

Finite convergence and weak sharpness for solutions of nonsmooth variational inequalities in Hilbert spaces

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Abstract

This paper deals with the study of weak sharp solutions for nonsmooth variational inequalities and finite convergence property of the proximal point method. We present several characterizations for weak sharpness of the solutions set of nonsmooth variational inequalities without using the gap functions. We show that under weak sharpness of the solutions set, the sequence generated by proximal point methods terminates after a finite number of iterations. We also give an upper bound for the number of iterations for which the sequence generated by the exact proximal point methods terminates.

Keywords: Convex programming, Nonsmooth variational inequalities, Weak sharp solutions, Finite convergence property, Pseudomonotone operators, Proximal point method.

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1. Introduction

The notion of weakly sharp solutions for convex optimization problems, introduced by Burke and Ferris [7, 10], is useful to study sensitivity analysis, error bounds and finite convergence of methods for solving optimization problems (see, e.g., [6, 10, 11]). Here, the finite convergence of a solution method means that the sequence

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generated by the solution method terminates after a finite number of iterations. Patricksson [20] generalized this concept for the solution set of variational inequalities VIP (2.3). Marcotte and Zhu [17] gave a characterization of weak sharpness of the solutions set of VIP in terms of a dual gap function. They also studied the finite convergence property of the descent algorithm for solving VIP under weak sharpness of the solutions set. Since then, the weak sharpness of solutions and its applications to the finite convergence property of algorithms for finding solutions of variational inequalities have been investigated by many authors, see, e.g., [4, 13, 16, 18, 19, 22, 23, 24] and the references therein. In the recent years, several authors have extended and studied the concept of weakly sharp solutions to general variational inequalities: set-valued variational inequalities [2, 21], variational-type inequalities [14], mixed variational inequalities [12], nonsmooth variational inequalities [3].

It is well-known that a variational inequality provides the first order necessary and sufficient optimality conditions for a solution of a convex and differentiable minimization problem. However, if the objective function of a convex minimization problem is not necessarily differentiable but has some kind of directional derivative, then the optimization problem can be solved by using a nonsmooth variational inequality. For a comprehensive study of nonsmooth variational inequalities and their applications, we refer [5, Chapter 6]. Recently, authors [3] introduced the concept of weakly sharp solutions for nonsmooth variational inequalities and gave a characterization of the weak sharpness of the solutions set in terms of error bound of the dual gap function. An application to the finite convergence of the gradient projection method for solving nonsmooth variational inequalities is also provided. The primal goal of this paper is to provide some new characterizations of the weak sharpness of the solutions set for nonsmooth variational inequalities without using gap functions. Applications to the finite convergence of proximal point methods for solving nonsmooth variational inequalities are also presented.

This paper is organized as follows. In the next section, we recall some known definitions and results which will be used in the sequel. In Section 3, we provide some characterizations of the weak sharpness of the solutions set of nonsmooth variational inequalities with or without pseudomonotonicity⁺ of the objective bifunctions. We also give some characterizations of pseudomonotonicity⁺ of bifunction. Section 4 is devoted to the finite convergence property of proximal point methods for finding a solution of nonsmooth variational inequalities. We also provide an upper bound for the number of iterations for which the sequence generated by the exact proximal point method terminates.

2. Formulations and preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We denote by $\mathbf{0}$ the zero vector of the corresponding vector space. For a subset C of H , we denote by $\text{co}C$ the convex hull of

C , by $\text{cl}C$ the closure of C and by $\text{cl}C^w$ the weak closure of C . The *polar* C° of C is defined by

$$C^\circ := \{x^* \in H : \langle x^*, x \rangle \leq 0 \text{ for all } x \in C\}.$$

For a given $x \in H$, the *distance* from x to C is defined by

$$\text{dist}(x, C) := \inf_{y \in C} \|y - x\|,$$

and the *projection* of x onto C is defined by

$$P_C(x) := \{y \in C : \|y - x\| = \text{dist}(x, C)\}.$$

It is well-known that $P_C(x)$ is a singleton set if C is nonempty, closed and convex. In this case, P_C is a nonexpansive mapping, that is,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \text{for all } x, y \in C.$$

Let X be a nonempty closed convex subset of H . The *tangent cone* to X at a point $x \in X$ is defined as

$$T_X(x) := \text{cl} \left(\bigcup_{\lambda > 0} \frac{X - x}{\lambda} \right).$$

The *normal cone* to X at $x \in X$ is defined by $N_X(x) := [T_X(x)]^\circ$. In other words,

$$N_X(x) = \{x^* \in H : \langle x^*, y - x \rangle \leq 0 \text{ for all } y \in X\}.$$

Definition 2.1. Let Λ be a subset of $[0, +\infty)$. A mapping $f : H \rightarrow \mathbb{R}$ is said to be

- (a) *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ for all $x \in H$ and $\lambda \geq 0$;
- (b) Λ -*subhomogeneous* if $f(\lambda x) \leq \lambda f(x)$ for all $x \in H$ and $\lambda \in \Lambda$;
- (c) *subadditive* if $f(x + y) \leq f(x) + f(y)$ for all $x, y \in H$;
- (d) *subodd* if $f(x) + f(-x) \geq 0$ for all $x \in H \setminus \{0\}$;
- (e) *proper subodd* if

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq 0,$$

for any $n \geq 2$ and $x_1, x_2, \dots, x_n \in H$ with $\sum_{i=1}^n x_i = 0$.

The relations among these notions have their own importance. We present here some facts.

Remark 2.1. Let $f : H \rightarrow \mathbb{R}$ be a mapping.

(a) If f is positively homogeneous, then f is Λ -subhomogeneous for any $\Lambda \subset [0, +\infty)$. The converse is not true. For instance, $f(x) = \|x\|^k$ for all $x \in H$, with $k > 1$, is $[0, 1]$ -subhomogeneous, but it is not positively homogeneous.

(b) If f is subadditive, then it is subodd and $f(\mathbf{0}) \geq 0$.

(c) If f is subadditive, \mathbb{N}^{-1} -subhomogeneous and lower semicontinuous, then f is $[0, +\infty)$ -subhomogeneous. Indeed, by the subadditivity, $f(nx) \leq nf(x)$ for all $x \in H$ and $n \in \mathbb{N}$. Then by \mathbb{N}^{-1} -subhomogeneity, $f(qx) \leq qf(x)$ for all $x \in H$ and $q \in \mathbb{Q}_+$. Finally, for any $\lambda \in [0, +\infty)$, there exists $\{q_n\} \subset \mathbb{Q}_+$ such that $q_n \rightarrow \lambda$ as $n \rightarrow \infty$. Since $f(q_n x) \leq q_n f(x)$, by the lower semicontinuity, we obtain $f(\lambda x) \leq \lambda f(x)$.

