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A new algorithm for concave quadratic programming

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Abstract The main outcomes of the paper are divided into two parts. First, we present a new dual for quadratic programs, in which, the dual variables are affine functions, and we prove strong duality. Since the new dual is intractable, we consider a modified version by restricting the feasible set. This leads to a new bound for quadratic programs. We demonstrate that the dual of the bound is a semi-definite relaxation of quadratic programs. In addition, we probe the relationship between this bound and the well-known bounds. In the second part, thanks to the new bound, we propose a branch and cut algorithm for concave quadratic programs. We establish that the algorithm enjoys global convergence. The effectiveness of the method is illustrated for numerical problem instances.

Keywords Non-convex quadratic programming · Duality · Semi-definite relaxation · Bound · Branch and cut method · Concave quadratic programming

1 Introduction

We consider the following quadratic program (QP):

$$\begin{aligned} &\min\ x^TQx + 2c^Tx \\ &s.t.\ Ax \leq b, \end{aligned} \tag{QP}$$

where Q is a real symmetric $n \times n$ matrix, A is a real $m \times n$ matrix, $c \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$. Moreover, throughout the paper, it is assumed that the feasible

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set, $X = \{x \in \mathbf{R}^n : Ax \leq b\}$, is nonempty and bounded. It is well-known that (QP) is solvable in polynomial time when Q is positive semi-definite. Nevertheless, indefinite QPs, in which Q is an indefinite matrix, are NP-hard even for rank-1 cases [34, 38]. In the paper, our focus is on non-convex QPs.

Duality plays a fundamental role in optimization, from both theoretical and numerical points of view [29]. It serves as a strong tool in stability and sensitivity analysis. For convex problems, duality is employed in some numerical methods for obtaining or verifying an optimal solution [4, 7].

It is well-known that the (Lagrangian) dual of a convex QP is also a convex QP, and satisfies strong duality. However, for non-convex case the dual of QPs might be meaningless because the objective function of the dual problem might be $-\infty$ while the primal has a finite optimal value.

The strong duality holds for convex QPs, though this property is still valid for some non-convex cases. Optimizing a quadratic function on the level set of a quadratic function is an archetype. S-lemma guarantees strong duality under Slater condition [35].

Global optimization methods for QPs are typically based on the convex relaxations and bounds. The most effective relaxations for QPs (with quadratic constraints) rest upon the semidefinite programming and the reformulation-linearization technique (RLT) [2, 39]. The semi-definite relaxations were first applied to some combinatorial problems [28]. Due to their efficiency, these methods have been extended to QPs with quadratic constraints [33]. For more discussion on semidefinite relaxations and their comparisons, we refer the reader to the recent survey [2]. Moreover, recently it has been shown that the combination of the semidefinite relaxations and RLT leads to stronger relaxations [1, 2].

In addition to the relaxation methods, scholars have proposed some bounds for classes of QPs [6]. Similar to the relaxation methods bounds give a lower bound. The most effective bounds for QPs are based on semidefinite programming [6].

Another method which is able to give a bound for QPs is the so-called Lasserre hierarchy [25]. In fact, this method provides optimal value. Lasserre hierarchy is able to tackle polynomial optimization problems (optimizing a polynomial function on a given semi-algebraic set). It is well-known that polytopes are Archimedean. Hence, due to the Putinar's Positivstellensatz theorem, optimal value of (QP) is obtained by solving the following convex optimization problem:

$$\begin{array}{l} \max \ell \\ s.t.\ x^TQx + 2c^Tx - \ell = \sigma_0(x) - \sum_{i=1}^m \sigma_i(x)(A_ix - b_i), \\ \quad \sigma_i \in \Sigma[x],\ i = 0, 1, ..., m, \end{array} \tag{1}$$

where $\Sigma[x]$ denotes the cone of polynomials which are sums of squares (SOS) [25]. By virtue of Lasserre hierarchy, the optimal value of (QP) can be obtained by solving the finite number of semi-definite programs. However, the dimension of semi-definite programs may increase dramatically [25].

Note that the Handelman's approximation hierarchy can be also used to produce a bound for QPs. Indeed, this method provides optimal value under some mild conditions [27]. In this approach, each subproblem is a linear program, though similar to the Lasserre hierarchy the dimension of linear programs may increase exponentially. For more details on the method, we refer the interested reader to [25, 27].

Concave QPs are important both theoretically and practically. Concave QPs appear in many applications including fixed charge and risk management problems and quadratic assignment problems [11, 15, 24]. In addition, it has been shown that some class of QPs can be reformulated as concave QPs [15,24]. It is well-known that a concave QP realizes its minimum at some vertices [15]. So the problem is equivalent to the combinatorial problem of optimizing a quadratic function on the vertices of a given polytope. This problem, as mentioned earlier, is NP-hard. Many avenues for tackling concave QPs have been pursued. Typical approaches are cutting plane methods, successive approximation methods and branch and bound approaches [19]. For more methods and details, see also [15, 19, 43].

One of the effective approach for solving QPs is mixed integer programming reformulation [44]. As there exist current state-of-the-art mixed integer programming solvers including GUROBI and CPLEX, this method may be very efficient. Note that some solvers including CPLEX takes advantage of this idea to handle QPs [20]. Semidefinite relaxations have been also employed in branch and bound method for solving QPs [10, 12].

Another approach which has deserve to be mentioned here is copositive programming method. It is known that a QP with quadratic constraints can be formulated as a linear program over the cone of completely positive matrices; See [8] and references therein. Although the latter is convex, the cone of completely positive matrices is intractable. In fact, it is also an NP-hard problem.

The paper is organized as follows. After reviewing terminologies and notations, the new dual for QPs is introduced in Section 2. Dual variables are affine functions and strong duality is proved. As the dual problem is intractable, we take into account a subset of the feasible set, leading to a new bound for QPs. We show that the bound is well-defined for QPs with bounded feasible set. Moreover, we establish that the bound is invariant under affine transformation and is independent of the algebraic representation of X.

In Section 3, we investigate the relationship between the new bound and the conventional bounds. We prove that the new bound is equivalent to the best proposed bound for standard QPs. Moreover, we show that for box constrained QPs the dual of the new bound is Shor relaxation with partial first-level RLT.

Section 4 is devoted to concave QPs. Thanks to the new bound, we introduce a new branch and cut method. In Section 5, we illustrate the effectiveness of the proposed method by presenting its numerical performance on some concave QPs.

1.1 Notation

The n-dimensional Euclidean space is denoted by $\mathbf{R}^n$. We denote the i^{th} row of a given matrix A by A_i. Vectors are considered to be column vectors and the superscript T denotes the transpose operation. W use e and and e_i to denote vector of ones and i^{th} unit vector, respectively. The nonnegative orthant is denoted by $\mathbf{R}^n_+$. Notation $A \succeq B$ means matrix $A - B$ is positive semidefinite. Furthermore, $A \bullet B$ denotes the inner product of A and B, i.e., $A \bullet B = trace(AB^T)$.

For a set $X \subseteq \mathbf{R}^n$, we use the notations $int(X)$ and $cone(X)$ for the interior and the convex conic hull of X, respectively. For a convex cone K, its dual cone is defined and denoted by $K^* = \{y : y^T x \geq 0, \ \forall x \in K\}$. Two notations $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ stand for the gradient and Hessian of smooth function f at $\bar{x}$. For the affine function $\alpha : \mathbf{R}^n \to \mathbf{R}$ given by $\alpha(x) = a^T x + a_0$, its norm is defined and denoted by $\|\alpha\| = \max_{i=0,1,\ldots,n} |a_i|$.

2 A new dual for quadratic programs

In this section, we present a new dual for QPs. Throughout the section, it is assumed that X is a bounded polyhedral. Inspired by Putinar's Positivstellensatz theorem [25], we propose the following convex optimization problem as a dual of (QP),

$$\begin{array}{l} \max \ \ell \\ s.t. \ x^T Q x + 2c^T x - \ell + \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \in P[x], \\ \quad \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \leq 0, \quad x \in X, \end{array} \tag{2}$$

where α_i, $i = 1, ..., m$, are affine functions and $P[x]$ denotes the set of nonnegative polynomials on $\mathbf{R}^n$. It is readily seen that the above problem is a convex problem with infinite constraints. Note that a quadratic function $q(x) = x^T Q x + 2c^T x + c_0$ is nonnegative on $\mathbf{R}^n$ if and only if $\begin{pmatrix} Q & c \\ c & c_0 \end{pmatrix} \succeq 0$. We prove that problem (2) is feasible and fulfills strong duality. Before we get to the proof, we need to present a lemma.

