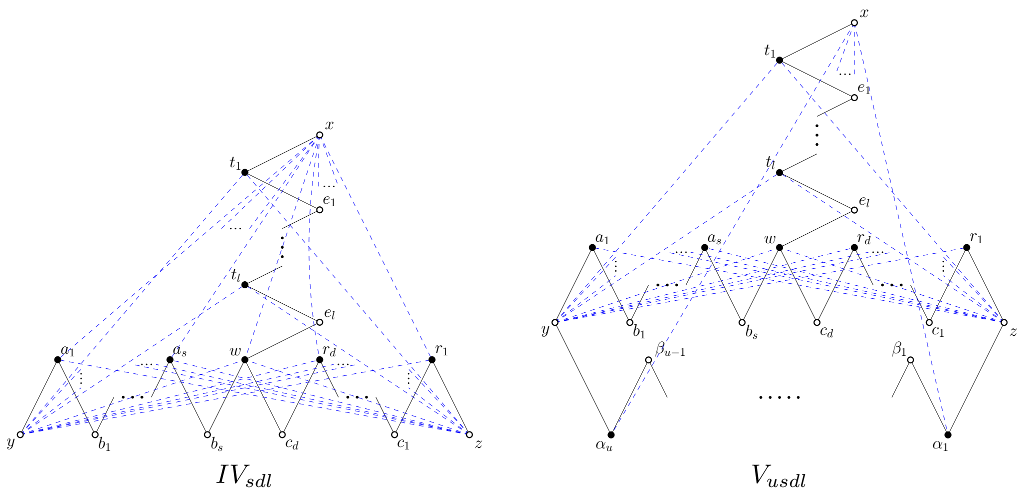

Figure 7: General graphs with a column asteroidal triple $(s, d, l, u \geq 1)$.

We define the following classes of inequalities (where $\tau = s + d + l$):

$$\mathcal{G}^1_{s,d,l} \circ X_{IJ} \leq 2\tau - 1 \qquad \forall\, I \in \rho^{\tau}, J \in \gamma^{\tau},\ s, d, l \geq 1 \tag{13}$$
$$\mathcal{G}^2_{s,d,l} \circ X_{IJ} \leq 2\tau - 1 \qquad \forall\, I \in \rho^{\tau}, J \in \gamma^{\tau+1},\ s, d, l \geq 1 \tag{14}$$
$$\mathcal{G}^3_{s,d,l} \circ X_{IJ} \leq 2\tau + 2 \qquad \forall\, I \in \rho^{\tau+1}, J \in \gamma^{k+3},\ s, d, l \geq 1 \tag{15}$$
$$\mathcal{G}^4_{u,s,d,l} \circ X_{IJ} \leq 2(u + \tau) + 2 \quad \forall\, I \in \rho^{u+\tau+1}, J \in \gamma^{u+\tau+2},\ u, s, d \geq 1, l \geq 0 \tag{16}$$

By adapting the strategy used in [7] to prove that (7-8) are facet defining, we have the following result.

Theorem 5 *Inequalities (13) define facets of* $P^{m,n}_{C1}$, $m, n \geq s + d + l$.

$$
\mathcal{G}^1_{3,3,3} = \begin{pmatrix}
+ & + & 0 & 0 & 0 & 0 & - & 0 & 0 \\
0 & + & + & 0 & 0 & 0 & - & 0 & 0 \\
0 & 0 & + & + & 0 & 0 & - & 0 & 0 \\
- & 0 & 0 & + & + & 0 & 0 & 0 & 0 \\
- & 0 & 0 & 0 & + & + & 0 & 0 & 0 \\
- & 0 & 0 & 0 & 0 & + & + & 0 & 0 \\
0 & 0 & 0 & - & 0 & 0 & + & + & 0 \\
0 & 0 & 0 & - & 0 & 0 & 0 & + & + \\
+ & 0 & 0 & - & 0 & 0 & 0 & 0 & +
\end{pmatrix}
\qquad
\bar{X} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \mathbf{1} & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\qquad
\hat{X} = \begin{pmatrix}
\mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Figure 8: The matrix $\mathcal{G}^1_{3,3,3}$ and two sample solutions.