(d) If f is subadditive and $[0, 1]$ -subhomogeneous, then f is convex. Indeed, let $x_1, x_2 \in H$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq f(\lambda x_1) + f((1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

(e) If $f(x) \geq 0$ for all x , then it is automatically proper subodd. A $[0, 1]$ -subhomogeneous and proper subodd may not be convex. Indeed, consider a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x_1, x_2) = \frac{x_1^4}{x_2^2 + 1}, \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

Since $f(x) \geq 0$ for all $x \in \mathbb{R}^2$, it is proper subodd. For any $\lambda \in [0, 1]$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we have

$$f(\lambda x) = \frac{\lambda^4 x_1^4}{\lambda^2 x_2^2 + 1} \leq \frac{\lambda^4 x_1^4}{\lambda^2 x_2^2 + \lambda^2} \leq \frac{\lambda^2 x_1^4}{x_2^2 + 1} \leq \lambda \frac{x_1^4}{x_2^2 + 1} = \lambda f(x).$$

Thus f is $[0, 1]$ -subhomogeneous. However, f is not convex since

$$\frac{1}{2}f(1, 0) + \frac{1}{2}f(1, 1) = \frac{3}{4} < \frac{4}{5} = f\left(1, \frac{1}{2}\right).$$

In this example, f is neither positively homogeneous.

Let X be a nonempty, closed and convex subset of H and $h: X \times H \rightarrow \mathbb{R}$ be a bifunction such that $h(x; \mathbf{0}) = 0$ for all $x \in X$. The *nonsmooth variational inequality problem* (in short, NVIP) is to find $x^* \in X$ such that

$$h(x^*; y - x^*) \geq 0, \quad \text{for all } y \in X. \quad (2.1)$$

A problem closely to NVIP (2.1) is the following *Minty type nonsmooth variational inequality problem* (in short, MNVIP): Find $x^* \in X$ such that

$$h(y; x^* - y) \leq 0, \quad \text{for all } y \in X. \quad (2.2)$$

We denote by X^* and X_* the solution sets of NVIP (2.1) and MNVIP (2.2), respectively.

When $h(x; y - x) = \langle F(x), y - x \rangle$ for all $x, y \in X$, where $F : X \rightarrow H$, then (2.1) coincides with the following classical *variational inequality problem* (in short, VIP): Find $x^* \in X$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \text{for all } y \in X. \quad (2.3)$$

For a comprehensive study of nonsmooth variational inequalities and their applications to nonsmooth optimization, we refer [5].

Definition 2.2. Let $X \subseteq H$ be a nonempty convex set. A function $f : X \rightarrow \mathbb{R}$ is said to be radially upper semicontinuous if for every pair of distinct points $x, y \in X$, the function f is upper semicontinuous on the line segment joining x and y .

Definition 2.3. Let $X \subseteq H$ be a nonempty convex set. A bifunction $h : X \times H \rightarrow \mathbb{R}$ is said to be upper sign continuous if for all $x, y \in X$,

$$h(y + \lambda(x - y); x - y) \leq 0 \text{ for all } \lambda \in (0, 1) \quad \Rightarrow \quad h(x; y - x) \geq 0.$$

Note that every radially upper semicontinuous function is upper sign continuous.

Definition 2.4. Let $X \subseteq H$ be a nonempty convex set and $h : X \times H \rightarrow \mathbb{R}$ be a bifunction with $h(x; \mathbf{0}) = 0$ for all $x \in X$. The bifunction h is said to be

(a) *monotone* on X if for all $x, y \in X$,

$$h(x; y - x) + h(y; x - y) \leq 0;$$

(b) *pseudomonotone* at $x \in X$ if for each $y \in X$,

$$h(x; y - x) \geq 0 \quad \Rightarrow \quad h(y; x - y) \leq 0;$$

(c) *pseudomonotone*⁺ at $x \in X$ if it is pseudomonotone at x and for each $y \in X$,

$$h(x; y - x) \geq 0 \text{ and } h(y; x - y) = 0 \quad \Rightarrow \quad h(x; d) = -h(y; -d), \quad \text{for all } d \in H;$$

(d) *pseudomonotone* (*pseudomonotone*⁺) on a set $C \subseteq X$ if it is pseudomonotone (*pseudomonotone*⁺) at each $x \in C$.

Remark 2.2. If h is monotone on X , then it is pseudomonotone on X . If h is pseudomonotone⁺ at $x \in X$, then $h(x; \cdot)$ is an odd function.

The following result provides the equivalence between the NVIP and the MNVIP under pseudomonotonicity and upper sign continuity assumptions.

Lemma 2.1. *[5] If h is pseudomonotone, then $X^* \subset X_*$. If h is positively homogeneous in the second argument and upper sign continuous, then $X_* \subset X^*$.*

The following lemma provides sufficient conditions for convexity and closedness of the solution set of NVIP (2.1) which is a refinement of [3, Proposition 2.1].

Lemma 2.2. *Assume that h is positively homogeneous in the second argument, upper sign continuous and pseudomonotone. If $x \mapsto h(y; x - y)$ is convex on X for each $y \in X$, then X^* is convex. If $x \mapsto h(y; x - y)$ is lower semicontinuous on X for each $y \in X$, then X^* is closed.*

Proof. By Lemma 2.1, $X^* = X_*$. Let $x_1, x_2 \in X^*$. Then, for all $y \in X$, we have $h(y; x_1 - y) \leq 0$ and $h(y; x_2 - y) \leq 0$. By the convexity of $x \mapsto h(y; x - y)$, we have

$$h(y; \lambda x_1 + (1 - \lambda)x_2 - y) \leq \lambda h(y; x_1 - y) + (1 - \lambda)h(y; x_2 - y) \leq 0, \quad \text{for all } \lambda \in [0, 1].$$

This means that $\lambda x_1 + (1 - \lambda)x_2 \in X_* = X^*$. Thus, X^* is convex. The closedness of X^* is proved in [3, Proposition 2.1]. \square

We now recall the definition of a primal gap function $\varphi(x)$ associated with NVIP (2.1) and a dual gap function $\Phi(x)$ associated with MNVIP (2.2) as follows: For $x \in X$,

$$\varphi(x) = \sup\{-h(x; y - x) : y \in X\},$$

and

$$\Phi(x) = \sup\{h(y; x - y) : y \in X\}.$$

These two functions are nonnegative on X . Related to φ and Φ , we have the following sets: For $x \in X$,

$$\Gamma(x) := \{y \in X : h(x; y - x) = -\varphi(x)\},$$

and

$$\Lambda(x) := \{y \in X : h(y; x - y) = \Phi(x)\}.$$

It is known (see, [1, Proposition 3]) that for $x \in X$,

$$x \in X^* \Leftrightarrow \varphi(x) = 0 \Leftrightarrow x \in \Gamma(x),$$

and

$$x \in X_* \Leftrightarrow \Phi(x) = 0 \Leftrightarrow x \in \Lambda(x).$$

Moreover, we have (see, [1, Proposition 5]) that

$$X^* \subset \Lambda(x_*), \quad \text{for all } x_* \in X_*,$$

and

$$X_* \subset \Gamma(x^*), \quad \text{for all } x^* \in X^*.$$

Finally, we recall some definitions and results which will be used in Section 4.