Lemma 1 *Let $X = \{x \in \mathbf{R}^n : Ax \leq b\}$ be a full-dimensional polytope and let $q(x) = x^T Q x + 2c^T x + c_0$ be a quadratic function. Then there exist affine functions α_i for $i = 1, ..., m$ such that*

$$x^T Q x + 2c^T x + c_0 = \sum_{i=1}^m \alpha_i(x)(A_i x - b_i).$$

Proof *The existence of affine functions α_i, $i = 1, ..., m$, satisfying the desired equality is equivalent to the consistency of the linear system*

$$\frac{1}{2} \sum_{i=1}^m \begin{pmatrix} d_i \\ f_i \end{pmatrix} \begin{pmatrix} A_i & -b_i \end{pmatrix} + \frac{1}{2} \sum_{i=1}^m \begin{pmatrix} A_i^T \\ -b_i \end{pmatrix} \begin{pmatrix} d_i^T & f_i \end{pmatrix} = \begin{pmatrix} Q & c \\ c & c_0 \end{pmatrix},$$

in which variables $\begin{pmatrix} d_i \\ f_i \end{pmatrix}$ *,* $i = 1, ..., m$*, are variables. Indeed,* $\begin{pmatrix} d_i \\ f_i \end{pmatrix}$ *is the representative of* α_i*. To prove the consistency, it is sufficient to show that the above system has full rank. On the contrary, suppose that the above linear system does not have full rank. So, there exists a non-zero symmetric matrix* D *such that*

$$\frac{1}{2}\sum_{i=1}^{m} D \bullet \begin{pmatrix} d_i \\ f_i \end{pmatrix} (A_i \ -b_i) + \frac{1}{2}\sum_{i=1}^{m} D \bullet \begin{pmatrix} A_i^T \\ -b_i \end{pmatrix} (d_i^T \ f_i) = 0$$

Since $A \bullet x^T y = x^T A y$*, we have*

$$\sum_{i=1}^{m} (A_i \ -b_i) \, D \begin{pmatrix} d_i \\ f_i \end{pmatrix} = 0.$$

As $\left\{ \begin{pmatrix} A_1^T \\ -b_1 \end{pmatrix}, ..., \begin{pmatrix} A_m^T \\ -b_m \end{pmatrix} \right\}$ *generates* $\mathbf{R}^{n+1}$*,* D *must be zero. This contradicts our assumption* $D \neq 0$ *and implies the consistency of the linear system.*

The following theorem shows that the proposed dual satisfies strong duality.

Theorem 1 *Let* X *be a polytope. The optimal values of problems* (QP) *and* (2) *are equal.*

Proof *Without loss of generality, we may assume that* X *is full-dimensional. Otherwise it is enough to consider* (QP) *on the affine space generated by* X*. Suppose that* $q^\star$ *is the optimal value of* (QP)*. By Lemma 1, there exists* $\bar{\alpha}_i$*,* $i = 1, ..., m$*, such that*

$$x^T Q x + 2b^T x - q^\star = -\sum_{i=1}^{m} \bar{\alpha}_i(x)(A_i x - b_i).$$

So, $\bar{\alpha}_i$*,* $i = 1, ..., m$*, and* $\bar{\ell} = q^\star$ *are feasible for* (2) *and the optimal value of problem* (2) *is greater than or equal to* $q^\star$*. The constraints of problem* (2) *imply that the optimal value of the dual problem is not greater than* $q^\star$*. So, the aforementioned feasible point is optimal solution for the dual problem and the proof is complete.*

Although problem (1) is convex, it is not tractable. This difficulty is caused by the number of constraints. So, to take advantages of this formulation, we need to adopt a procedure which one is able to handle the dual problem. A natural approach is the maximization of the objective function on a subset of the feasible set, i.e. restricting the feasible set of the dual problem. In the rest of the paper, we restrict the variables ,α_i, to nonnegative affine functions on X.

If the vertices of X are also available one can consider the following set which includes the aforementioned set. Let $v_1, ..., v_k$ denote the vertices of X.

It is easily seen that the affine functions $\alpha_i,\ i=1,...,m$, which fulfill the following inequalities are feasible for problem (2),

$$\sum_{i=1}^{m}(A_i v_j - b_i)\alpha_i(x) \le 0,\ x\in X, j=1,...,k. \tag{3}$$

Non-homogenous Farkas' Lemma provides an explicit form of the affine functions which satisfy (3). So, it can be formulated by a finite number of linear inequalities. If $cone(\{Ax-b : x\in X\}) = -\mathbf{R}^m_+$, then $\alpha_i,\ i=1,...,m$, that fulfill (3) are nonnegative affine functions on X.

Let $\mathcal{A}_+(X)$ denote the set of nonnegative affine functions on non-empty polytope X. By non-homogenous Farkas' Lemma, $\alpha(x)=d^Tx+f$ is nonnegative on nonempty polyhedron X if and only if there exist nonnegative scalars $\lambda_i,\ i=0,...,m$, with $\alpha(x)=\lambda_0+\sum_{i=1}^m\lambda_i(b_i-A_ix)$; See Theorem 8.4.2 in [31]. It is easily seen that $\mathcal{A}_+(X)$ is a polyhedral cone with nonempty interior [31]. To tackle problem (2), we consider the following restricted problem,

$$\begin{array}{l} \max\ \ell \\ s.t.\ x^TQx+2c^Tx-\ell+\sum_{i=1}^m\alpha_i(x)(A_ix-b_i)\in P[x], \\ \qquad \alpha_i\in\mathcal{A}_+(X),\ i=1,...,m. \end{array} \tag{4}$$

Considering affine functions instead of scalars for Lagrange multipliers can be found in the literature. To extend S-lemma, Sturm et al. applied an affine function as a Lagrange multiplier [42]. To make the point clear, consider the optimization problem

$$\begin{array}{l} \min\ x^TQ_1x+2c_1^Tx \\ s.t.\ x^TQ_2x+2c_2^Tx+b_2\le 0, \\ \qquad c_3^Tx+b_3\le 0, \end{array} \tag{5}$$

where $Q_2\succeq 0$ and $\mathcal{X}=\{x : x^TQ_2x+2c_2^Tx+b_2\le 0, c_3^T\bar{x}+b_3\le 0\}$ has nonempty interior. They show that the optimal values of problem (5) and the following problem are the same,

$$\begin{array}{l} \max\ \ell \\ s.t.\ x^TQ_1x+2c_1^Tx-\ell+t(x^TQ_2x+2c_2^Tx+b_2)+(e^Tx+f)(c_3^Tx+b_3)\in P[x], \\ \qquad t\ge 0,\ \begin{pmatrix} d\\ f\end{pmatrix}\in\mathcal{H}^*, \end{array}$$

where the convex cone $\mathcal{H}=\left\{\begin{pmatrix} x\\ x_0\end{pmatrix} : x^TQ_2x+2x_0c_2^Tx+b_2x_0^2\le 0, 2c_2^Tx+b_2x_0\le 0, x_0\ge 0\right\}$. As seen, the Lagrange multiplier of the linear constraint is an affine function. Unlike problem (4), which the feasible affine functions are characterized by the feasible set, only the quadratic constraint determines a feasible affine function. It may be of interest to verify the validity of Sturm et al.'s result when the constraint $\begin{pmatrix} d\\ f\end{pmatrix}\in\mathcal{H}^*$ is replaced with $d^Tx+f\in\mathcal{A}_+(\mathcal{X})$. The following proposition says that the result also holds under these conditions.

Proposition 1 *If $Q_2 \succeq 0$ and $int(\mathcal{X}) \neq \emptyset$, then the optimal value of the following problem is equal to that of problem* (5),

$$\begin{aligned} &\max\ \ell \\ &s.t.\ x^T Q_1 x + 2c_1^T x - \ell + t(x^T Q_2 x + 2c_2^T x + b_2) + (d^T x + f)(c_3^T x + b_3) \in P[x], \\ &\quad t \geq 0,\ d^T x + f \in \mathcal{A}_+(\mathcal{X}). \end{aligned} \tag{6}$$

Proof *It is seen that the optimal value of problem* (6) *is less than or equal to that of problem* (5). *By Sturm et al.'s result, to show equality, it suffices to prove the inclusion* $\left\{d^T x + f : \begin{pmatrix} d \\ f \end{pmatrix} \in \mathcal{H}^*\right\} \subseteq \mathcal{A}_+(\mathcal{X})$. *Assume on the contrary, there exists* $\begin{pmatrix} \bar{d} \\ \bar{f} \end{pmatrix} \in \mathcal{H}^*$, *but* $\bar{d}^T x + \bar{f} \notin \mathcal{A}_+(\mathcal{X})$. *So there exists* $\bar{x} \in \mathcal{X}$ *with* $\bar{d}^T \bar{x} + \bar{f} < 0$. *The semi-positive definiteness of* Q_2 *and* $\bar{x} \in \mathcal{X}$ *imply that* $2c_2^T \bar{x} + b_2 \leq -\bar{x}^T Q_2 \bar{x} \leq 0$. *Therefore,* $\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in \mathcal{H}$. *Since* $\begin{pmatrix} \bar{d} \\ \bar{f} \end{pmatrix} \in \mathcal{H}^*$, *we must have* $\bar{d}^T \bar{x} + \bar{f} \geq 0$, *which contradicts the assumption that* $\bar{d}^T \bar{x} + \bar{f} < 0$ *and completes the proof.*

Note that under the assumptions of Proposition 1 affine function α is non-negative on $\mathcal{X}$ if and only if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ with

$$\alpha(x) + \lambda_1(x^T Q_2 x + 2c_2^T x + b_2) + \lambda_2(c_3^T x + b_3) \in P[x];$$

See [31] for proof. Thus, problem (6) can be formulated as a semi-definite program.

