On a reduction of the weighted induced bipartite subgraph problem to the weighted independent set problem

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Abstract

We study the weighted induced bipartite subgraph problem (WIBSP). The goal of WIBSP is, given a graph and nonnegative weights for the nodes, to find a set W of nodes with the maximum total weight such that a subgraph induced by W is bipartite. WIBSP is also referred as to the graph bipartization problem or the odd cycle transversal problem. In this paper, we show that WIBSP can be reduced to the weighted independent set problem (WISP) where the number of nodes becomes twice and the maximum degree increases by 1. WISP is a well-studied combinatorial optimization problem. Thus, by using the reduction and results about WISP, we can obtain nontrivial approximation and exact algorithms for WIBSP.

Keywords: Weighted induced bipartite subgraph problem, Graph bipartization problem, Odd cycle transversal problem, Independent set problem

1. Introduction

Let $G = (V, E)$ be an undirected graph. For $W \subseteq V$, define $E(W) = \{\{u, v\} \in E \mid u, \ v \in W\}$. The graph $G(W) = (W, E(W))$ is said to be the subgraph of G induced by W. A subset of nodes $W \subseteq V$ is called an independent set of G if $E(W) = \phi$. A graph G is called bipartite if all the nodes can be partitioned into disjoint sets V_1 and V_2 such that no two

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nodes within the same set are adjacent. Given a graph $G = (V, E)$ and nonnegative weights w_v $(v \in V)$, the weighted induced bipartite subgraph problem (WIBSP) is to find $W \subseteq V$ with the maximum total weight such that the subgraph $G(W)$ of G induced by W is bipartite. WIBSP is also refereed as to the graph bipartization problem or the odd cycle transversal problem. The weighted independent set problem (WISP) is to find an independent set W of G with the maximum total weight.

WIBSP has many applications such as VLSI design and the DNA sequencing problem [1]. For complexity, Choi et al. [2] show that WIBSP is NP-hard even when a graph is planer and the maximum degree is at most 4. Recently, Baïou and Barahona [3] show that WIBSP can be solved in polynomial time when a graph is planer and the maximum degree is at most 3. WIBSP is also studied from multiple viewpoints such as exponential-time algorithms [4], approximation algorithms [5, 6, 7] and fixed parameter algorithms [8, 9].

It is known that WISP can be easily reduced to WIBSP [1]. In this paper, we show that WIBSP can also be reduced to WISP where the number of nodes becomes twice and the maximum degree increases by 1. Since WISP is a well-studied combinatorial optimization problem, some results about WISP can be used for WIBSP by this reduction. As a result, we can obtain nontrivial algorithms for WIBSP. First, we present a $(\frac{3}{\Delta+3})$-approximation algorithm for WIBSP, where Δ is the maximum degree of the graph. To our knowledge, this is the first approximation algorithm for WIBSP while there are some approximation algorithms for the minimization version of WIBSP. Second, we propose a $1.439^n n^{O(1)}$-time algorithm for WIBSP which improves the previous best result of $1.62^n n^{O(1)}$ by Raman et al. [4]. Details about these algorithms and related works are shown in Section 3.

2. Reduction to the weighted independent set problem

In this section, we show that WIBSP can be reduced to WISP.

Theorem 1. *Given a graph $G = (V, E)$ $(V = \{1, \ldots, n\})$ and weights w_v $(v \in V)$, we define the graph $H(G) = (V^1 \cup V^2, E^1 \cup E^2 \cup E^3)$ and weights for the nodes $V^1 \cup V^2$ where*

- $V^1 = \{v_1^1, \ldots, v_n^1\}$

- $V^2 = \{v_1^2, \ldots, v_n^2\}$

- $E^1 = \{\{v_i^1, v_j^1\} \ (\forall i < j) \mid \{v_i, v_j\} \in E\}$

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- $E^2 = \{\{v_i^2, v_j^2\} \ (\forall i < j) \mid \{v_i, v_j\} \in E\}$

- $E^3 = \{\{v_i^1, v_i^2\} \mid i \in \{1 \dots n\}\}$

- $w_{v_i^1} = w_{v_i^2} = w_{v_i} \ (\forall i \in \{1 \dots n\}).$

Then, any independent set W^I of $H(G)$ can be transformed into a set $W^B \subseteq V$ such that $G(W^B)$ is bipartite whose total weight does not change, and vice versa.
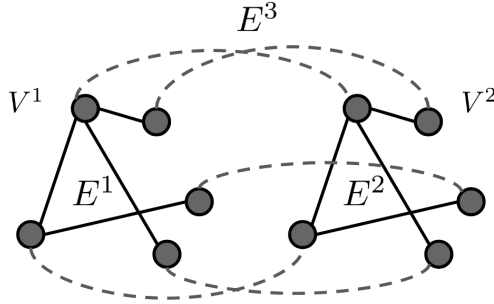