Definition 2.5. Let $X \subseteq H$ be a nonempty convex set. A bifunction $h : X \times H \rightarrow \mathbb{R}$ is said to be *0-diagonally convex* if for any finite set $\{x, x_1, x_2, \dots, x_n\} \subseteq X$ and for any $\lambda_i \geq 0, i = 1, 2, \dots, n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i x_i = x$, we have

$$\sum_{i=1}^n \lambda_i h(x; x_i - x) \geq 0.$$

Remark 2.3. (a) The concept of α -diagonal convexity for a bifunction is defined in [25]. It can be seen that if $h : X \times H \rightarrow \mathbb{R}$ is $[0, 1]$ -subhomogeneous and proper subodd in the second argument, then it is 0-diagonally convex.

(b) For each $y \in X$, if $x \mapsto h(y; x - y)$ is convex, then h is also 0-diagonally convex.

(c) There exists $h : X \times H \rightarrow \mathbb{R}$ such that it is $[0, 1]$ -subhomogeneous and proper subodd in the second argument, but $x \mapsto h(y; x - y)$ is not convex for some $y \in X$. For example, based on the example in Remark 2.1 (e), the function $h : [0, 1]^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$h(x; d) = (x_1 + x_2) \frac{d_1^4}{d_2^2 + 1}, \quad \text{for all } x = (x_1, x_2) \in [0, 1]^2, d = (d_1, d_2) \in \mathbb{R}^2,$$

is not convex, but it is $[0, 1]$ -subhomogeneous and proper subodd in the second argument, and therefore, it is 0-diagonally convex.

Definition 2.6 (Knaster–Kuratowski–Mazurkiewicz). Let X be a Hausdorff topological real linear space and $M \subseteq X$. The set-valued mapping $G : M \rightrightarrows X$ is called a *KKM mapping*, if for every finite number of elements $x_1, x_2, \dots, x_n \in M$, one has

$$\text{co}\{x_1, x_2, \dots, x_n\} \subseteq \bigcup_{i=1}^n G(x_i).$$

Lemma 2.3 (Fan-KKM Lemma). (see [9, 15]) Let X be a Banach space, $M \subseteq X$ be a nonempty set and $G : M \rightrightarrows X$ be a KKM mapping. If $G(x)$ is weakly (sequentially) closed for every $x \in M$ and there exists $x_0 \in M$ such that $G(x_0)$ is weakly compact, then

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

3. Weak sharp solutions for nonsmooth variational inequalities

We first recall the definition of weak sharp solutions for NVIP (2.1) which was introduced in [3].

Definition 3.1. The solution set X^* of NVIP (2.1) is said to be *weakly sharp* if there exists a constant $\alpha > 0$ such that, for all $x^* \in X^*$,

$$h(x^*; d) \geq \alpha \|d\|, \quad \text{for all } d \in T_X(x^*) \cap N_{X^*}(x^*).$$

The constant α is called the *modulus of weak sharpness* of the solution set X^* .

Example 3.1. Let $H = \mathbb{R}^2$ and $X = [0, 1]^2$ and $h : X \times H \rightarrow \mathbb{R}$ be defined by

$$h(x; d) = \frac{x_1 d_1^3}{d_2^2 + 1} + d_1 + d_2, \quad \text{for all } x = (x_1, x_2) \in X, d = (d_1, d_2) \in H.$$

Assume that $x^* = (x_1^*, x_2^*) \in X^*$ is a solution of NVIP (2.1). Then,

$$\frac{x_1^* (y_1 - x_1^*)^3}{(y_2 - x_2^*)^2 + 1} + (y_1 - x_1^*) + (y_2 - x_2^*) \geq 0, \quad \text{for all } (y_1, y_2) \in [0, 1]^2.$$

So, x^* must be $(0, 0)$. Thus, $X^* = \{(0, 0)\}$. We have $T_X(0, 0) = \mathbb{R}_+^2$ and $N_{X^*}(0, 0) = \mathbb{R}^2$. Hence,

$$T_X(0, 0) \cap N_{X^*}(0, 0) = \mathbb{R}_+^2.$$

For any $d = (d_1, d_2) \in T_X(0, 0) \cap N_{X^*}(0, 0)$, one has

$$h(0; d) = d_1 + d_2 \geq \sqrt{d_1^2 + d_2^2} = \|d\|.$$

Thus, X^* is weakly sharp with modulus 1.

In [3], Al-Homidan, Ansari and Nguyen characterized the weak sharpness of the solutions set X^* in term of the dual gap function under, among others, the compactness of X . In this section, we present some other characterizations of the weak sharpness of X^* without using gap functions and we do not require the compactness of X .

3.1. Weak sharpness under pseudomonotonicity⁺

In this subsection, we characterize the weak sharpness of X^* when the bifunction h is pseudomonotone⁺. We first provide the characterization of the pseudomonotonicity⁺ of h on the solution set X^* .

Proposition 3.1. *Assume that h is subadditive in the second argument and X^* is nonempty. Then the following assertions are equivalent:*

- (a) h is pseudomonotone⁺ on X^* .
- (b) $X^* \subseteq X_*$ and for all $x^* \in X^*$, $h(x^*; d) = -h(x; -d)$ and $h(x; d) = h(y; d)$ for all $x, y \in \Lambda(x^*)$ and for all $d \in H$.
- (c) $X^* \subseteq X_*$, $X^* = \Lambda(x^*)$ for each $x^* \in X^*$ and $h(x; d) = h(y; d)$ and $h(x; d) = -h(y; -d)$ for all $x, y \in X^*$ and for all $d \in H$.

Proof. (a) \Rightarrow (b). Let h be pseudomonotone⁺ on X^* and $x^* \in X^*$. Then, $h(x^*; z - x^*) \geq 0$ for all $z \in X$, and by pseudomonotonicity⁺ of h on X^* , we have $h(z; x^* - z) \leq 0$ for all $z \in X$, i.e., $x^* \in X_*$. Thus, $X^* \subseteq X_*$ and $\Phi(x^*) = 0$. For $x \in \Lambda(x^*)$, we have $h(x; x^* - x) = \Phi(x^*) = 0$ and $h(x^*; x - x^*) \geq 0$. Hence, by pseudomonotone⁺ of h on X^* , we have $h(x^*; d) = -h(x; -d)$ for all $d \in H$. Since x is arbitrary in $\Lambda(x^*)$, we conclude that $h(x; d) = h(y; d)$ for all $x, y \in \Lambda(x^*)$ and all $d \in H$.

(b) \Rightarrow (c). It is sufficient to show that $X^* = \Lambda(x^*)$ for all $x^* \in X^*$. This can be done in the same way as in Proposition 3.2 in [3] where h is assumed to be pseudomonotone⁺ on X . We present the proof here for the reader's convenience. Fix $x^* \in X^*$. For any $y^* \in X^*$, we have $h(y^*; x^* - y^*) \geq 0$. Since $X^* \subseteq X_*$, $h(y^*; x^* - y^*) \leq 0$. Hence, $h(y^*; x^* - y^*) = 0$ which implies that $y^* \in \Lambda(x^*)$. Thus, $X^* \subseteq \Lambda(x^*)$.