Typically, SOS polynomials are used as Lagrange multipliers in polynomial optimization [13,22,26]. However, to the best knowledge of author, affine functions have not been applied in this manner as dual variables in finite optimization theory.

Let us return back to problem (4). When X is nonempty, (4) can be formulated as the semi-definite program:

$$\begin{aligned} &\max\ \ell \\ &s.t.\ \tfrac{1}{2}\textstyle\sum_{i=1}^n \begin{pmatrix} d_i \\ f_i \end{pmatrix} \begin{pmatrix} A_i & -b_i \end{pmatrix} + \tfrac{1}{2}\sum_{i=1}^n \begin{pmatrix} A_i^T \\ -b_i \end{pmatrix} \begin{pmatrix} d_i^T & f_i \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0, \\ &\quad \begin{pmatrix} d_i \\ f_i \end{pmatrix} \in cone\left\{ \begin{pmatrix} -A_j^T \\ b_j \end{pmatrix}, 1 \leq j \leq m, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},\ i = 1, ..., m. \end{aligned} \tag{7}$$

Note that if X is empty, the second constraint of (7) is not necessarily equivalent to constraint $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, ..., m$. As the second constraint of (7) gives α_i explicitly, the above problem can be written as

$$\begin{aligned} &\max\ \ell \\ &s.t.\ \frac{1}{2}\begin{pmatrix} -A^T Y A - A^T Y^T A & A^T(Yb + Y^T b + y) \\ (Yb + Y^T b + y)^T A & -2y^T b - 2b^T Y b \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0, \\ &\quad Y \geq 0,\ y \geq 0, \end{aligned} \tag{8}$$

where $Y \in \mathbf{R}^{m\times m}$ and $y \in \mathbf{R}^m$. In the above semi-definite-program, the matrix

$$\Gamma_{ij} = \frac{-1}{2}\begin{pmatrix} A_i^T A_j + A_j^T A_i & -b_i A_j^T - b_j A_i^T \\ -b_i A_j - b_j A_i & 2b_i b_j \end{pmatrix}$$

is the coefficient of y_{ij}. As coefficients of y_{ij} and y_{ji} are the same, we can assume that Y is symmetric. So, problem (8) is reformulated as

$$\begin{aligned} \max\ & \ell \\ s.t.\ & \begin{pmatrix} -A^T Y A & A^T(Yb + 0.5y) \\ (Yb + 0.5y)^T A & -y^T b - b^T Y b \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0, \\ & Y \geq 0,\ y \geq 0,\ Y^T = Y. \end{aligned} \tag{9}$$

The next lemma shows that, under the boundedness of X, problem (4) is feasible.

Proposition 2 *If X is a polytope, then problem (4) is feasible.*

Proof *As X is bounded, there exist scalars f_i, $i = 1, ..., m$, such that $\alpha_i(x) = A_i x + f_i$ is positive on X and $\alpha_i \in \mathcal{A}_+(X)$. Due to the boundedness of X, the system $Ad \leq 0$ does not have any non-zero solution. Thus, $H = \sum_{i=1}^m A_i^T A_i$ is positive-definite. By choosing γ sufficiently large and a suitable choice of ℓ, $\gamma\alpha_i$, $i = 1, ..., m$, accompanying ℓ satisfy all constraints of problem (4).*

Proposition 2 does not hold necessarily for QPs with unbounded feasible set. The following example illustrates the point.

Example 1 Consider the QP

$$\begin{aligned} \min\ & x_1 x_2 \\ s.t.\ & x_1 = 0. \end{aligned}$$

In this example, problem (4) is infeasible while the optimal value is zero.

Problem (4) can be also interpreted via S-lemma. Let $int(X) \neq \emptyset$. For given $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, ..., m$, we have $X \subseteq \{x : \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \leq 0\}$, which is an overestimation of X with a level set of a quadratic function. By S-lemma, the optimal value of $x^T Q x + 2c^T x$ on this set is obtained by solving the problem

$$\{\max \ell : x^T Q x + 2c^T x - \ell + t(\sum_{i=1}^m \alpha_i(x)(A_i x - b_i)) \in P[x],\ t \geq 0\}.$$

As $\mathcal{A}_+(X)$ is a cone, the first constraint of problem (4) is merely S-lemma and problem (4) provides the greatest optimal value $x^T Q x + 2c^T x$ on the level sets of quadratic functions which include X.

In the sequel, we denote the set of $\{\sum_{i=1}^m \alpha_i(x)(A_i x - b_i) : \alpha_i(x) \in \mathcal{A}_+(X)\}$ by $\mathcal{N}(X)$. One can infer from the proof of Proposition 2 that $\mathcal{N}(X)$ is nonempty and involves a strictly convex function. It is easily seen that $\mathcal{N}(X)$ is a closed, convex cone with nonempty interior. Moreover, if the interior of X is nonempty, then $\mathcal{N}(X)$ is pointed, i.e. $\mathcal{N}(X) \cap -\mathcal{N}(X) = \{0\}$.

The cone $\mathcal{N}(X)$ can be applied for obtaining a convex underestimator of a quadratic function on a given polytope. For (QP), one can consider $F : X \to \mathbf{R}$, defined by

$$F(x) = \{\max\ x^T Qx + 2c^T x + p(x) : p \in \mathcal{N}(X), \nabla^2 p + 2Q \succeq 0\},$$

as a convex underestimator. Another problem in which $\mathcal{N}(X)$ might be useful is the approximation of Löwner-John ellipsoid for a given polytope. We refer the interested reader to [4, 7] for more details about Löwner-John ellipsoid.

As we consider a subset of the feasible set of dual problem, the optimal value of (4) may be strictly less than that of (2) or equivalently (QP). So, we can regard problem (4) as a new bound for QPs. The following proposition states this fact.

Proposition 3 *Let $\bar{x}$ and $\bar{\alpha}_i$ $(i = 1, ..., m)$, $\bar{\ell}$ be feasible for* (QP) *and problem* (4)*, respectively. Then $\bar{x}^T Q\bar{x} + 2c^T \bar{x} \geq \bar{\ell}$.*

Proof The first constraint of problem (4) implies $\bar{x}^T Q\bar{x}+2c^T \bar{x}-\bar{\ell}+\sum_{i=1}^m \bar{\alpha}_i(\bar{x})(A_i\bar{x}-b_i) \geq 0$. Since $-\sum_{i=1}^m \bar{\alpha}_i(\bar{x})(A_i\bar{x} - b_i) \geq 0$, we get $\bar{x}^T Q\bar{x} + 2c^T \bar{x} \geq \bar{\ell}$, which is the desired inequality.

The next theorem provides sufficient conditions under which bound (4) is exact. Let $I(x)$ denote active constraints at x, i.e. $I(x) = \{i : A_i x = b_i\}$.

Theorem 2 *Let X be a polytope. The optimal values of* (QP) *and problem* (4) *are the same if there exist $\bar{x} \in argmin_{x\in X} x^T Qx+2c^T x$ and $d_i^T x+f_i \in \mathcal{A}_+(X)$ for $i = 1, 2, ..., m$ such that*

$$Q + \frac{1}{2}\sum_{i=1}^m d_i A_i + \frac{1}{2}\sum_{i=1}^m A_i^T d_i^T \succeq 0,$$

$$Q\bar{x} + c + \sum_{i\in I(\bar{x})} (d_i^T \bar{x} + f_i)A_i^T + \sum_{i\in\{1,2,...,m\}\setminus I(\bar{x})} (A_i\bar{x} - b_i)d_i = 0,$$

and

$$d_i^T \bar{x} + f_i = 0,\ \forall i \in \{1, 2, ..., m\} \setminus I(\bar{x}).$$

Proof *Consider the quadratic function $q(x) = x^T Qx + 2c^T x + \sum_{i=1}^m (d_i^T x + f_i)(A_i x - b_i)$. The first condition implies that q is convex. We can infer from the second and third conditions that $\nabla q(\bar{x}) = 0$. By the sufficient optimality conditions for convex quadratic functions, $\bar{x}$ is an optimal solution problem $\{\min q(x) : x \in \mathbf{R}^n\}$ with the optimal value $\bar{x}^T Q\bar{x} + 2c^T \bar{x}$. Additionally, $\alpha_i(x) = d_i^T x+f_i$, $i = 1, 2, ..., m$, and $\ell = \bar{x}^T Q\bar{x}+2c^T \bar{x}$ are feasible for problem* (4)*. By virtue of Proposition 3, the optimal value of both problems are equal and the proof is complete.*

We notice that checking the sufficient conditions provided in Theorem 2 is not difficult. In fact, it can be done by testing the feasibility of a semi-definite program, which there are polynomial time algorithms to do it. Under convexity, the above theorem holds for any optimal solution, so the bound is exact in the convex case. However, in general, the assumptions in Theorem 2 may not hold.

