# Best case exponential running time of a branch-and-bound algorithm using an optimal semidefinite relaxation

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**Abstract**

Chvatal (1980) has given a simple example of a knapsack problem for which a branch-and-bound algorithm using dominance and linear relaxations to eliminate subproblems will use an exponential number of steps in the best case. In this short note it is shown that Chvatals result remains true when the LP relaxation is replaced with a semidefinite relaxation that is best possible in a certain sense.

**Key words:** Semidefinite relaxation, branch-and-bound algorithm, exponential running time.

## 1 Introduction

In 1980, Chvatal [1] presented a class of instances of the knapsack problem that has so little structure to work with that a branch-and-bound algorithm to solve the knapsack problem using dominance and linear relaxations (LP bounds) to eliminate subproblems will need an exponential running time for any instance in this class. Since semidefinite programming has augmented the toolbox of available algorithms in the mean time, this short note shows that also when including semidefinite relaxations for these problems, the result by Chvatal remains valid. Nevertheless the analysis is far away from covering all known algorithmic approaches for the knapsack problem. In particular, the use of Chvatal-Gomory cutting planes [3] in a branch-and-cut framework is not considered.

## 2 The result by Chvatal

The knapsack problem is considered in the following binary linear form:

$$\min \ c^T x \ \mid \ a^T x \geq \beta, \ x \in \{0,1\}^n \tag{1}$$

where $a, c \in \mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\alpha > 0$.

### 2.1 A branch-and-bound framework for solving (1)

The branch and bound framework outlined below is based on the following definition of dominance:

A subproblem of (1) is defined by selecting a subset $J$ of the variables $x_i$ ($1 \leq i \leq n$) and fixing the variables in $J$ to some values in $\{0, 1\}$.

Assume the variables

$$x_{i_1} = \ldots = x_{i_k} = 1, \quad \text{and} \quad x_{i_{k+1}} = \ldots = x_{i_p} = 0$$

are fixed in a first subproblem and the variables

$$x_{j_1} = \ldots = x_{j_{k'}} = 1, \quad \text{and} \quad x_{j_{k'+1}} = \ldots = x_{j_{p'}} = 0$$

are fixed in a second subproblem. Then, the first subproblem *dominates* the second if $\{i_1, \ldots, i_p\} \subseteq \{j_1, \ldots, j_{p'}\}$ and if

$$\begin{array}{rrcl} & c_{i_1} + \ldots + c_{i_k} & \leq & c_{j_1} + \ldots + c_{j_{k'}} \\ \text{and} & a_{i_1} + \ldots + a_{i_k} & \geq & a_{j_1} + \ldots + a_{j_{k'}}. \end{array}$$

(The second problem fixes possibly more variables, and it does so in a way that is not better, both with respect to the objective value as well as with respect to the constraint.)

The following algorithm will be considered:

**Algorithm 1 (Branch-and-bound, also using dominance)**

1. Initialization:

   Set $\ell := 1$ and define the collection of open subproblems $\mathcal{O}$ as the set containing only the subproblem with $J = \emptyset$ (no variables fixed yet).

   Set $\omega_{best} := \infty$ (the lowest value of $c^T x$ found so far for a feasible $x$ of (1)) and $x^{best} := e$.

2. Choose a problem from $\mathcal{O}$. Assume the variables

   $$x_{i_1} = \ldots = x_{i_k} = 1, \quad \text{and} \quad x_{i_{k+1}} = \ldots = x_{i_p} = 0 \tag{2}$$

   are fixed in the chosen subproblem. Let $J_\ell := \{x_{i_1}, \ldots, x_{i_p}\}$.

3. Set

   $$\omega_\ell := \begin{cases} \infty & \text{if } \sum_{j \notin \{i_{k+1}, \ldots, i_p\}} a_j < \beta \\ \min\{c^T x \mid x \in [0,1]^n \text{ satisfies (2) and } a^T x \geq \beta\} & \text{else} \end{cases}$$

   the LP lower bound for the optimal value of (1) with the variables in $J_\ell$ fixed in (2).

