

Dynamic Scheduling of Home Health Care Patients to Medical Providers

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Home care provides personalized medical care and social support to patients within their own home. Our work proposes a dynamic scheduling framework to assist in the assignment of patients to health practitioners (HPs) at a single home care agency. We model the decision of which patients to assign to HPs as a discrete-time Markov decision process. Due to the curse of dimensionality and the difficult combinatorial structure associated with an HP's daily travel, we propose an approximate dynamic programming approach (ADP). Our method is based on a one-step policy improvement heuristic and includes both predictive and prescriptive components. In particular, the ADP policy is formulated as a stochastic program that estimates future HP assignments using patient features, such as the type of treatment that is required and the region of residence. We show how to obtain these estimates using supervised learning techniques or by solving a mathematical program. We then solve the stochastic program using a novel adaption of the integer L-shaped algorithm and present valid inequalities to speed up convergence. We investigate several extensions to account for multiple patient-HP assignments, patients who return for service, and periodic care. Our solution methodology is compared to existing policies in a discrete-event simulation using data from a Canadian home care provider.

1. Introduction

Home care aims to provide personalized medical care and social support to patients within their own home (Genet et al., 2013). Care is provided by a licensed health practitioner (HP), such as a registered nurse, physical therapist, and/or personal support worker. HPs can administer medical treatment, provide assisted-living support, and manage palliative care plans, thereby either preventing or postponing institutionalization (Rechel et al., 2010). Home care is currently regarded as an essential service in patient-centric health systems. The National Association for Home Care and Hospice reports that, in 2010, 12 million patients received home care services from almost 33,000 agencies in North America (NAHC, 2010). In 2014, home care was the fastest-growing U.S. industry with a projected growth of almost 5% per year through 2024 (U.S. Bureau of Labor Statistics, 2015). In

Canada, over 2 million people rely on home care services, which represents approximately 5.5% of the total population, and this figure is growing ([Canadian Home Care Association, 2010](#)).

The assignment of patients to HPs is one of the most critical operational decisions a home care agency (HCA) can make. It involves quality-of-service constraints, cost considerations, and transit time management. In particular, agencies employ HPs with a wide-range of skill sets in order to serve patients with a diverse set of health requirements. An HCA must guarantee that HPs have the proper qualifications and expertise to deliver high-quality care to their patients. Continuity of care (i.e., consistency in personnel) is a top priority. The ability to facilitate an enduring contact between a patient and their HP leads to better patient outcomes ([Kistler and Drickamer, 2018](#)) and is a key performance guideline ([Community Care Access Centre, 2016](#)). An HP's shift is stressful and unpredictable; overtime work is commonplace and is a potentially costly occurrence ([Denton et al., 2002](#)). Class-action lawsuits ([Richards, 2016](#)) and recent legislation (e.g., the Patient Protection and Affordable Care Act) have mandated that HCAs must pay employees for time spent in transit ([Department of Labor's Wage and Hour Division, 2013](#)). This is not an inconsequential cost; each year HPs travel twice the distance of United Parcel Service (UPS) delivery drivers ([NAHC, 2010](#)).

The current study is motivated by a partnership with a company that provides scheduling solutions for HCAs in Ontario, Canada, a typical home care setting in North America. In such a system, patients enter the home care network through a referral by a certified medical practitioner. The referral process consists of sending a requisition form to the Local Health Integration Network (LHIN) operating in the patient's region of residence. The LHIN is responsible for overseeing care for all patients receiving home care services. Upon receiving a requisition form, the LHIN processes and forwards it to both private and public HCAs operating within the patients region of residence.

Our research focuses on the scheduling decisions made by an HCA. Their objective is to maximize revenue while minimizing service and transit costs, which are associated with providing medical care to a geographically dispersed population. In each day of operation, an HCA will receive a random number of referrals from the LHIN. A referral is associated with a health requirement, expected duration of care, and region of residence. An HCA must then decide whether or not to provide care for the arriving patients and, if so, which HP should be assigned. If a patient is accepted, the HCA receives a fixed remuneration amount per visit, as outlined by the Ontario Health Insurance Plan (OHIP). In return, the HCA must ensure that each patient receives at-home medical service or social support for the duration the patient spends in the system. The HCA may also decide not to accept a referred patient, perhaps at some opportunity cost. Nonetheless, maintaining a healthy relationship with the LHIN is important, and rejecting too many patients may strain this relationship over the long term. Discussions with scheduling personnel from several HCAs have indicated that the

assignment of patients to HPs is currently done in an ad-hoc manner. At best, HCAs assign patients to HPs using a myopic approach; they do not consider demand that may arrive in future periods.

We propose a dynamic scheduling framework for an HCA facing uncertain demand. The objective is to determine whether referred patients should be accepted by an HCA and, if so, how they should be assigned to HPs while observing continuity-of-care and quality-of-service constraints. We model a single agency's admission-and-assignment problem as a discrete-time Markov decision process. To overcome the curse of dimensionality, we use an approximate dynamic programming (ADP) approach based on a policy improvement technique (e.g., Liu et al., 2010). In contrast to existing myopic approaches for home care scheduling and routing, our ADP technique incorporates a predictive component that estimates the expected number of patients, their health requirements, durations of care, and regions of residence, assigned to HPs in future periods. We show that such estimation can be performed using, e.g., supervised learning techniques over historic HP allocations or a mathematical program. The estimates are applied in a policy improvement heuristic that maximizes the agency's infinite-horizon discounted reward, which we reformulate as a finite-horizon stochastic program.

One of the key features of our proposed methodology is its flexibility with regards to estimating transit time. This is one of the main sources of complexity in home care scheduling as it is both time consuming and costly. We formulate an HP's travel costs in future periods as an instance of a *probabilistic traveling salesperson problem* (PTSP), a variant of the classical traveling salesperson problem (TSP) where regions are associated with a probability of being visited (Bertsimas et al., 1990). This allows us to write the underlying optimization model as a mixed-integer linear program (MILP) albeit with exponentially many constraints. We solve the model using a novel adaption of the integer L-shaped algorithm (Laporte and Louveaux, 1993) which iteratively converges to the optimal solution of the policy improvement heuristic. The benefit of this approach is that it provides feasible solutions and optimization bounds at each iteration. Thus, it can be stopped at any desired optimality gap. We develop new valid inequalities that speed up convergence and show that a restricted version can be solved in polynomial time. Finally, we discuss several extensions, such as when patients return for service, need periodic service (e.g., every other day), and must be assigned to multiple HPs.

Our research extends the home care scheduling literature by addressing many of the shortcomings identified in the review paper by Fikar and Hirsch (2017); namely, stochastic routing and scheduling, multi-stage and multi-period planning, and multimodality. Although we focus on a Canadian HCA, our model is general enough to be applied to HCAs in the United States (Welch et al., 1996, Harris-Kojetin et al., 2013) and in many European countries (Genet et al., 2011, Van Eenoo et al., 2015). We present several structural results and, using a discrete-event simulation parameterized by a data set from an HCA operating in Ontario, Canada, demonstrate the novelty of our approach by exploring the value of incorporating our scheduling framework into practice.

Our analysis reveals several important managerial insights. First, we observe that myopic scheduling practices, represented by routing optimization models that do not consider future referrals, may result in poor assignments that forfeit potential profits. Second, our approach is particularly beneficial for HCAs operating in large geographic regions or dense urban centers with low average travel speeds. These examples represent settings with non-trivial travel times. Finally, in systems with HPs that are severely under- or over-utilized, little can be done to improve profitability. In all other cases, our solution approach is adept at avoiding excess costs while maintaining suitably high HP utilization.

2. Literature Review and Contribution

Our paper is related to several streams of research in the assignment, routing, and dynamic scheduling literature. The most well-known is the *nurse rostering* problem, where health workers are scheduled to cover hospital shifts while observing legal requirements (e.g., maximum/minimum shift lengths). The goal is to minimize rostering costs, such as the idle time and overtime of nurses, and has been subject to extensive study; see, e.g., the survey by [Van den Bergh et al. \(2013\)](#). In recent years, several papers have addressed the nurse rostering problem for home care services (e.g., [Bertels and Fahle, 2006](#), [Cattafi et al., 2015](#), [Wirnitzer et al., 2016](#)). Nevertheless, this stream of research does not address routing decisions and typically assumes that an HP's patient panel is known and static.

Several models examine home care systems where the set of patients, their health requirements, and regions of residence are known and fixed. [Eveborn et al. \(2006\)](#) and [Steeg and Schröder \(2008\)](#) solve a combination of a vehicle routing problem (VRP) and a nurse rostering problem. They incorporate hard and soft assignment constraints (e.g., skill-set qualifications) and routing restrictions, (e.g., time windows that limit when a patient can be visited). A similar setup is addressed by [Nickel et al. \(2012\)](#), who combine a constraint programming model and heuristics to solve a variant of the VRP with nurse rostering constraints. [Rasmussen et al. \(2012\)](#) propose a branch-and-price algorithm to find the minimum-cost assignment of HPs with various skill sets to patients. Similarly, [Mankowska et al. \(2014\)](#) formulate a MILP for routing HPs in a single day and incorporate interdependencies between service operations. Finally, [Cappanera and Scutellà \(2014\)](#) present a pattern-based approach for the joint assignment, scheduling, and routing of HPs to patients. Although this literature is extensive, it does not address the dynamic and stochastic nature of home care scheduling.

Few papers have investigated the multi-period HP assignment and routing problem with deterministic demand. [Yalçındağ et al. \(2012\)](#) propose a rolling-horizon, two-stage approach for the multi-period problem. The first stage uses a MILP to define patient-HP assignments while the second stage establishes the optimal tours for each each HP. In [Bard et al. \(2014\)](#), patients with different visit preferences are assigned to HPs with varying skill sets over a single week. A branch-and-price-and-cut algorithm embedded in a rolling-horizon framework incrementally constructs a weekly schedule.

Torres-Ramos et al. (2014) propose a multi-period MILP that finds optimal tours for HPs assigned to patients with heterogeneous work requirements (e.g., number of visits). The goal is to minimize the operating time for an HP who begins and ends their shift at home. Lin et al. (2016) present a MILP that determines optimal HP schedules by maximizing the assignment rate of new patients while minimizing the reassignment rate of existing ones. Our research extends this literature by relaxing the assumption that demand is deterministic and the set of patients to be assigned are fixed. We also assume that demand for service arises randomly and account for different durations of care.

The home care rostering problem with demand uncertainty has been addressed using two distinct approaches. Lanzarone and Matta (2012) formulate a stochastic program with the goal of balancing HP workload where a single arriving patient is assigned to an HP in each period. The objective is to minimize a quadratic cost function and the uncertainty in future demand is modeled with known distributions (triangular/uniform). Lanzarone et al. (2012) later incorporate demand uncertainty by optimizing the expected workload over several scenarios and explore different interpretations of continuity of care. The second approach formulates the problem as a Markov decision process (MDP), the same methodology we use to define our problem. Koeleman et al. (2012) use a continuous-time MDP to determine whether a single referred patient should be rejected, put on a wait list, or assigned to an HP. However, the model does not take into account the region of residence for referred patients. In Bennett and Erera (2011), patients arrive stochastically, need periodic service, have various medical requirements, and remain in the system for a set number of weeks. The objective is to maximize the number of patients served per period subject to assignment constraints. The benefits of a myopic, rolling-horizon policy is explored. For a single HP, Demirbilek et al. (2018) demonstrate that a scenario-based approach of simulating future demand can outperform a myopic policy.

As compared to previous literature on home care rostering with demand uncertainty, our model is dynamic and allows for multiple patient-HP assignments per period. We consider the travel time and duration that patients spend in the system, ensure that all assignments adhere to a continuity of care constraint, and allow an HCA to reject unprofitable referrals. Our formulation also makes few assumptions about the structure of the reward function. As a result, our approach is data-driven and can be applied to agencies with different operating procedures and compensation structures.

Our scheduling model uses a policy improvement heuristic, an ADP technique employed in dynamic settings. Zenios et al. (2000) use it to derive fair allocation policies to maximize quality-adjusted life-years for patients on the kidney transplant list. Opp et al. (2005) and Argon et al. (2009) apply the technique to route customers in a queueing network. Liu et al. (2010) use the approach to compare scheduling policies in a primary care practice. Our paper extends this research by applying ADP techniques to a stochastic dynamic scheduling problem where obtaining feasible actions entail solving a model with a complex combinatorial structure (i.e., HPs travel in tours to serve their patients).

Our approach utilizes an advanced optimization technique, i.e., the integer L-shaped algorithm (Laporte and Louveaux, 1993). The literature applying L-shaped algorithms in home care has focused on two-stage problems with uncertain service times (Zhan et al., 2015). It is also used more generally for routing applications; see, e.g., Gendreau et al. (1995), Laporte et al. (2002). Our formulation differs in the type of decomposition that is applied. Namely, in previous work, the problem is formulated on the edge-variable space of a network, i.e., incorporating variables that indicate if a vehicle must travel from one region to another. In contrast, our approach determines assignment decisions by abstracting the routing component. Once the assignment variables are fixed, our subproblems decompose into separate PTSPs for each HP and decision epoch. We then use PTSP solution techniques to derive bounding functions that speed-up convergence when there are multiple periods and nonlinear rewards.

3. Model Formulation

We propose a dynamic scheduling framework for a home care agency (HCA) to assign patients to home care practitioners (HPs), such as registered nurses and personal support workers. We formulate the problem as a discrete-time, rolling-horizon, Markov decision process (MDP).

3.1. Patient Types

At the beginning of each decision epoch (i.e., end of operating day), the HCA receives a number of patient referrals, each with certain characteristics. Patients are heterogeneous with respect to their health requirements, service durations, and their geographic region of residence, as follows.

Health Requirements: Patients have a health requirement from the set $\mathcal{K} = \{1, \dots, K\}$ that indicates the type of care that they need (e.g., wound treatment, personal support, or any combination of medical procedures). We assume that health requirements do not change during treatment.

Service Duration: A patient with a health requirement $k \in \mathcal{K}$ remains in the system for $d \in \{1, \dots, D_k\}$ periods, where D_k is the maximum duration before reassessment for requirement k .