Another point concerning Theorem 2 is that it provides sufficient conditions for global optimality. Some sufficient global optimality conditions by virtue of semi-definite relaxation can be found in the literature, see e.g., [45] and the references therein. As we will see in the sequel the dual of problem (4) is a semi-definite relaxation of (QP). So, most of the derived results with a little modification can be applied to problem (4). We refer the interested reader to [5, 15] for some necessary and sufficient global optimality conditions for QPs.

In general, when the feasible set of a semi-definite program is unbounded, it is probable that it does not realize its optimal value [4]. In the following proposition, we prove that problem (4) achieves its optimal value while the feasible set of (4) may be unbounded.

Proposition 4 *If X be a polytope, then problem* (4) *achieves its optimal value.*

Proof *Without loss of generality, we can assume $int(X) \neq \emptyset$. Otherwise, it is enough to consider the polytope on the affine space generated by itself.*
By Proposition 3, the optimal value of problem (4)*, denoted by $\bar{\ell}$, is finite. There exist sequences $\{\alpha_i^k\} \subseteq \mathcal{A}_+(X)$,$i = 1, ..., m$, and $\{\ell^k\}$ such that $\ell^k \to \bar{\ell}$ and $x^T Qx + 2c^T x - \ell^k + \sum_{i=1}^m \alpha_i^k(x)(A_i x - b_i) \in P[x]$. If the sequences $\{\alpha_i^k\}$, $i = 1, ..., m$, are bounded, then due to the closedness of $P[x]$ and $\mathcal{A}_+(X)$, the proof is complete. Otherwise, by setting $\mu_i^k = \frac{\alpha_i^k}{t_k}$ for $i = 1, ..., m$ with $t_k = \max_{1\leq i\leq m} \|\alpha_i^k\|$, and choosing appropriate subsequences if necessary, we can assume that $\mu_i^k \to \bar{\mu}_i$ for $i = 1, ..., m$ and there exists j such that $\bar{\mu}_j \neq 0$. Due to the closedness of the set of nonnegative polynomials, we have $q(x) = \sum_{i=1}^m \bar{\mu}_i(x)(A_i x - b_i) \in P[x]$. As $int(X) \neq \emptyset$, there exists some $\bar{x} \in X$ such that $\bar{\mu}_j(\bar{x}) > 0$ and $A_j \bar{x} < b_i$. Therefore, $q(\bar{x}) < 0$, which contradicts $q(x) \in P[x]$.*

It follows from the proof of Proposition 4 that all optimal solutions of problem (4) are bounded when $int(X) \neq \emptyset$. One important question may arise about (4) is that: "Is the optimal value of problem (4) independent of the representation of X?". Next theorem gives the affirmative answer to the question.

Theorem 3 *Let $\{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$ and $\{x \in \mathbf{R}^n : \hat{A}x \leq \hat{b}\}$ be two different representations of the polytope X. Then the optimal values of problem* (4) *corresponding to these representations are equal.*

Proof *Suppose that $\hat{\ell}$ and $\bar{\ell}$ are the optimal values of problem* (4) *corresponding to the representations $\{x \in \mathbf{R}^n : \hat{A}x \leq \hat{b}\}$ and $\{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$, respectively,* *with* $\hat{A} \in \mathbf{R}^{\hat{m}\times n}$ *and* $\bar{A} \in \mathbf{R}^{\bar{m}\times n}$. *By Proposition 4, there exist* $\hat{\alpha}_i$ $(i = 1, ..., \hat{m})$ *which are optimal for*

$$\begin{array}{ll} \max \ell & \\ s.t.\ x^T Qx + 2c^T x - \ell + \sum_{i=1}^{\hat{m}} \alpha_i(x)(\hat{A}_i x - \hat{b}_i) \in P[x], & \\ \quad\ \alpha_i \in \mathcal{A}_+(X),\ i = 1, ..., \hat{m}. & \end{array} \tag{10}$$

As $\hat{b_i} - \hat{A}_i x \in \mathcal{A}_+(X)$, *there exist nonnegative scalars* Y_{ij}, $j = 0, 1, ..., \bar{m}$, *such that*

$$\hat{A}_i x - \hat{b_i} = \sum_{j=1}^{\bar{m}} Y_{ij}(\bar{A}_j x - \bar{b_j}) - Y_{i0}.$$

Similarly, according to $\hat{\alpha}_i \in \mathcal{A}_+(X)$, *there exist nonnegative scalars* W_{ij}, $j = 0, 1, ..., \bar{m}$, *with* $\hat{\alpha}_i(x) = \sum_{j=1}^{\bar{m}} -W_{ij}(\bar{A}_j x - \bar{b_j}) + W_{i0}$. *By the first constraint of problem* (10), *we get*

$$\begin{aligned} & x^T Qx + 2c^T x - \hat{\ell} + \sum_{i=1}^{\hat{m}} \hat{\alpha}_i(x)\Big(\sum_{j=1}^{\bar{m}} Y_{ij}(\bar{A}_j x - \bar{b_j}) - Y_{i0}\Big) = \\ & x^T Qx + 2c^T x - \hat{\ell} + \sum_{j=1}^{\bar{m}} (\bar{A}_j x - \bar{b_j})\Big(\sum_{i=1}^{\hat{m}} Y_{ij}\hat{\alpha}_i(x)\Big) - \sum_{i=1}^{\hat{m}} Y_{i0}\hat{\alpha}_i(x) = \\ & x^T Qx + 2c^T x - (\hat{\ell} + \sum_{i=1}^{\hat{m}} Y_{i0}W_{i0}) + \sum_{j=1}^{\bar{m}} (\bar{A}_j x - \bar{b_j})\Big(\sum_{i=1}^{\hat{m}} (Y_{ij}\hat{\alpha}_i(x) + W_{ij}Y_{i0})\Big) \\ & \in P[x]. \end{aligned}$$

As $\mathcal{A}_+(X)$ *is a convex cone,* $\tilde{\alpha}_i(x) = \sum_{i=1}^{\hat{m}} (Y_{ij}\hat{\alpha}_i(x) + W_{ij}Y_{i0}) \in \mathcal{A}_+(X)$ *for* $i = 1, ..., \bar{m}$. *So,* $\tilde{\ell} = \hat{\ell} + \sum_{i=1}^{\hat{m}} Y_{i0}W_{i0}$ *and* $\tilde{\alpha}_i$, $i = 1, ..., \bar{m}$, *are feasible for problem* (4) *corresponding to the representation* $X = \{x \in \mathbf{R}^n : \bar{A}x \le \bar{b}\}$. *This implies* $\bar{\ell} \ge \hat{\ell}$, *because* $Y_{i0}W_{i0} \ge 0$. *By a similar argument, one can establish* $\hat{\ell} \ge \bar{\ell}$. *Therefore,* $\bar{\ell} = \hat{\ell}$ *and the proof is complete.*

Similar to the proof of Theorem 3, one can prove that if polytope X_1 is a subset of polytope X_2, then $opt(X_1, Q, c) \le opt(X_2, Q, c)$. (Let $opt(X, Q, c)$ denote the optimal value of problem (4) corresponding to polytope X, matrix Q and vector c). We call this property as inclusion property. In the following proposition, we show that the optimal value of problem (4) is constant under invertible affine transformation.

Proposition 5 *The optimal value of problem* (4) *is invariant under affine invertible transformations.*

Proof *Let* T *be an invertible affine transformation on* $\mathbf{R}^n$. *We set* $Y = T^{-1}(X)$ *and assume that* $T(y) = Hy + h$ *for some invertible matrix* H *and* $h \in \mathbf{R}^n$. *For* $\alpha_i \in \mathcal{A}_+(X)$, $i = 1, ..., m$, *and* ℓ *satisfying*

$$x^T Qx + 2c^T x - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \in P[x],$$

we have

$$y^T\bar{Q}y + 2\bar{c}^Ty + c_0 - \ell + \sum_{i=1}^{m}\mu_i(y)(\bar{A}_iy - \bar{b}_i) \in P[y],$$

where $\mu_i = \alpha_i T \in \mathcal{A}_+(Y)$*,* $\bar{Q} = H^TQH$*,* $\bar{c} = Hc + H^TQh$*,* $c_0 = h^TQh + 2c^Th$*,* $\bar{A} = AH$ *and* $\bar{b} = b - Hh$*. This statement implies* $opt(X, Q, c) \le opt(Y, \bar{Q}, \bar{c}) - c_0$*. By the same argument, one can derive that* $opt(Y, \bar{Q}, \bar{c}) - c_0 \le opt(X, Q, c)$*. This completes the proof.*

The following proposition states if a polytope is singleton, then bound (4) is exact.