Let $z^* \in \Lambda(x^*)$ be arbitrary. Then, $h(z^*; x^* - z^*) = 0$. By subadditivity of h , we have

$$0 \leq h(x^*; x - x^*) \leq h(x^*; z^* - x^*) + h(x^*; x - z^*) = h(x^*; x - z^*), \quad \text{for all } x \in X.$$

Since $h(x^*; x - z^*) = -h(z^*; z^* - x)$, it follows from the latter inequality that $h(z^*; z^* - x) \leq 0$. As h is subodd in the second argument, $-h(z^*; x - z^*) \leq h(z^*; z^* - x) \leq 0$. Thus, $h(z^*; x - z^*) \geq 0$. This means that $z^* \in X^*$. Hence, $\Lambda(x^*) \subseteq X^*$.

(c) \Rightarrow (a). Let $x^* \in X^*$. Then, $h(x^*; z - x^*) \geq 0$ for all $z \in X$. Since $X^* \subseteq X_*$, the latter inequality implies $h(z; x^* - z) \leq 0$ for all $z \in X$. Thus, F is pseudomonotone. Now, let $z \in X$ be such that $h(z; x^* - z) = 0$. Then, $z \in \Lambda(x^*) = X^*$. Thus, by assumption, $h(x^*; d) = -h(z; -d)$ for all $d \in H$. Therefore, h is pseudomonotone⁺ on X^* . \square

Proposition 3.2. *Assume that h is positively homogeneous and subadditive in the second argument. If h is pseudomonotone⁺ on X^* , then there exists a function $f : H \rightarrow \mathbb{R}$ such that for each $x^* \in X^*$, $f(x^*) = 0$, $f'(x^*; d) = h(x^*; d)$ for all $d \in H$ and*

$$f(x) \geq h(z; x - z), \quad \text{for all } (z, x) \in \Lambda(x^*) \times H.$$

Moreover, if h is continuous with respect to the second argument, then f is Lipschitz near $x^ \in X^*$.*

Proof. Consider a function $f : H \rightarrow \mathbb{R}$ defined by

$$f(x) := \sup\{h(z; x - z) : z \in X^*\}, \quad \text{for all } x \in H.$$

Let $x^* \in X^*$. Since $\Lambda(x^*) = X^*$, by the definition of f , we have

$$f(x) \geq h(z; x - z), \quad \text{for all } (z, x) \in \Lambda(x^*) \times H.$$

In particular,

$$f(x) \geq h(x^*; x - x^*), \quad \text{for all } x \in H. \quad (3.1)$$

Moreover, for all $(z, x) \in X^* \times H$, by subadditivity of h , we have

$$h(z; x - z) = h(z; x - x^* + x^* - z) \leq h(z; x - x^*) + h(z; x^* - z).$$

Since $x^* \in X^* \subseteq X_*$, we have $h(z; x^* - z) \leq 0$, and thus, $h(z; x - z) \leq h(z; x - x^*)$ for all $(z, x) \in X^* \times H$. Since h is pseudomonotone⁺, by Proposition 3.1, we have $h(z; x - x^*) = h(x^*; x - x^*)$. Hence, $h(z; x - z) \leq h(x^*; x - x^*)$ for all $(z, x) \in X^* \times H$. Thus,

$$f(x) \leq h(x^*; x - x^*), \quad \text{for all } x \in H. \quad (3.2)$$

It follows from (3.1) and (3.2) that $f(x) = h(x^*; x - x^*)$ for all $x \in H$. This implies that $f(x^*) = h(x^*; \mathbf{0}) = 0$.

Now, for $d \in H$ and $t > 0$, by positive homogeneity of h , we have

$$\frac{f(x^* + td) - f(x^*)}{t} = \frac{h(x^*; x^* + td - x^*)}{t} = h(x^*; d).$$

Letting $t \rightarrow 0^+$, we obtain $f'(x^*; d) = h(x^*; d)$.

Since h is positive homogeneous and subadditive in the second argument, it is convex with respect to the second argument. Hence, $f(x) = h(x^*; x - x^*)$ is convex. This, together with the continuity of h and $f(x^*) = 0$, implies that f is Lipschitz near x^* . \square

We are now ready to present a characterization of the weak sharpness of X^* under the pseudomonotonicity⁺ of h .

Theorem 3.1. *Assume that h is positively homogeneous, subadditive and continuous in the second argument, and it is pseudomonotone⁺ on X^* . Then the following assertions are equivalent:*

(a) X^* is weakly sharp.

(b) There exists a function $f : H \rightarrow \mathbb{R}$ such that, for each $x^* \in X$, $f(x^*) = 0$, $f'(x^*; d) = h(x^*; d)$ for all $d \in H$, f is Lipschitz near x^* and satisfies

$$f(x) \geq h(z; x - z), \quad \text{for all } (z, x) \in \Lambda(x^*) \times H, \quad (3.3)$$

and there exists $\alpha > 0$ such that $\alpha \text{dist}(x, X^*) \leq h(x^*; x - x^*)$ for all $x \in X$.

(c) There exists a function $f : H \rightarrow \mathbb{R}$ such that, for each $x^* \in X$, $f(x^*) = 0$, $f'(x^*; d) = h(x^*; d)$ for all $d \in H$, f is Lipschitz near x^* and satisfies (3.3), and there exists $\alpha > 0$ such that

$$\alpha \text{dist}(x, X^*) \leq f(x), \quad \text{for all } x \in X. \quad (3.4)$$

Proof. (a) \Rightarrow (b). The existence of a function $f : H \rightarrow \mathbb{R}$, which satisfies all the conditions in (b), follows from Proposition 3.2. For $x \in X$, let $y^* = P_{X^*}(x)$. Then, $\|x - y^*\| = \text{dist}(x, X^*)$ and $x - y^* \in T_X(y^*) \cap N_{X^*}(y^*)$. By weak sharpness of X^* , there exists $\alpha > 0$ such that

$$\alpha \text{dist}(x, X^*) = \alpha \|x - y^*\| \leq h(y^*; x - y^*). \quad (3.5)$$

By subadditivity of h , we have

$$h(y^*; x - y^*) = h(y^*; x - x^*) + h(y^*; x^* - y^*) \leq h(y^*; x - x^*),$$

since $h(y^*; x^* - y^*) \leq 0$. It follows from Proposition 3.1 that $h(y^*; x - x^*) = h(x^*; x - x^*)$. Thus, $h(y^*; x - y^*) \leq h(x^*; x - x^*)$. This, together with (3.5), leads to

$$\alpha \text{dist}(x, X^*) \leq h(x^*; x - x^*).$$

(b) \Rightarrow (c). We only need to show (3.4). Let $x^* \in X^*$. By (3.3), for each $z \in \Lambda(x^*)$, we have that $h(z; x - z) \leq f(x)$ for all $x \in H$. In particular, $f(x) \geq h(x^*; x - x^*)$ for all $x \in H$. Therefore,

$$\alpha \text{dist}(x, X^*) \leq h(x^*; x - x^*) \leq f(x), \quad \text{for all } x \in X.$$