Geographic Regions: The HCA serves a finite number of geographic regions $\mathcal{M} = \{1, \dots, M\}$ indicating, e.g., addresses, postal codes, or zoning districts. The number of regions depends on the HCAs desired spatial granularity. Each patient belongs to a single geographic region $m \in \mathcal{M}$.

Referrals are realizations $\vec{\theta}$ of a random vector Θ describing the number of arriving patients and their characteristics. That is, at the beginning of each decision epoch, the HCA observes the vector $\vec{\theta} \equiv \{\theta_{kdm}\}_{\forall k,d,m}$, where θ_{kdm} is the number of referred patients with health requirement k , that need d periods of service, and reside in region m ; $\mathbb{E}[\Theta]$ is the expected number of referrals per period.

3.2. Home Care Agency Actions

We assume an HCA employs J HPs and that this number, and the skills each HP possesses, remains fixed over the planning horizon, i.e., we do not model employee turnover or attrition. An HP can provide home care services to patients with \mathcal{K}_j health requirements, where $\mathcal{K}_j \subseteq \mathcal{K}$ for HP $j = 1, \dots, J$.

In each decision epoch, an HCA observes the set of patients currently assigned to each HP as well as the number and characteristics of referred patients. Then, two classes of decisions are made: (i) assign referred patients to one of the J HPs currently on staff; or (ii) reject some of the referred patients who will eventually be assigned to some other HCA. The HCAs actions are constrained by: *Resource Constraint*: A patient with health requirement k can be assigned to HP j only if $k \in \mathcal{K}_j$.

That is, an HP must possess the necessary skills to manage the patient's health requirement.

Continuity of Care: If a patient is assigned to HP j 's panel, then we assume that HP j must provide service for the duration that the patient spends in the system.

Distance/Time Consideration: In each period, HP j must travel to her assigned patients' region of residence, allocating the necessary service time to address the patient's health requirement.

3.3. State Variables, Action Variables, and System Dynamics

The state of the system is represented by x_{kdmj} , the number of patients that have health requirement k , must spend d more periods in the system, live in region m , and are currently being served by HP j , for all k, d, m , and j . We denote $\vec{x} = \{x_{kdmj}\}_{\forall k,d,m,j}$ as the state variable that tracks the number of patients of each type that the HCA is currently providing service to.

At each decision epoch, the HCA decides how many patients of each type will be assigned to the available HPs and how many patients of each type will be rejected. Let $a_{kdmj} \in \mathbb{Z}_+$ be an action variable that is equal to the number of type (k, d, m) -patients that are assigned to HP j , for all k, d, m , and j . That is, $\vec{a} = \{a_{kdmj}\}_{\forall k,d,m,j}$ is the set of referred patients assigned to HPs. Analogously, we let b_{kdm} be the number of patients of type (k, d, m) that are rejected, with $\vec{b} = \{b_{kdm}\}_{\forall k,d,m}$.

After assignment and rejection decisions are made, the current period rewards are realized, i.e., the revenue and costs associated with providing service during the next day are accrued. The state space then rolls over by a single period and we define a special operator, \oplus , to indicate this behavior:

$$\vec{x} \oplus \vec{a} \equiv \{ (x_{k1mj}, x_{k2mj}, \dots, x_{kD_{\max}mj}) \rightarrow (x_{k2mj} + a_{k2m}, \dots, x_{kD_{\max}mj} + a_{kD_{\max}mj}, 0) \}_{\forall k,m,j},$$

where $D_{\max} = \max_{k=1,\dots,K} \{D_k\}$ is the maximum duration a patient can be referred before being reassessed. That is, after applying action (\vec{a}, \vec{b}) to a state \vec{x} and realizing the rewards, the system transitions to a new state where, for all HPs j and regions m , the number of (k, m) -patients assigned to HP j that currently require $d - 1$ periods of service is equal to the number of patients that previously required d periods of service (i.e., x_{kdmj}) plus the new patients that were assigned to HP j (i.e., a_{kdmj}). The rolling horizon framework assumes patients are visited by an HP once per period, an assumption that is relaxed in Section 8. Further, patients assigned to HP j cannot be reassigned to other HPs in a future period. Hence, the continuity of care requirement is satisfied by construction.

3.4. Action Set Constraints

Let $\mathcal{J}(k) = \{j : k \in \mathcal{K}_j\}$ be the set of HPs that can serve patients with health requirement $k \in \mathcal{K}$. The set of feasible actions (\vec{a}, \vec{b}) with respect to a realization $\vec{\theta}$ is given by $\mathcal{A}(\vec{\theta})$, where

$$\mathcal{A}(\vec{\theta}) := \left\{ (\vec{a}, \vec{b}) : \sum_{j \in \mathcal{J}(k)} a_{kdmj} + b_{kdm} = \theta_{kdm}, \quad \forall k, d, m \right\}, \quad (1)$$

where \vec{a} and \vec{b} are of appropriate dimension. Constraint (1) ensures that all referred patients are either assigned to an HP with appropriate skill or are rejected.

3.5. Health Practitioner (HP) Characteristics

The time per period that an HP spends traveling and providing service is characterized as follows:

Service Time: Given a state \vec{x} , let $F_S^j(\vec{x})$ be the total time needed for HP j to serve all her patients.

We assume that $F_S^j(\vec{x})$ is non-decreasing in \vec{x} for each j . Although time with a patient can exhibit some randomness, we assume scheduling decisions are based on the expected service time only.

Transit Time: Given a state \vec{x} , let $F_T^j(\vec{x})$ be the total time required for HP j to travel to all her patients. We assume that $F_T^j(\vec{x})$ is non-decreasing in \vec{x} for each j that the transit time between any two regions is known. A valid $F_T^j(\vec{x})$ could be the minimum-length traveling salesperson tour that visits each region, possibly departing and returning to the HP's residence or a hospital.

3.6. Rewards and Value Function

Let A_{kdmj} be the per period revenue associated with accepting a type (k, d, m) -patient and assigning them to HP j . The HCA incurs a wage cost $G_j : \mathbb{R} \rightarrow \mathbb{R}$ associated with the per period transit and service time of each HP. We assume $G_j(\cdot)$ is non-decreasing and convex. This ensures that, as the service or transit time increases, the cost (weakly) increases at a non-decreasing rate. There is also a cost, B_{kdm} , for each referred patient of type (k, d, m) that is rejected.

Defining $\rho \in [0, 1)$ to be the per-period discount factor of future rewards, and given the system state \vec{x} and actions (\vec{a}, \vec{b}) , the current period reward function is $R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a})$. The first term is

$$R(\vec{a}, \vec{b}) := \sum_{k=1}^K \sum_{d=1}^{D_k} \sum_{m=1}^M \sum_{j=1}^J \left(\frac{1 - \rho^d}{1 - \rho} A_{kdmj} \right) a_{kdmj} - \sum_{k=1}^K \sum_{d=1}^{D_k} \sum_{m=1}^M B_{kdm} b_{kdm}$$

which captures the per period revenue attributed to HP j by serving patients in her panel minus the rejection cost of not accepting unprofitable referrals. Since there is no uncertainty as to when accepted patients will leave the system, the total revenue associated with serving a new patient can be computed immediately upon acceptance with the appropriate discounting. We choose to represent revenue in this way because it simplifies the exposition in later sections. The second term is

$$G(\vec{x} + \vec{a}) := \sum_{j=1}^J G_j \left(F_S^j(\vec{x} + \vec{a}) + F_T^j(\vec{x} + \vec{a}) \right)$$

which captures the total wage cost for each HP j . This cost is incurred by the HCA and is a function of the total time each HP spends traveling and providing care to their patients.

Let $V(\vec{x})$ be the value of state \vec{x} . The objective is to determine a scheduling policy that maximizes the expected, discounted reward over an infinite horizon. Thus, the Bellman equation is

$$V(\vec{x}) = \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E}[V(\vec{x} \oplus \vec{a})] \right\}. \quad (2)$$

3.7. Model Discussion

We have made several choices in the formulation of our model. First, although we assume the referral distribution is time stationary and state independent, we do not make any assumptions about the relationship between the number of referred patients and their characteristics. As a result, the referral distribution can be independent or contain complicated correlation structures. We assume the HCA receives a fixed remuneration amount per visit. This follows from the fee-for-service compensation structure of many medical systems, such as the one in Ontario (Simpson, 2015). We define a unique service and transit time function for each HP. This reinforces the generality of our framework so that it can be directly applied to agencies where HPs may complete similar tasks at different rates, where a portion of the HPs begin their shifts from their own home versus a central location (e.g., a hospital), and multi-modal organizations (e.g., Hiermann et al., 2015). Wage costs are assumed to be separable in j as HPs may be paid at different rates even though they may provide the same service. Wage differences, for example, can be attributed to an HP’s professional designation (e.g., physiotherapist, chiropractor) or their level of seniority. Finally, the convexity, non-decreasing, and nonlinear shape of $G_j(\cdot)$ ensures that the HCA is appropriately penalized if an HP accrues overtime.

By decoupling the service time function (F_S^j) from the transit time function (F_T^j), we implicitly assume they are independent, which allows us to analyze their cost contributions separately. If this assumption does not hold, e.g., if transit times are time dependent or if time-window constraints must be considered, the resulting rewards characterize valid bounds. This is done for several reasons. First, discussions with our industry partner indicate that the order in which patients are visited frequently changes. This can be attributed to changes in an HP’s panel (i.e., new patients and those that finish service), ad-hoc decisions made by the HCA, or because patient preferences regarding when they would like to be seen can vary. Travel times can also be highly idiosyncratic; they may depend on the time-of-day, day-of-week, and month-of-year. As a result, when an HP visits a patient (and in what order) may vary greatly, even over short time horizons. Thus, although the MDP can be adapted to address more complex, time-dependent formulations, we focus our analysis on a high-level understanding of an HP’s time commitments. Our formulation allows an HCA to make data-driven accept/reject decisions in highly stochastic environments that are subject to change. While patient

preferences and time-varying transit times are important issues to consider in practice, the inclusion of such behavior would make this problem more difficult to systematically analyze.

The model includes decision variables for rejecting a referred patient and the associated cost may be dependent on an individual’s health requirement, expected duration, or region of residence. For example, patients that need service for long durations may represent a larger revenue stream, and incur a larger rejection cost, than patients with short durations. An HCA may reject a patient because they cannot ensure appropriate care given the available personnel. It may also be more costly to reject patients from rural regions than from urban areas (Kitts and Cook, 2014). Our model captures the differential costs associated with rejecting a referral. Interviews with home care managers also suggest that rejecting a patient carries an opportunity cost. For example, if too many patients are rejected, the LHIN may decide to refer fewer patients to the HCA in the future. Because the rejection cost is hard to accurately quantify, we investigate its effect on performance in Section 7.

Our model assumes that patients require service once per period and that each patient is assigned to a single HP. It can, however, be extended in several directions. We examine three of the most common in Section 8, i.e., return for service, periodic service, and multiple assignments.

4. Approximate Dynamic Program

In this section, we develop a data-driven, approximate dynamic program (ADP) approach to assign patients to HPs that addresses both the underlying combinatorial structure of the wage cost $G_j(\cdot)$, encoding patient visits as traveling salesperson (TSP) tours, and the intractable state-space size of the formulation (2). Our method is based on a one-step policy improvement heuristic (PIH), which gives a finite-horizon approximation that incorporates user-specified look-ahead information.

4.1. Policy Improvement Heuristic

The PIH is an approach where optimal actions in the current period are determined with respect to a value function approximation, U , that is applied only to future periods. That is, as opposed to computing $V(\vec{x})$ as in (2), we determine an approximate value $\tilde{V}(\vec{x})$ for a state \vec{x} defined as

$$\tilde{V}(\vec{x}) = \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E}[U(\vec{x} \oplus \vec{a})] \right\}, \quad (3)$$

where U replaces V only in the expectation term of (2).

We design U by considering a *basis policy* \mathcal{P} for which the expected reward $\mathbb{E}[U(\vec{x} \oplus \vec{a})]$ in (3) can be characterized efficiently. In our context, we assume that U corresponds to the application of a time stationary, state independent, and probabilistic policy. While the basis policy does not depend on the time period or system state, the probability of an assignment can be based on a patient’s type (e.g., health requirement, duration of care, region of residence) and an HP’s unique characteristics. Thus,

given the current state of the system \vec{x} and the assignment \vec{a} of new patients to HPs, U represents the value of using the basis policy for assigning patients to HPs in all future periods.

Formally, let $\vec{q} = \{q_{kdmj}\}_{\forall k,d,m,j}$ where $q_{kdmj} \in [0, 1]$ is the probability of assigning a patient of type (k, d, m) to HP j . In each period, the actions $\vec{a}^{\mathcal{P}}$ applied by basis policy \mathcal{P} are defined as

$$(a_{kdm1}^{\mathcal{P}}, a_{kdm2}^{\mathcal{P}}, \dots, a_{kdmJ}^{\mathcal{P}}, b_{kdm}^{\mathcal{P}}) \sim \text{Multinomial}(\theta_{kdm}, q_{kdm1}, q_{kdm2}, \dots, q_{kdmJ}), \quad \forall k, d, m, \quad (4)$$

for any realization $\theta_{kdm} \sim \Theta$. In (4), each patient assignment is mutually exclusive and we constrain $q_{kdmj} = 0$ for $j \notin \mathcal{J}(k)$. We also ensure $\sum_{j \in \mathcal{J}(k)} q_{kdmj} \leq 1$ for every k, d and m , so that $b_{kdm}^{\mathcal{P}} := \theta_{kdm} - \sum_{j=1}^J a_{kdmj}^{\mathcal{P}}$. Thus, $(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) \in \mathcal{A}(\vec{\theta})$, i.e., all actions satisfy the feasibility conditions as per (1).

Based on the form of (4), many such policies can be considered for \mathcal{P} , such as those derived from supervised learning techniques or from mathematical programming models. We present three alternatives in Section 4.2. Moreover, the basis policy \mathcal{P} yields an approximate value function

$$U(\vec{x}) = R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) - G(\vec{x} + \vec{a}^{\mathcal{P}}) + \rho \mathbb{E}[U(\vec{x} \oplus \vec{a}^{\mathcal{P}})], \quad (5)$$

where the expectation is defined both in terms of Θ and the distribution associated with \mathcal{P} .