Proposition 6 *If* $X = \{\bar{x}\}$*, then the optimal values of* (QP) *and* (4) *are equal.*

Proof *According to Proposition 5, the optimal value of problem* (4) *is independent of translation. So, without loss of generality* $\bar{x} = 0$*. In addition, by virtue of Theorem 3, we assume* $X = \{x : x = 0\}$*. We define* $\alpha_i, \mu_i \in \mathcal{A}_+(X)$ *for* $i = 1, ..., n$*, corresponding to* (QP)*, as follows*

$$\alpha_i(x) = \begin{cases} 0 & c_i \ge 0 \\ (-Qx)_i - 2c_i & c_i < 0 \end{cases} \quad \mu_i(x) = \begin{cases} (Qx)_i + 2c_i & c_i \ge 0 \\ 0 & c_i < 0 \end{cases}$$

Thus, $x^TQx + 2c^Tx + \sum_{i=1}^m \alpha_i(x)(x_i) + \sum_{i=1}^m \mu_i(x)(-x_i) = 0$*. On account of Proposition 3, the optimal value of* (4) *is zero, which is the desired conclusion.*

Since (4) is a convex optimization problem, it is natural to ask about its dual. To get the dual of problem (4), we consider problem (9). The dual of problem (9) is formulated as follows,

$$\begin{aligned}
\min\ & \begin{pmatrix} Q & c \\ c^T & 0 \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \\
s.t.\ & \begin{pmatrix} A_i^TA_j & -b_jA_i^T \\ -b_jA_i & -b_ib_j \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \ge 0,\ i \le j = 1, ..., n \\
& \begin{pmatrix} 0 & -.5A_i^T \\ -.5A_i & b_i \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \ge 0,\ i = 1, ..., n \\
& \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} = 1, \\
& \begin{pmatrix} X & x \\ x^T & x_0 \end{pmatrix} \succeq 0.
\end{aligned} \tag{11}$$

By $A_i^TA_j \bullet X = A_iXA_j^T$ and Shor decomposition, the dual may be rewritten as

$$\begin{aligned}
\min\ & Q \bullet X + 2c^Tx \\
s.t.\ & AXA^T - b^TAx - x^TA^Tb \ge bb^T, \\
& Ax \le b \\
& X \succeq xx^T.
\end{aligned} \tag{12}$$

Problem (12) is a semi-definite relaxation of (QP). Indeed, let $\bar{x}$ be a feasible point of (QP), It is easily seen that the matrix $\begin{pmatrix} \bar{x}\bar{x}^T & \bar{x} \\ \bar{x}^T & 1 \end{pmatrix}$ is feasible for (12) and $\bar{x}^T Q\bar{x} + 2c^T\bar{x} = Q \bullet \bar{x}^T\bar{x} + 2c^T\bar{x}$.

It is worth noting that as problem (4) is feasible for any quadratic function, so the semi-definite program (4) is strongly feasible. By Theorem 3.2.6 in [36], the strong duality holds, that is, the optimal values of both problems (4) and (12) are equal.

As problem (12) is the dual of problem (4), it follows from Proposition 3 that it is bounded from below, and there is no need to add more constraints. In general, semidefinite relaxations are not necessarily bounded from below [18]. This relaxation is also called a strong relaxation by some scholars [9]. One may wonder to know about the relationship between this relaxation with other semidefinite relaxations existing in the literature. It is seen that problem (15) is Shor's relaxation of

$$\begin{array}{ll} \min & x^TQx + 2c^Tx \\ s.t. & -Ax \geq -b, \\ & (A_ix - b_i)(A_jx - b_j) \geq 0,\ i \leq j = 1, ..., m \end{array}$$

which is exactly (QP) with $0.5m(m+1)$ redundant constraints [41]. Most relaxation methods, including RLT, use redundant constraints [40]. Gorge et al. [18] use convex combination of these redundant constraints as a cut for semi-definite relaxations. We refer the reader to [18] for more information on the applications of these redundant constraints in QPs. Note that the constraint $Ax \leq b$ is redundant for (12) and so it can be removed [40].

One important inquiry about bounds is how one can reduce the gap. In this context, one idea may be the replacement of nonnegative affine functions with nonnegative convex quadratic functions on the given polytope. Similar to the affine case, a new bound can be formulated as a semi-definite program. Furthermore, one can show that most presented results in Section 2 hold in this case. However, the number of variables is of the order of m^3, and makes this formulation less attractive. In addition, numerical implementations showed that the gap improvement was not considerable compared to the affine case. Pursuing this method by polynomials with degree greater than or equal to three is not practical since checking positivity of such a polynomial on a given polytope is not easy [25]. However, the procedure can be followed by restricting to some classes of polynomials [21].

As mentioned earlier, problem (4) can be written as

$$\max_{q \in \mathcal{N}(X)} \{\min x^TQx + 2c^Tx : q(x) \leq 0\},$$

and the bound (4) is exact provided $x^TQx + 2c^Tx - q^\star \in P[x] - \mathcal{N}(X)$ ($q^\star$ denotes the optimal value of (QP)). As a result, enlargement of $\mathcal{N}(X)$ may lead to the reduction of gap.

Let $\bar{x} \in int(X)$ and let $d \in \mathbf{R}^n$ be an arbitrary non-zero vector. The polytope X can be partitioned in two polytopes $X_1 = \{d^T(x-\bar{x}) \leq 0, x \in X\}$ and $X_2 = \{d^T(x-\bar{x}) \geq 0, x \in X\}$. It is readily seen that

$$\mathcal{N}(X) \subseteq N(X_1) \cap N(X_2).$$

It is likely that cone $\mathcal{N}(X)$ is strictly included in $\mathcal{N}(X_1) \cap \mathcal{N}(X_2)$ when the bound is not exact. By the replacement of $\mathcal{N}(X)$ with $\mathcal{N}(X_1) \cap \mathcal{N}(X_2)$ in (4), the number of variables and constraints will be two times more than the former case. It is easily seen the optimal value of (4) depends on the choice of $\bar{x}$ and d. Note that partitioning the feasible set is a wide-spread method for reducing the duality gap; See [14] and references therein. We will take advantage of this idea to develop a branch and cut algorithm for concave QPs.

In the same line, one can partition X to k polytopes and fattens $\mathcal{N}(X)$. In this case, the number of variables will be of $O(km^2)$. For instance, for $\bar{x} \in int(X)$, one could consider two different hyperplanes which pass through the give point. As a result, the polytope is divided into four polytopes.

Although problem (4) provides a lower bound for (QP), the number of variables is of $O(m^2)$, which makes this semi-definite program time-consuming in some cases. In the sequel, we propose the bounds whose decision variables are less than that of problem (4). However, this problem does not equip us with a better lower bound.

Consider problem (QP). Let L denote the subspace generated by the eigenvectors of Q corresponding to the negative eigenvalues. We propose the following bound for (QP),

$$\begin{array}{l} \max\ \ell \\ s.t.\ x^TQx + 2c^Tx - \ell + \sum_{i=1}^m \alpha_i(x)(A_ix - b_i) \in P[x], \\ \quad \alpha_i \in \mathcal{A}_+(X),\ i = 1, ..., m, \\ \quad \nabla\alpha_i \in L,\ i = 1, ..., m. \end{array} \tag{13}$$

The above problem is reduced to problem (4) if Q is negative definite. Nevertheless, for the class of problems which Q has only one negative eigenvalue, problem (13) has $2m$ variables. As a result, it would be more beneficial from time aspect to tackle this problem instead of (4). In the following proposition, we prove that the optimal value of problem (13) is finite.

Proposition 7 *Let X be a polytope. Then problem* (13) *has finite optimal value.*

Proof *By Proposition 3, the optimal value of* (13) *is either finite or minus infinity. So, it suffices to prove the existence of a feasible point. Let $\{v_1, ..., v_k\}$ be a basis for L. As X is bounded, by virtue of Farkas' Lemma, there exist nonnegative constants Y_{ij}, $j = 1, ..., k$, such that $v_j = \sum_{i=1}^m Y_{ij}A_i^T$. Moreover, there are f_j, $j = 1, ..., k$, with $v_j^Tx + f_j \in \mathcal{A}_+(X)$. For γ sufficiently large, the matrix $Q + \gamma\sum_{j=1}^k v_jv_j^T$ is positive semi-definite. As X is bounded, for suitable choice of ℓ the affine functions $\alpha_i(x) = \gamma\sum_{j=1}^k Y_{ij}(v_j^Tx + f_j)$, $i = 1, ..., m$, fulfill all constraints of* (13).

The following example demonstrates that the optimal value of (13) may be strictly less than that of (4).

Example 2 Consider the QP,

$$\begin{aligned} &\min\ 2x_1x_2 \\ &s.t.\ x_1, x_2 \leq 1 \\ &\quad -x_1, -x_2 \leq 0. \end{aligned}$$

The optimal values of this QP and (4) is zero (Take into account $\alpha_1(x) = 0, \alpha_2(x) = 0, \alpha_3(x) = x_2, \alpha_4(x) = x_1, \ell = 0$). It is seen that $\alpha \in \mathcal{A}_+(X)$ and $\nabla\alpha \in L$ if and only if $\alpha \in cone(\{x_1 - x_2 + 1, -x_1 + x_2 + 1\})$. By solving problem (13), for any feasible point, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ell$, we have $\ell \leq \frac{-1}{8}$.

One can also formulate a semi-definite program for box constrained QPs with fewer variables in comparison with (4). In this case, one could consider the affine coefficient of constraint $x_k \leq u_k$ $(-x_k \leq -l_k)$, α_k, in the form

$$\alpha_k(x) = d_k x_k + f_i,\ \ \alpha_k \in \mathcal{A}_+(X),$$

where $d_k \in \mathbf{R}$. Similar to Proposition 2, it is proved that the bound is finite in this case as well.