(c) \Rightarrow (a). Assume that f is a function as in (b) and $\alpha > 0$ such that $\alpha \text{dist}(x, X^*) \leq f(x)$ for all $x \in X$. We show that α satisfies

$$\alpha \|d\| \leq h(x^*; d), \quad \text{for all } x^* \in X^* \text{ and } d \in T_X(x^*) \cap N_{X^*}(x^*). \quad (3.6)$$

Let $x^* \in X^*$ and $d \in T_X(x^*) \cap N_{X^*}(x^*)$. If $d = \mathbf{0}$, then (3.6) trivially holds. Assume that $d \neq \mathbf{0}$. Since $d \in N_{X^*}(x^*)$,

$$\langle d, y^* - x^* \rangle \leq 0, \quad \text{for all } y^* \in X^*.$$

Additionally, $\langle d, d \rangle > 0$. Thus, X^* is separated from $x^* + d$ by the hyperplane

$$H_d = \{x \in H : \langle d, x - x^* \rangle = 0\}.$$

Since $d \in T_X(x^*)$, for each sequence of positive real numbers $\{t_k\}$ converging to 0, there exists a sequence $\{d_k\}$ converging to d such that $x^* + t_k d_k \in X$ for sufficiently large k . Hence, we have $\langle d, d_k \rangle > 0$ for sufficiently large k . Since $\langle d, x^* + t_k d_k - x^* \rangle = t_k \langle d, d_k \rangle > 0$ for k large enough, $x^* + t_k d_k$ belongs to the open set $\{x \in H : \langle d, x - x^* \rangle > 0\}$, which is separated from X^* by H_d . Therefore, for sufficiently large k , we have

$$\text{dist}(x^* + t_k d_k, X^*) \geq \text{dist}(x^* + t_k d_k, H_d) = \frac{t_k \langle d, d_k \rangle}{\|d\|}. \quad (3.7)$$

Using (3.4), we have

$$f(x^* + t_k d_k) \geq \alpha \text{dist}(x^* + t_k d_k, X^*) \geq \frac{\alpha t_k \langle d, d_k \rangle}{\|d\|}.$$

Since $f(x^*) = 0$, $f'(x^*; d) = h(x^*; d)$ and f is Lipschitz near x^* , we have

$$\begin{aligned} h(x^*; d) &= f'(x^*; d) = \lim_{k \rightarrow \infty} \frac{f(x^* + t_k d) - f(x^*)}{t_k} \\ &= \lim_{k \rightarrow \infty} \frac{f(x^* + t_k d_k) - f(x^*)}{t_k} \geq \alpha \|d\|. \end{aligned}$$

The proof is complete. □

Remark 3.1. In [3], Al-Homidan, Ansari and Nguyen characterized the weak sharpness of X^* in terms of the dual gap function Φ (see, Theorem 4.1 in [3]) under pseudomonotonicity⁺ of h . Note that under the assumptions of Theorem 4.1 in [3], the dual gap function satisfies all the conditions that f satisfies of Theorem 3.1 (b). Thus, Theorem 4.1 in [3] is a consequence of Theorem 3.1.

Remark 3.2. From the proof of (a) \Rightarrow (b) of Theorem 3.1, we can see that if h is pseudomonotone⁺ on X^* and h is subadditive in the second argument and X^* is weakly sharp, then there exists $\alpha > 0$ such that for each $x^* \in X^*$,

$$\alpha \text{dist}(x, X^*) \leq h(x^*; x - x^*), \quad \text{for all } x \in X.$$

We can then characterize the weak sharpness of the solutions set X^* as follows.

Corollary 3.1. *Assume that h is positively homogeneous, subadditive and continuous in the second argument, and it is pseudomonotone⁺ on X^* . Then, X^* is weakly sharp if and only if there exist $\alpha > 0$ and $x^* \in X^*$ such that $\alpha \text{dist}(x, X^*) \leq h(x^*; x - x^*)$ for all $x \in X$.*

Alshahrani *et al.* [1] introduced and studied minimum and maximum principle sufficiency properties for non-smooth variational inequalities and provided several characterizations. To conclude this subsection, we show that if X^* is weakly sharp, then NVIP (2.1) has the *minimum principle sufficiency property*, i.e., $\Gamma(x^*) = X^*$ for all $x^* \in X^*$. More precisely, we have the following proposition.

Proposition 3.3. *Assume that h is subadditive in the second argument, and it is pseudomonotone⁺ on X^* . If X^* is weakly sharp, then NVIP (2.1) has the minimum principle sufficiency property.*

Proof. Since h is pseudomonotone⁺ on X^* , we have $X^* \subset X_*$. Moreover, $X_* \subseteq \Gamma(x^*)$ for all $x^* \in X^*$. Thus, $X^* \subset \Gamma(x^*)$ for all $x^* \in X^*$. We prove that $\Gamma(x^*) \subseteq X^*$ for all $x^* \in X^*$. Letting $x^* \in X^*$ and $\hat{x} \in \Gamma(x^*)$, we have

$$h(x^*; y - x^*) \geq h(x^*; \hat{x} - x^*), \quad \text{for all } y \in X.$$

In particular, taking $y := x^*$ in the previous inequality, we get $h(x^*; \hat{x} - x^*) \leq 0$. Since X^* is weakly sharp, as in Remark 3.2, we have for some $\alpha > 0$ that

$$\alpha \text{dist}(\hat{x}, X^*) \leq h(x^*; \hat{x} - x^*) \leq 0,$$

which implies that $\hat{x} \in X^*$. Thus, $\Gamma(x^*) \subseteq X^*$. The proof is complete. \square

Remark 3.3. It is still open whether we can characterize the weak sharpness of X^* using the minimum principle sufficiency property (see [1, Question 1]).

3.2. Weak sharpness without pseudomonotonicity⁺

We are now going to characterize the weak sharpness of X^* without the pseudomonotonicity⁺ of h . Our results generalize some results for variational inequalities in [4].

Proposition 3.4. *Assume that X^* is weakly sharp with modulus α . Then,*

$$h(P_{X^*}(x); x - P_{X^*}(x)) \geq \alpha \text{dist}(x, X^*), \quad \text{for all } x \in X. \quad (3.8)$$

If h is monotone, then

$$-h(x; P_{X^*}(x) - x) \geq \alpha \text{dist}(x, X^*), \quad \text{for all } x \in X. \quad (3.9)$$

Proof. For any $x \in X$, we have $\|x - P_{X^*}(x)\| = \text{dist}(x, X^*)$, and

$$x - P_{X^*}(x) \in T_X(P_{X^*}(x)) \cap N_{X^*}(P_{X^*}(x)).$$

Thus, by weak sharpness of X^* , one has

$$h(P_{X^*}(x); x - P_{X^*}(x)) \geq \alpha \|x - P_{X^*}(x)\| = \alpha \text{dist}(x, X^*).$$

If h is monotone, then

$$-h(x; P_{X^*}(x) - x) \geq h(P_{X^*}(x), x - P_{X^*}(x)) \geq \alpha \text{dist}(x, X^*).$$

The proof is complete. \square

Theorem 3.2. *Assume that h is $[0, 1]$ -subhomogeneous in the second argument and upper semicontinuous in both arguments. If either*

(i) (3.8) holds, or

(ii) (3.9) holds and h is subodd in the second argument,

then X^* is weakly sharp with modulus α .