We now show that \tilde{V} can be written as a finite-horizon problem. For a realization $\theta_{kdm} \sim \Theta$, let $\tau := \max\{d : \theta_{kdm} > 0\}$ be the maximum duration of patients that arrive in the current period.

PROPOSITION 1. *For a value function U designed using a basis policy \mathcal{P} , a feasible action $(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})$ optimally solves (3) if and only if it is optimal for*

$$V_{PIH}(\vec{x}) = \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\theta)} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \rho^t \mathbb{E}_t [G(\vec{y}_t + \vec{a}^{\mathcal{P}})] \right\}, \quad (6)$$

where \vec{y}_t is the state of the system at the end of the t -th decision epoch, assuming that policy (\vec{a}, \vec{b}) is applied in the current period and the basis policy defined by (4) is applied at $t \geq 1$, i.e.,

$$\vec{y}_t = \begin{cases} \vec{x} \oplus \vec{a} & : t = 1, \\ \vec{y}_{t-1} \oplus \vec{a}^{\mathcal{P}} & : t > 1, \end{cases}$$

and $\mathbb{E}_t[\cdot]$ is the expectation corresponding to the arrival process Θ and basis policy \mathcal{P} at period t .

Proposition 1 demonstrates that the PIH policy derived from V_{PIH} , a finite-horizon problem, is equivalent to the optimal policy that could be derived using \tilde{V} , an infinite-horizon problem, as long as U is designed using a basis policy \mathcal{P} . Further, the resulting PIH policy yields a lower bound when applied to the true value function V since it represents a subset of all the feasible actions in this case.

We note that if the discount rate is $\rho = 0$, the optimal solution for (6) is a myopic policy that considers the service and transit time for the current period only, thereby encompassing several of the deterministic models discussed in Section 2 for appropriate choices of $F_S^j(\vec{x})$ and $F_T^j(\vec{x})$. When $\rho > 0$, the optimal policy for (6) incorporates look-ahead information. The trade-off between myopic and expected future estimates can therefore be adjusted by setting ρ as desired.

4.2. Basis Policies

The basis policy \mathcal{P} is a key component of the one-step policy improvement heuristic. When such a policy is chosen sensibly, it is likely that the PIH will perform well (Liu et al., 2010). In this regard, we propose and test three alternative basis policies by defining how the probabilities \vec{q} are computed.

Equitable Assignment (EA) Policy. The EA policy assumes that all HPs who have the skill set to satisfy a referred patient with health requirement k have an equal probability of being assigned. This policy does not consider the duration a patient will spend in the system nor does it account for the distance between patients and HPs. However, it ensures that HPs who provide the same care will have identical service-time workloads in expectation. It is also the simplest basis policy to implement.

Supervised Learning (SL) Policy. The SL policy estimates the q_{kdmj} values by applying machine learning (e.g., Naïve Bayes) to historical data. That is, the SL policy learns the parameters for the basis policy by mining the HCA's previous scheduling decisions. The result is a state-independent approximation of how the agency has scheduled in the past. The SL policy makes few assumptions about the decision-making process and the estimation technique can be chosen by the practitioner.

Optimal Time-Stationary (OTS) Policy. The OTS policy is derived from an optimization model that exploits the structure of (3). The idea is to estimate the long-run average number of patients in the system, which is used in a deterministic problem that is solved to optimality. The corresponding optimal patient-HP assignments can then be used to find q_{kdmj} for all k, d, m and j . Specifically, define \mathbb{P}_{kdm} to be the probability that a patient of type (k, d, m) is referred to the HCA. Let $\mathbb{E}[\Theta]$ be the mean number of patients that arrive per period and define $\sigma_{kdm} := \mathbb{P}_{kdm}\mathbb{E}[\Theta]$. Let $\tilde{\sigma}_{ktm}$ be the expected number of patients of type (k, m) in period t , where

$$\tilde{\sigma}_{k1m} = \sum_{d=1}^{D_k} \sigma_{kdm}, \quad \tilde{\sigma}_{ktm} = \tilde{\sigma}_{k(t-1)m} + \tilde{\sigma}_{k1m} - \sum_{d=1}^{\min\{t, D_k\}} \sigma_{k(t-d)m} \text{ for } t > 1.$$

The following proposition demonstrates that the above quantity converges to a steady-state.

PROPOSITION 2. *For all $t > D_{\max} + 1$, $\tilde{\sigma}_{ktm} = \tilde{\sigma}_{km}$ for some $\tilde{\sigma}_{km} < \infty$.*

From Proposition 2, we let $\tilde{\sigma}_{km}$ be the average number of patients with health requirement k and who reside in region m . The objective of the OTS policy is to cover the expected number of patients $\{\tilde{\sigma}_{km}\}_{\forall k, m}$ with minimum cost. To this end, we assume that $\lceil \tilde{\sigma}_{km} \rceil$ patients with duration $d = 1$ must be scheduled to HPs with empty panels, where $\lceil \cdot \rceil$ rounds to the closest integer. Solving this to optimality, \vec{q} is then proportional to the assignment provided by \vec{a} . That is,

$$q_{kdmj} = \frac{a_{kdmj}}{\sum_{k=1}^K \sum_{m=1}^M \lceil \tilde{\sigma}_{km} \rceil}, \quad \forall k, m, d, j.$$

5. Stochastic Optimization Model

In this section, we formulate a stochastic optimization model that characterizes the PIH-based policies by defining representative service and transit time functions. Namely, we assume that service times are linear functions of the health requirements for each patient, while transit times represent minimum-length TSP tours for each HP. From the perspective of an HCA, such functions estimate the time required for each HP to visit the patients in her panel and are key in strategic planning.

The PIH depends on the structure of the wage cost $\mathbb{E}[G_j(\cdot)]$ and is difficult to characterize in general. Thus, to facilitate its tractable evaluation, we propose the value function approximation V_{APIH} where we instead compute $G_j(\mathbb{E}[\cdot])$. This provides separate estimates of the expected service and transit times. The benefits are threefold. First, it efficiently approximates the stochastic service and transit times in future periods. Second, the approach is more easily implementable for practitioners while still providing superior assignments. Finally, we show that V_{APIH} is an upper bound to V_{PIH} in (6) and its policy gives a lower bound to the true value function V in (2).

5.1. Optimization Model

Suppose that \vec{x} and (\vec{a}, \vec{b}) are the state and actions applied in the current decision epoch. Let $S_{tj}(\vec{x}, \vec{a})$ and $T_{tj}(\vec{x}, \vec{a})$ be defined as the expected service and travel times for HP j in period $t \geq 1$, respectively. That is, for the state \vec{y}_t as defined in Proposition 1, we have $S_{tj}(\vec{x}, \vec{a}) := \mathbb{E}_t \left[F_S^j(\vec{y}_t + \vec{a}^P) \right]$ and $T_{tj}(\vec{x}, \vec{a}) := \mathbb{E}_t \left[F_T^j(\vec{y}_t + \vec{a}^P) \right]$. We consider a new value function, $V_{APIH} \geq V_{PIH}$, that evaluates the wage cost function of the expected service and transit times. This is formalized below.

PROPOSITION 3. *For any state \vec{x} , the value function*

$$V_{APIH}(\vec{x}) = \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\theta)} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \sum_{j=1}^J \rho^t G_j \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right) \right\} \quad (7)$$

is an upper bound to $V_{PIH}(\vec{x})$ in (6).

The action pair (\vec{a}, \vec{b}) that solves (7) to optimality is the form of the PIH that we implement. When the policy is applied to the true value function V , it yields a lower bound as it is an optimizer of a different value function. Further, equation (7) can be equivalently recast as

$$\max_{\vec{a}, \vec{b}} \quad R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \sum_{j=1}^J \rho^t G_j \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right) \quad (\text{SOP})$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}(k)} a_{kdmj} + b_{kdm} = \theta_{kdm} \quad \forall k, d, m, \quad (8)$$

$$a_{kdmj}, b_{kdm} \in \mathbb{Z}_+, \quad \forall k, d, m, j. \quad (9)$$

The objective function of (SOP) represents the value function in (7). Constraint (8) ensures that actions are feasible with respect to $\mathcal{A}(\theta)$. Constraint (9) is the integrality and nonnegativity condition. In subsequent sections, we derive functional structures for the service and transit time. Then, in Section 6, we propose a solution method that solves (SOP) to optimality.

5.2. Expected Service Time

We first characterize the service time function $S_{tj}(\vec{x}, \vec{a})$. Here, we assume that $F_S^j(\vec{x})$ is the sum of linear functions that depend solely on the number of patients with each health requirement assigned to HP j . This is consistent with the home care literature (Canadian Home Care Association, 2010) and our observations of practice. Formally, let $h_{kj} \geq 0$ be the service time of a patient with health requirement $k \in \mathcal{K}$ that is assigned to HP j . Then,

$$F_S^j(\vec{x}) := \sum_{k=1}^K \sum_{d=1}^{D_k} \sum_{m=1}^M h_{kj} x_{kdmj}. \quad (10)$$

The expected service time for each HP is therefore equivalent to the service time for the expected number of patients assigned to each HP for each $t = 1, \dots, \tau - 1$. Thus, $S_{tj}(\cdot) = \mathbb{E}_t [F_S^j(\cdot)] = F_S^j(\mathbb{E}_t[\cdot])$.

To calculate $S_{tj}(\cdot)$ based on (10), recall from Section 4.1 that $q_{kdmj} \in [0, 1]$ is the probability of assigning a patient of type (k, d, m) to HP j . Define ω_{kdj} to be the expected number of patients who have health requirement k , need service for d periods, and are assigned to HP j . Based on the action set of the basis policy given by (4), ω_{kdj} is calculated as

$$\omega_{kdj} = \left(\sum_{m=1}^M \mathbb{P}_{kdm} q_{kdmj} \right) \mathbb{E}[\Theta], \quad (11)$$

where \mathbb{P}_{kdm} is the probability that a patient of type (k, d, m) arrives in the system and $\mathbb{E}[\Theta]$ is the expected number of patients referred per period. Both values are obtained from the underlying arrival distribution and can be estimated from our data (see Section 7.1).

PROPOSITION 4. *Suppose that in the current decision epoch, action \vec{a} is performed in state \vec{x} . The expected time required for HP j to provide service to patients in her panel in period $t = 1, \dots, \tau - 1$ is*

$$S_{tj}(\vec{x}, \vec{a}) = \sum_{k=1}^K h_{kj} \left(\tilde{\omega}_{ktj} + \sum_{m=1}^M \sum_{d=t+1}^{\tau} (x_{kdmj} + a_{kdmj}) \right), \quad (12)$$

where $\tilde{\omega}_{ktj}$ is the expected number of patients with health requirement k who are assigned to HP j in period t using basis policy \mathcal{P} . The values of $\tilde{\omega}_{ktj}$ are computed using the following set of relations:

$$\tilde{\omega}_{k1j} = \sum_{d=1}^{D_k} \omega_{kdj}, \quad \tilde{\omega}_{ktj} = \tilde{\omega}_{k(t-1)j} + \tilde{\omega}_{k1j} - \sum_{d=1}^{\min\{t, D_k\}} \omega_{k(t-d)j}, \quad t > 1. \quad (13)$$

Note that, because the probabilities q_{kdmj} are given, the terms $\{\tilde{\omega}_{ktj}\}_{\forall k, t, j}$ can be computed a priori.

5.3. Expected Transit Time

We next characterize the transit time function $T_{tj}(\vec{x}, \vec{a})$. In practice, the route by which patients are visited may be decided by the HCA on the day of service. Thus, we present an approach that estimates the expected transit time for each HP. It may also be used to provide valid bounds if additional operational constraints (e.g., time windows and time-dependent travel times) are incorporated.

Our estimate assumes that geographic regions are defined on a two-dimensional metric space representing, for example, locations on a map. Such a space is represented by a graph $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$ for each HP j in the system. A vertex in the set \mathcal{V}_j designates a region $m \in \{1, \dots, M\}$ that an HP may visit (HPs may not serve patients from every region) and is denoted by v_{mj} . The vertex set also includes a special vertex, $v_j^0 \in \mathcal{V}_j$, representing the region HP j starts and ends her shift. The edge set \mathcal{E}_j encodes the connections between regions. With each edge $(v_{mj}, v_{\tilde{m}j}) \in \mathcal{E}_j$ we associate a distance function that provides an estimate of the travel time between regions m and \tilde{m} for HP j . Since \mathcal{G}_j is embedded in a metric space, any three vertices satisfy the triangle inequality. This representation also assumes that the travel time between two patients in the same region is negligible.

For the purposes of estimating $T_{tj}(\vec{x}, \vec{a})$, we construct a graph \mathcal{G}_j with the same vertex set \mathcal{V}_j and edge set \mathcal{E}_j for all $t = 1, \dots, \tau - 1$. Each vertex $v_{mj} \in \mathcal{V}_j$ is said to be active (i.e., must be visited) with probability $\tilde{\phi}_{mtj}$ and inactive (i.e., does not have to be visited) with probability $1 - \tilde{\phi}_{mtj}$, where $\tilde{\phi}_{mtj}$ is interpreted as the probability that at least one patient from region m is served by HP j in period t . Probabilities $\tilde{\phi}_{mtj}$ are computed using the equation

$$\tilde{\phi}_{mtj} = \sum_{k=1}^K \sum_{d=t}^{D_k} \phi_{kdm} q_{kdmj}, \quad (14)$$

with ϕ_{kdm} as the probability that at least one patient with health requirement k , from region m , and with an expected duration of at least d periods, is in HP j 's panel. As in Section 5.2, $\tilde{\phi}_{mtj}$ is estimated from the arrival probability distribution (see Section 7.1). Notice that the vertices and edges of \mathcal{G}_j are fixed for each HP while the probability a vertex is active is period-dependent.