3 Comparison with existing bounds

One important question here is the relationship between the bound (4) and the conventional lower bounds for QPs. For comparison, we focus on two classes of QPs, standard quadratic programs and box constrained quadratic programs. Let us first concentrate on standard quadratic programs. Consider the standard quadratic program,

$$\begin{aligned} &\min\ x^TQx \\ &s.t.\ \textstyle\sum_{i=1}^n x_i = 1, \\ &\quad x \geq 0. \end{aligned} \tag{StQP}$$

It is well-known that (StQP) is solvable in polynomial time if Q is either positive semi-definite or negative semi-definite on standard simplex (denoted by Δ hereafter). In general, (StQP) is NP-hard [6].

We denote the optimal value of (StQP) by ℓ_Q. Note that optimizing a quadratic function on standard simplex can be casted as (StQP). This follows form the fact that for each $x \in \Delta$, we have $x^TQx + 2c^Tx = x^T(Q + ec^T + ce^T)x$. It is seen that $\ell_{Q+tee^T} = \ell_Q + t$ for each $t \in \mathbf{R}$. So, in the current section we assume that Q is nonnegative in (StQP).

Bomze et al. [6] have proposed the best (quadratic) convex underestimation bound as follows

$$\ell_Q^{conv} = \sup\{\ell_S : S \succeq 0, Q - S \geq 0, diag(S) = diag(Q)\}, \tag{14}$$

where $diag(S)$ denotes the diagonal of S. They show that the above problem gives better bound in comparison with other quadratic bounds. Problem (4) is formulated for (StQP) as follows,

$$\begin{array}{l} \max\ \ell \\ s.t.\ x^TQx - \ell + \sum_{i=1}^{n} \alpha_i(x)(-x_i) + \alpha_{n+1}(x)(e^Tx - 1) \in P[x], \\ \quad \alpha_i \in \mathcal{A}_+(\varDelta),\ i = 1, ..., n. \end{array} \tag{15}$$

Next theorem shows that bounds (14) and (15) are equivalent.

Theorem 4 *Problems* (14) *and* (15) *give the same bound.*

Proof *Let* $\bar{\ell}$ *denote the optimal value problem* (15). *First we show that* $\ell_Q^{conv} \le \bar{\ell}$. *Without loss of generality, we may assume that* (14) *admits an optimal solution* S. *As* $(Q - S)_i \ge 0$, $\bar{\alpha}_i(x) = (Q - S)_i x \in \mathcal{A}_+(\varDelta)$, $i = 1, ..., n$, *and*

$$x^TQx - x^TSx + \sum_{i=1}^{n} \bar{\alpha}_i(x)(-x_i) = 0.$$

Invoking the optimality conditions for convex QPs, there are nonnegative scalars β_i, $i = 1, ..., n$ *and* β_{n+1} *such that*

$$x^TSx - \ell_Q^{conv} + \sum_{i=1}^{n} \beta_i(-x_i) + \beta_{n+1}(e^Tx - 1) \in P[x].$$

By above equalities, it is seen that $\alpha_i(x) = \bar{\alpha}_i(x) + \beta_i$, $i = 1, ..., n + 1$, *and* $\ell = \ell_Q^{conv}$ *are feasible for problem* (15). *So* $\ell_Q^{conv} \le \bar{\ell}$.
Now, let $\bar{\ell}$ *and* $\bar{\alpha}_i = a_i^Tx + a_i^0$, $i = 1, ..., n + 1$, *be optimal for* (15). *We get*

$$x^TQx - \bar{\ell} - \sum_{i=1}^{n} x_i\alpha_i(x) + (e^Tx - 1)\alpha_{n+1}(x) = (x - \bar{x})^TS(x - \bar{x}), \tag{16}$$

where S *is a positive semi-definite matrix and* $\bar{x} \in \mathbf{R}^n$. *Let* $e^Tx \neq 0$. *By replacing* x *with* $(e^Tx)^{-1}x$ *and multiplying both sides of* (16) *by* $(e^Tx)^2$, *we get*

$$x^T(Q - \bar{\ell}ee^T)x - \sum_{i=1}^{n} x^Te_i(a_i + a_i^0e)x = (x - (e^Tx)\bar{x})^TS(x - (e^Tx)\bar{x}).$$

Since $\alpha_i \in \mathcal{A}_+(\varDelta)$, $a_i^j + a_i^0 \ge 0$ *for* $i = 1, ..., n$ *and* $j = 1, ..., n$. *So, matrix* $N = \sum_{i=1}^{n} e_i(a_i + a_i^0e) \ge 0$. *By continuity, the homogenous quadratic function* $(x - (e^Tx)\bar{x})^TS(x - (e^Tx)\bar{x})$ *is nonnegative on* $\mathbf{R}^n$. *Thus,* $(x - (e^Tx)\bar{x})^TS(x - (e^Tx)\bar{x}) = x^T\bar{S}x$ *for some* $\bar{S} \succeq 0$. *Hence,*

$$\{\max \ell : Q - \ell ee^T \succeq N,\ N \ge 0\} \ge \bar{\ell}.$$

Since $\ell_Q^{conv} = \{\max \ell : Q - \ell ee^T \succeq N,\ N \ge 0\}$ *(see Section 6 in [6]),* $\bar{\ell} \le \ell_Q^{conv}$ *and the proof is complete.*

In the rest of the section, we continue our discussion for box constrained QPs. Consider the box constrained quadratic program

$$\begin{aligned} \min\ & x^T Q_0 x + 2c_0^T x \\ & a_i^T x = d_i, \qquad\qquad i = 1, ..., m \\ & l \le x \le u, \end{aligned} \tag{17}$$

where $n > m$. For convenience, we may assume that $l = 0$ and $u = e$. By the combination of semidefinite programming relaxation (SDP) and RLT, some scholars have proposed new relaxations for problem (17) [1,2]. One of the most effective relaxations in this category is Shor relaxation with partial first-level RLT (SRLT). This bound is formulated as

$$\begin{aligned} \min\ & Q_0 \bullet X + 2c_0^T x \\ s.t.\ & X a_i^T = d_i x, \qquad\qquad i = 1, ..., m \\ & a_i^T x = d_i, \qquad\qquad i = 1, ..., m \\ & X \ge 0, \\ & ex^T - X \ge 0, \\ & X - ex^T - xe^T + ee^T \ge 0, \\ & X - xx^T \succeq 0. \end{aligned} \tag{18}$$

Bao et al. [2] have provided a full comparison of relaxation methods for box constrained QPs. They show that SRLT dominates the other relaxations. By the discussion made in the former section on the dual of (4), it is straightforward to see that SRLT is the dual of (4) for box constrained quadratic programs, see (12). As strong duality holds for problem (4), SRLT snd (4) are equivalent.

One important issue with quadratic programs is how to convert a relaxation solution to an approximate solution [30]. As problem (4) not only provides a lower bound for quadratic programs, but also gives a convex underestimator, one may obtain an approximate solution by optimizing the given function on the feasible set. We use this strategy in the next section.

We conclude the section by addressing an interesting point about bound (4). This bound can be regarded as a special case of the following bound

$$\begin{aligned} \max\ & \ell \\ s.t.\ & x^T Q x + 2c^T x - \ell - \sum_{\tau \in \mathbb{N}_d^d} \lambda_\tau \prod_{i=1}^{d} (b_i - A_i x)^{\tau_i} = \sigma(x), \\ & \sigma \in \Sigma[x],\ \lambda \ge 0, \end{aligned} \tag{19}$$

where $\mathbb{N}_d^d = \{\tau \in \mathbb{N}^d : \sum_{i=1}^d \tau_i \le d\}$. One may regard the above-mentioned bound as a combination of Lasserre hierarchy and Handelman's approximation hierarchy. One can obtain bound (4) by setting $d = 2$ in (19). In fact, this follows from Non-homogenous Farkas' Lemma. Recently, some scholars have taken advantage of this idea and proposed new bounds for general polynomial optimization problems; See [26] for more details.

4 A new algorithm for concave quadratic optimization

In this section, by virtue of the newly introduced bound, we propose a new algorithm for concave QPs. Throughout the section, it is assumed that X is a polytope with nonempty interior and Q is negative semi-definite. We introduce a branch and cut (B&C) algorithm. We use Konno's cut in the cutting step. Before we go into the details of the algorithm, let us remind a definition.

Definition 1 Let $\hat{x} \in X$ be a vertex. This vertex is called a local optimal if the value of the objective function at this point is less than or equal to that at adjacent vertices.

As mentioned before, we are developing a B&C algorithm for concave QPs. The main steps of a B&C method are branching, bounding, fathoming and cutting. A typical branching approach for QPs is partitioning and for bounding is linear program relaxation based on RLT; See [19,39] for more details. Recently, some scholars have employed KKT optimality condition and semi-definite relaxation for branching and bounding, respectively [10, 12]. In the sequel, we present the details of the steps.