Proof Inequalities are valid, by Theorem 3. By Theorem 1, we just need to show the assert for minimal $m = n = s + d + l$ and, moreover, we only need to consider one possible row/column order to determine I and J, for example the same used to build $\mathcal{G}^1_{s,d,l}$, which we can fix to the one obtained starting from y and following the cycle in Figure 6 up to t_l: let $a^T x \leq a_0$ denote this inequality, and let $F = \{x \mid a^T x = a_0\}$ be the induced face. By Theorem 1, $P^{m,n}_{C1}$ is full dimensional, hence inequalities inducing the same facet are the same up to multiplication by a positive scalar. Let $b^T x \leq b_0$ the inequality inducing a facet $\bar{F}$ that contains F. We will prove that $\exists\ \lambda > 0 : b = \lambda a$ and, hence, that F is a facet. The strategy is to show that (i) all the coefficients of b corresponding to $+1$ in a are equal to a number β, (ii) all the coefficients of b corresponding to -1 in a are equal to $-\beta$, and (iii) remaining coefficients of b are 0.

For each (p, q)-entry of $\mathcal{G}^1_{s,d,l}$ equal to $+1$, let the vector $x^{p\,q}$ denote a solution having $x^{p\,q}_{ij} = 1$ if the (i, j)-entry of $\mathcal{G}^1_{s,d,l}$ is $+1$ and $(i, j) \neq (p, q)$, and $x^{p\,q}_{ij} = 0$ otherwise. Clearly, $a^T x^{p\,q} = 2(s + d + l) = a_0$, and the matrix $X^{p\,q}$ associated to $x^{p\,q}$ is C1 (the associated bipartite graph is a path, which does not contain asteroidal triples). Thus, $x^{p\,q} \in F$ and, since $F \subseteq \bar{F}$, we have $b^T x^{p\,q} = b_0$. For two different $+1$-entries (p, q) and (r, t) of $\mathcal{G}^1_{s,d,l}$, we have $0 = b_0 - b_0 = b^T x^{p\,q} - b^T x^{r\,t} = b_{rt} - b_{pq}$, that is $b_{rt} = b_{pq} = \beta$.

Let us now consider "0"-entries. According to the chosen row and column order, we have, e.g. for $s = d = l = 3$, the matrix $\mathcal{G}^1_{3,3,3}$ reported in Figure 8.

Let a solution $\bar{x}$ be obtained from $x^{s+d+l\,1}$ by switching to 1 the 0 entries related to the rows of r_d and t_1 and the columns of $e_1 \dots e_{l-1}$ (e.g., matrix $\bar{X}$ of Figure 8). Clearly, $\bar{x} \in F$ (in particular, the matrix $\bar{X}$ associated to $\bar{x}$ is C1). Let $\bar{x}'$ be obtained from $\bar{x}$ by switching to 1 a 0-entry in position (i, j) next to a 1-entry and such that i corresponds to a row vertex in $\{a_1 \dots a_s\}$. We have that $\bar{x}' \in F$ and, hence, $0 = b_0 - b_0 = b^T \bar{x} - b^T \bar{x}' = -b_{ij}$, that is $b_{ij} = 0$. If we move the column $s+d+1$ of $\bar{x}$ (where coefficients -1 appear in rows $a_1 \dots a_s$) in the last position, we can recourse the same arguments with a further 0-entry and show that, for the rows $a_1 \dots a_s$, all coefficients of b corresponding to 0 in a are 0. Similar arguments show that this true even for the remaining rows.

Finally, let us consider "-1"-entries, one for each row of $\mathcal{G}^1_{s,d,l}$. Let $\hat{x}$ be obtained from x^{11} by setting to 1 all the entries in the first row (e.g., matrix $\hat{X}$ in Figure 8). We have $x^{11}, \hat{x} \in F$ (to see that they are C1 we just need to move the first column in the last position). Taking into account that $b_{ij} = 0$ whenever $a_{ij} = 0$, we have that $0 = b_0 - b_0 = b^T x^{11} - b^T \hat{x} = -b_{1\,1} - b_{1\,s+d+1}$, that is $b_{1\,s+d+1} = -b_{1\,1} = -\beta$. The same argument applied to every rows shows that $b_{ij} = -\beta$ whenever $a_{ij} = -1$.

