On the Relation Between MPECs and Optimization Problems in Abs-Normal Form

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Abstract

We show that the problem of unconstrained minimization of a function in abs-normal form is equivalent to identifying a certain stationary point of a counterpart Mathematical Program with Equilibrium Constraints (MPEC). Hence, concepts introduced for the abs-normal forms turn out to be closely related to established concepts in the theory of MPECs. We give a number of proofs of equivalence or implication for the kink qualifications LIKQ and MFKQ. We also show that the counterpart MPEC always satisfies MPEC-ACQ. We then consider non-smooth nonlinear optimization problems (NLPs) where both the objective function and the constraints are presented in abs-normal form. We show that this extended problem class also has a counterpart MPEC problem.

Keywords: Non-smooth optimization, abs-normal form, MPECs, constraint qualifications, stationarity conditions

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1 Introduction

Non-smooth finite-dimensional optimization models arise in many application problems from engineering, economics, and other areas. One typical source of non-smoothness are equilibrium conditions or complementarity conditions. Another typical source are models with piecewise definitions and models involving the absolute value, maximum, and minimum functions. Many of the practical applications essentially lead to NLPs with finitely many kinks, which gives rise to more general standard problem classes like MPECs [8] and (unconstrained) optimization problems in abs-normal form [4]. In this paper we provide a systematic comparison of the two problem classes in terms of constraint qualifications and stationarity concepts. It turns out that the two classes are intimately related and that the extended class of *constrained* optimization problems in abs-normal form is in fact equivalent to the class of MPECs.

Notation. We write $\bar{n} := \{1, \dots, n\}$ for some $n \in \mathbb{N}$, denote by ∂_i the partial derivative w.r.t. the i-th argument, and by $\partial_{i,j}$ the partial derivative w.r.t. the j-th component of the i-th argument vector. For a matrix or vector A, the bracket $[A]_{i\in\mathcal{S}}$ is the submatrix or subvector composed from rows with indices in the set $\mathcal{S}$.

MPEC variables will be denoted y (smooth), u and v (complementarity) and the symbol $\perp$ indicates complementarity between two vectors of unknowns. Lagrange multipliers are denoted by Greek lowercase letters.

Abs-normal form variables will be denoted x (smooth) and z (non-differentiable). The bracket $[x]^+ := \max(x, 0)$ denotes the nonnegative part, and $[x]^- := \max(-x, 0)$ denotes the modulus of the non-positive part. For signatures $\sigma \in \{-1, 0, 1\}$, we use the partial order $\sigma \succeq \hat{\sigma} :\Leftrightarrow \sigma\hat{\sigma} \geq \hat{\sigma}^2$, i.e., σ is arbitrary if $\hat{\sigma} = 0$, and $\sigma = \hat{\sigma}$ otherwise.

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Definition 1 (Abs-Normal Form)**.** For an open set $D \subset \mathbb{R}^n$ we say that a function $\phi \in \mathcal{C}^0(\bar{D}, \mathbb{R})$ is in *abs-normal form* if functions $f \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^s_{\geq 0}, \mathbb{R})$ and $F \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^s_{\geq 0}, \mathbb{R}^s)$ exist such that

$$\phi(x) = f(x, |z|) \quad \text{for all } x \in \bar{D}, \tag{1a}$$

$$z = F(x, |z|) \quad \text{with } L := \partial_2 F(x, |z|) \text{ strictly lower triangular.} \tag{1b}$$

Note that (1b) defines $z(x)$ implicitly while in fact the strict lower triangular form of the Jacobian L allows to compute the components of z one by one from previously computed ones, $z_i = F_i(x, |z_1|, \dots, |z_{i-1}|)$ for $i \in \bar{s}$. The abs-normal form is an ingenious way of exposing structured non-smoothness. It allows for comparably easy generation of algorithmic derivatives and subdifferentials as has been shown in [4, 5]. In particular, one defines

$$a := \partial_1 f(x, |z|), \quad b := \partial_2 f(x, |z|), \quad Z := \partial_1 F(x, |z|), \quad L := \partial_2 F(x, |z|), \tag{2}$$

and introduces the following notation.

Definition 2 (Signature of z)**.** With $\sigma(z_i) := \operatorname{sign}(z_i) \in \{-1, 0, 1\}$ we define the signature vector $\sigma(z) := (\sigma(z_1), \dots, \sigma(z_s))^T$ and signature matrix $\Sigma(z) := \operatorname{diag}(\sigma(z))$. A signature vector $\sigma \in \{-1, 1\}^s$ is called *definite*, otherwise we call σ *indefinite*.

Note that $|z| = \Sigma z$ and that the system $z = F(x, \Sigma z)$ has a locally unique solution $z(x)$ for fixed Σ by the implicit function theorem, where $(I - \partial_2 F(x, |z|)\Sigma)\partial_x z(x) = \partial_1 F(x, |z|)$. Using the chain rule for $\partial_x \phi(x)$, one then obtains

$$\partial_x \phi(x) = \partial_1 f(x, |z|) + \partial_2 f(x, |z|)\partial_z \text{abs}(z) \left[\partial_1 F(x, |z|) + \partial_2 F(x, |z|)\partial_z \text{abs}(z)\partial_x z(x)\right].$$

Using (2) and Def. 2, this yields

$$\begin{aligned} \partial_x \phi(x) &= a^T + b^T \Sigma[Z + L\Sigma \partial_x z(x)] \\ &= a^T + b^T \Sigma \partial_x z(x) \\ &= a^T + b^T \Sigma (I - L\Sigma)^{-1} Z. \end{aligned} \tag{3}$$

We say that a component of $z \in \mathbb{R}^s$ is *active* at $x \in \bar{D}$ if $z_i(x) = 0$. We denote the index set of active components of $z(x)$ by $\alpha(x) = \{i \in \bar{s} \colon \sigma(z_i(x)) = 0\}$.

Definition 3 (Kink Qualifications)**.** We say that a point $x \in \bar{D}$ satisfies LIKQ if the matrix $[\partial_x z(x)]_{i \in \alpha}$ has full row rank $|\alpha|$.

We say that a point $x \in \bar{D}$ satisfies MFKQ if for all *definite* $\sigma \succeq \sigma(z(x))$ the linear inequality system $[\Sigma \partial_x z(x)]_{i \in \alpha} w > 0$ admits a solution $w \in \mathbb{R}^n$, unless $[\Sigma \partial_x z(x)]_{i \in \alpha} w \geq 0$ admits only the solution $w = 0$.

Indefinite signatures $\sigma \succeq \sigma(z(x))$ must be excluded in the definition of MFKQ: if $\sigma_k = 0$, then $k \in \alpha(x)$ and $[\Sigma \partial_x z(x)]_{i \in \alpha} w > 0$ cannot admit a solution since row k of the matrix is zero.