Given a graph \mathcal{G}_j , we estimate the expected transit time of HP j by formulating the problem as a *probabilistic traveling salesperson problem* (PTSP) or *a priori TSP* (Jaillet, 1985), defined as follows. Let $\pi_j \in 2^{\mathcal{V}}$ be a permutation of \mathcal{V}_j . For a realization of vertices $\mathcal{V} \subseteq \mathcal{V}_j$ and a period t , we define $L_{tj}(\mathcal{V}, \pi_j)$ to be the length of a Hamiltonian tour that visits each vertex in \mathcal{V} according to the order imposed by π_j ; that is, we skip vertices that have not been realized. The probability that \mathcal{V} occurs in period t is $\delta_{tj}(\mathcal{V}) = \prod_{v_{mj} \in \mathcal{V}} \tilde{\phi}_{mtj} \prod_{v_{mj} \notin \mathcal{V}} (1 - \tilde{\phi}_{mtj})$ and the expected tour length for HP j in period t is $\sum_{\mathcal{V} \subseteq \mathcal{V}_j} \delta_{tj}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j)$. The objective of the PTSP is to find a permutation π_j^* with the minimum expected length, i.e., $\sum_{\mathcal{V} \subseteq \mathcal{V}_j} \delta_{tj}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) \leq \sum_{\mathcal{V} \subseteq \mathcal{V}_j} \delta_{tj}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j)$ for all $\pi_j \in 2^{\mathcal{V}}$.

We use the PTSP to represent the expected transit time, $T_{tj}(\vec{x}, \vec{a})$, given that the patient's panel of HP j in period t is uncertain. Specifically, let $\mathcal{Q} \subseteq \mathcal{V}_j$ be a subset of vertices that must be visited. That is, vertices $v_{mj} \in \mathcal{Q}$ are active with probability 1, while vertices $v_{mj} \notin \mathcal{Q}$ are active with probability $\tilde{\phi}_{mtj}$. Define $\text{PTSP}_{tj} : 2^{\mathcal{V}_j} \rightarrow \mathbb{R}_{\geq 0}$ to be a function such that $\text{PTSP}_{tj}(\mathcal{Q})$ returns the optimal expected length of the PTSP tour given vertices $\mathcal{Q} \subseteq \mathcal{V}_j$. Thus, given probabilities $\tilde{\phi}_{mtj}$ for each vertex in \mathcal{V}_j ,

$$T_{tj}(\vec{x}, \vec{a}) = \text{PTSP}_{tj} \left(\left\{ v_{mj} \in \mathcal{V}_j : \sum_{k=1}^K \sum_{d=t+1}^{\tau} (x_{kdmj} + a_{kdmj}) \geq 1 \right\} \right), \quad (15)$$

The argument of $\text{PTSP}_{tj}(\cdot)$ are the geographic regions where at least one patient is assigned to HP j in period t (patients are either in the panel from previous periods or are newly allocated in the current period). Since these regions represent vertices that are active with probability one, the minimum expected PTSP tour length will account for these vertices while assuming that the remaining vertices are active with probability $\tilde{\phi}_{mtj}$. The following result demonstrates that $\text{PTSP}_{tj}(\cdot)$ is non-decreasing in its argument, which makes it particularly useful for optimization.

PROPOSITION 5. For any period t and vertex sets $\mathcal{Q}_1 \subseteq \mathcal{Q}_2 \subseteq \mathcal{V}_j$, $\text{PTSP}_{tj}(\mathcal{Q}_1) \leq \text{PTSP}_{tj}(\mathcal{Q}_2)$.

6. Solution Approach

In this section, we present a solution approach to (SOP) using an iterative cutting-plane approach known as the integer L-shaped method (Laporte and Louveaux, 1993). The algorithm maintains a deterministic mathematical program, denoted by the *master problem*, that approximates the wage cost for each HP from below. At every iteration, the costs are evaluated exactly and the wage cost approximation is made more accurate through the addition of *optimality cuts* to the master problem. The procedure is repeated until the optimal is obtained or a desired optimality gap is achieved.

To define the master problem, let $\eta = \sum_{k=1}^K \sum_{d=1}^{D_k} \sum_{m=1}^M \theta_{kdm}$ be the total number of arriving patients for a realization $\vec{\theta} \sim \Theta$ in a given period. We associate each arriving patient with a distinct index $i = 1, \dots, \eta$. Patients of the same type are identified by the set $\mathcal{H}(k, d, m)$, which returns the indices associated with a patient who has health requirement k , duration d , and region of residence m . We also introduce a binary vector $\vec{\mu} \in \{0, 1\}^{\eta \times J}$ that encodes a solution (\vec{a}, \vec{b}) to (SOP) when

$$\sum_{i \in \mathcal{H}(k, d, m)} \mu_{ij} = a_{kdmj}$$

and $b_{kdm} = \theta_{kdm} - \sum_{j=1}^J a_{kdmj}$ for all k, d , and m . That is, $\mu_{ij} = 1$ if and only if patient i is assigned to HP j . This redundant representation plays a key role in the convergence of our solution procedure. Namely, at an arbitrary iteration ℓ , we consider the following mathematical program as the master problem, which introduces continuous variables c_j to represent the discounted wage cost of HP j :

$$\max_{\vec{\mu}, \vec{a}, \vec{b}, \vec{c}} R(\vec{a}, \vec{b}) - \sum_{j=1}^J c_j \quad (\text{MP})$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}(k)} a_{kdmj} + b_{kdm} = \theta_{kdm} \quad \forall k, d, m, \quad (16)$$

$$\sum_{i \in \mathcal{H}(k, d, m)} \mu_{ij} - a_{kdmj} = 0 \quad \forall k, d, m, \quad (17)$$

$$c_j \geq 0, b_{kdm} \in \mathbb{Z}_+, \mu_{ij} \in \{0, 1\} \quad \forall k, d, m, i, j, \quad (18)$$

$$c_j \geq \text{Optimality Cut}_\ell(\vec{\mu}) \quad \forall \ell. \quad (19)$$

The objective function of (MP) minimizes the expected revenue minus the rejection and wage costs over time horizon τ . Constraint (16) ensures that the optimal action (\vec{a}, \vec{b}) remains feasible. The assignment variables \vec{a} are linked to the binary variables $\vec{\mu}$ by constraint (17). Constraints (18) defines the domains of $\vec{\mu}$, \vec{b} , and \vec{c} ; the domain of \vec{a} is implied by $\vec{\mu}$. Finally, constraint (19) represents the optimality cuts accumulated from the ℓ previous iterations. Note that, because wage costs are non-negative and increasing in their arguments, (MP) provides an upper bound to (SOP).

Constraint (19) must relate $\vec{\mu}$ and \vec{c} in order to improve the estimation of the wage cost for HP j in each iteration. Thus, consider an optimal solution $(\vec{\mu}^\ell, \vec{a}^\ell)$ to (MP) in iteration ℓ . The wage cost for HP j in iteration $\ell + 1$ should be at least as large as the cost when assigning the same patients, implied by $(\vec{\mu}^\ell, \vec{a}^\ell)$, to that HP j . The optimality cuts are, therefore, written as the linear inequalities

$$c_j \geq \left(1 + \sum_{\{i|\mu_{ij}^\ell=1\}} \mu_{ij} - \sum_{i=1}^{\eta} \mu_{ij}^\ell \right) \cdot \left[G_j \left(F_S^j(\vec{x} + \vec{a}^\ell) + F_T^j(\vec{x} + \vec{a}^\ell) \right) + \sum_{t=1}^{\tau-1} \rho^t G_j \left(S_{tj}(\vec{x}, \vec{a}^\ell) + T_{tj}(\vec{x}, \vec{a}^\ell) \right) \right], \quad \forall j. \quad (20)$$

The next proposition proves that the algorithm converges to the optimum of (SOP) in a finite number of iterations, while also providing a sequence of valid upper bounds.

PROPOSITION 6. *Let z^ℓ be the optimal solution value of (MP) in iteration ℓ with optimality cuts defined as in (20). For a sufficiently large but finite $\bar{\ell}$, we have that $z^1 \geq z^2 \geq \dots \geq z^{\bar{\ell}} = z^*$, where z^* is the optimal objective function value of problem (SOP).*

Because we compute the real cost of a feasible assignment $\vec{\mu}$ at each iteration, we also obtain a sequence of valid lower bounds to (SOP). Since the upper bounds become tighter as ℓ increases, the solution procedure can be terminated when the desired optimality gap has been achieved.

6.1. Computing Optimality Cuts

Given an optimal solution $(\vec{\mu}^\ell, \vec{a}^\ell)$ to (MP) in iteration ℓ , the cut (20) requires the evaluation of the service time $F_S^j(\vec{x})$, transit time $F_T^j(\vec{x})$, and their corresponding stochastic variants, $S_{tj}(\vec{x}, \vec{a})$ and $T_{tj}(\vec{x}, \vec{a})$, for each HP j and period t . These quantities are then used as arguments to $G_j(\cdot)$ and the corresponding cut is generated. The terms $F_S^j(\vec{x})$ and $S_{tj}(\vec{x}, \vec{a})$ are obtained using the closed-form expression (12). Given an assignment of patients to HP j , the function $F_T^j(\vec{x})$ can be computed exactly by solving a deterministic minimum-length TSP (e.g., Applegate et al. (2006)). The function $T_{tj}(\vec{x}, \vec{a})$ requires the solution of a PTSP as defined in (15), for which we propose two alternatives.

Exact Method: Several exact solution approaches for the PTSP exist, e.g., Henchiri and Bellalouna (2014) and Amar et al. (2017). We consider the branch-and-cut technique by Laporte et al. (1994) that has been shown to be successful for up to 50 regions. The model introduces a binary variable

$r_{m\tilde{m}}$ for each $v_{m_j}, v_{\tilde{m}_j} \in \mathcal{V}_j$, which is equal to one if and only if v_m immediately precedes $v_{\tilde{m}}$ in the PTSP sequence. If $E_j(m, \tilde{m})$ is the edge length between vertices v_{m_j} and $v_{\tilde{m}_j}$ for HP j , then

$$\begin{aligned} & \min_{\vec{r}, \xi} \quad \sum_{v_{m_j}, v_{\tilde{m}_j}} E_j(m, \tilde{m}) \cdot r_{m\tilde{m}} + \xi && \text{(MPTSP)} \\ \text{subject to} & \quad \sum_{v_{m_j}} r_{m\tilde{m}} = \sum_{v_{\tilde{m}_j}} r_{\tilde{m}m} = 1 && \forall v_{\tilde{m}_j} \in \mathcal{V}_j, \quad (21) \\ & \quad \sum_{v_{m_j}, v_{\tilde{m}_j} \in \mathcal{S}} r_{m\tilde{m}} \leq |\mathcal{S}| - 1 && \forall \mathcal{S} \subseteq \mathcal{V}_j, 3 \leq |\mathcal{S}| \leq |\mathcal{V}_j| - 3, \quad (22) \\ & \quad r_{m\tilde{m}} \in \{0, 1\}, \quad \xi \text{ is unrestricted} && \forall v_{m_j}, v_{\tilde{m}_j} \in \mathcal{V}_j. \quad (23) \end{aligned}$$

The objective of (MPTSP) minimizes the TSP tour length plus a correction factor ξ that adjusts for the PTSP objective. Constraint (21) ensures that each region is assigned a position in the optimal tour while constraint (22) are the subtour elimination constraints as classically defined for the TSP.

Formulation (MPTSP) is solved using a branch-and-cut approach where the subtour elimination constraints (22) are separated at fractional nodes. For nodes where an integer feasible solution is found, a special cut is added to adjust the correction factor ξ . The strength of the cut depends on several factors but can be enhanced by leveraging other lower bounds for ξ ; we refer to Laporte et al. (1994) for a list of available cuts and cut-strengthening techniques.

Approximate Method: Exact techniques for the PTSP require an involved implementation to be efficient and may be associated with large computational costs. Thus, we also investigate a heuristic approach that has a well-defined approximation guarantee. Specifically, we apply the randomized 4-approximation heuristic by Shmoys and Talwar (2008). The procedure first samples the vertices according to their probabilities and then computes a 2-approximation to the deterministic TSP based on a minimum spanning tree algorithm (see, e.g., Cook et al. 1998). The worst-case time complexity is $O(|\mathcal{V}_j| \log |\mathcal{V}_j|)$. It is repeated several times (up to 50) to reach the desired approximation accuracy. For each sampled solution, we apply a local search technique (i.e., 2-p-opt and 1-shift) to improve the objective as in Bianchi and Campbell (2007). The final complexity of the procedure is $O(|\mathcal{V}_j|^2)$.

The approach yields an upper bound to the real PTSP that is polynomial-time solvable, which can be beneficial when a large number of cuts are generated. While the L-shaped method with cuts derived from such upper bounds may converge to a non-optimal solution for (SOP), we show in Section 7.2 that the values between the real and approximate PTSP are often close.

6.2. Enhancing the L-Shaped Method

We now discuss several enhancements to the L-shaped method that significantly improve the convergence of the procedure, specifically for the case of larger HCAs.

Warm-start. An initial heuristic solution can be computed efficiently and provide a meaningful lower bound. It can be also used by mathematical programming solvers to rule out suboptimal solutions. We apply a simple insertion heuristic where, given a sequence of patients ordered by their duration, we consecutively add patients to the HP that gives the largest reward.

Swaps. Each iteration of the L-shaped method provides a feasible solution to (MP). We can heuristically swap patients between HPs to verify if any further improvements can be made. For our model, we sort HPs in decreasing order of their wage costs. Next, for each HP in the sequence, we remove one patient from that HP and add them to another HP based on the classical nearest-neighbor heuristic, i.e., the HP with minimum distance to that patient. The solution quality is then evaluated.

Valid Inequalities. A key technique to improve convergence is to add valid linear inequalities to the master problem, thereby improving its upper bound and ruling out unnecessary assignments. We propose a two-step approximation to derive inequalities that bound c_j from below. First, given $N > 0$, we apply the procedure by Yang and Goh (1997) to obtain a piecewise linear function

$$\max_{n=1,\dots,N} \left\{ \beta_{nj}^{(1)} + \beta_{nj}^{(2)} \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right) \right\}$$

that bounds $G_j(\cdot)$ from below for coefficients $\beta_{nj}^{(1)}$ and $\beta_{nj}^{(2)}$. This can be applied for each HP j and decision epoch t . Piecewise functions can be linearized and are addressed by special techniques in state-of-the-art solvers (e.g., Gurobi Optimization, 2018). We include additional details in the appendix concerning the computation of the function. This leads to the valid inequality

$$c_j \geq \sum_{t=1}^{\tau-1} \rho^t \left(\max_{n=1,\dots,N} \left\{ \beta_{nj}^{(1)} + \beta_n^{(2)} \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right) \right\} \right). \quad (24)$$

The term $S_{tj}(\vec{x}, \vec{a})$ is the expression (12) and can be explicitly incorporated into (24). The term $T_{tj}(\vec{x}, \vec{a})$ can be replaced by a linear inequality that also bounds the PTSP from below. Thus, in the second step, we derive an inequality based on the marginal effect of assigning a patient to an HP.