Summarizing, $b = \beta a$. The scalar β cannot be 0 (otherwise $b^T x \leq b_0$ would not define a facet) and cannot be negative. In fact, since $P^{m,n}_{C1}$ is full dimensional, there exists $\tilde{x} \in P^{m,n}_{C1} : a\tilde{x} < a_0$ and $\tilde{x}$ would violate the inequality if $\beta < 0$. □

With similar arguments (see [5] for details) we have the following result.

Theorem 6 *Inequalities (14) define facets of $P^{m,n}_{C1}$, $m \geq s+d+l$ and $n \geq s+d+l+1$; inequalities (15) define facets of $P^{m,n}_{C1}$, $m \geq s+d+l+1$ and $n \geq s+d+l+3$; inequalities (16) define facets of $P^{m,n}_{C1}$, $m \geq u+s+d+l+1$ and $n \geq u+s+d+l+2$.*

It is easy to check that the new class of facets (13) forbids Tucker submatrices T^1_k and T^2_k, (14) forbids T^3_k, (15) forbids T^4, and (16) forbids T^5. Hence, (13-16) yields a new valid formulation for $P^{m,n}_{C1}$ made of facet defining inequalities. Furthermore, the classes of inequalities (13), (14) and (15) generalize (7), (8) and (9), respectively, since $\mathcal{F}^1_k = \mathcal{G}^1_{s,d,1}$ with $k = s+d-1$, $\mathcal{F}^2_k = \mathcal{G}^2_{k,1,1}$, and $\mathcal{F}^3 = \mathcal{G}^3_{1,1,1}$. Concerning $\mathcal{F}^4$, it is not a special case of any of the proposed new classes of inequalities: in fact, if we consider the bipartite graphs associated to matrices that violate (10), there is no column asteroidal triple contained in all of them, which would be necessary to define matrices $\mathcal{G}$. Notice also that $\mathcal{G}^1_V = \mathcal{G}^4_{1,1,1,0}$, meaning that $\mathcal{G}^4_{u,s,d,l}$ generalizes $\mathcal{G}^1_V$. From the discussion above, we have the following result.

Corollary 7 *The valid formulation for $P^{m,n}_{C1}$ obtained from (13-15) and (10) is stronger than formulation (7-10), and it can be further strengthened by (16) and the inequalities introduced by Theorem 4 for $t = 2 \dots 4$.*

3 Computational Experiments

We propose a branch-and-cut algorithm for solving problems involving C1 matrices, in particular C1P and GMLP, formulated as follows (see [7] and [4]):

$$\text{(C1P)} \qquad \min \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - 2\, a_{ij})\, x_{ij} \tag{17}$$

$$\text{s.t. } X \text{ is C1} \tag{18}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\, i = 1 \ldots m,\ j = 1 \ldots n \tag{19}$$

$$\text{(GMLP)} \qquad \min \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \tag{20}$$

$$\text{s.t. } X \text{ is C1} \tag{21}$$

$$x_{ij} \geq a_{ij} \qquad \forall\, i = 1 \ldots m,\ j = 1 \ldots n \tag{22}$$

$$\sum_{i=1}^{m} x_{ij} \leq \lambda \qquad \forall\, j = 1 \ldots n \tag{23}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\, i = 1 \ldots m,\ j = 1 \ldots n \tag{24}$$