4. If $\omega_\ell \geq \omega_{best}$ delete the subproblem (2) from $\mathcal{O}$ and go to Step 8.

5. If the chosen subproblem dominates a subproblem from $\mathcal{O}$ then delete the dominated problem from $\mathcal{O}$.

6. If the chosen subproblem is dominated by a subproblem from $\mathcal{O}$, then delete the chosen subproblem from $\mathcal{O}$ and go to Step 8.

7. If the solution $x$ associated with the computation of $\omega_\ell$ is integer and $\omega_\ell < \omega_{best}$ set $\omega_{best} := \omega_\ell$ and $x^{best} := x$. Delete the chosen subproblem from $\mathcal{O}$ and go to Step 8.

8. Choose $x_i \notin J_\ell$. In $\mathcal{O}$ replace the chosen subproblem (2) with two new subproblems where, in addition to (2), the variable $x_i$ is set to one, respectively to zero.

9. Set $\ell := \ell + 1$. If $\mathcal{O}$ is empty stop, else go to Step 2.

## 2.2 Exponential running time

Chvatal [1] considered the case of problem (1) where $\beta := \left\lfloor \frac{1}{2} \sum_{i=1}^n a_i \right\rfloor$ and $a = c \ > 0$ is chosen such that

$$\begin{aligned} i) \qquad & \sum_{i \in J_1} a_i \neq \sum_{i \in J_2} a_i \quad \text{whenever } J_1 \neq J_2, \\ ii) \qquad & \sum_{i \in J} a_i \geq \beta \quad \text{whenever } |J| \geq 0.9\, n \\ iii) \qquad & \nexists J \text{ with } \sum_{i \in J} a_i = \beta. \end{aligned}$$

Chvatal actually considered a further condition that also excludes a step he called “improving the current solution” when $|J_\ell| \leq n/10$, and he showed that when choosing moderately large random integer numbers $a_i$ all these conditions are satisfied with high probability. (The further condition considered by Chvatal is omitted here for brevity; it does not change the analysis of this short note.)

It is then noted in [1] that under condition $i$), there can never be dominance, because dominance would imply

$$a_{i_1} + \ldots + a_{i_k} = a_{j_1} + \ldots + a_{j_{k'}}.$$

Moreover, when $n$ is a multiple of 10 and when $|J_\ell| \leq n/10$ in Algorithm 1, the LP-relaxation in Step 3. is always feasible because of condition $ii$) and then also $\omega_\ell = \beta$ holds true. On the other hand, even if a local search is included to improve $\omega_{best}$ and the associated feasible solution $x^{best}$ following Step 3., because of $iii$), it follows that $\omega_{best} > \beta$ holds at all times, so that for $|J_\ell| \leq n/10$ subproblem (2) will not be deleted from $\mathcal{O}$ in Step 4. of Algorithm 1. This implies that all subsets $|J_\ell| \leq n/10$ will be considered in the algorithm. By Stirlings formula, there are about $\sqrt{\frac{50}{9\pi n}}\, 2^{.469n}$ sets with $|J_\ell| = n/10$ and for each set there are $2^{n/10}$ different fixations of the variables in $J_\ell$. This leads to the result of Chvatal [1]:

**Lemma 2.1** *The overall computational effort of Algorithm 1 for solving any instance of problem (1) under assumptions i) – iii) is at least $O(2^{n/2})$.*

# 3 Including semidefinite programs

In this section it is shown that for $|J_\ell| \leq \frac{n}{10}$, a subproblem (2) will not be deleted from $\mathcal{O}$ in Step 4. of Algorithm 1 even if the linear relaxation of Step 3. is replaced with an optimal semidefinite relaxation as outlined below. More precisely, it is shown that for $|J_\ell| \leq \frac{n}{10}$, same as the linear relaxation, also the semidefinite relaxation yields an optimal value $\omega_\ell = \beta$ in Step 3. Thus, in particular, the lower bound of Lemma 2.1 remains true.