PROPOSITION 7. *For each patient $i = 1, \dots, \eta$, define $\vec{a}^i = \{a_{kdmj}^i\}_{\forall k,d,m,j}$ such that $a_{kdmj}^i = 1$ for every j if $i \in \mathcal{H}(k,d,m)$ and zero otherwise. The following is valid for any feasible $(\vec{a}, \vec{\mu})$ of (MP).*

$$T_{tj}(\vec{x}, \vec{a}) \geq \frac{1}{\eta} \cdot \sum_{i=1}^{\eta} T_{tj}(\vec{x}, \vec{a}^i) \mu_{ij}, \quad \forall t, j \quad (25)$$

Proposition 7 can be applied for cases where $T_{tj}(\vec{x}, \vec{a}^i)$ is computationally inexpensive to obtain, such as when HPs can only visit a small number of regions (as is typical of home care systems). Nonetheless, if a heuristic with an α -approximation guarantee is provided, such as in the case of Shmoys and Talwar (2008), we can also derive a valid cut to (MP), as follows:

COROLLARY 1. Let w_{tj}^i be an α -approximate solution to $T_{tj}(\vec{x}, \vec{a}^i)$, where \vec{a}^i is defined as in Proposition 7. The following inequalities are valid for any feasible solution $(\vec{a}, \vec{\mu})$ of (MP).

$$T_{tj}(\vec{x}, \vec{a}) \geq \frac{1}{\alpha \cdot \eta} \cdot \sum_{i=1}^{\eta} w_{tj}^i \cdot \mu_{ij}, \quad \forall t, j \quad (26)$$

We note that an added benefit of approximating $G_j(\cdot)$ using a piecewise-linear function is that the first iteration of the solution approach, when $N = 1$, can be solved in polynomial time.

PROPOSITION 8. If the piecewise linear function is defined by a single segment (i.e., $N = 1$) and each referral is from a different geographic region, then the first iteration of the L-shaped algorithm can be solved in polynomial time on the number of referred patients.

If it is rare for multiple patients to be referred from the same region in the same period, the estimate may perform well. In fact, even if $\sum_{k=1}^K \sum_{d=1}^{D_k} \theta_{kdm} \geq 2$ for some m , the bound can still be used. In this case, region m can be partitioned into multiple subregions, each with the same cost.

7. Numerical Analysis

In this section, we present a numerical study of our methodology using a discrete-event simulation of the HCA. We focus on understanding how our approach affects the profitability of the HCA as a function of the arrival rate, average travel speed, available number of HPs, and the rejection cost. We also assess the quality of the PTSP approximation and convergence of the L-shaped algorithm.

7.1. Data Set Description and Probability Estimation

We obtained data on 1,435 patients referred to a home care agency operating in the Halton region, a large municipality in southern Ontario, Canada. All patients were referred between August 2015 and June 2016. Each entry in the data set includes the date the patient was referred, their mailing code (e.g., zip code), the start date, the projected number of visits, the set of service requirements (i.e., the skill set), the employees assigned to their care, and the corresponding employee's mailing code.

The data set contains five health requirements: personal support, nursing, wound care, continence, and enterostomal therapy. Nurses can perform all activities while personal support workers (PSWs) only provide personal support. Although we experimented with other hierarchical structures and found qualitatively similar results, we focus on this nested structure as it represents the setup of our industry partner. The proportion of patients that need each health requirement is given in Table 1.

We find the geometric distribution most closely models the number of referrals per period; the Ljung-Box test also suggests the data is stationary ($\chi^2 = 534.27$, $p < 0.0001$). A lognormal distribution best describes the expected amount of time a patient spends in the system. The parameters of the lognormal distribution for each health requirement are presented at the bottom of Table 1. A one-way ANOVA confirms that health requirements affect the expected duration ($F(4) = 6.3528$, $p < 0.0001$).

	Continance	Enterostomal	Nurse	Personal Support	Wound
Proportion	0.0071	0.0519	0.3154	0.5683	0.0574
Location	1.4050	1.2934	1.1192	1.7173	1.0376
Scale	1.4881	1.4343	1.3002	1.5920	1.2432
Mean	12.3314	10.1952	7.1312	19.7773	6.1123
99-Percentile	132.1568	102.7057	63.0684	226.0731	50.9336

Table 1 Proportion of patients for each health requirement and the duration distribution parameters.

We use the patient’s mailing code to compute the spatial distribution of referrals and the distance in kilometers between regions. This is similar to what the HCA does to estimate its travel times. We find weak evidence to suggest that the mailing code and health requirement are correlated ($\chi^2 = 211.62$, $p = 0.033$, Cramer’s $-V = 0.193$) and hence assume pairwise independence. Our data set contains 67 mailing codes and the probability a patient is referred from a geographic region is the ratio between the number of patients referred with that mailing code divided by the total number of referrals.

The data set includes a list of employees, their skill sets, and their region of residence. We assume the HCA incurs a regular wage cost of \$40/hour for nurses ([Registered Nurses Association of Ontario, 2017](#)) and \$20/hour for PSWs ([Ontario Ministry of Health, 2015](#)). Overtime pay is typically 1.5 times the regular hourly rate ([Registered Nurses Association of Ontario, 2017](#)). However, after two hours of overtime, we exponentially increase the hourly rate to discourage excessively long shifts. This is done to ensure that the scheduling behavior of the HCA in the simulation coincides with practice. On average, an HCA is reimbursed approximately \$45/visit ([Ontario Community Care Access Centres, 2015](#)). Since there are five health requirements, we order them from the highest to the lowest reimbursement rate, i.e., Continance, Enterostomal, Wound, Nurse, Personal Support. We then set the weighted average (reimbursement rate multiplied by the proportion of its occurrence from [Table 1](#)) equal to \$45. We assume all appointments are 30 minutes in length with no variability and that employees normally work 6 hours per day; this includes both transit and service time.

The rejection cost is set to the sum of three quantities. The first represents the total revenue that the patient would bring to the system, minus the minimum cost associated with their service (i.e., service plus direct travel time). This is the direct cost of not providing service to that patient. The second quantity is a penalty associated with the number of periods the patient is expected to remain in the system. This represents a “loss-of-goodwill” cost associated with the strain that rejecting a patient places on the relationship between the HCA and the LHIN. Finally, there is an additional penalty if the patient lives in a remote region (i.e., no HP lives in that region). We vary these values to investigate the sensitivity of our results to the magnitude of the rejection cost.

The parameters of the EA policy are chosen such that all HPs that have the skill set to satisfy health requirement k have an equal chance of being assigned to the patient i.e., $q_{kdmj} = 1/|\mathcal{J}(k)|$. For the SL policy, we employ a two-stage machine learning procedure to estimate the q_{kdmj} values.

In the first stage, we impute missing values of the duration distribution and patient mailing code using a non-parametric random forest technique (Stekhoven and Bühlmann, 2012). The approach can handle mixed-type data, nonlinear relationships, and has been shown to be faster and more accurate than other imputation procedures (Shah et al., 2014). In the second stage, we use the imputed data in a Naïve Bayes classification algorithm. A multinomial distribution is used for discrete features and a lognormal distribution is used for continuous data. Although other classifiers may give more accurate probability estimates, it is generally known that Naïve Bayes classifiers outperform other machine learning models (e.g., logistic regression) when there is less training data. They are also easy to implement, which makes them well-suited for use by practitioners.

To obtain the OTS policy parameters and to implement the PIH-based policies, we first define the relevant probabilistic and expected quantities. Recall that $\mathbb{P}_{kdm} := \mathbb{P}(k)\mathbb{P}(d|k)\mathbb{P}(m)$ is the probability that a patient with health requirement k , who stays d periods in the system, and is from region m , is referred to the HCA. Let λ_{\max} be the largest number of patients that can be referred in any period. If $\mathbb{P}(l)$ is the probability that l patients are referred in a given period, then

$$\phi_{kdm} = 1 - \sum_{l=0}^{\lambda_{\max}} \left(1 - \sum_{k=1}^K \sum_{\bar{d}=d}^{D_k} \mathbb{P}_{k\bar{d}m} q_{k\bar{d}m,j} \right)^l \mathbb{P}(l) \quad (27)$$

is the probability that at least one patient of type (k, d, m) is referred to the HCA. Since the arrival distribution is geometric with probability parameter λ , (27) can be written in closed-form.

$$\phi_{kdm} = 1 - \frac{\lambda \left(1 - (1 - \lambda)^{1 + \lambda_{\max}} \left(\sum_{k=1}^K \sum_{\bar{d}=d}^{D_k} \mathbb{P}_{k\bar{d}m} q_{k\bar{d}m,j} \right)^{1 + \lambda_{\max}} \right)}{\lambda + (1 - \lambda) \left(\sum_{k=1}^K \sum_{\bar{d}=d}^{D_k} \mathbb{P}_{k\bar{d}m} q_{k\bar{d}m,j} \right)}.$$

For the myopic policy, patients are assigned to HPs such that the reward in the current period is maximized. That is, we solve (SOP) considering only referred patients and the current HPs' panel, using a minimum-length TSP tour to determine the transit time for each HP. Computing the TSP tour is also used for the comparative statistics presented in Section 7.2.

To ensure the summations and probability vectors are finite, we set $\lambda_{\max} = 34$ and $D_{\max} = 226$. This represents the 99th percentile for the number of arrivals and expected duration, respectively. We discretize the lognormal distribution using the methods described in (Chakraborty, 2015). This ensures proper mass is used for each realization of the duration and that the summations over discrete values of d are unbiased. All other quantities, i.e., $\tilde{\omega}_{mtj}$ and $\tilde{\phi}_{mtj}$, can be computed once ω_{kdmj} and ϕ_{kdmj} are known for all k, m, d, j . Finally, we use a discount factor of $\rho = 0.8$.

7.2. Discrete-Event Simulation Results

We employed the batch-means method with 105 batches of 365 consecutive days (i.e., one calendar year) of operation. The first 5 batches were used as the warm-up period, while the remaining 100

were used to compute relevant simulation statistics. Mathematical programs were solved with Gurobi 8.0 (Gurobi Optimization, 2018) on a 3.2 GHz processor with 16 GB RAM and 2 cores/process. For each model, the solver terminated if the relative optimality gap was less than 5% or a time limit of 30 minutes was reached. We also used the approximate PTSP approach of Section 6.1 for all simulations; the quality of such bounds is subject to analysis below. We use EA, SL, and OTS to denote (MP) with basis policies EA, SL, and OTS, respectively; *Myopic* denotes the myopic policy.

Analysis of Arrival Rates and Travel Speed. We first investigate how our approach affects the profitability of the HCA as a function of the average arrival rate and travel speed. We consider the 20 most active HPs in the HCA; eight are PSWs and twelve are nurses. We evaluate six arrival rates, six vehicle speeds, and four different rejection costs. For each scenario, we recorded the average value and standard error of the daily reward, the work time per HP per day, the mean number of patients per HP per day, and the ratio between the time spent traveling versus providing service.

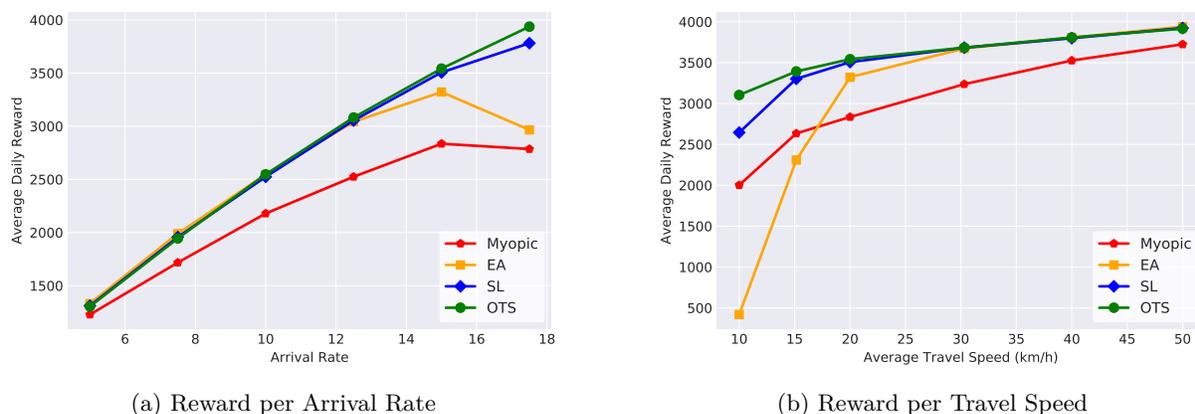


Figure 1 Average agency daily reward as a function of the arrival rate in (a) and the average travel speed in (b). For (a), the average travel speed is 20 km/h, and for (b) the average arrival rate is 15 patients/day.

Figure 1a shows the average daily profit for distinct arrival rates and for a travel speed of 20 km/h (detailed tables are presented in the appendix). We observe that, when the arrival rate is low, there is no statistical difference between the policies. This is because HPs are underutilized, with an average shift length that is far less than the standard 8-hour work day. As the arrival rate increases, HPs are responsible for more patients and the HCA's scheduling practices impact the amount of overtime. For an arrival rate of 12.5 patients/day, the SL and OTS policies generate approximately 19% more profit as compared to Myopic. This grows to 35% for an arrival rate of 17.5 patients/day due to the escalating, nonlinear cost of overtime. Moreover, the PIH-based policies enable a more equitable distribution of workload over the long term by considering not only the duration of care, but the probability that patients from different regions will be referred in the future. This indicates that HPs

serve more patients, work fewer hours on average, and travel less. Finally, we note that OTS and SL significantly outperform EA for more congested systems as they both assume that patients in future periods are more likely to be assigned to HPs in nearby geographic regions.