Contributions. In this article, we consider the unconstrained non-smooth minimization problem

$$\min_{x \in \mathbb{R}^n} \phi(x) = f(x, |z(x)|) \tag{4}$$

and its extension to an abs-normal constrained minimization problem of the form

$$\min_{x \in \mathbb{R}^n} f(x, |z(x)|) \quad \text{s.t.} \quad g(x, |z(x)|) = 0, \quad h(x, |z(x)|) \geq 0.$$

We show that both problem classes may identically be cast as mathematical programs with equilibrium constraints and that, in case of the unconstrained problem (4), constraint qualifications and optimality conditions presented in [5, 7, 6] have counterparts in the established theory of MPECs. In particular, we prove equivalence between LIKQ and MPEC-LICQ in Lemma 15, equivalence between MFCQ for all branch problems and MPEC-MFCQ for the counterpart MPEC of an abs-normal form in Lemma 16, and the fact that all MPEC counterpart problems of abs-normal forms satisfy MPEC-ACQ in Lemma 18. Then, we proceed to show that minimizers of abs-normal forms under LIKQ are strongly stationary points in Prop. 19, and that they are M-stationary points otherwise in Prop. 20.

Structure. The remainder of this article is structured as follows. In Section 2 we present some prerequisites about MPECs. In Section 3 we consider the unconstrained setting of minimizing a function in abs-normal form, and prove connections to the theory of MPECs concerning constraint qualifications and stationarity concepts. In Section 4, we consider a class of non-smooth NLPs where the objective function and constraints are presented in abs-normal form, and show equivalence with a certain counterpart MPEC. We conclude in Section 5 and provide an appendix with auxiliary results.

2 Some Prerequisites about MPECs

In this section, we briefly review some necessary prerequisites about MPECs. For details, proofs, and literature, the reader may wish to consult, e.g., [8]. As smooth MPEC constraints we consider only equalities $c(y, u, v) = 0$ since inequalities are not needed.

Definition 4 (Mathematical Program with Equilibrium Constraints)**.** An optimization problem of the form

$$\boxed{\begin{aligned} \min_{y,u,v} \quad & \varphi(y, u, v) && && \text{(5a)} \\ \text{s.t.} \quad & c(y, u, v) = 0, && \mid \lambda && \text{(5b)} \\ & 0 \le u \perp v \ge 0, && \mid \mu_\text{u},\ \mu_\text{v} && \text{(5c)} \end{aligned}}$$

with $\varphi \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^s_{\ge 0} \times \mathbb{R}^s_{\ge 0}, \mathbb{R})$, $c \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^s_{\ge 0} \times \mathbb{R}^s_{\ge 0}, \mathbb{R}^{n_c})$ is called a *Mathematical Program with Equilibrium Constraints* (MPEC).

Definition 5 (Index Sets)**.** We denote by $\mathcal{U}_0 := \{i \in \bar{s}\colon u_i = 0\}$ the set of indices of active inequalities $u_i \ge 0$, and by $\mathcal{U}_+ := \{i \in \bar{s}\colon u_i > 0\}$ the set of indices of inactive inequalities $u_i \ge 0$. Analogous definitions hold of $\mathcal{V}_0$ and $\mathcal{V}_+$. By $\mathcal{D} := \mathcal{U}_0 \cap \mathcal{V}_0$ we denote the set of indices of non-strict (degenerate) complementarity pairs.

We deviate from contemporary MPEC literature, which frequently makes reference to the sets

$$\mathcal{I}_{+0} = \mathcal{U}_+ \cap \mathcal{V}_0, \quad \mathcal{I}_{0+} = \mathcal{U}_0 \cap \mathcal{V}_+, \quad \mathcal{I}_{00} = \mathcal{U}_0 \cap \mathcal{V}_0.$$

Note that by complementarity we have $\mathcal{U}_+ \cap \mathcal{V}_0 = \mathcal{U}_+$, $\mathcal{U}_0 \cap \mathcal{V}_+ = \mathcal{V}_+$, $\mathcal{U}_+ \cap \mathcal{V}_+ = \emptyset$, and hence the partitioning $\bar{s} = \mathcal{D} \cup \mathcal{U}_+ \cup \mathcal{V}_+$.

Definition 6 (MPEC Constraint Qualifications)**.** We say that a feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) satisfies *MPEC-LICQ* if

$$\operatorname{rank} \begin{bmatrix} \partial_y c(\hat{y}, \hat{u}, \hat{v}) & \partial_{\mathcal{U}_+} c(\hat{y}, \hat{u}, \hat{v}) & \partial_{\mathcal{V}_+} c(\hat{y}, \hat{u}, \hat{v}) \end{bmatrix} = n_c. \tag{6}$$

We say that a feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) satisfies *MPEC-MFCQ* if the linear system

$$\partial_y c(\hat{y}, \hat{u}, \hat{v}) d_\text{y} + \partial_{\mathcal{U}_+} c(\hat{y}, \hat{u}, \hat{v}) d_\text{u} + \partial_{\mathcal{V}_+} c(\hat{y}, \hat{u}, \hat{v}) d_\text{v} = 0 \tag{7}$$

admits a solution $(d_\text{y}, d_\text{u}, d_\text{v}) \in \mathbb{R}^n \times \mathbb{R}^{|\mathcal{U}_+|} \times \mathbb{R}^{|\mathcal{V}_+|}$, and additionally (6) holds.

We say that a feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) satisfies *MPEC-ACQ* if the linearized cone and the tangential cone of the tightened NLP (see Appendix A) are identical.

Note that the above definitions are already specialized to our particular MPEC formulation, see Appendix A for details. In general LICQ implies, but is stronger than, MPEC-LICQ, which in turn implies, but is stronger than, MPEC-MFCQ. The MPEC in our purely equality constrained case $c = 0$ is an exception: here MPEC-MFCQ is equivalent to MPEC-LICQ because the tightened NLP has no active inequalities and (6) implies that (7) has a solution. Finally, when considering MPEC-ACQ we use results from the literature and do not need the two cones of the definition.

Definition 7. A feasible point $(\hat{y}, \hat{u}, \hat{v})$ is a *strongly stationary or S-stationary point* if $(\hat{y}, \hat{u}, \hat{v})$ is a minimizer of the *relaxed NLP* defined as

$$\boxed{\begin{aligned} \min_{y,u,v} \quad & \varphi(y,u,v) && \text{(8a)}\\ \text{s.t.} \quad & c(y,u,v) = 0, && \text{(8b)}\\ & 0 = u_i,\ 0 \le v_i \quad \text{if } \hat{u}_i = 0,\ \hat{v}_i > 0\ (i \in \mathcal{V}_+), && \text{(8c)}\\ & 0 \le u_i,\ 0 = v_i \quad \text{if } \hat{u}_i > 0,\ \hat{v}_i = 0\ (i \in \mathcal{U}_+), && \text{(8d)}\\ & 0 \le u_i,\ 0 \le v_i \quad \text{if } \hat{u}_i = 0,\ \hat{v}_i = 0\ (i \in \mathcal{D}). && \text{(8e)} \end{aligned}}$$

Proposition 8 (Strongly Stationary Point)**.** *A feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) is an S-stationary point if there exist MPEC multipliers $(\lambda, \mu_\mathrm{u}, \mu_\mathrm{v})$ satisfying*

$$\begin{aligned} & 0 = \partial_{y,u,v}\mathcal{L}_\perp(\hat{y}, \hat{u}, \hat{v}, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) && \textit{(stationarity)},\\ & 0 = \lambda^T c(\hat{y}, \hat{u}, \hat{v}),\ \ 0 = \mu_\mathrm{u}^T \hat{u},\ \ 0 = \mu_\mathrm{v}^T \hat{v} && \textit{(complementary slackness)},\\ & 0 \le \mu_{\mathrm{u},i},\ \ 0 \le \mu_{\mathrm{v},i}\ \forall i \in \mathcal{D}(\hat{u}, \hat{v}) && \textit{(strong stationarity)}. \end{aligned}$$

Herein, $\mathcal{L}_\perp(y, u, v, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v})$ is the MPEC-Lagrangian function associated with (5),

$$\mathcal{L}_\perp(y, u, v, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) := \varphi(y, u, v) - \lambda^T c(y, u, v) - \mu_\mathrm{u}^T u - \mu_\mathrm{v}^T v.$$

Proof. Immediate by comparison to the first order necessary conditions of (8). □

Proposition 9. *Under MPEC-LICQ, all local minimizers of (5) are S-stationary points.*

Proof. A proof may be found in [3]. □

In absence of MPEC-LICQ, local minimizers may exist that do not qualify as strongly stationary points. Such points are characterized as follows.