where A is an input matrix, λ a threshold value (related to the area covered by the electronic circuit). We formulate the constraint "X is C1" by inequalities (13-16). The huge number of constraints imposes a cutting plane approach to tackle the linear relaxation, and requires the separation of both integer and fractional solutions. Given a solution $\bar{X}$ to be separated, the task is to determine, for each class of facets, the sequences I and J of rows and columns of $\bar{X}$ that define a (maximally) violated inequality. Following the idea in [7, 8], this can be done by computing paths on suitably weighted complete bipartite graphs with row and columns as vertex sets. Let us consider, for example, (13). For each column node h, we associate to each edge $\{r, c\}$ a weight $w^h_{rc} = 1 - \bar{x}_{rc} + 0.5\, \bar{x}_{rh}$ and we compute the shortest path for each column pair (i, j). We thus obtain the length Π^h_{ij} of the shortest path between column vertices i and j "according to" h. We then consider all possible column triples x, y and z: if $\Pi^x_{yz} + \Pi^y_{zx} + \Pi^z_{xy} < 1$, then we can build the sequences I and J from the shortest paths; if no such a triple is found, then no violated (13) exists. The overall complexity of the procedure is $O(n(n + m)^3)$. Similar procedures separate (14), (15) and (16), using for edge $\{r, c\}$ the weight $w^{gh}_{rc} = 1 - \bar{x}_{rc} + 0.5\, \bar{x}_{rg} + 0.5\, \bar{x}_{rh}$, for each column pair (g, h). We refer to [5] for further details. Here, we just observe that the same node may belong to more than one shortest path. This means that the above procedures (and the ones in [7, 8]) may return a violated inequality with column/row repetitions. Since a matrix A is C1 if and only if the matrix A' obtained by row/columns repetitions is C1, the inequality is still valid. However, it is not facet defining, meaning that the described procedures, as well as the ones proposed in [7, 8], may return non-facet defining inequalities. For this reason, we define a new exact separation procedure based on a multi-commodity flow (MCF) ILP model that simultaneously finds the best combination of paths, under the constraint that each vertex belongs to at most one path.

The proposed separation routines have been embedded in a branch-and-cut algorithm, implemented in C, using Cplex 12.2 Callable Libraries. In particular, the separation, triggered by the Cplex callback mechanism, works as follows: first, the path-based separation is called and, in case it finds a path with vertex repetitions, the corresponding inequality is discarded (it is not facet defining) and we resort to the MCF separator to obtain facet defining cutting planes. The Cplex default branching strategy has been used. At most 10 inequalities per class and per round are added. In case the current solution is integer, we use the PQ-Tree algorithm to preliminary verify if the solutions itself is C1 before applying separation. We remark that both the path-based and the MCF-based separation procedures are able to separate fractional solutions, i.e., it is not necessary to round the solution of the current linear relaxation to a 0-1 vector. Tests have been performed on a 1.0 GHz Intel Dual Core CPU with 4 GB RAM.

We compare our method to the branch-and-cut algorithm in [4], where the constraint "X is C1" is represented by (7-10). The separation procedure is similar to the one proposed in [8] and works on (in case rounded) integer solutions: it uses the PQ-Tree algorithm to find a non-C1 minimal submatrix, and adds related inequality. Notice that the procedure used in [8] is heuristic if the solution to cut-off is fractional, exact otherwise. An advanced branching rule is used, based on betweenness considerations. The algorithm has been implemented in C++ using the SCIP 2.1 framework and Cplex 12.4 as LP solver, and run on a 1.6 GHz Intel Centrino CPU with 4 GB RAM.

Preliminary results referred to a set of real GMLP instances [6] are presented in Table 3. The first three columns give instance name and size, then results of our procedure (running time in seconds, number of branch-and-bound nodes, number of cuts), and the one presented in [4] (running time and number of cuts) are shown. For both algorithms, Cplex preprocessing and cuts have been disabled. Notwithstanding its straightforward implementation, the algorithm based on the new classes of facet defining inequalities seems to be

Table 1: Computational results on real GMLP instances

			This work			Branch-and-cut [4]	
inst.	m	n	sec.$^\flat$	Nodes	Cuts	sec.$^\natural$	cuts
Wli	11	10	**0.1**	1	**113**	5	171
W1	18	21	71.6	2	1 491	**4**	**694**
v4000	10	17	**27.1**	334	**3 834**	42	23 716
v4050	13	16	**13.3**	10	1 192	23	**793**
v4090	23	27	–	1	**6 943**	–	318 964

– 1 hour time limit

very competitive, since running time is always smaller, but in one instance.