Proof. It is sufficient to show that, for each $x^* \in X^*$ and $\mathbf{0} \neq d \in T_X(x^*) \cap N_{X^*}(x^*)$,

$$h(x^*; d) \geq \alpha \|d\|.$$

As in the proof of (c) \Rightarrow (a) in Theorem 3.1, for each $x^* \in X^*$, $\mathbf{0} \neq d \in T_X(x^*) \cap N_{X^*}(x^*)$ and for any sequence of positive real numbers $\{t_k\}$ converging to 0, there exists a sequence $\{d_k\}$ converging to d such that $x^* + t_k d_k \in X$ for sufficiently large k . In addition, for sufficiently large k , one has

$$\text{dist}(x^* + t_k d_k, X^*) \geq \frac{t_k \langle d, d_k \rangle}{\|d\|}. \quad (3.10)$$

(i) Since (3.8) holds, we have, for sufficiently large k , that

$$h(P_{X^*}(x^* + t_k d_k); x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \geq \alpha \text{dist}(x^* + t_k d_k, X^*) \geq \alpha \frac{t_k \langle d, d_k \rangle}{\|d\|}. \quad (3.11)$$

We may assume that $t_k \in (0, 1)$ for all k . Since h is $[0, 1]$ -subhomogeneous in the second argument, one has

$$h(P_{X^*}(x^* + t_k d_k); x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \leq t_k h\left(P_{X^*}(x^* + t_k d_k); \frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k}\right).$$

By combining this with (3.11), we get

$$\alpha \frac{\langle d, d_k \rangle}{\|d\|} \leq h\left(P_{X^*}(x^* + t_k d_k); \frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k}\right). \quad (3.12)$$

Since $t_k > 0$ and $d \in N_{X^*}(x^*)$, we have $x^* = P_{X^*}(x^* + t_k d)$. By nonexpansiveness of the projection mapping, we have

$$\begin{aligned} \left\| \frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k} - d \right\| &= \left\| d_k - d + \frac{P_{X^*}(x^* + t_k d) - P_{X^*}(x^* + t_k d_k)}{t_k} \right\| \\ &\leq \|d_k - d\| + \|d - d_k\| \\ &= 2\|d_k - d\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Thus,

$$\frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k} \rightarrow d \quad \text{as } k \rightarrow \infty.$$

Since $d_k \rightarrow d$ and $t_k \rightarrow 0$ as $k \rightarrow \infty$, one has $x^* + t_k d_k \rightarrow x^*$ as $k \rightarrow \infty$. By continuity of h and P_{X^*} , and taking limit as $k \rightarrow \infty$ both sides of (3.12), we obtain

$$h(x^*; d) \geq \alpha \|d\|.$$

Thus, X^* is weakly sharp with modulus α .

(ii) Since (3.9) holds, we have, for sufficiently large k , that

$$-h(x^* + t_k d_k, P_{X^*}(x^* + t_k d_k) - x^* - t_k d_k) \geq \alpha \frac{t_k \langle d, d_k \rangle}{\|d\|}. \quad (3.13)$$

Since h is subodd in the second argument, we obtain

$$h(x^* + t_k d_k, x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \geq -h(x^* + t_k d_k, P_{X^*}(x^* + t_k d_k) - x^* - t_k d_k).$$

It follows from (3.13) that

$$h(x^* + t_k d_k, x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \geq \alpha \frac{t_k \langle d, d_k \rangle}{\|d\|}.$$

By $[0, 1]$ -subhomogeneousness of h in the second argument, we have

$$h(x^* + t_k d_k, x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \leq t_k h\left(x^* + t_k d_k, \frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k}\right).$$

Hence,

$$h\left(x^* + t_k d_k, \frac{x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)}{t_k}\right) \geq \alpha \frac{\langle d, d_k \rangle}{\|d\|}.$$

As in the proof of part (i), letting $k \rightarrow \infty$ in the latter inequality, we get

$$h(x^*; d) \geq \alpha \|d\|.$$

This ends the proof. □

4. Finite termination property for proximal point methods

In this section, we study the finite termination property for the sequence generated by proximal point methods under the weak sharpness of the solutions set X^* of NVIP (2.1). Let us consider the following inexact proximal point method: For $x_1 \in H$,

$$h(x_{n+1}; y - x_{n+1}) + \frac{1}{\gamma_n} \langle y - x_{n+1}, x_{n+1} - x_n + e_n \rangle \geq 0, \quad \text{for all } y \in X, \quad (4.1)$$

where $\{\gamma_n\}$ is a sequence of positive real numbers and $\{e_n\}$ is a sequence in H .

The method (4.1) is well-defined under suitable assumptions. For example, we recall the following well-known result.

For each $x \in H$ and $r > 0$, we define

$$Q_r(x) = \left\{ z \in X : h(z; y - z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in X \right\}. \quad (4.2)$$

Lemma 4.1. (see, [8, Lemma 2.12]) *Let X be a nonempty, closed and convex subset of H and $h : X \times H \rightarrow \mathbb{R}$ satisfy the following conditions:*

- (i) $h(x; \mathbf{0}) = 0$.
- (ii) h is monotone on X .
- (iii) For each $x \in X$, $y \mapsto h(x; y - x)$ is convex and lower semicontinuous on X .
- (iv) $\lim_{t \downarrow 0} h(tz + (1-t)x; y - [tz + (1-t)x]) \leq h(x; y - x)$ for any $x, y, z \in X$.

Then, for any $x \in H$ and $r > 0$, $Q_r(x)$, defined by (4.2), is single-valued and the solutions set X^* of NVIP (2.1) is closed and convex.

A more general form of Lemma 4.1 is given in the form of Lemma 2.12 in [8]. From the proof of [8, Lemma 2.12], one can see that if h is monotone, then $Q_r(x)$ has no more than one element for any $x \in X$ and $r > 0$. Moreover, if $Q_r(x)$ is single-valued for all $x \in X$, then the solution set X^* of NVIP (2.1) is closed and convex. In the next lemma, we provide other conditions for the nonemptiness of $Q_r(x)$ under general convexity.

Lemma 4.2. *Let X be a nonempty, closed and convex subset of H and $h : X \times H \rightarrow \mathbb{R}$ satisfy the following conditions:*

- (i) h is 0-diagonally convex;
- (ii) if $\{x_n\} \subseteq K$ converges weakly to $x \in K$, then $\liminf_{n \rightarrow \infty} h(x_n; y - x_n) \leq h(x; y - x)$ for all $y \in K$.