Proposition 10 (M-Stationary Point)**.** *Let $(\hat{y}, \hat{u}, \hat{v})$ be a local minimizer of (5), and let MPEC-ACQ be satisfied. Then there exist MPEC multipliers $(\lambda, \mu_\mathrm{u}, \mu_\mathrm{v})$ satisfying*

$$\begin{aligned} & 0 = \partial_{y,u,v}\mathcal{L}_\perp(\hat{y}, \hat{u}, \hat{v}, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) && \textit{(stationarity)},\\ & 0 = \lambda^T c(\hat{y}, \hat{u}, \hat{v}),\ \ 0 = \mu_\mathrm{u}^T \hat{u},\ \ 0 = \mu_\mathrm{v}^T \hat{v}, && \textit{(complementary slackness)},\\ & (0 < \mu_{\mathrm{u},i},\ \ 0 < \mu_{\mathrm{v},i})\ \vee\ (0 = \mu_{\mathrm{u},i}\mu_{\mathrm{v},i})\ \forall i \in \mathcal{D}(\hat{u}, \hat{v}) && \textit{(M-stationarity)}. \end{aligned}$$

The point $(\hat{y}, \hat{u}, \hat{v})$ is called a Mordukhovich stationary or M-stationary *point.*

Proof. A proof may be found in, e.g., Theorem 5.25 in [1]. □

Finally, the strongest necessary condition based on first order information that holds in absence of a constraint qualification is characterized as follows.

Definition 11 (Bouligand Stationary Point)**.** A feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) is called a *Bouligand stationary or B-stationary point* if $(\hat{y}, \hat{u}, \hat{v})$ is a minimizer of all *branch problems* for the subsets $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$. For a subset $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$ and denoting the complement of $\mathcal{P}$ in the set $\mathcal{D}(\hat{u}, \hat{v})$ by $\bar{\mathcal{P}}$, the associated branch problem NLP($\mathcal{P}$) is defined by

$$\boxed{\begin{aligned} \min_{y,u,v} \quad & \varphi(y,u,v) && \text{(9a)}\\ \text{s.t.} \quad & c(y,u,v) = 0, && \text{(9b)}\\ & 0 \le u_i,\ 0 = v_i\ \forall i \in \mathcal{U}_+ \cup \mathcal{P}, && \text{(9c)}\\ & 0 = u_i,\ 0 \le v_i\ \forall i \in \mathcal{V}_+ \cup \bar{\mathcal{P}}. && \text{(9d)} \end{aligned}}$$

If $(\hat{y}, \hat{u}, \hat{v})$ solves the tightened NLP (see appendix), then $(\hat{y}, \hat{u}, \hat{v})$ solves all branch problems. Hence, strong stationarity under MPEC-LICQ is a sufficient condition for Bouligand stationarity. There are $2^{|\mathcal{D}|}$ branch problems and, if this sufficient condition does not hold, verifying Bouligand stationary has exponential effort in the number of non-strict complementarities.

Proposition 12. *A feasible point $(\hat{y}, \hat{u}, \hat{v})$ of (5) is a Bouligand stationary point if for all subsets $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$, there exist MPEC multipliers $(\lambda, \mu_{\rm u}, \mu_{\rm v})$ satisfying*

$$\begin{aligned} &0 = \partial_{y,u,v}\mathcal{L}_\perp(\hat{y}, \hat{u}, \hat{v}, \lambda, \mu_{\rm u}, \mu_{\rm v}) && \textit{(stationarity)},\\ &0 = \lambda^T c(\hat{y}, \hat{u}, \hat{v}),\ \ 0 = \mu_{\rm u}^T \hat{u},\ \ 0 = \mu_{\rm v}^T \hat{v} && \textit{(complementary slackness)},\\ &0 \le \mu_{\rm u,i}\ \forall i \in \mathcal{P},\ \ 0 \le \mu_{\rm v,i}\ \forall i \in \bar{\mathcal{P}} && \textit{(Bouligand stationarity)}. \end{aligned}$$

Proof. Again immediate by comparison to the first order necessary conditions of (9) for all subsets $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$. □

Proposition 13. *Local minimizers of (5) are Bouligand stationary points of (5).*

Proof. See, for example, [1]. □

3 Optimizing the Abs-Normal Form is a Subclass of MPECs

In this section, we model the problem of unconstrained minimization of a function given in abs-normal form by a particular counterpart MPEC. We show that the kink qualification LIKQ is equivalent to MPEC-LICQ and MPEC-MFCQ for the counterpart MPEC. We also investigate the relation of the kink qualification MFKQ with MPEC-MFCQ and MPEC-ACQ. We then show that under LIKQ, local minimizers of the function in abs-normal form coincide with strongly stationary points of the counterpart MPEC. If LIKQ fails to hold, local minimizers of the function in abs-normal form coincide with M-stationary points of the counterpart MPEC.

Definition 14. We call the MPEC

$$\boxed{\begin{aligned} \min_{y,u,v}\quad & f(y, u + v) && \\ \text{s.t.}\quad & u - v - F(y, u + v) = 0, && \mid \lambda \\ & 0 \le u \perp v \ge 0, && \mid \mu_{\rm u},\ \mu_{\rm v} \end{aligned}} \qquad \begin{aligned}&(10a)\\&(10b)\\&(10c)\end{aligned}$$

the *counterpart MPEC* of the minimization problem (4) in abs-normal form (1).

For future reference, the MPEC-Lagrangian function associated with (10) is

$$\mathcal{L}_\perp(y, u, v, \lambda, \mu_{\rm u}, \mu_{\rm v}) = f(y, u + v) - \lambda^T(u - v - F(y, u + v)) - \mu_{\rm u}^T u - \mu_{\rm v}^T v.$$

At a feasible point $(\hat{y}, \hat{u}, \hat{v})$, the signature components of (10) are $\hat{\sigma}_i = +1$ for $i \in \mathcal{U}_+$, $\hat{\sigma}_i = -1$ for $i \in \mathcal{V}_+$, and $\hat{\sigma}_i = 0$ for indices i in the active set, $i \in \mathcal{D} \equiv \alpha$. At a feasible point $(\hat{y}, \hat{u}, \hat{v})$ with at least one pair of non-strict complementarities, $\hat{u}_i = \hat{v}_i = 0$, the signature $\hat{\Sigma}$ has at least one zero entry $\hat{\sigma}_i = 0$. The family of branch problems for (10) then gives rise to all definite signatures $\Sigma_{\mathcal{P}} \succeq \hat{\Sigma}$.