The proposed method regards (QP) as the root node of the B&C tree. Let the following quadratic program be the subproblem of a node,

$$\begin{array}{l} \min \ x^T Q x + 2c^T x \\ s.t. \ \bar{A}x \leq \bar{b}, \end{array}$$

and let $\bar{X}$ denote the feasible set of the above problem. To get a lower bound for the node, we formulate the semi-definite program

$$\begin{array}{ll} \max & \ell \\ s.t. & \begin{pmatrix} -\bar{A}^T Y \bar{A} & \bar{A}^T(Y\bar{b} + 0.5y) \\ (Y\bar{b} + 0.5y)^T \bar{A} & -y^T\bar{b} - \bar{b}^T Y \bar{b} \end{pmatrix} + \begin{pmatrix} Q & c \\ c^T & -\ell \end{pmatrix} \succeq 0, \\ & \ell \leq u \\ & Y \geq 0, \ Y = Y^T, \ y \geq 0, \end{array} \tag{20}$$

where u is the best upper bound obtained by the algorithm so far. Let $\bar{Y}$ and $\bar{y}$ be optimal solutions to (20). Then, we formulate the convex QP,

$$\begin{array}{l} \min \ x^T(Q - \bar{A}^T \bar{Y} \bar{A})x + 2(c + \bar{A}^T \bar{Y} \bar{b} + 0.5A^T \bar{y})^T x \\ s.t. \ Ax \leq b. \end{array} \tag{21}$$

Let $\hat{x}$ be a solution of problem (21). Next, a local vertex optimal point $\bar{x} \in X$ is chosen such that $\bar{x}^T Q \bar{x} + 2c^T \bar{x} \leq \hat{x}^T Q \hat{x} + 2c^T \hat{x}$. This step is not time consuming. Indeed, there are some efficient approaches for computing $\bar{x}$ [43]. We use the value $\bar{x}^T Q \bar{x} + 2c^T \bar{x}$ to update the upper bound, u. We use $\bar{l}$ and other lower bounds obtained from fathomed nodes and child nodes to update the lower bound.
After the bounding step, if the difference of ℓ and u is less than the prescribed

tolerance, ϵ, the node will be fathomed. Otherwise, the algorithm solves problem (21) corresponding to the feasible set of the subproblem, $\bar{X}$, and computes a local optimal point $\bar{x} \in \bar{X}$ for concave QP $\{\min x^T Qx + 2c^T x : x \in \bar{X}\}$.

In the next step, the algorithm produces a cut. Let us go into the details of the step. As mentioned above, we employ Konno's cut. For convenience, let $\bar{x} = 0$ be a non-degenerated local optimal vertex of $\{\min x^T Qx + 2c^T x : x \in \bar{X}\}$. Suppose that vectors e_i, $i = 1, ..., n$, denote extreme directions at $\bar{x} \in \bar{X}$. Let $q(x) = x^T Qx + 2c^T x$. To compute Konno's cut, first we need to obtain Tuy's cut given by

$$\sum_{i=1}^{n} \frac{x_i}{t_i} \geq 1,$$

where $t_i := \max\{\theta : q(\bar{x} + \theta e_i) \geq u - \epsilon\}$. Then, the algorithm computes $y^i \in \text{argmin}\{\bar{x}^T Qy + 2c^T y : x \in X, \sum_{i=1}^{n} \frac{x_i}{t_i} \geq 1\}$. If the following inequality holds it will continue the cutting step, for otherwise it goes to the branching step.

$$\min_{i=1,\ldots,n} \{q(y^i)\} \geq u - \epsilon.$$

We call $\bar{x}$ an eligible vertex if we have the above inequality. Let $\bar{x}$ be eligible. The method will add the valid cut $\sum_{i=1}^{n} \frac{x_i}{s_i} \geq 1$, where

$$s_i = \max\{\theta : -A^T + t^{-1}\mu - Q_i^T \theta = c, -b^T \lambda + \mu + c_i \theta \geq u - \epsilon, \lambda \geq 0, \mu \geq 0\},$$

for $i = 1, ..., n$ and $t^{-1} = \left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right)^T$. We refer the interested reader to [23, 43] for more information on Konno's cut.

After computing Konno's cut, the method updates the feasible set, that is, set $\bar{X} = \{x \in \bar{X} : \sum_{i=1}^{n} \frac{x_i}{s_i} \geq 1\}$. If the feasible set is empty, the node will be fathomed. Otherwise, the method goes to the branching step.

For branching, the algorithm divides the polytope $\bar{X}$ into two partitions X_1 and X_2 such that $X_1 = \{x : x \in \bar{X}, d^T x \leq d^T x_c\}$ and $X_2 = \{x : x \in \bar{X}, d^T x \geq d^T x_c\}$ where x_c is a Chebyshev center of $\bar{X}$ and $d \neq 0$ is a random vector in $\mathbf{R}^n$. It is worth noting that a Chebyshev center of a polytope is computed by solving a linear program [7].

Updating the upper bound is straightforward (it is enough to consider the minimum of the provided upper bounds). To update the lower bound, we must consider the lower bound of all fathomed and child nodes. Strictly speaking, it is seen that for father node n_p and its two child nodes, n_{p_1} and n_{p_2}, we have $l_p \leq \min\{l_{p_1}, l_{p_2}\}$, where l_p, l_{p_1} and l_{p_2} denote the generated lower bound of n_p, n_{p_1} and n_{p_2}, respectively. Therefore, if $\{l_i^f\}_{i=1}^k$ and $\{l_i^c\}_{i=1}^o$ are the lower bound fathomed and new nodes, respectively, then the new lower bound will be obtained by the following formula

$$l = \min\{\min_{i=1}^{k} l_i^f, \min_{i=1}^{o} l_i^c\}. \tag{22}$$

It can be seen that the lower bound is increasing throughout the algorithm. All steps of the method are presented in Algorithm 1.

Algorithm 1 Branch and Cut Algorithm

Initialization:
  $k=1, l=-\infty, u=\infty, s=1, \epsilon>0, L=\{\}, \Pi_1=\{(P_1,X)\}$ and $\Pi_2=\{\}$.
while s = 1 **do**
  while $\Pi_1 \neq \emptyset$ **do**
    $(P_k, X_k) \leftarrow select(\Pi_1)$ and $\Pi_1 \setminus (P_k, X_k)$.
    Solve semi-definite program (4) corresponding to (P_k, X_k) to get l_k.
    $L \leftarrow l_k$ and delete its father lower bound if exists in L.
    Solve convex QP (21) and obtain a local vertex optimal $\bar{x}$.
    $u \leftarrow \min(u, \bar{x}^T Q\bar{x} + 2c^T\bar{x})$.
    if $l_k + \epsilon < u$, **then**
      Solve convex QP (21) corresponding to X_k and obtain a local vertex optimal $\bar{x} \in \{\min x^TQx + 2c^Tx : x \in \bar{X}\}$.
      if $\bar{x}$ is eligible, **then**
        Compute Konno's cut $\sum_{i=1}^n \frac{x_i}{t_i} \geq 1$ at $\bar{x}$.
        $X_k \leftarrow \{X_k : \sum_{i=1}^n \frac{x_i}{t_i} \geq 1\}$
      end if
      if $X_k \neq \emptyset$, **then**
        Branch (P_k, X_k) to two nodes (P_{k+1}, X_{k+1}) and (P_{k+2}, X_{k+2}).
        $\Pi_2 \leftarrow \{(P_{k+1}, X_{k+1}), (P_{k+2}, X_{k+2})\}$ and $k \leftarrow k+2$.
      end if
    end if
  end while
  update l by formula (22).
  if $\Pi_2 = \emptyset$ or stopping criteria are satisfied, **then**
    $s=0$.
  else
    $\Pi_1 \leftarrow \Pi_2$ and $\Pi_2 \leftarrow \{\}$.
  end if
end while

Here, Π_1 and Π_2 denote the set of nodes and the set L contains lower bounds of fathomed and child nodes. Stoping criteria which one may use in Algorithm 1 can be absolute gap tolerance, a limit on the maximum running time, etc.

In the rest of the section, we investigate the finite convergent of the algorithm with the stopping accuracy $\epsilon > 0$. To this end, we consider function $\Phi : \mathbf{R}^n \times \mathbf{R}^n_+ \to \mathbf{R}$ where $\Phi(y,d)$ is defined as the optimal value

$$\begin{aligned} &\max\ \ell \\ &s.t.\ x^TQx + 2c^Tx - \ell + \sum_{i=1}^n [\alpha_i(x)(x - y_i - d_i) + \mu_i(x)(-x + y_i - d_i)] \in P[x] \\ &\qquad \alpha_i, \mu_i \in \mathcal{A}_+(X(y,d)), \end{aligned} \tag{23}$$

where $X(y,d) = \{x : y - d \leq x \leq y + d\}$. The well-definedness of Φ on its domain follows from Proposition 2. The next lemma lists some properties of Φ.

Lemma 2 *The function* $\Phi : \mathbf{R}^n \times \mathbf{R}^n_+ \to \mathbf{R}$ *has the following properties.*

(i) $\Phi(y,0) = y^TQy + 2c^Ty, \quad \forall y \in \mathbf{R}^n$*;*

(ii) Φ *is continuous on* $X \times \mathbf{R}^n_+$.