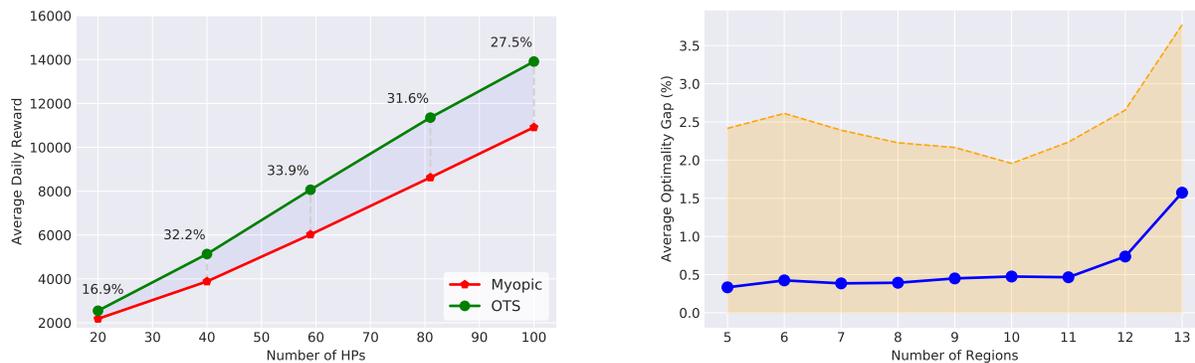
Figure 1b shows the average daily profit for distinct travel speeds with an arrival rate of 15 patients/day. We observe that, as the average travel speed decreases, the gap between the performance of the PIH-based policies and the myopic policy increases. This occurs as, given a certain average workload, the ratio of the travel to service time increases. In regions with average travel speeds around 10 km/h (Trigg, 2015), average daily rewards for SL and OTS are approximately 32% better than Myopic. We note that all policies, including Myopic, are superior to EA for low travel speeds, as it captures little information about expected patient assignments in future periods.

Rej. Scale	Policy	Reward/Day		# Patients/HP		Shift Length (hours)		Travel/Service	
0x	Myopic	3,234.02	(27.35)	11.43	(0.083)	7.68	(0.043)	0.35	(0.003)
	OTS	3,996.75	(31.24)	11.90	(0.090)	7.14	(0.046)	0.20	(0.002)
1x	Myopic	2,785.65	(71.01)	12.11	(0.090)	8.30	(0.047)	0.38	(0.003)
	OTS	3,936.89	(42.71)	12.19	(0.093)	7.47	(0.049)	0.23	(0.002)
10x	Myopic	2,322.89	(245.78)	12.14	(0.096)	8.29	(0.052)	0.37	(0.003)
	OTS	3,892.94	(208.14)	12.26	(0.096)	7.54	(0.051)	0.23	(0.002)
100x	Myopic	4,452.77	(1,861.88)	12.15	(0.097)	8.33	(0.053)	0.38	(0.003)
	OTS	6,156.50	(1,865.10)	12.18	(0.095)	7.55	(0.051)	0.24	(0.002)

Table 2 Simulation results for varying rejection scale factors. The travel speed is 20 km/h with an arrival rate of 17.5 patients/day. Standard errors are in parenthesis.

Analysis of Rejection Costs. In Table 2, we analyze how sensitive our policies are to the estimated cost of rejecting patients. The first column represents the scale of the rejection cost, i.e., the ratio between the average rejection cost and the average reward per patient. As expected, our solution methodology becomes more valuable as the cost of rejection increases; our other results remain qualitatively similar. Conversely, as the cost of rejection decreases, all policies have approximately equal performance. This is because when there is no rejection penalty, each policy will accept only the most beneficial patients. However, even if rejecting patients is costless, the PIH-based policies still result in shorter shifts and less HP travel time. Similar results are obtained with SL and EA.

Analysis of Larger Systems. We now investigate how our scheduling approach scales as the size of the HCA (i.e., J) increases. We consider the most active 20, 40, 60, 80, and 100 HPs in the HCA and note that the proportion of PSWs is approximately 50% in each case. The travel speed is set to 20 km/h and the daily arrival rate is fixed to $J/2$, i.e., half of the number of HPs. We observe, in Figure 2a, that as the size of the HCA increases, the profitability with respect to the myopic policy also increases. The performance of the PIH-based policies scale with the scenario size as each HP is individually allocated patients more effectively. As a result, additional HPs (for a proportional arrival rate) will positively contribute to the total cost. Similar results are obtained with SL.



(a) Reward vs. Number of HPs

(b) PTSP: Instance size vs. Avg. Optimality Gap

Figure 2 (a) Average daily reward as a function of the number of HPs and (b) quality of the PTSP solutions.

PTSP Evaluation. We next evaluate the quality of the approximate PTSP solutions in comparison to the exact method as described in Section 6.1. We collected distinct PTSP instances generated during a representative simulation, namely, a scenario with 20 HPs, an arrival rate of 17.5 (resulting in larger HP panels), and travel speeds of 20 km/h. The probabilities were computed using SL so that ϕ_{kdm} was derived from our data. For a 105-batch run, 1,509,309 distinct PTSP instances were generated. We collected the relative gap of the approximation to the optimal solution value.

Figure 2b presents the average optimality gap as a function of the number of regions with nonzero probability. The instance sizes are relatively small (up to 13 regions), but sufficiently large for simple enumeration to be prohibitive when generating cuts. This is a key feature in home care, as HPs typically travel to a few regions per day. We find that the approximation performs better than its theoretical guarantee: it matched the optimal in 78% of runs, only 0.6% instances had an optimality gap above 5%, and the worst-case run had an optimality gap of 22%. Finally, solution times for the exact PTSP approach varied greatly with instance characteristics (such as the probabilities), ranging from a few seconds to minutes, while the approximate PTSP required a few milliseconds in all cases.

Performance of the Integer L-Shaped Method. Figure 3a presents the average convergence time for different HCA sizes (i.e., J) under the approximate PTSP regime, obtained from the simulations with a travel speed of 20 km/h and an arrival rate of $J/2$. Solution times are less than one minute for 91% of the runs. In particular, all runs with 20 and 40 HPs were solved to the desired optimality gap within the time limit and the enhancements (Section 6.2) were key to such performance. Figure 3b depicts a typical instance and demonstrates that convergence of the enhanced L-Shaped method is orders of magnitude faster than the model without valid inequalities. For the hardest problems, i.e., scenarios with 100 HPs, Figure 3c plots the solution time for instances that converged within the time limit of 30 minutes as a function of the number of referred patients. Note these times are on average below 400 seconds. For the approximately 5% of instances that were not solved within the

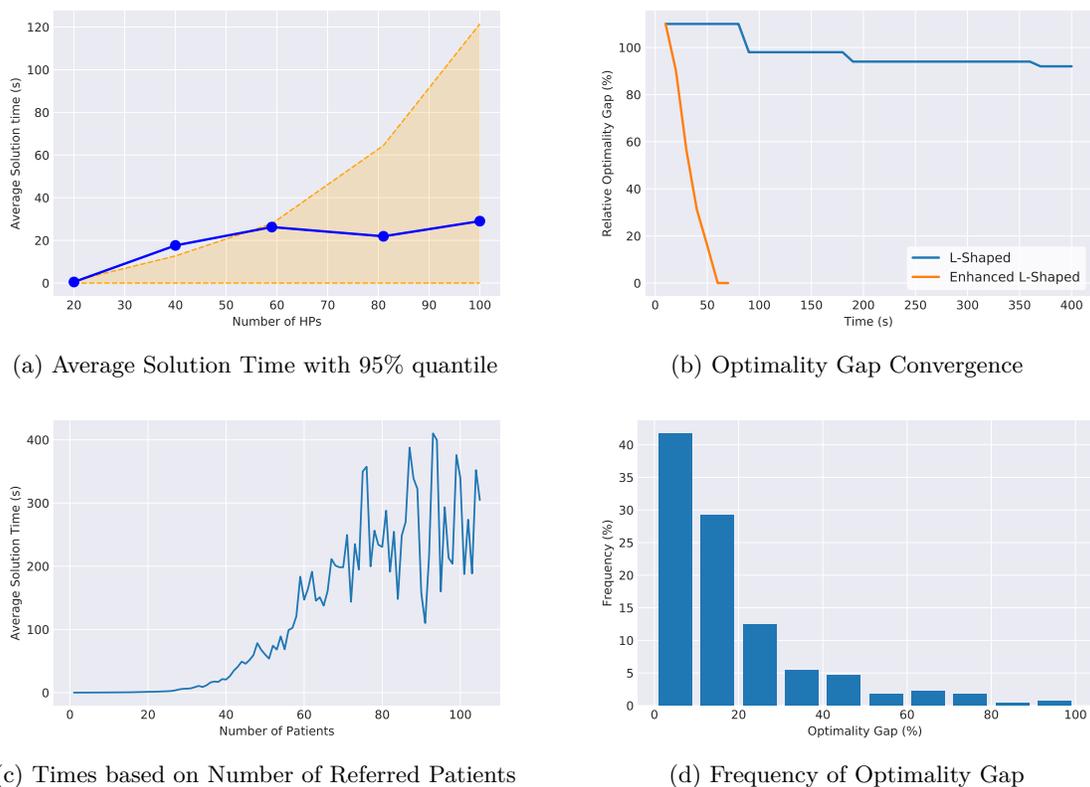


Figure 3 L-Shaped algorithm performance considering: (a) HCA size; (b) optimality gap convergence; (c) convergence times for solved instances; and (d) optimality gap for instances reaching the time limit.

time limit, Figure 3d shows the frequency of the resulting optimality gap. These results suggest that instances where the lower bounds cannot be assessed, albeit expected due to the long-tail solution time of NP-Hard problems, are still rare in comparison to the mean run for the short time limit.

8. Extensions

In this section, we expand our formulation to include three features commonly observed in practice: patients who return for additional service, periodic visits, and multiple HP-patient assignments.

8.1. Returning Patients

We previously assumed that upon referral, the number of service periods that a patient requires is known. Throughout the course of treatment, however, this requirement may change. Although it is rare that patients need less service, it is common to require care for longer than was expected. Thus, we now assume that some patients return to the system after their duration of care has completed.

If returning patients can be assigned to any HP (i.e., the continuity-of-care constraint is relaxed) and if they can also be assigned to a different HCA (i.e., rejected), the MILP formulation and the integer L-shaped algorithm remain the same. The only modification is that the set of referrals is augmented by the set of returning patients. We refer to this model as (SOP1) and its objective

function value as z_{SOP1} . If returning patients must be served by the same HP (i.e., the continuity-of-care constraint is enforced), the model is identical to (SOP); none of the decision variables change. The only difference is that the current panel of patients served by HP j will be augmented by the set of returning patients from HP j . However, in this case, returning patients do not affect the set of feasible actions. We refer to this model as (SOP2) and its objective function value as z_{SOP2} .

PROPOSITION 9. *Let z_{SOP1}^* , z_{SOP2}^* , and z_{SOP}^* be the optimal objective function values of (SOP1), (SOP2), and (SOP), respectively. Then, $z_{SOP}^* = z_{SOP1}^* \geq z_{SOP2}^*$.*

Note that even if returning patients have different duration distributions as compared to new patients with the same health requirement, the scheduling framework can be easily adjusted.

8.2. Periodic Service

The original model assumes that patients need service in every period. However, a patient may need home care every second or third day. To account for periodic service, the model can be adjusted to accommodate patients who must be visited in different periods.

Let \mathcal{F} represent the set of frequencies. For instance, $\mathcal{F} = \{1, 2, 3\}$ indicates that patients are seen by an HP every day, second day, or third day. A patient now arrives with service frequency $f \in \mathcal{F}$, health requirement k , region of residence m , and duration d , which is the number of periods that they are scheduled to be in the system. Now, a patient remains in the system for d periods but receives $\lceil d \div f \rceil$ periods of service. For example, suppose a patient arrives in the current period with $f = 2$ and $d = 9$. She will be served in periods $t = 0, 2, 4, \dots, 8$, and thus receive five periods of service. We then incorporate f into the state description by augmenting the state variable so that $\vec{x} = \{x_{kdmfj}\}_{\forall k,d,m,f,j}$. The set $\mathcal{H}(k, d, m)$ is also redefined as $\mathcal{H}(k, d, m, f)$.

Although the state dynamics and the action set constraints (1) are unchanged, a key difference in the model with periodic service is that only patients in state element x_{kdmfj} , where $(d \bmod f) = 0$, are considered when calculating the wage cost. Thus, we add an extra condition, $(t \bmod f) = 0$, to the calculation of the expected wage cost and the L-shaped method remains the same. Finally, if $\mathcal{F} = \{1\}$, we recover the original model where all schedules have a periodicity of 1.

8.3. Multiple Providers

In practice, multiple HPs with the same skill set may need to be assigned to a patient. For example, consider a palliative patient who lives alone with no support network. This individual may need an HP to wake her up in the morning and prepare breakfast, and a second HP to provide tuck-in service at days end. The example also illustrates how the continuity-of-care constraint for multiple providers requires that the same *set* of HPs serve the patient. In this case, we require that (1) be replaced by

$$\mathcal{A}(\vec{\theta}) := \left\{ (\vec{a}, \vec{b}) : \sum_{j \in \mathcal{J}(k)} a_{kdmj} + H_{kdm} b_{kdm} = H_{kdm} \theta_{kdm}, \quad \forall k, d, m \right\}, \quad (28)$$

where H_{kdm} is the number of HPs required to serve a patient of type (k, d, m) . The constraint ensures that if a patient is accepted, i.e., $b_{kdm} = 0$, exactly H_{kdm} HPs must be assigned to provide service.

If, instead, a patient needs multiple HPs each with different skill sets, this can be modeled by creating several copies of that patient; one for every skill set that is required. The HCA can then decide whether to provide a subset of all the services required by the patient. This approach is appropriate because an HCA may only be able to provide care for a subset of the patients health requirements. For example, an HCA may decide to accept a patient who requires multiple personal support workers and refer them to another HCA to fulfill their nursing requirements.