A second set of experiments concerns the C1P. In this case we apply a different solution approach, based on row generation. Again, we consider inequalities (13-16) to formulate the C1 constraint and an iterative approach is implemented. At each iteration an ILP model containing a subset of (13-16) inequalities (starting from the empty set) is solved to integrality. Then we use the PQ-tree algorithm to test if the solution corresponds to a C1 matrix, in this case the algorithm stops, otherwise the path-based separation is used, resorting to the MCF model if necessary, and the procedure iterates. At most 100 cuts per class and per round are added. The algorithm has been implemented in C using Cplex 12.2 as ILP solver, and run on a 1.0 Ghz Intel Dual Core CPU with 4 GB RAM.

We compare our results to the branch-and-cut in [8]. It uses the formulation (7-10) and both the separation procedure starting from rounded fractional solutions, and path-based ones. A primal heuristic provides upper bounds by rounding a fractional solution and using the PQ-Trees to repair it if necessary. The algorithm has been implemented using ABACUS framework and Cplex 6.5.3 as LP solver, and run on a Sun Ultra 10 with UltraSPARC IIi 440 MHz CPU.

A benchmark of random C1P instances has been taken into account, made of $(n \times n)$ input matrices with $n = 9, 10, 11, 12$. For each n and each density $d = 0.2, 0.3 \dots 0.9$, 10 random instances are created. All the instances have been solved to optimality by both methods, and a comparison of the average running time (in seconds) is presented in Figure 9: solid lines ("1") refer to our row generation approach, dashed lines ("2") to the procedure in [8]. Even if the tests has been performed on different architecture, we can reasonably conclude that the simple row generation approach is more efficient than the branch-and-cut algorithm.

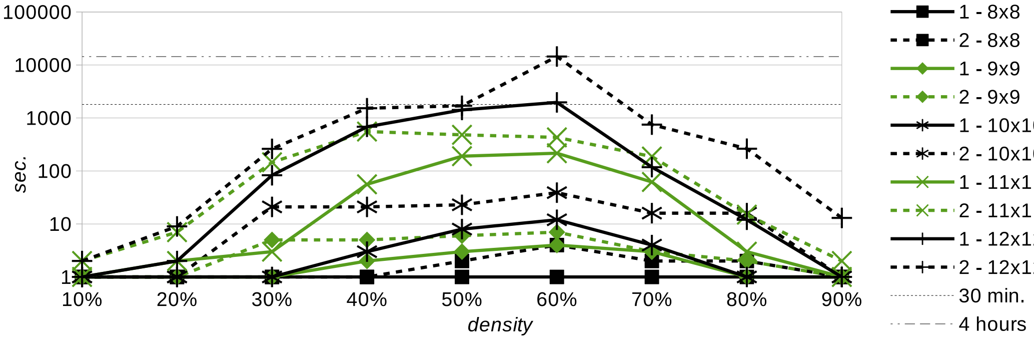

Figure 9: Computational results on random C1P instances

4 Conclusions and future work

The paper has mainly focused on a new method to obtain strong valid inequalities for the Consecutive ones polytope, exploiting relations between C1 matrices and column asteroidal triple-free graphs. The method is very general, and its application to Tucker graphs yields facet defining inequalities to further strengthen the formulations of C1P and other problems related to C1 matrices. The proposed inequalities seems also effective in practice, since preliminary implementation of cutting plane algorithms often outperforms previous more sophisticated branch-and-cut approaches, and there is still room for improvement.

From a theoretical point of view, the proposed method, explicitly based on forbidden subgraphs, offers the opportunity of deriving further valid (and possibly facet defining) inequalities (for example by generalizing $\mathcal{G}_V^4$). Moreover, conditions on the generating graph in order to guarantee facet defining inequalities, as well as more general methods, e.g. allowing multiple column asteroidal triple, are the object of current research.

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