In this section, we are first concerned with constraint qualifications. Figure 1 gives an overview of the interrelations of the different constraint qualifications that are known from the literature cited, or that are proved in this article. We start by showing that the kink qualification LIKQ for (1) is equivalent to MPEC-LICQ for (10).

Lemma 15 (Equivalence of LIKQ and MPEC-LICQ)**.** *A feasible point $(\hat{x}, \hat{z})$ of (1) satisfies LIKQ if and only if the point $(\hat{y}, \hat{u}, \hat{v}) = (\hat{x}, [\hat{z}]^+, [\hat{z}]^-)$ of (10) satisfies MPEC-LICQ.*

Proof. For the MPEC (10) with $c(y, u, v) = u - v - F(y, u + v) = 0$, MPEC-LICQ means that

$$\operatorname{rank}\begin{bmatrix}-Z & (I - L)P_{\mathcal{U}_+} & (-I - L)P_{\mathcal{V}_+}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-Z & (I - L)P_{\mathcal{U}_+} & (I - (-L))P_{\mathcal{V}_+}\end{bmatrix} = s.$$

Here $P_{\mathcal{S}}$ denote projectors onto the subset of variables u_i and v_i with indices $i \in \mathcal{S} \subseteq \bar{s}$. By complementarity, any index i is in at most one of the sets $\mathcal{U}_+$ and $\mathcal{V}_+$, and because of the regularity of $I - L\Sigma$, cf. also Eq. (27) in [5], we may equivalently ask that

$$\operatorname{rank}\begin{bmatrix}-Z & (I - L\Sigma)P_{\mathcal{U}_+ \cup \mathcal{V}_+}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-(I - L\Sigma)^{-1}Z & P_{\mathcal{U}_+ \cup \mathcal{V}_+}\end{bmatrix} = s.$$

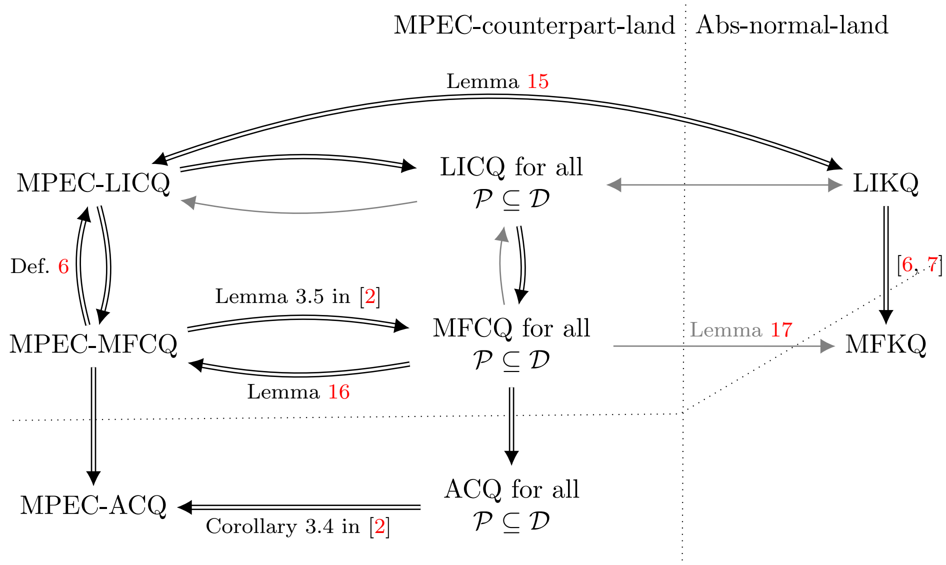

Figure 1: Relations between constraint qualifications for the abs-normal form (right) and MPEC constraint qualifications for the particular class of counterpart problems (left). Vertical implications may be found in, e.g., [2, 8] for MPECs and in [6, 7] for the abs-normal form. Grey single-lined arrows denote implied relations that need not be proved explicitly. Note that MPEC-ACQ always holds without prerequisite.

For the strict indices $i \in \mathcal{U}_+ \cup \mathcal{V}_+$ we have exactly one unit entry in every row $[P_{\mathcal{U}_+ \cup \mathcal{V}_+}]_i$ and at most one entry per column. For the non-strict indices $i \in \mathcal{D}$ we have $[P_{\mathcal{U}_+ \cup \mathcal{V}_+}]_i = 0$ and hence the condition is equivalent to asking that

$$[(I - L\Sigma)^{-1} Z]_{i \in \mathcal{D}}$$

has full rank. Because $\mathcal{D} = \alpha(x)$, we find the condition to be LIKQ. $\square$

Notably, the additional regularity provided by the abs-normal form, i.e., existence of the inverse $(I - L\Sigma)^{-1}$, enables the definition of LIKQ, which is seemingly weaker in only making a regularity statement for active components $i \in \alpha$. The regularity for the inactive components is afforded by the lower triangular structure of L, and both are required to satisfy MPEC-LICQ in the counterpart MPEC.

It is known that MPEC-MFCQ implies MFCQ for all branch problems, cf. [2, Lemma 3.5]. The converse is not true in general, but holds for the counterpart MPEC of an abs-normal form. We observe in the following lemma and proof that MFCQ for all branch problems can only hold if $s \le n$. This case may be considered rare.

Lemma 16 (MFCQ for all abs-normal counterpart branch problems implies MPEC-MFCQ)**.** *Let* $(\hat{y}, \hat{u}, \hat{v})$ *be a feasible point of* (10) *and let the branch problems* (9) *satisfy MFCQ for all* $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$*. Then MPEC-MFCQ holds at* $(\hat{y}, \hat{u}, \hat{v})$*.*

Proof. MFCQ for all branch problems means that for all $\mathcal{P} \in \mathcal{D}$,

1. the Jacobian of the equality constraints has full row rank,

$$\operatorname{rank} \begin{bmatrix} -Z & I - L & -I - L \\ & P_{\mathcal{V}_+ \cup \bar{\mathcal{P}}} & \\ & & P_{\mathcal{U}_+ \cup \mathcal{P}} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -Z & I - L & I - (-L) \\ & P_{\mathcal{V}_+ \cup \bar{\mathcal{P}}} & \\ & & -P_{\mathcal{U}_+ \cup \mathcal{P}} \end{bmatrix} = 2s.$$

Observing that each of the last s rows contains precisely one unit entry in $P_{\mathcal{V}_+ \cup \bar{\mathcal{P}}}$ or $P_{\mathcal{U}_+ \cup \mathcal{P}}$ and keeping the s remaining columns associated with variables $P_{\mathcal{U}_+ \cup \mathcal{P}} u$ and $P_{\mathcal{V}_+ \cup \bar{\mathcal{P}}} v$, this is equivalent to

$$\operatorname{rank}\begin{bmatrix}-Z & I-\Sigma L\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-(I-\Sigma L)^{-1}Z & I\end{bmatrix} = s,$$

which is always satisfied.