Assume that either

(iii)(a) X is bounded; or

(iii)(b) For any $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset X$ and $y_x \in X$ such that, for any $z \in X \setminus D_x$,

$$h(z; y_x - z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0.$$

Then, for each $r > 0$ and $x \in H$, the set $Q_r(x)$, defined by (4.2), is nonempty.

Proof. Let $r > 0$ and $x \in H$. Consider the set-valued mapping $P : X \rightrightarrows X$ defined by

$$P(y) := \left\{ z \in X : h(z; y - z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \right\}, \quad \text{for all } y \in X.$$

One can see that for each $y \in X$, $P(y)$ is nonempty since $y \in P(y)$. We shall show that P is a KKM mapping. Now, let $\{y_1, y_2, \dots, y_m\}$ be a finite subset of X and $y \in \text{co}\{y_1, y_2, \dots, y_m\}$. We show that $y \in \bigcup_{i=1}^m P(y_i)$. Indeed, assume to the contrary that $y \notin P(y_i)$ for all $i = 1, 2, \dots, m$, that is,

$$h(y; y_i - y) + \frac{1}{r} \langle y_i - y, y_i - x \rangle < 0, \quad (4.3)$$

for each $i = 1, 2, \dots, m$. Since $y \in \text{co}\{y_1, y_2, \dots, y_m\}$, there exist α_i , $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$ such that $y = \sum_{i=1}^m \alpha_i y_i$. Thus, it follows (4.3) that

$$\sum_{i=1}^m \alpha_i h(y; y_i - y) < 0,$$

which contradicts to the 0-diagonal convexity of h . Thus, P is a KMM mapping.

Observe that the weak closure $\text{cl}P(y)^w$ of $P(y)$ is a weakly closed subset of X for each $y \in X$. If (iii)(a) holds, then $\text{cl}P(y)^w$ is weakly compact for each $y \in X$. If (iii)(b) holds, then for $x \in H$, there exist a bounded subset D_x of X and $y_x \in X$ such that for any $z \in X \setminus D_x$,

$$h(z; y_x - z) + \frac{1}{r} \langle y_x - z, z - z \rangle < 0.$$

This implies that

$$P(y_x) = \left\{ z \in X : h(z; y_x - z) + \frac{1}{r} \langle y_x - z, z - z \rangle \geq 0 \right\} \subset D_x,$$

and thus, $\text{cl}P(y_x)^w$ is weakly compact. Hence, in both cases, there exists $y_0 \in X$ such that $\text{cl}P(y_0)^w$ is weakly compact. We shall show that $P(y) = \text{cl}P(y)^w$ for all $y \in X$. For $y \in X$ and $z \in \text{cl}P(y_0)^w$, let $\{z_n\} \subseteq P(y)$ be such that $\{z_n\}$ converges weakly to some z . By (ii), we have

$$\liminf_{n \rightarrow \infty} h(z_n; y - z_n) \leq h(z; y - z).$$

Moreover, since $\|\cdot\|^2$ is weakly lower semicontinuous, we have

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \langle y - z_n, z_n - x \rangle &= \liminf_{n \rightarrow \infty} (\langle z_n - y, x \rangle + \langle y, z_n \rangle - \|z_n\|^2) \\
&= \lim_{n \rightarrow \infty} \langle z_n - y, x \rangle + \lim_{n \rightarrow \infty} \langle y, z_n \rangle - \limsup_{n \rightarrow \infty} \|z_n\|^2 \\
&\leq \lim_{n \rightarrow \infty} \langle z_n - y, x \rangle + \lim_{n \rightarrow \infty} \langle y, z_n \rangle - \liminf_{n \rightarrow \infty} \|z_n\|^2 \\
&\leq \langle z - y, x \rangle + \langle y, z \rangle - \|z\|^2 \\
&= \langle y - z, z - x \rangle.
\end{aligned}$$

It then follows from the fact $z_n \in P(y)$ that

$$\begin{aligned}
0 &\leq \liminf_{n \rightarrow \infty} h(z_n; y - z_n) + \frac{1}{r} \langle y - z_n, z_n - x \rangle \\
&\leq h(z; y - z) + \frac{1}{r} \langle y - z, z - x \rangle,
\end{aligned}$$

i.e., $z \in P(y)$. Hence, $P(y)$ is weakly closed for all $y \in X$, and there exists $y_0 \in X$ such that $P(y_0)$ is weakly compact.

Applying Lemma 2.3, we have $Q_r(x) = \bigcap_{y \in X} P(y) \neq \emptyset$. \square

Remark 4.1. If for each $y \in X$, $x \mapsto h(x; y - x)$ is weakly upper semicontinuous, then condition (ii) in Lemma 4.2 holds.

Remark 4.2. In Lemma 4.2, if, in addition, h is monotone, then $Q_r(x)$ is single-valued and the solutions set X^* of NVIP (2.1) is closed and convex.

From now on, we always assume that the solutions set X^* of NVIP (2.1) is nonempty, closed and convex, and method (4.1) is well-defined. We have the following result about the finite termination property of the sequence generated by the inexact proximal point method under the weak sharpness of X^* .

Theorem 4.1. *Let h be monotone and let $\{x_n\}$ be a sequence generated by inexact proximal point method (4.1) with $\liminf_{n \rightarrow \infty} \gamma_n > 0$. Suppose that the following conditions hold:*

- (i) *The sequence $\{x_{n+1} - x_n\}$ converges strongly to 0;*
- (ii) *The sequence $\{e_n\}$ converges strongly to 0.*

If X^ is weakly sharp, then $x_n \in X^*$ for all n large enough.*

Proof. It follows from (4.1) that

$$\begin{aligned}
-h(x_{n+1}; y - x_{n+1}) &\leq \frac{1}{\gamma_n} (\langle y - x_{n+1}, x_{n+1} - x_n \rangle + \langle e_n, y - x_{n+1} \rangle) \\
&\leq \frac{1}{\gamma_n} (\|y - x_{n+1}\| \|x_n - x_{n+1}\| + \|e_n\| \|y - x_{n+1}\|), \tag{4.4}
\end{aligned}$$

for all $y \in X$.