Let $e$ denote the all-one-vector and set $z := 2x - e \in \{\pm 1\}^n$. It follows that $a^T z = 2a^T x - a^T e$. Consider the problem of testing whether the optimal value of the knapsack problem (1) is equal to $\beta$ when fixing $a = c$ as in Section 2.2. This can be written as solving the binary equation $a^T z = 2\beta - a^T e$. Defining an extended vector $a \in \mathbb{R}^{n+1}$ by setting $a_{n+1} := a^T e - 2\beta \geq 0$ and adding a variable $z_{n+1} \in \{\pm 1\}$, this is equivalent to solving

$$\min\ z^T(aa^T)z \quad \text{for} \quad z \in \{\pm 1\}^{n+1}. \tag{3}$$

Indeed, using the fact that $z$ and $-z$ return the same objective value in (3) we may assume without loss of generality that $z_{n+1} = 1$, and thus, the first $n$ components of $x := \frac{1}{2}(z + e)$ solve the knapsack problem (1) with objective value $\beta$ if and only if the objective value of $z$ in (3) is zero.

Problem (3) is a binary quadratic program and it coincides with the max-cut-problem with edge weights $a_i a_j \geq 0$. (The Laplacian for this max-cut problem has off-diagonal entries $-a_i a_j$ (see e.g. [2]), and when changing to a minimization problem we obtain problem (3) except from a change in the diagonal entries which translates to a constant term in the objective value.)

For (3) the standard semidefinite relaxation is given by

$$\min\ (aa^T) \bullet X \quad \text{for} \quad X \succeq 0, \quad \operatorname{diag}(X) = e. \tag{4}$$

Here, the scalar product of two symmetric matrices $X, S$ is given by $X \bullet S := \operatorname{trace}(XS)$ and $\operatorname{diag}(X)$ denotes the vector with the diagonal entries of $X$. The randomization technique by Goemans and Williamson [4] is based on (4) and is known to be optimal (in the sense that there is no polynomial time algorithm with better performance guaranty) if the Unique Games Conjecture is true [6]. In this sense, the semidefinite approximation (4) provides a much stronger relaxation of (3) than a linear approximation as used in Step 3. of Algorithm 1 or than an approximation based on the metric polytope.

Here, it is shown that nevertheless, for $|J_\ell| \leq n/10$ the semidefinite relaxation does not improve the lower bound $z_\ell = \beta$ in Step 3. of Algorithm 1, so that the run time estimate in Lemma 2.1 remains valid. To this end note that an optimal value $\omega_\ell = \beta$ in Step 3. of the branch-and-bound algorithm translates to an optimal value zero for problem (4).

Indeed, when fixing at most $n/10$ of the variables, the $a_i$ associated with the variables fixed to one are subtracted from the right hand side $\beta$, and one obtains a new knapsack problem of the same form (i.e. with $a = c$) and with fewer variables, say with $p-1$ variables. By *ii*), the new problem is solvable when the constraint $x \in \{0,1\}^{p-1}$ is relaxed to $x \in [0,1]^{p-1}$. Translating to the variable $z \in \{\pm 1\}^p$ this implies that none of the entries of the extended vector $a \in \mathbb{R}^p$ is larger in absolute value than the sum of the absolute values of the remaining components of $a$. The next lemma states that under this assumption the optimal value of (4) is zero (and hence, the chosen subproblem cannot be deleted in Algorithm 1).

**Lemma 3.1** *The optimal value of (4) is zero if, and only if,* $|a_i| \le \sum_{j \ne i} |a_j|$ *for all* $i \in \{1, \ldots, n\}$.

**Proof.** The proof of this lemma is given, for example, in Theorem 1 in [5]. □

# 4 Final remark

The analysis in this short note uses the semidefinite relaxation based on which Goemans and Williamson proposed their randomization technique [4]. The approximation guaranty of this randomization technique cannot be improved provided the Unique Games Conjecture is true, indicating that the semidefinite relaxation is best possible in a certain sense. This however, does not imply that the relaxation cannot be improved at all; in fact it is well known that adding triangle inequalities or other inequalities to the semidefinite relaxation does lead to a better relaxation for many instances of the max-cut problem. Neither does this note consider inequalities that are derived from techniques such as Gomory cuts.

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