2. there is a vector $d = (d_\mathrm{y}, d_\mathrm{u}, d_\mathrm{v})$, possibly dependent on the particular set $\mathcal{P}$, such that

$$\begin{bmatrix}-Z & I-L & -I-L\\ & P_{\mathcal{V}_+\cup\bar{\mathcal{P}}} & \\ & & P_{\mathcal{U}_+\cup\mathcal{P}}\end{bmatrix} d = 0, \quad \begin{bmatrix}0 & P_{\mathcal{U}_+\cup\mathcal{P}} & 0\\ 0 & 0 & P_{\mathcal{V}_+\cup\bar{\mathcal{P}}}\end{bmatrix} d > 0.$$

This means $[d_\mathrm{u}]_{\mathcal{V}_+\cup\bar{\mathcal{P}}} = 0$, $[d_\mathrm{v}]_{\mathcal{U}_+\cup\mathcal{P}} = 0$, $[d_\mathrm{u}]_{\mathcal{U}_+\cup\mathcal{P}} > 0$, and $[d_\mathrm{v}]_{\mathcal{V}_+\cup\bar{\mathcal{P}}} > 0$. By complementarity feasibility, the index sets of the nonzero components of d_u and $-d_\mathrm{v}$ are disjoint. Hence, we may combine the s nonzero components of d_u and $-d_\mathrm{v}$ in a compound vector $d_\mathrm{uv} \in \mathbb{R}^s$, and may write

$$\begin{aligned} & 0 = -Zd_\mathrm{y} + (I-L)d_\mathrm{u} + (I-(-L))(-d_\mathrm{v}) \\ \iff\ & 0 < \Sigma d_\mathrm{uv} = (I-\Sigma L)^{-1} Z d_\mathrm{y}. \end{aligned}$$

Thus, MFCQ for only one branch problem already implies that $Z \in \mathbb{R}^{s\times n}$ is surjective on $\mathbb{R}^s$, hence $\operatorname{rank}(Z) = s \le n$.

To obtain MPEC-MFCQ for the counterpart MPEC (10), we need to show that

$$\operatorname{rank}\begin{bmatrix}-Z & (I-L)P_{\mathcal{U}_+} & (-I-L)P_{\mathcal{V}_+}\end{bmatrix} = s,$$

which is satisfied if $Z \in \mathbb{R}^{s\times n}$ with $s \le n$ has full row rank. As mentioned, this is implied by MFCQ for all branch problems, and MPEC-MFCQ holds. □

With similar tools, we may investigate the relation of MFKQ for the abs-normal form (1) to MPEC-MFCQ for the counterpart MPEC (10). As MPEC-MFCQ, MPEC-LICQ and LIKQ are equivalent in the case of abs-normal forms, but LIKQ and MFKQ are not, equivalence cannot hold. We show that MFKQ is implied by MPEC-MFCQ via MFCQ for all branch problems. Note that the following Lemma 17 need not have been proven separately, as it is implied by the chain of Lemma 16, Def. 6, and Lemma 15.

Lemma 17 (MFCQ for all MPEC branch problems implies MFKQ)**.** *A feasible point* $(\hat{x}, \hat{z})$ *of* (1) *satisfies MFKQ if the point* $(\hat{y}, \hat{u}, \hat{v}) = (\hat{x}, [\hat{z}]^+, [\hat{z}]^-)$ *of* (10) *satisfies MFCQ for all MPEC branch problems.*

Proof. We first consider MFCQ for a branch problem $\mathcal{P} \subseteq \mathcal{D}(\hat{u}, \hat{v})$ of (10). The substantial condition is that there is a direction $d = (d_\mathrm{y}, d_\mathrm{u}, d_\mathrm{v})$ such that

$$\begin{aligned} & -Zd_\mathrm{y} + (I-L)P_{\mathcal{U}_+\cup\mathcal{P}}d_\mathrm{u} + (-I-L)P_{\mathcal{V}_+\cup\bar{\mathcal{P}}}d_\mathrm{v} = 0,\ P_{\mathcal{P}}d_\mathrm{u} > 0,\ P_{\bar{\mathcal{P}}}d_\mathrm{v} > 0 \\ \iff\ & -Zd_\mathrm{y} + (I-L\hat{\Sigma})P_{\mathcal{U}_+\cup\mathcal{P}}d_\mathrm{u} + (I-L\hat{\Sigma})P_{\mathcal{V}_+\cup\bar{\mathcal{P}}}d_\mathrm{v} = 0,\ P_{\mathcal{P}}d_\mathrm{u} > 0,\ P_{\bar{\mathcal{P}}}d_\mathrm{v} < 0, \end{aligned}$$

where $\hat{\Sigma} = \Sigma(\hat{z})$ only depends on $\hat{z} = z(\hat{x})$. The sets $\mathcal{U}_+ \cup \mathcal{P}$ and $\mathcal{V}_+ \cup \bar{\mathcal{P}}$ are disjoint, so merging vectors d_u and d_v as above and using the regularity of the abs-normal form, we may equivalently ask that there is a direction $d = (d_\mathrm{y}, d_\mathrm{uv}) \in \mathbb{R}^n \times \mathbb{R}^s$ such that

$$\begin{cases} [-(I-L\hat{\Sigma})^{-1}Zd_\mathrm{y} + d_\mathrm{uv}]_{i\in\mathcal{U}_+\cup\mathcal{V}_+} = 0, \\ [-(I-L\hat{\Sigma})^{-1}Zd_\mathrm{y} + d_\mathrm{uv}]_{i\in\mathcal{D}} = 0 \text{ and } \Sigma d_\mathrm{uv} > 0, \end{cases}$$

where we have written the condition separately for the strict complementarities and for the non-strict set $\mathcal{D} = \mathcal{P} \cup \bar{\mathcal{P}}$.

The first condition can always be satisfied by appropriate choice of $[d_\mathrm{uv}]_i$, $i \in \mathcal{U}_+ \cup \mathcal{V}_+$.

In the second condition, the signature matrix $\Sigma = \mathrm{diag}(\sigma_i)$ depends on the branch $\mathcal{P}$ selected and is defined by $\sigma_i = +1$ if $i \in \mathcal{P}$, $\sigma_i = -1$ if $i \in \bar{\mathcal{P}}$, and $\sigma_i = \hat{\sigma}_i \neq 0$ if $i \notin \mathcal{D}$. It is, in general, different from $\hat{\Sigma}$. The second condition is thus equivalent to asking that d_{y} satisfy

$$[\Sigma(I - L\hat{\Sigma})^{-1} Z d_{\mathrm{y}}]_{i\in\mathcal{D}} = [\Sigma d_{\mathrm{uv}}]_{i\in\mathcal{D}} > 0 \iff [\Sigma \partial_x z(x) d_{\mathrm{y}}]_{i\in\mathcal{D}} > 0.$$

Now, since $\mathcal{D} = \alpha(\hat{x})$, the right hand side condition becomes

$$[\Sigma \partial_x z(x)]_{i\in\alpha} d_{\mathrm{y}} > 0.$$

Finally the signatures σ associated with the branch problems $\mathcal{P}$ are precisely the *definite* signatures that satisfy $\sigma \succeq \hat{\sigma} = \sigma(\hat{z})$. Thus, MFCQ for all MPEC branch problems $\mathcal{P} \subseteq \mathcal{D}$ implies MFKQ. □

Note that, because of the above relation to definite signatures, the MPEC branch problems NLP($\mathcal{P}$) are precisely the branch NLPs of the abs-normal problem (1).