Suppose that the conclusion of the theorem is false. Then, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin X^*$ for all k . For each k , set $y_{n_k} = P_{X^*}(x_{n_k})$. Then, $y_{n_k} \neq x_{n_k}$ for all k . Since X^* is weakly sharp and h is monotone, it follows from Proposition 3.4 that

$$-h(x_{n_k}; y_{n_k} - x_{n_k}) \geq \alpha \text{dist}(x_{n_k}, X^*) = \alpha \|x_{n_k} - y_{n_k}\|. \quad (4.5)$$

By combining (4.4) and (4.5), we obtain

$$\alpha \|x_{n_k+1} - y_{n_k+1}\| \leq \frac{1}{\gamma_{n_k}} (\|x_{n_k+1} - y_{n_k+1}\| \|x_{n_k+1} - x_{n_k}\| + \|e_{n_k}\| \|x_{n_k+1} - y_{n_k+1}\|).$$

Dividing both the sides of the latter inequality by $\|x_{n_k+1} - y_{n_k+1}\| > 0$, we get

$$\alpha \leq \frac{1}{\gamma_{n_k}} (\|x_{n_k+1} - x_{n_k}\| + \|e_{n_k}\|).$$

Since $\liminf_{n \rightarrow \infty} \gamma_n, e_n \rightarrow 0$ and $x_{n+1} - x_n \rightarrow 0$ as $k \rightarrow \infty$, the latter inequality yields $\alpha \leq 0$ which contradicts to $\alpha > 0$. Therefore, $x_n \in X^*$ for all n large enough. \square

Now, taking $e_n = 0$ for all n in (4.1), we have the following exact proximal point method:

$$h(x_{n+1}; y - x_{n+1}) + \frac{1}{\gamma_n} \langle y - x_{n+1}, x_{n+1} - x_n \rangle \geq 0, \quad \text{for all } y \in X. \quad (4.6)$$

Theorem 4.2. *Assume that h is monotone and X^* is weakly sharp with modulus $\alpha > 0$. Let $\{x_n\}$ be a sequence generated by (4.6) with $\lambda_n \in [\lambda, +\infty)$ for some $\lambda > 0$. Then, $\{x_n\}$ converges to a point in X^* in at most $\kappa + 1$ iterations with*

$$\kappa \leq \frac{\text{dist}(x_1, X^*)}{\gamma^2 \alpha^2} + 1. \quad (4.7)$$

Proof. The inequality (4.6) can be written as

$$h(x_{n+1}; y - x_{n+1}) + \frac{1}{2\gamma_n} \|y - x_n\|^2 - \frac{1}{2\gamma_n} \|y - x_{n+1}\|^2 - \frac{1}{2\gamma_n} \|x_n - x_{n+1}\|^2 \geq 0, \quad \text{for all } y \in X.$$

Let x^* be a solution of NVIP (2.1). Putting $y = x^*$ in the latter inequality, we get

$$h(x_{n+1}; x^* - x_{n+1}) + \frac{1}{2\gamma_n} \|x^* - x_n\|^2 - \frac{1}{2\gamma_n} \|x^* - x_{n+1}\|^2 - \frac{1}{2\gamma_n} \|x_n - x_{n+1}\|^2 \geq 0. \quad (4.8)$$

Since $x^* \in X^*$, $h(x^*; x_{n+1} - x^*) \geq 0$. By the monotonicity of h , we have

$$h(x_{n+1}; x^* - x_{n+1}) \leq -h(x^*; x_{n+1} - x^*) \leq 0.$$

Thus, it follows from (4.8) that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2, \quad \text{for all } n. \quad (4.9)$$

This implies that $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists and

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Applying Theorem 4.1, we have $x_n \in X^*$ for all n large enough. Let $N > 1$ be a positive integer. From (4.9), one has

$$\begin{aligned} \|x_1 - x^*\|^2 &\geq \|x_2 - x^*\|^2 + \|x_2 - x_1\|^2 \\ &\geq \|x_3 - x^*\|^2 + \|x_3 - x_2\|^2 + \|x_2 - x_1\|^2 \\ &\quad \vdots \\ &\geq \|x_{N+1} - x^*\|^2 + \sum_{i=1}^N \|x_{i+1} - x_i\|^2 \\ &\geq \sum_{i=1}^N \|x_{i+1} - x_i\|^2. \end{aligned}$$

Thus, for all $1 < N \in \mathbb{N}$,

$$\text{dist}(x_1, X^*)^2 = \inf_{x^* \in X^*} \|x_1 - x^*\|^2 \geq \sum_{i=1}^N \|x_{i+1} - x_i\|^2. \quad (4.10)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, we let κ be the smallest integer such that

$$\|x_{\kappa+1} - x_\kappa\| < \gamma\alpha. \quad (4.11)$$

If $x_{\kappa+1} \notin X^*$, then we set $y_{\kappa+1} = P_{X^*}(x_{\kappa+1})$. By Proposition 3.4 and (4.6), one has

$$\begin{aligned} \alpha \|x_{\kappa+1} - y_{\kappa+1}\| &\leq \alpha \text{dist}(x_{\kappa+1}, X^*) \\ &\leq -h(x_{\kappa+1}; y_{\kappa+1} - x_{\kappa+1}) \\ &\leq \frac{1}{\gamma_\kappa} \langle y_{\kappa+1} - x_{\kappa+1}, x_{\kappa+1} - x_\kappa \rangle \\ &\leq \frac{1}{\gamma_\kappa} \|y_{\kappa+1} - x_{\kappa+1}\| \|x_{\kappa+1} - x_\kappa\|. \end{aligned} \quad (4.12)$$

Since $y_{\kappa+1} \neq x_{\kappa+1}$, we have from (4.11) and (4.12) that

$$\gamma_\kappa \alpha \leq \|x_{\kappa+1} - x_\kappa\| < \gamma\alpha,$$

which contradicts to the assumption that $\gamma_n \geq \gamma$ for all n . Thus $x_{\kappa+1} \in X^*$. Moreover,

$$\text{dist}(x_1, X^*)^2 \geq \sum_{i=1}^{\kappa-1} \|x_{i+1} - x_i\|^2 \geq (\kappa-1)\gamma^2 \alpha^2.$$

Therefore,

$$\kappa \leq \frac{\text{dist}(x_1, X^*)}{\gamma^2 \alpha^2} + 1.$$

This completes the proof. □

Remark 4.3. If X is bounded, then for any choice of the initial point x_1 , it follows from (4.7) that

$$\kappa \leq \frac{\text{diam}(X)}{\gamma^2 \alpha^2} + 1,$$

where $\text{diam}(X) = \sup\{\|x - y\| : x, y \in X\}$. It means that we have an upper bound of the number of iterations without knowing of the solutions set X^* .

Finally, we provide an example to illustrate Theorem 4.2.

Example 4.1. Let $H = \mathbb{R}$, $X = [0, 1]$ and $h : X \times H \rightarrow \mathbb{R}$ be defined by

$$h(x; d) = xd^3 + d, \quad \text{for all } (x, d) \in X \times H.$$

Assume that $x^* \in X^*$ is a solution of NVIP (2.1). Then, $h(x^*; y - x^*) \geq 0$ for all $y \in X$, i.e.,

$$x^*(y - x^*)^3 + (y - x^*) \geq 0, \quad \text{for all } y \in [0, 1].$$

This implies that $x^* = 0$. Thus, $X^* = \{0\}$.

We have $T_X(0) = \mathbb{R}_+$ and $N_{X^*}(0) = \mathbb{R}$. Hence, $T_X(0) \cap N_{X^*}(0) = \mathbb{R}_+$. Moreover,

$$h(0; d) = d = \|d\|, \quad \text{for all } d \in T_X(0) \cap N_{X^*}(0).$$

Thus, X^* is weakly sharp with modulus $\alpha = 1$. We can easily check that h is monotone on X . So, in Theorem 4.2, if we take $\gamma > 1$, then method (4.6) terminates after one iteration for any choice of x_1 .

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