9. Conclusion and Managerial Insights

Home care is one of the fastest-growing industries, and it is projected that by 2020, revenues will top \$300 billion (FranchiseHelp, 2017). Previous work in home care typically focuses on myopic scheduling policies based on variants of the deterministic vehicle routing problem. Our analysis, however, suggests that there is value in considering the service and transit time of patients that may arrive in the future. We demonstrate that such an approach, using a policy improvement heuristic, can significantly outperform such myopic policies that are standard in practice. Further, many easily implementable basis policies yield substantial improvements in profit, which can then be incorporated into existing vehicle routing systems. Finally, the integer L-shaped algorithm we develop allows us to include agency-specific, nonlinear cost specifications. Thus, our scheduling approach is flexible, can accommodate complex operational structures, and can be generalized to cases where patients return for additional service, need periodic service, or patients who need to be assigned multiple HPs.

Our analysis has uncovered four important managerial observations. First, myopic scheduling practices that do not account for future demand and HP availability may result in decreased profitability. Since this is the standard industry approach, new gains may be obtained. Second, our scheduling framework is particularly beneficial for systems in which HPs do not have vast amounts of idle time. Since most practitioners have contracts that assure them a minimum number of hours, it is in the agency's best interest to maximize the number of working hours per HP without incurring large amounts of overtime. Third, our approach is crucial for HCAs where inter-patient travel time is a nontrivial component of an HP's shift (e.g., large geographic regions, dense urban centers with low travel speeds). If travel time is negligible, the majority of an HP's shift is already spent providing service. Finally, the increase in profit that our policy generates is highly dependent on the rejection cost. HCAs that incur no penalties for rejecting patients may choose to forgo an investment in better scheduling. However, HCAs with nontrivial rejection costs will benefit substantially from our framework. These observations can help HCAs better manage their scheduling operations.

The estimation method for the supervised learning (SL) policy assumes HPs who provided service in the past will still provide service in the future. This may not be a good assumption if an HCA has

a high turnover rate and our work would benefit from the implementation of a data-driven employee management system. Nevertheless, to implement the SL policy, one can find an existing HP whose service regions are similar and use those parameters in the model. Further, all other basis policies can be applied directly as long as the set of health requirements the HCA serves remain fixed.

Finally, we highlight a few limitations of our study that should be addressed in future work. First, our data includes limited information regarding patients' health requirements. More detailed information would allow us establish a direct link between the services provided by the agency and the reimbursement rates per employee type. Second, our model makes some assumptions about the HCA's fee structure and posits a cost for rejecting patients. Each jurisdiction may have their own localized revenue agreements, and the rejection cost may be a complex function of these details. As a result, it may be a nontrivial undertaking to estimate this cost. Third, we focus on a high-level understanding of an HP's time commitments and as a result, do not consider patient preferences, time-varying transit times, and stochastic service times. These represent additional sources of uncertainty and future work can investigate how to methodically include these complexities when making assignment decisions. Finally, we enforce a strict interpretation of the continuity-of-care regulation. Although this constraint can be relaxed, evidence suggests that higher levels of care continuity result in better patient outcomes (Russell et al., 2011). Future work should investigate the trade-off between the rigidity of the continuity of care regulation and the long-term health outcomes of home care patients.

Despite these limitations, our methodology and numerical results provide important insights into the management of home care agencies. Demand for home care services is predicted to increase substantially and the future patient mix may be drastically different. This study, along with systematic estimation procedures, provides a quantitative framework to aid in dynamic scheduling decisions.

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Appendix: Tables, Proofs, and Additional Results

A.1. Details of Numerical Analysis

Arrival Rate	Policy	Reward/Day	# Patients/HP	Shift Length (hrs)	Travel/Service
5	Myopic	1,227.95 (15.85)	4.94 (0.043)	3.05 (0.024)	0.24 (0.003)
	EA	1,331.03 (16.31)	4.87 (0.042)	2.82 (0.023)	0.16 (0.002)
	SL	1,310.81 (16.08)	4.96 (0.044)	2.91 (0.024)	0.18 (0.002)
	OTS	1,304.33 (15.91)	5.06 (0.044)	2.97 (0.024)	0.18 (0.002)
7.5	Myopic	1,717.17 (21.32)	6.51 (0.055)	4.22 (0.031)	0.30 (0.003)
	EA	1,989.61 (21.89)	6.48 (0.053)	3.77 (0.029)	0.17 (0.002)
	SL	1,956.79 (21.26)	6.83 (0.057)	4.01 (0.032)	0.18 (0.002)
	OTS	1,944.38 (21.03)	6.87 (0.057)	4.05 (0.032)	0.18 (0.002)
10	Myopic	2,179.22 (26.78)	7.65 (0.062)	5.12 (0.035)	0.34 (0.003)
	EA	2,546.13 (27.53)	7.66 (0.061)	4.62 (0.034)	0.21 (0.002)
	SL	2,524.14 (26.20)	8.38 (0.066)	5.03 (0.037)	0.20 (0.002)
	OTS	2,546.98 (26.22)	8.37 (0.066)	5.00 (0.037)	0.20 (0.002)
12.5	Myopic	2,524.66 (30.24)	8.79 (0.071)	6.09 (0.040)	0.39 (0.003)
	EA	3,042.13 (32.06)	8.86 (0.070)	5.49 (0.038)	0.24 (0.002)
	SL	3,054.43 (30.09)	9.45 (0.073)	5.76 (0.040)	0.22 (0.002)
	OTS	3,081.88 (30.02)	9.58 (0.076)	5.81 (0.041)	0.22 (0.002)
15	Myopic	2,835.28 (32.57)	10.47 (0.083)	7.25 (0.045)	0.39 (0.003)
	EA	3,321.88 (34.22)	10.45 (0.081)	6.74 (0.045)	0.29 (0.002)
	SL	3,506.23 (33.17)	10.59 (0.081)	6.56 (0.044)	0.24 (0.002)
	OTS	3,542.13 (33.27)	10.72 (0.084)	6.61 (0.045)	0.24 (0.002)
17.5	Myopic	2,785.65 (71.01)	12.11 (0.090)	8.30 (0.047)	0.38 (0.003)
	EA	2,965.86 (71.77)	11.77 (0.077)	7.83 (0.040)	0.34 (0.002)
	SL	3,781.57 (46.27)	12.16 (0.092)	7.52 (0.049)	0.24 (0.002)
	OTS	3,936.89 (42.71)	12.19 (0.093)	7.47 (0.049)	0.23 (0.002)

Table A.1 Results for varying arrival rates with a travel speed of 20 km/h. Standard errors are in parenthesis.

Avg. Speed (km/h)	Policy	Reward/Day		# Patients/HP		Shift Length (hours)		Travel/Service	
10	Myopic	2,001.94	(70.09)	10.27	(0.079)	7.84	(0.046)	0.54	(0.004)
	EA	418.82	(137.31)	9.04	(0.054)	6.67	(0.032)	0.48	(0.004)
	SL	2,644.97	(60.72)	10.41	(0.079)	7.27	(0.046)	0.40	(0.003)
	OTS	3,103.67	(59.49)	10.74	(0.082)	7.09	(0.046)	0.32	(0.002)
15	Myopic	2,633.79	(41.79)	10.37	(0.083)	7.35	(0.045)	0.43	(0.003)
	EA	2,309.21	(49.61)	10.05	(0.073)	7.23	(0.039)	0.45	(0.003)
	SL	3,301.95	(39.43)	10.54	(0.082)	6.70	(0.045)	0.28	(0.002)
	OTS	3,392.34	(39.42)	10.68	(0.084)	6.73	(0.045)	0.27	(0.002)
20	Myopic	2,835.28	(32.57)	10.47	(0.083)	7.25	(0.045)	0.39	(0.003)
	EA	3,321.88	(34.22)	10.45	(0.081)	6.74	(0.045)	0.29	(0.002)
	SL	3,506.23	(33.17)	10.59	(0.081)	6.56	(0.044)	0.24	(0.002)
	OTS	3,542.13	(33.27)	10.72	(0.084)	6.61	(0.045)	0.24	(0.002)
30	Myopic	3,236.62	(31.40)	10.41	(0.084)	6.79	(0.045)	0.31	(0.002)
	EA	3,673.44	(33.71)	10.40	(0.084)	6.33	(0.044)	0.22	(0.002)
	SL	3,683.19	(32.19)	10.53	(0.082)	6.32	(0.044)	0.20	(0.002)
	OTS	3,683.18	(32.30)	10.53	(0.084)	6.33	(0.045)	0.21	(0.002)
40	Myopic	3,525.85	(31.80)	10.42	(0.084)	6.49	(0.045)	0.25	(0.002)
	EA	3,805.45	(34.36)	10.42	(0.084)	6.21	(0.044)	0.20	(0.002)
	SL	3,798.81	(32.93)	10.50	(0.082)	6.19	(0.044)	0.18	(0.001)
	OTS	3,808.12	(32.35)	10.56	(0.084)	6.21	(0.045)	0.18	(0.001)
50	Myopic	3,723.98	(32.38)	10.43	(0.084)	6.28	(0.045)	0.21	(0.001)
	EA	3,939.90	(34.58)	10.42	(0.084)	6.05	(0.044)	0.16	(0.001)
	SL	3,923.57	(33.51)	10.50	(0.082)	6.06	(0.043)	0.16	(0.001)
	OTS	3,914.71	(33.12)	10.48	(0.084)	6.05	(0.044)	0.16	(0.001)

Table A.2 Results for varying travel speeds in km/h and an arrival rate of 15 patients/day. Standard errors are in parenthesis.

J	Policy	Reward/Day		# Patients/HP		Shift Length (hrs)		Travel/Service		Sol. Time (s)	
20	Myopic	2,179.22	(26.78)	7.65	(0.062)	5.12	(0.035)	0.34	(0.003)	0.01	(0.003)
	OTS	2,546.98	(26.22)	8.37	(0.066)	5.00	(0.037)	0.20	(0.002)	0.49	(0.161)
40	Myopic	3,883.27	(46.24)	7.42	(0.056)	5.30	(0.030)	0.44	(0.003)	0.05	(0.009)
	OTS	5,133.68	(43.10)	9.22	(0.069)	5.43	(0.037)	0.18	(0.002)	17.67	(6.410)
60	Myopic	6,023.33	(68.34)	7.59	(0.056)	5.33	(0.030)	0.41	(0.003)	0.17	(0.070)
	OTS	8,063.35	(66.04)	9.20	(0.070)	5.20	(0.037)	0.13	(0.001)	26.25	(8.487)
80	Myopic	8,624.83	(93.29)	7.73	(0.055)	5.29	(0.029)	0.38	(0.003)	0.45	(0.237)
	OTS	11,353.88	(90.68)	8.73	(0.065)	4.82	(0.034)	0.11	(0.001)	21.77	(5.678)
100	Myopic	10,910.75	(113.22)	7.76	(0.055)	5.20	(0.029)	0.35	(0.002)	0.64	(0.400)
	OTS	13,909.99	(109.43)	8.31	(0.060)	4.54	(0.031)	0.09	(0.001)	29.01	(5.204)

Table A.3 Results for varying number of HPs employed (J). Standard errors are in parenthesis.

A.2. Proofs of Statements

Proof of Proposition 1: From equations (3), (5), and the definition of \vec{y}_t ,

$$\begin{aligned}
\tilde{V}(\vec{x}) &= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E}[U(\vec{x} \oplus \vec{a})] \right\} \\
&= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E} \left[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) - G((\vec{x} \oplus \vec{a}) + \vec{a}^{\mathcal{P}}) + \rho \mathbb{E}[U((\vec{x} \oplus \vec{a}) \oplus \vec{a}^{\mathcal{P}})] \right] \right\} \\
&= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E} \left[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) - G(\vec{y}_1 + \vec{a}^{\mathcal{P}}) + \rho \mathbb{E}[U(\vec{y}_1 \oplus \vec{a}^{\mathcal{P}})] \right] \right\} \\
&= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E} \left[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) - G(\vec{y}_1 + \vec{a}^{\mathcal{P}}) + \rho \mathbb{E}[U(\vec{y}_2)] \right] \right\} \\
&= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E} \left[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) \right] - \rho \mathbb{E}[G(\vec{y}_1 + \vec{a}^{\mathcal{P}})] + \rho^2 \mathbb{E}[\mathbb{E}[U(\vec{y}_2)]] \right\} \\
&= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) + \rho \mathbb{E}_1 \left[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}) \right] - \rho \mathbb{E}_1[G(\vec{y}_1 + \vec{a}^{\mathcal{P}})] + \rho^2 \mathbb{E}_2[U(\vec{y}_2)] \right\},
\end{aligned}$$

where, for simplicity, we have defined the conditional expectations as $\mathbb{E}_1[\cdot] = \mathbb{E}[\cdot | \vec{x}, \vec{a}, \vec{b}]$ and $\mathbb{E}_2[U(\vec{y}_2)] = \mathbb{E}[\mathbb{E}[U(\vec{y}_2)]] = \mathbb{E}[\mathbb{E}[U(\vec{y}_2) | \vec{y}_1, \vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}}] | \vec{x}, \vec{a}, \vec{b}]$. Because $\rho \mathbb{E}_1[R(\vec{a}^{\mathcal{P}}, \vec{b}^{\mathcal{P}})]$ is a constant that does not depend on the decision variables (\vec{a}, \vec{b}) in the current period, it can be removed from the objective. Applying the same argument for τ periods, we obtain

$$\max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\vec{\theta})} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \rho^t \mathbb{E}_t[G(\vec{y}_t + \vec{a}^{\mathcal{P}})] + \rho^\tau \mathbb{E}_\tau[U(\vec{y}_\tau)] \right\}.$$

Notice, however, that the final term is a constant that does not depend on the decision variables (\vec{a}, \vec{b}) in the current period. This is because at the end of period $(\tau - 1)$, all arriving patients assigned at $t = 0$ have left the system and all decisions over the last τ periods were made using basis policy \mathcal{P} . Thus, $\mathbb{E}_\tau[\cdot]$ can be deconditioned with respect to the decision variables (\vec{a}, \vec{b}) and subsequently removed from the objective; this term is now a constant. As a result, we obtain (6), as required. ■