The converse of Lemma 17 holds only if $|\alpha| \leq n$, i.e., not too many indefinite signatures are present, and if the additional regularity condition $\mathrm{rank}(Z) = |\alpha|$ is satisfied. As a consequence, MFKQ and LIKQ are identical for such problems.

Next, one would be interested in learning of an MPEC constraint qualification that is implied by MFKQ, such that it holds in absence of LIKQ. Unfortunately, we do not know how to obtain a result based on MFKQ to this end.

Finally, the following lemma proves that Abadie's constraint qualification (MPEC-ACQ) even holds without prerequisite. Key is, again, the absence of inequality constraints from (10), besides the complementarities.

Lemma 18 (MPEC-ACQ holds)**.** *Any feasible point* $(\hat{y}, \hat{u}, \hat{v})$ *of* (10) *satisfies MPEC-ACQ.*

Proof. We introduce the following representations of the MPEC-linearized cone and the tangent cone of the counterpart MPEC at $(\hat{y}, \hat{u}, \hat{v})$,

$$\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}}(\hat{y}, \hat{u}, \hat{v}) := \left\{ d = (d_{\mathrm{y}}, d_{\mathrm{u}}, d_{\mathrm{v}}) \;\middle|\; \begin{array}{l} 0 \leq d_{\mathrm{u}i},\ 0 = d_{\mathrm{v}i} \ \text{ if } i \in \mathcal{U}_+ \\ 0 = d_{\mathrm{u}i},\ 0 \leq d_{\mathrm{v}i} \ \text{ if } i \in \mathcal{V}_+ \\ 0 \leq d_{\mathrm{u}i} \perp d_{\mathrm{v}i} \geq 0 \ \text{ if } i \in \mathcal{D}(\hat{u}, \hat{v}) \\ 0 = d_{\mathrm{u}} - d_{\mathrm{v}} - Z d_{\mathrm{y}} - L(d_{\mathrm{u}} + d_{\mathrm{v}}) \end{array} \right\},$$

$$\mathcal{T}(\hat{y}, \hat{u}, \hat{v}) := \left\{ d = (d_{\mathrm{y}}, d_{\mathrm{u}}, d_{\mathrm{v}}) \;\middle|\; \begin{array}{l} \exists\, (y^k, u^k, v^k)_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathrm{MPEC}},\ t_k \searrow 0: \\ (y^k, u^k, v^k) \to (\hat{y}, \hat{u}, \hat{v}), \\ (t^k)^{-1}(y^k - \hat{y}, u^k - \hat{u}, v^k - \hat{v}) \to d \end{array} \right\},$$

$$\text{with } \mathcal{F}_{\mathrm{MPEC}} := \left\{ (y, u, v) \;\middle|\; \begin{array}{l} 0 \leq u_i,\ 0 = v_i \ \text{ if } i \in \mathcal{U}_+ \\ 0 = u_i,\ 0 \leq v_i \ \text{ if } i \in \mathcal{V}_+ \\ 0 \leq u_i \perp v_i \geq 0 \ \text{ if } i \in \mathcal{D}(\hat{u}, \hat{v}) \\ 0 = u - v - F(y, u + v) \end{array} \right\}.$$

To show MPEC-ACQ, i.e., $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}} = \mathcal{T}$, it suffices to show $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}} \subseteq \mathcal{T}$ as the reverse inclusion is well known to hold, cf. [2]. A direction $d \in \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}}$ satisfies $(I - L\hat{\Sigma})^{-1} Z d_{\mathrm{y}} = d_{\mathrm{u}} - d_{\mathrm{v}}$ and $\hat{\Sigma}(d_{\mathrm{u}} - d_{\mathrm{v}}) \geq 0$.

Now, fix a direction $d = (d_{\mathrm{y}}, d_{\mathrm{u}}, d_{\mathrm{v}}) \in \mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}}$, choose $t^k = 1/k$, and choose a sequence $(y^k) \subset \mathbb{R}^n$ such that $y^k \to \hat{y}$ and $(y^k - \hat{y})/t^k \to d_{\mathrm{y}}$.

The solution map $z(x) = F(x, \hat{\Sigma} z)$ (see Def. 2) is locally unique, i.e., there is a small number $\epsilon > 0$ and a number k_1 such that we have a unique $z^k = F(y^k, \hat{\Sigma} z^k)$ for all $k \geq k_1$. Defining $(u^k, v^k) := ([z^k]^+, [z^k]^-)$, we have $(y^k, u^k, v^k) \in \mathcal{F}_{\mathrm{MPEC}}$ since u^k, v^k have feasible signs and are complementary by construction. We also have $z^k \to \hat{z}$ by continuity of F and hence obtain a convergent sequence $(u^k, v^k) \to (\hat{u}, \hat{v})$. Finally, since

$$\begin{aligned} z^k - \hat{z} &= F(y^k, \hat{\Sigma} z^k) - F(\hat{y}, \hat{\Sigma} \hat{z}) \\ &= Z(y^k - \hat{y}) + L\hat{\Sigma}(z^k - \hat{z}) + o(\|(y^k - \hat{y}, z^k - \hat{z})\|), \end{aligned}$$

we obtain

$$\frac{(u^k - \hat{u})}{t^k} - \frac{(v^k - \hat{v})}{t^k} - (I - L\hat{\Sigma})^{-1} Z \frac{y^k - \hat{y}}{t^k} \to 0, \tag{11}$$

which means $(t^k)^{-1}((u^k - \hat{u}) - (v^k - \hat{v})) \to d_\mathrm{u} - d_\mathrm{v}$. It remains to show the separate convergence $(t^k)^{-1}(u^k - \hat{u}, v^k - \hat{v}) \to (d_\mathrm{u}, d_\mathrm{v})$. We argue separately for components i of the vectors:

- For components with $\hat{u}_i > 0$, we have $\hat{v}_i = 0$ by feasibility and from the definition of $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}}$ we have $d_{\mathrm{u}i} \geq 0$ and $d_{\mathrm{v}i} = 0$, and there is an index $k_2 \geq k_1$ such that $u_i^k > 0$ and by complementarity $v_i^k = 0$ for all $k \geq k_2$. Then we have $(t^k)^{-1}(u_i^k - \hat{u}_i) - (v_i^k - \hat{v}_i)) = (t^k)^{-1}(u_i^k - \hat{u}_i)$. By uniqueness of the solution map, $(t^k)^{-1}(u_i^k - \hat{u}_i) \to d_{\mathrm{u}i}$ must hold.

- Similar reasoning holds for components with $\hat{v}_i > 0$.

- For components with $\hat{u}_i = \hat{v}_i = 0$, for any direction with either $d_{\mathrm{u}i} > 0$ or $d_{\mathrm{v}i} > 0$ (both cannot hold simultaneously by definition of $\mathcal{T}^{\mathrm{lin}}_{\mathrm{MPEC}}$) we must have $(t^k)^{-1}(u_i^k - v_i^k) \to d_{\mathrm{u}i} - d_{\mathrm{v}i}$ and because of signs $(t^k)^{-1}u_i^k \to d_{\mathrm{u}i}$ and $(t^k)^{-1}v_i^k \to d_{\mathrm{v}i}$.