Figure 1: Graph $H(G) = (V^1 \cup V^2, E^1 \cup E^2 \cup E^3)$

Proof. An example of $H(G)$ is shown in Figure 1. Let $W^I \subseteq V^1 \cup V^2$ be an independent set of $H(G)$. Let $W_1^I = W^I \cap V^1$ and $W_2^I = W^I \cap V^2$. We define $W^B = W_1^B \cup W_2^B$ where $W_1^B = \{v_i \in V \mid v_i^1 \in W_1^I\}$ and $W_2^B = \{v_i \in V \mid v_i^2 \in W_2^I\}$. From Figure 2, we can easily see that W_1^B and W_2^B are disjoint and $G(W_1^B)$ and $G(W_2^B)$ are independent sets of G. Therefore, $G(W^B)$ is bipartite. We can easily see that the total weight does not change in this transformation.

Next, we will show the converse. Let W^B be a subset of V such that $G(W^B)$ is bipartite. Since $G(W^B)$ is bipartite, there exist two independent sets W_1^B and W_2^B of G, such that $W_1^B \cup W_2^B = W^B$ and $W_1^B \cap W_2^B = \phi$. We define $W_1^I = \{v_i^1 \in V^1 \mid v_i \in W_1^B\}$ and $W_2^I = \{v_i^2 \in V^2 \mid v_i \in W_1^B\}$. Then, $W^I = W_1^I \cup W_2^I$ is an independent set of $H(G)$. The total weight also does not change. $\square$

Note that the number of nodes increase by 2 times and the maximum degree increases by 1 in the reduction of WIBSP to WISP.
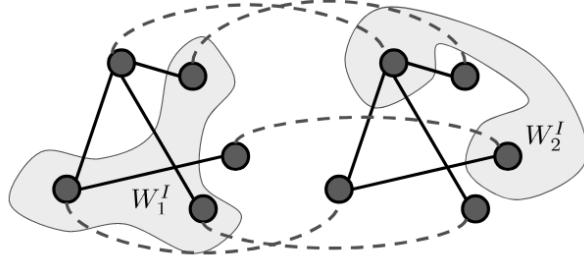
Figure 2: W_1^I and W_2^I

3. Algorithms

In this section, by using Theorem 1 we obtain approximation and exact algorithms for WIBSP.

Approximation algorithm

An algorithm is called an α-approximation algorithm for a minimization problem (resp., a maximization problem) if it runs in polynomial time and produces a solution whose objective value is less (resp., greater) than or equal to α times the optimal value. There are some approximation algorithms for the minimization version of MWBSP where we find $W \subseteq V$ with the minimum total weight such that $G(V \setminus W)$ is bipartite. Klein et al. [7] present a $O(\log^3 n)$-approximation algorithm. Garg et al. [6] propose an improved $O(\log n)$-approximation algorithm. Goemans and Williamson [5] give a $\frac{9}{4}$-approximation algorithm when a graph is planar. To the best of our knowledge, no approximation algorithms are proposed for WIBSP. For WISP, a $\frac{3}{\Delta+2}$-approximation algorithm is proposed by Halldrsson and Lau [10], where Δ is the maximum degree. By using their algorithm for WISP and Theorem 1, we have the following corollary since the maximum degree increase by 1 in the reduction. Note that for any approximation algorithm for WISP [11], we can obtain an approximation algorithm for WIBSP.

Corollary 1. *There is a $(\frac{3}{\Delta+3})$-approximation algorithm for the weighted induced bipartite subgraph problem, where Δ is the maximum degree of the graph.*

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Exact algorithm

A $1.1996^n n^{O(1)}$-time algorithm for the (unweighted) independent set problem is recently presented by Xiao and Nagamochi [12], where n is the number of the nodes. By using this algorithm and Theorem 1, we easily get an exact algorithm for the unweighted induced bipartite subgraph problem, which is a special case of WIBSP where all weights are uniform. In the reduction of Theorem 1, the number of the nodes becomes twice. Thus, by using the algorithm by Xiao and Nagamochi, we have a $1.1996^{2n} n^{O(1)} \simeq 1.439^n n^{O(1)}$ time algorithm which improves the previous best result of $1.62^n n^{O(1)}$ by Raman et al. [4].

Corollary 2. *There is a $1.439^n n^{O(1)}$-time algorithm for the unweighted induced bipartite subgraph problem, where n is the number of the nodes.*

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