Proof of Proposition 2: Consider the element $\tilde{\sigma}_{ktm}$ for $t + 1 > D_{\max}$. By definition,

$$\begin{aligned}
\tilde{\sigma}_{k(t+1)m} &= \tilde{\sigma}_{ktm} + \tilde{\sigma}_{k1m} - \sum_{d=1}^{\min\{t, D_k\}} \sigma_{k(t-d)m}, \\
&= \tilde{\sigma}_{ktm} + \tilde{\sigma}_{k1m} - \sum_{d=1}^{D_k} \sigma_{k(t-d)m}, \\
&= \tilde{\sigma}_{ktm} + \sum_{d=1}^{D_k} \sigma_{kdm} - \sum_{d=1}^{D_k} \sigma_{k(t-d)m}, \\
&= \tilde{\sigma}_{ktm},
\end{aligned}$$

where the second equality follows because $t \geq D_k$, and the third equality follows by definition. ■

Proof of Proposition 3: First, because $G_j(\cdot)$ is assumed to be convex, Jensen's inequality implies that, for any $t \geq 0$ and state \vec{x} , we have

$$\begin{aligned} \mathbb{E}_t [G(\vec{y}_t + \vec{a}^P)] &= \mathbb{E}_t \left[\sum_{j=1}^J G_j \left(F_S^j(\vec{y}_t + \vec{a}^P) + F_T^j(\vec{y}_t + \vec{a}^P) \right) \right] \\ &\geq \sum_{j=1}^J G_j \left(\mathbb{E}_t \left[F_S^j(\vec{y}_t + \vec{a}^P) + F_T^j(\vec{y}_t + \vec{a}^P) \right] \right) \\ &= \sum_{j=1}^J G_j \left(\mathbb{E}_t \left[F_S^j(\vec{y}_t + \vec{a}^P) \right] + \mathbb{E}_t \left[F_T^j(\vec{y}_t + \vec{a}^P) \right] \right) \\ &= \sum_{j=1}^J G_j \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right). \end{aligned}$$

Substituting the above inequality in (6), we obtain

$$\begin{aligned} V_{PIH}(\vec{x}) &= \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\theta)} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \rho^t \mathbb{E}_t [G(\vec{y}_t + \vec{a}^P)] \right\} \\ &\leq \max_{(\vec{a}, \vec{b}) \in \mathcal{A}(\theta)} \left\{ R(\vec{a}, \vec{b}) - G(\vec{x} + \vec{a}) - \sum_{t=1}^{\tau-1} \sum_{j=1}^J \rho^t G_j \left(S_{tj}(\vec{x}, \vec{a}) + T_{tj}(\vec{x}, \vec{a}) \right) \right\} \\ &= V_{APIH}(\vec{x}), \end{aligned}$$

as desired. ■

Proof of Proposition 4: We first elaborate on relations (13). The first term for $t > 1$, $\tilde{\omega}_{k(t-1)j}$, captures the expected number of patients with health requirement k who were assigned to HP j from previous periods. The expected number of referred patients assigned to HP j in the current period is $\tilde{\omega}_{k1j}$, i.e., the second term. The third term represents the number of patients who leave the system after being referred in period d and need $t - d$ periods of service.

For the expected number of patients who have health requirement k and receive service in period $t = 1, \dots, \tau - 1$ in (12), $\tilde{\omega}_{ktj}$ is added to the patients currently on HP j 's panel, $\sum_{m=1}^M \sum_{d=t+1}^{\tau} x_{kdmj}$, plus new patients assigned to HP j that need at least $t + 1$ periods of service, $\sum_{m=1}^M \sum_{d=t+1}^{\tau} a_{kdmj}$. This results in the equality presented in (4). ■

Proof of Proposition 5: Let $\{\tilde{\phi}_{mtj}^{(\mathcal{Q}_1)}\}_{m,j}$ and $\{\tilde{\phi}_{mtj}^{(\mathcal{Q}_2)}\}_{m,j}$ be vertex probabilities considered in the evaluation of $\text{PTSP}_{tj}(\mathcal{Q}_1)$ and $\text{PTSP}_{tj}(\mathcal{Q}_2)$, respectively. We denote the probabilities of a scenario $\mathcal{V} \subseteq \mathcal{V}_j$ by $\delta^{(\mathcal{Q}_1)}(\mathcal{V})$ and $\delta^{(\mathcal{Q}_2)}(\mathcal{V})$ in each case. Since $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, we can partition the set \mathcal{V}_j as

$$\mathcal{V}_j = \left(\mathcal{V}_j \setminus \mathcal{Q}_2 \right) \cup \mathcal{Q}_1 \cup \left(\mathcal{Q}_2 \setminus \mathcal{Q}_1 \right).$$

Hence, for the optimal permutation π_j^* of vertices in \mathcal{Q}_2 , we can partition the scenarios so that

$$\begin{aligned} \text{PTSP}_{tj}(\mathcal{Q}_2) &= \sum_{\mathcal{V} \subseteq \mathcal{V}_j} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*), \\ &= \sum_{\mathcal{V} \subseteq \mathcal{V}_j \setminus \mathcal{Q}_2} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) \\ &= \sum_{\mathcal{V} \subseteq \mathcal{V}_j \setminus \mathcal{Q}_2} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*), \end{aligned}$$

where the second equality follows by definition and the third equality follows since $\delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) = \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V})$ for all $\mathcal{V} \subseteq \mathcal{V}_j \setminus \mathcal{Q}_2$ and $\mathcal{V} \subseteq \mathcal{Q}_1$. Next, notice that $\delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) = 1$ and $\delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) < 1$ for all $\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1$, since all vertices in \mathcal{Q}_2 have a probability of 1 and must be active. Thus,

$$\sum_{\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_2)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) \geq \sum_{\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*).$$

As a result,

$$\begin{aligned} \text{PTSP}_{tj}(\mathcal{Q}_2) &\geq \sum_{\mathcal{V} \subseteq \mathcal{V}_j \setminus \mathcal{Q}_2} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) + \sum_{\mathcal{V} \subseteq \mathcal{Q}_2 \setminus \mathcal{Q}_1} \delta_{tj}^{(\mathcal{Q}_1)}(\mathcal{V}) L_{tj}(\mathcal{V}, \pi_j^*) \\ &\geq \text{PTSP}_{tj}(\mathcal{Q}_1), \end{aligned}$$

where the last inequality follows as π_j^* may not be optimal for $\text{PTSP}_{tj}(\mathcal{Q}_1)$. \blacksquare

Proof of Proposition 6. Because cuts are accumulated in (MP) as ℓ increases, $z^\ell \geq z^{\ell+1}$ for any ℓ . Since there are only a finite number of possible assignments to $\vec{\mu}$, there exists some sufficiently large $\bar{\ell}$ where any cut of the form (20) added in an iteration $\ell \geq \bar{\ell}$ is redundant, i.e., $z^\ell = z^{\bar{\ell}}$. Thus, if $\vec{c}^{\bar{\ell}}$ is optimal to (MP) in iteration $\bar{\ell}$, then for each HP j there exists some $\ell_j < \bar{\ell}$ where

$$\vec{c}_j^{\bar{\ell}} = G_j \left(F_S^j(\vec{x} + \vec{a}^{\ell_j}) + F_T^j(\vec{x} + \vec{a}^{\ell_j}) \right) + \sum_{t=1}^{\tau-1} \rho^t G_j \left(S_{tj}(\vec{x}, \vec{a}^{\ell_j}) + T_{tj}(\vec{x}, \vec{a}^{\ell_j}) \right).$$

Thus, we found an optimal solution to (MP) that evaluates to the same objective value as in (SOP) for HP j . Since every iteration of (MP) is a relaxation of (SOP), the result follows. \blacksquare

Proof of Proposition 7. Let $(\vec{a}, \vec{\mu})$ be a feasible solution to (MP) and fix j and t . Because the PTSP function is increasing (Proposition 5), we have that

$$T_{tj}(\vec{x}, \vec{a}) \geq T_{tj}(\vec{x}, \vec{a}^i),$$

for all i where $\mu_{ij} = 1$. Summing over the above inequalities for each i , we obtain

$$|\{i : \mu_{ij} = 1\}| \cdot T_{tj}(\vec{x}, \vec{a}) \geq \sum_{\{i : \mu_{ij} = 1\}} T_{tj}(\vec{x}, \vec{a}^i) = \sum_{i=1}^{\eta} T_{tj}(\vec{x}, \vec{a}^i) \mu_{ij}.$$

Since $|\{i : \mu_{ij} = 1\}| \leq \eta$ we have that

$$\eta T_{tj}(\vec{x}, \vec{a}) \geq |\{i : \mu_{ij} = 1\}| \cdot T_{tj}(\vec{x}, \vec{a}) \geq \sum_{i=1}^{\eta} T_{tj}(\vec{x}, \vec{a}^i) \mu_{ij}.$$

The proposition statement then follows by dividing the inequality by $\eta > 0$. ■

Proof of Corollary 1. The result follows from applying the definition of an α -approximation solution, $w_{ij}^i \leq \alpha T_{tj}(\vec{x}, \vec{a}^i)$, to inequality (25). ■

Proof of Proposition 8: For a single piecewise linear function and a single region per patient, we can decompose c_j per patient; i.e., $c_j = \sum_{i=1}^{\eta} c_{ij}$, where

$$c_{ij} = \sum_{t=1}^{\tau-1} \rho^t \left(\beta_{1j}^{(1)} + \beta_{1n}^{(2)} \left(S_{tj}(\vec{x}, \vec{a}^i) + T_{tj}(\vec{x}, \vec{a}^i) \right) \right),$$

where \vec{a}^i is defined as in Proposition 7. The same can be done with the reward $R(\vec{a}, \vec{b})$. Thus, the objective function becomes separable in $\vec{\mu}$, and we can rewrite the first iteration as a classical matching problem which is solvable in polynomial time (Ahuja et. al, 2008). ■

Proof of Proposition 9: (SOP1) and (SOP2) have the same objective functions but (SOP1) has a larger action set. That is, every feasible solution to (SOP2) is also a feasible solution in (SOP1). However, there exist feasible solutions to (SOP1) where returning patients are either rejected or switch HPs. These solutions are infeasible in (SOP2). Thus, (SOP1) is a relaxation of (SOP2). ■

A.3. Additional Details - Enhanced L-Shaped Method

A.3.1. Implementation of the Piecewise Linear Approximation for $G_j(\cdot)$

We now describe the piecewise linear approximation by [Yang and Goh \(1997\)](#) to approximate the wage cost $G_j(\cdot)$, discussed in [Section 6.2](#).

1. Define points $o_1 < o_2 < \dots < o_{N+1}$, where o_1 and o_{N+1} are the minimum and maximum values of the domain for $G_j(\cdot)$, respectively.
2. For each interval $[o_n, o_{n+1}]$, solve the following optimization problem

$$\min \left\{ G_j(o) - o \cdot \frac{G_j(o_{n+1}) - G_j(o_n)}{o_{n+1} - o_n} : o_n \leq o \leq o_{n+1} \right\}. \quad (\text{A.1})$$

The problem is convex and can be solved efficiently ([Boyd and Lieven, 2004](#)), e.g., binary search if $G_j(\cdot)$ is discrete or Newton's method if $G_j(\cdot)$ is continuous and differentiable.

3. Let o_n^* be the optimal solution to [\(A.1\)](#) for $n = 1, \dots, N$. The following coefficients ensure that [\(24\)](#) is satisfied ([Yang and Goh, 1997](#)):

$$\alpha_n = G_j(o_n^*) - o_n^* \cdot \frac{G_j(o_{n+1}) - G_j(o_n)}{o_{n+1} - o_n}, \quad \beta_n = \frac{G_j(o_{n+1}) - G_j(o_n)}{o_{n+1} - o_n}.$$

A.3.2. Additional details on valid inequalities

Suppose that the sequence which optimizes $T_{tj}(\vec{x}, \vec{0})$ is available where $\vec{0}$ is a vector of zeros of appropriate dimension. Based on [Proposition 7](#), we can compute an alternative lower bound to $T_{tj}(\vec{x}, \vec{a}^i)$ in order to derive a valid inequality in the form of [\(7\)](#). Namely, as in [Section 5.3](#), let \mathcal{Q} be the set of vertices covered by \vec{x} for HP j in decision epoch t , and assume that \vec{a}^i adds one additional vertex $v_{mj} \notin \mathcal{Q}$. We have $T_{tj}(\vec{x}, \vec{a}^i) \geq T_{tj}(\vec{x}, \vec{0}) + \gamma$, where

$$\begin{aligned} \gamma \geq & 2 \cdot \min_{v_{\tilde{m}j} \in \mathcal{V}} \left\{ \tilde{\phi}_{mtj} \tilde{\phi}_{\tilde{m}tj} E_j(m, \tilde{m}) + \frac{1}{2} \tilde{\phi}_{mtj} (1 - \tilde{\phi}_{m'tj}) \min_{v_{\tilde{m}j} \in \mathcal{Q} \setminus \{v_{mj}, v_{\tilde{m}j}\}} E_j(m, \tilde{m}) \right. \\ & \left. + \frac{1}{2} (1 - \tilde{\phi}_{mtj}) (\tilde{\phi}_{\tilde{m}tj}) \min_{v_{\tilde{m}j} \in \mathcal{Q} \setminus \{v_{mj}, v_{\tilde{m}j}\}} E(\tilde{m}, \tilde{m}) \right\}. \end{aligned}$$

The quantity γ represents a lower bound on the expected distance traveled from v_{mj} to the other vertices $v_{\tilde{m}j}$ currently in the tour. The minimization searches for the edge $(v_{mj}, v_{\tilde{m}j})$ that will increase the expected PTSP the least. In particular, the first term is the expected distance when both v_{mj} and $v_{\tilde{m}j}$ are in the tour, the second term is the expected distance when only v_{mj} is present, and the third term is the expected distance when only $v_{\tilde{m}j}$ is present. We multiply the minimum value by 2 because including v_{mj} will add at least two edges in any tour where it is realized. Notice that this result generalizes [Proposition 3](#) from [Laporte et al. \(1994\)](#).

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