- Finally, for components with $\hat{u}_i = \hat{v}_i = 0$ and $d_{\mathrm{u}i} = d_{\mathrm{v}i} = 0$, we have

  $$(t^k)^{-1}(u_i^k - v_i^k) \to 0.$$

  Because of signs and complementarity, this can only be true if $(t^k)^{-1}u_i^k \to 0,\ (t^k)^{-1}v_i^k \to 0$ also holds.

This proves separate convergence of $u^k - \hat{u} \to d_\mathrm{u}$ and $v^k - \hat{v} \to d_\mathrm{v}$, and hence membership of d in the tangential cone $\mathcal{T}$. □

Having discussed constraint qualifications, we now proceed by showing that a local minimizer of an abs-normal form satisfying LIKQ defines a corresponding strongly stationary point of the counterpart MPEC satisfying MPEC-LICQ.

Proposition 19 (Strongly Stationary Points and Minimizers of the Abs-Normal Form)**.** *Let* (1) *satisfy LIKQ. If* (x, z) *is a local minimizer of* (1)*, then* $(y, u, v) = (x, [z]^+, [z]^-)$ *is a strongly stationary point of* (10).

Proof. First, let (x, z) be a local minimizer of (1). Then, we need to have

$$0 \in \partial_x \phi(x) = \partial_1 f(x, |z|) + \partial_2 f(x, |z|)\Sigma(z)\partial_x z(x) \tag{12a}$$

$$\iff -\partial_1 f(x, |z|) \in \partial_2 f(x, |z|)\Sigma(z)\partial_x z(x) \tag{12b}$$

Second, a strongly stationary point of (10) satisfies

$$\begin{aligned} 0 &= \partial_y \mathcal{L}_\perp(y, u, v, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) = \partial_1 f(y, u + v) + \lambda^T \partial_y z(y), \\ 0 &= \partial_u \mathcal{L}_\perp(y, u, v, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) = \partial_2 f(y, u + v) - \lambda^T I - \mu_\mathrm{u}^T I, \\ 0 &= \partial_v \mathcal{L}_\perp(y, u, v, \lambda, \mu_\mathrm{u}, \mu_\mathrm{v}) = \partial_2 f(y, u + v) + \lambda^T I - \mu_\mathrm{v}^T I. \end{aligned}$$

Identifying (x, z) with $(y, u - v)$ and using LIKQ, we have $\lambda^T = -\partial_1 f(x, |z|)(\partial_x z(x))^{-1}$. We now distinguish the case of active and inactive components $z_i = u_i - v_i$:

- In the inactive case, $u_i > 0$ and $v_i = 0$, or $u_i = 0$ and $v_i > 0$. Strong stationarity then requires $\mu_{u,i} = 0$ or $\mu_{v,i} = 0$, respectively. We present the first case; the second one follows analogously.

  $$0 = \mu_{\mathrm{u},i} = \partial_{2,i} f(y, u + v) - \lambda_i$$

  $$\iff \partial_{2,i} f(y, u + v) = -[\partial_1 f(x, |z|)(\partial_x z(x))^{-1}]_i \tag{13}$$

- In the active case $0 = u_i - v_i = z_i = |z_i|$, strong stationarity requires

$$\mu_{\mathrm{u},i} = \partial_{2,i} f(y, u+v) - \lambda_i \geq 0,$$
$$\mu_{\mathrm{v},i} = \partial_{2,i} f(y, u+v) + \lambda_i \geq 0.$$

  Substituting $u + v = |z|$, we obtain $0 \leq \partial_{2,i} f(x, |z|)$ by adding up, and

$$-\partial_{2,i} f(x, |z|) \leq \pm\lambda_i \leq +\partial_{2,i} f(x, |z|).$$

  After substituting $\lambda_i = -[\partial_1 f(x, |z|)(\partial_x z(x))^{-1}]_i$, we find that

$$-\partial_{2,i} f(x, |z|) \leq \pm[\partial_1 f(x, |z|)(\partial_x z(x))^{-1}]_i \leq \partial_{2,i} f(x, |z|) \tag{14}$$

Collecting (13) and (14) for all $i \in \bar{s}$ and using Σ of Def. 2, we may write this more compactly,

$$-\partial_1 f(x, |z|)(\partial_x z(x))^{-1} \in \partial_{2,i} f(x, |z|)\Sigma.$$

This condition is equivalent to (12b). □

We finally show that in absence of LIKQ but in presence of MFKQ, a local minimizer of (1) is an M-stationary point of (10).

Proposition 20 (M-Stationary Points and Minimizers of the Abs-Normal Form)**.** *If a feasible point* $(\hat{x}, \hat{z})$ *of* (1) *is a local minimizer, the point* $(\hat{y}, \hat{u}, \hat{v}) = (\hat{y}, [\hat{z}]^+, [\hat{z}]^-)$ *of* (10) *is an M-stationary point.*

Proof. We have shown in Lemma 18 that MPEC-ACQ holds for (10). It was shown in [1] that M-stationarity is a necessary condition for optimality under MPEC-ACQ. □

A necessary condition that only requires the weaker constraint qualification Guignard CQ to hold for all branch problems is Bouligand stationarity, cf. [1]. The effort of verification of B-stationarity is, in general, exponential in the number $|\mathcal{D}(\hat{u}, \hat{v})| = |\{i \mid \sigma_i(\hat{x}) = 0\}|$ of non-strict complementarity pairs or indefinite signatures. Interestingly, since MPEC-ACQ always holds, this computationally costly case only arises when verifying first order optimality of an M-stationary point of a counterpart MPEC for an abs-normal form that violates MFKQ.

4 Abs-Normal NLPs

In this section, we extend the idea of minimizing a function in abs-normal form to non-smooth NLPs with objective function, equality and inequality constraints in abs-normal form. Likewise, we now consider general MPECs that have smooth equality and inequality constraints:

$$\min_{y,u,v} \quad \varphi(y, u, v) \tag{15a}$$
$$\text{s.t.} \quad c_{\mathcal{E}}(y, u, v) = 0, \tag{15b}$$
$$c_{\mathcal{I}}(y, u, v) \geq 0, \tag{15c}$$
$$0 \leq u \perp v \geq 0, \tag{15d}$$

with $\varphi \in \mathcal{C}^1(W, \mathbb{R})$, $c_{\mathcal{E}} \in \mathcal{C}^1(W, \mathbb{R}^{n_{\mathcal{E}}})$, $c_{\mathcal{I}} \in \mathcal{C}^1(W, \mathbb{R}^{n_{\mathcal{I}}})$, $W = \mathbb{R}^n \times \mathbb{R}^s_{\geq 0} \times \mathbb{R}^s_{\geq 0}$.

Definition 21 (Abs-Normal NLP)**.** We say that a non-smooth NLP is in abs-normal form if functions $f \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^n_{\geq 0}, \mathbb{R})$, $g \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^n_{\geq 0}, \mathbb{R}^{n_g})$, $h \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^n_{\geq 0}, \mathbb{R}^{n_h})$, and $F \in \mathcal{C}^1(\bar{D} \times \mathbb{R}^s_{\geq 0}, \mathbb{R}^s)$ exist such that the problem reads as follows.

$$\min_{x,z} \quad f(x, |z|) \tag{16a}$$
$$\text{s.t.} \quad g(x, |z|) = 0, \tag{16b}$$
$$h(x, |z|) \geq 0, \tag{16c}$$
$$z = F(x, |z|) \quad \text{with } L := \partial_2 F(x, |z|) \text{ strictly lower triangular.} \tag{16d}$$

Similar to what has been presented for the unconstrained case in §2, the MPEC counterpart problem to (16) may be defined as follows.

Definition 22 (Counterpart MPEC of Abs-Normal NLP)**.** The *counterpart MPEC* of (16) reads as follows.

$$\begin{aligned} \min_{y,u,v} \quad & f(y, u+v) && (17a)\\ \text{s.t.} \quad & g(y, u+v) = 0, && (17b)\\ & h(y, u+v) \geq 0, && (17c)\\ & u - v - F(y, u+v) = 0, && (17d)\\ & 0 \leq u \perp v \geq 0. && (17e) \end{aligned}$$

Here $g = 0$ with $u - v - F = 0$ corresponds to $c_{\mathcal{E}} = 0$ in (15), and $h \geq 0$ corresponds to $c_{\mathcal{I}} \geq 0$ in (15).

Conversely, we can rewrite every MPEC as an abs-normal NLP.

Definition 23 (Counterpart Abs-Normal NLP of an MPEC)**.** With smooth variables $x = (x_y, x_u, x_v) = (y, u, v)$, the *counterpart abs-normal NLP* of the MPEC (15) reads

$$\begin{aligned} \min_{x,z} \quad & \varphi(x) && (18a)\\ \text{s.t.} \quad & c_{\mathcal{E}}(x) = 0, && (18b)\\ & c_{\mathcal{I}}(x) \geq 0, && (18c)\\ & x_u + x_v - |z| = 0, && (18d)\\ & z = x_u - x_v. && (18e) \end{aligned}$$

Here (18d) with $z = x_u - x_v$ is the standard reformulation of $\min(x_u, x_v) = 0$, which in turn is equivalent to the complementarity requirement $0 \leq x_u \perp x_v \geq 0$. Thus $c_{\mathcal{E}} = 0$ with $x_u + x_v - |z| = 0$ corresponds to $g = 0$, $c_{\mathcal{I}} \geq 0$ corresponds to $h \geq 0$, and $z = x_u - x_v$ corresponds to $z = F$ (with $L = 0$) in (16).

It is thus apparent that the problem class (16) of non-smooth NLPs admitting an abs-normal form is equivalent to the problem class (15) of general MPECs. Close relations between respective constraint qualifications and stationarity concepts for MPECs and for the abs-normal NLPs are likely and a subject of ongoing research.

5 Conclusion

The abs-normal form is a convenient way of exposing structured non-smoothness in objective function and constraints. It allows for comparably easy generation of algorithmic derivatives and subdifferentials. In this article, we have shown that under LIKQ (or MPEC-LICQ), there are no theoretical differences between unconstrained problems in abs-normal form and certain MPECs. The abs-normal form however automatically implies regularity properties for inactive components, such that kink qualifications can be formulated with seemingly less restrictive requirements than is the case for MPECs. Moreover, we have shown MFKQ to be weaker than asking that MFCQ holds for all branch problems. In absence of LIKQ, we have shown that M-stationarity is a necessary condition for a local minimizer of the abs-normal counterpart MPEC. We have also proposed an extension of the problem class to abs-normal NLPs, and have suggested equivalent MPEC counterpart problems that are likely to expose similarly close links between both settings. This is a subject of ongoing research.

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A On the definition of MPEC-LICQ and MPEC-MFCQ

Here we provide a brief derivation of MPEC-LICQ and MPEC-MFCQ in Definition 6. Given a feasible point $(\hat{x}, \hat{u}, \hat{v})$ of the MPEC (5) with associated active sets $\mathcal{U}_0, \mathcal{V}_0$, the *tightened NLP* is defined as

$$\min_{y,u,v} \quad \varphi(y, u, v) \tag{19a}$$

$$\text{s.t.} \quad c(y, u, v) = 0, \tag{19b}$$

$$0 = u_i,\ 0 \le v_i \quad \text{if } \hat{u}_i = 0,\ \hat{v}_i > 0\ (i \in \mathcal{V}_+), \tag{19c}$$

$$0 \le u_i,\ 0 = v_i \quad \text{if } \hat{u}_i > 0,\ \hat{v}_i = 0\ (i \in \mathcal{U}_+), \tag{19d}$$

$$0 = u_i,\ 0 = v_i \quad \text{if } \hat{u}_i = 0,\ \hat{v}_i = 0\ (i \in \mathcal{D}). \tag{19e}$$

Its equality constraints read

$$c(y, u, v) = 0, \qquad u_i = 0\ \forall i \in \mathcal{U}_0, \qquad v_i = 0\ \forall i \in \mathcal{V}_0,$$

and the inequality constraints read

$$u_i \ge 0\ \forall i \in \mathcal{U}_+, \qquad v_i \ge 0\ \forall i \in \mathcal{V}_+.$$

The active set at $(y, u, v) = (\hat{x}, \hat{u}, \hat{v})$ is $\bar{n}_c \cup \mathcal{U}_0 \cup \mathcal{V}_0$. Note that there are *no active inequalities* at $(y, u, v) = (\hat{x}, \hat{u}, \hat{v})$, and in particular not for the counterpart MPEC (10).

Thus, MPEC-LICQ at $(y, u, v) = (\hat{x}, \hat{u}, \hat{v})$, which is defined as LICQ for the tightened NLP, requires linear independence of the following constraint derivatives:

$$\partial_{y,u,v} c(y, u, v), \quad P_{\mathcal{U}_0} u = I, \quad P_{\mathcal{V}_0} v = I.$$

This is equivalent to

$$\operatorname{rank}\begin{bmatrix}\partial_y c(y,u,v) & \partial_{\mathcal{U}_+} c(y,u,v) & \partial_{\mathcal{V}_+} c(y,u,v) & \partial_{\mathcal{U}_0} c(y,u,v) & \partial_{\mathcal{V}_0} c(y,u,v)\\ & & & I & \\ & & & & I\end{bmatrix} = n_c + |\mathcal{U}_0| + |\mathcal{V}_0|,$$

where we have reordered the columns of the matrix according to indices in sets $\mathcal{U}_+$, $\mathcal{V}_+$, $\mathcal{U}_0$, $\mathcal{V}_0$, respectively. Exploiting the identity block on the lower right, one obtains the formulation in Definition 6.

Similarly, MPEC-MFCQ at $(y,u,v) = (\hat{x},\hat{u},\hat{v})$, which is defined as MFCQ for the tightened NLP, requires the existence of a vector $(d_{\mathrm{y}}, d_{\mathrm{u}}, d_{\mathrm{v}})$ that satisfies

$$\begin{aligned}\partial_y c(y,u,v) d_{\mathrm{y}} + \partial_u c(y,u,v) d_{\mathrm{u}} + \partial_v c(y,u,v) d_{\mathrm{v}} &= 0,\\ P_{\mathcal{U}_0} d_{\mathrm{u}} &= 0,\\ P_{\mathcal{V}_0} d_{\mathrm{v}} &= 0,\end{aligned}$$

where $\operatorname{rank}[\partial_{y,u,v} c(y,u,v)] = n_c$ is additionally required. Clearly, this is equivalent to the formulation in Definition 6.