Proof *The first part follows immediately from Proposition 6.*
For the second part, first we prove the lower semi-continuity of Φ. *Let* $(\bar{y}, \bar{d}) \in \mathbf{R}^n_+$ *and* $\bar{\alpha}_i, \bar{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$ *and* $\bar{\ell}$ *are optimal for* (23). *For every* $\epsilon > 0$, *there are* $\hat{\alpha}_i, \hat{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$ *such that the quadratic function*

$$x^T Qx+2c^T x-(\bar{\ell}-\epsilon)+\sum_{i=1}^n(\bar{\alpha}_i(x)+\hat{\alpha}_i(x))(x-\bar{y}_i-\bar{d}_i)+(\bar{\mu}_i(x)+\hat{\mu}_i(x))(-x+\bar{y}_i-\bar{d}_i),$$

is strictly convex and positive on $\mathbf{R}^n$. *As a result, for small perturbations of* $\bar{\alpha}_i, \bar{\mu}_i, \bar{y}$ *and* $\bar{d}$, *the above-mentioned quadratic function belongs to* $P[x]$, *which implies*

$$\liminf_{(y,d)\to(\bar{y},\bar{d})} \Phi(y, d) \geq \bar{\ell} - \epsilon.$$

As the above atatement holds for each $\epsilon > 0$, Φ *is lower semi-continuous at* $(\bar{y}, \bar{d})$. *Now, we prove the upper semi-continuity of* Φ. *First, we consider the case* $\bar{d} \in int(\mathbf{R}^n_+)$. *Let the sequence* $\{(y_k, d_k)\} \subseteq X \times \mathbf{R}^n_+$ *tends to* $(\bar{y}, \bar{d})$. *Suppose that* α_i^k, μ_i^k *and* ℓ^k *are optimal for problem* (23) *corresponding to* (y_k, d_k). *If for each* $i = 1, ..., n$ *the sequences* $\{\alpha_i^k\}, \{\mu_i^k\} \subseteq \mathbf{R}^{n+1}$ *are bounded, then without loss of generality we may assume that* $\alpha_i^k \to \bar{\alpha}_i$, $\mu_i^k \to \bar{\mu}_i$ *and* $\ell_k \to \bar{\ell}$. *In addition, due to the lower semi-continuity of the set-valued mapping* $X(.,.)$, *we have* $\bar{\alpha}_i, \bar{\mu}_i \in \mathcal{A}_+(X(\bar{y}, \bar{d}))$ *and*

$$x^T Qx + 2c^T x - \bar{\ell} + \sum_{i=1}^n[\bar{\alpha}_i(x)(x - \bar{y}_i - \bar{d}_i) + \bar{\mu}_i(x)(-x + \bar{y}_i - \bar{d}_i)] \in P[x],$$

which implies upper semi-continuity in this case. For the case of the existence of some unbounded sequences, without loss of generality we may assume that $t_k^{-1}\alpha_i^k \to \bar{\alpha}_i$ *and* $t_k^{-1}\mu_i^k \to \bar{\mu}_i$, *where* $t_k = \max_{1\leq i\leq m}\{\|\alpha_i^k\|, \|\mu_i^k\|\}$. *Moreover, there exists* $\bar{\alpha}_i \neq 0$ *and*

$$q(x) = \sum_{i=1}^n[\bar{\alpha}_i(x)(x - \bar{y}_i - \bar{d}_i) + \bar{\mu}_i(x)(-x + \bar{y}_i - \bar{d}_i)] \in P[x].$$

Similar to the proof of Proposition 4, because of $int(X(\bar{y}, \bar{d})) \neq \emptyset$, *there is* $\bar{x}$ *such that* $q(\bar{x}) < 0$, *which contradicts the nonnegativity of* q. *So, the unboundedness case cannot occur, and in this case the upper semi-continuity of* Φ *is derived. Likewise, one can prove it for the case that some components of* $\bar{d}$ *are zero,* Φ *is upper semi-continuous at* $(\bar{y}, \bar{d})$ *on* $X \times \mathbf{R}^n_{\bar{d}}$, *where* $\mathbf{R}^n_{\bar{d}} = \{x \in \mathbf{R}^n_+ : x_i = 0 \ if \ \bar{d}_i = 0\}$. *Suppose that the sequence* $\{(y_k, d_k)\} \subseteq X \times \mathbf{R}^n_+$ *tends to* $(\bar{y}, \bar{d})$. *We decompose the sequence* $\{d_k\}$ *as* $d_k = d_k^1 + d_k^2$ *where* d_k^1 *is the projection of* d_k *on* $\mathbf{R}^n_{\bar{d}}$. *It is seen that* $d_k^1 \to \bar{d}$. *So, we have*

$$\limsup_{k\to\infty} \Phi(y_k, d_k) \leq \limsup_{k\to\infty} \Phi(y_k, d_k^1) \leq \Phi(\bar{y}, \bar{d}).$$

The first inequality results from the inclusion property. Therefore, Φ *is continuous on* $X \times \mathbf{R}^n_+$ *and the proof is complete.*

Theorem 5 *Algorithm 1 is finitely convergent with the stopping accuracy $\epsilon > 0$.*

Proof *Consider the function Φ on the compact set $X \times (B \cap \mathbf{R}^n_+)$, where B stands for the closed unit ball. On account of Lemma 2, we can infer uniform continuity of Φ on the given domain. Additionally, we have the following property*

$$\forall \epsilon > 0,\ \exists \delta > 0, \forall x \in X, \forall d \in \mathbf{R}^n_+;\ \|d\|_\infty < 2\delta \ \Rightarrow |\Phi(x, d) - x^T Qx - 2c^T x| < \epsilon.$$

Since X is compact, the objective function is Lipschitz continuous on it. Without loss of generality, let Lipschitz modulus be one. Let $\Delta \subseteq X$ be a polytope with diameter less than $0.5 \min\{\epsilon, \delta\}$, that is, $\max_{x_1, x_2 \in \Delta} \|x_1 - x_2\| < 0.5 \min\{\epsilon, \delta\}$. As a result, there are $\bar{y} \in X$ and $\bar{d} \in \mathbf{R}^n_+$ such that $\Delta \subseteq X(\bar{y}, \bar{d})$ and $\|\bar{d}\|_\infty < 2\delta$. Due to the inclusion property and the provided results, we have

$$\begin{aligned}
&|opt(\Delta, Q, c) - \min_{x \in \Delta}(x^T Qx + 2c^T x)| \leq |\Phi(\bar{y}, \bar{d}) - \min_{x \in \Delta}(x^T Qx + 2c^T x)| \\
&\leq |\Phi(\bar{y}, \bar{d}) - \bar{y}^T Q\bar{y} - 2c^T \bar{y}| + |\bar{y}^T Q\bar{y} + 2c^T \bar{y} - \min_{x \in \Delta}(x^T Qx + 2c^T x)| \\
&\leq \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Since after finite number of branching, the feasible set of subproblems, Δ, is included in $X(y, d)$ for some $y \in X$ and $d \in \mathbf{R}^n_+$ ($\|d\|_\infty < \min\{\epsilon, \delta\}$), all nodes will be fathomed and algorithm will stop after finite steps.

It is worth noting that for having the finite convergent, one should adopt a branching procedure which guarantees the diameters of generated polytopes tend to zero. However, if the method selects nonzero vector d randomly, with probability of one, the diameters of generated polytopes will converge to zero.

5 Computational results

In this section, we illustrate numerical performance of Algorithm 1 on four groups of test problems. The code and the test problems are publicly available at https://github.com/molsemzamani/quadproga.

We implemented the algorithm using MATLAB 2018b. The computations were run on a Windows PC with Intel Core i7 CPU, 3.4 GHz, and 16GB of RAM. To solve semi-definite program (4), we employed MOSEK [32]. To solve convex QP (21), we employed CPLEX's function cplexqp. In addition, CPLEX was used for solving linear programs.

To evaluate the performance of Algorithm 1, we compared the numerical results with three non-convex quadratic optimization solvers: BARON 18.11.12, Couenne v. 1.0 and CPLEX 12.8 [3, 20, 37]. All solvers were run on MATLAB 2018b and we applied AMPL to pass the problems to BARON and Couenne [16].

In our numerical experiments, we used two stopping criteria, absolute gap

Table 1 Globallib instances

Instance	n	BARON	Couenne	CPLEX	Algorithm 1
st-qpc-m1	5	0.08	0.12	0.04	0.11
st-bsj4	6	0.23	0.09	0.05	0.31
ex2-1-6	10	0.29	0.19	0.05	0.56
st-fp5	10	0.14	0.11	0.07	0.15
st-qpk3	11	0.39	1.19	0.07	0.14
qudlin	12	0.13	0.1	0.01	0.12
ex2-1-7	20	0.76	10.06	0.09	1.38
st-fp7a	20	0.55	0.58	0.08	0.52
st-fp7b	20	0.45	0.89	0.07	0.48
st-fp7d	20	0.31	0.46	0.06	0.18
st-fp7e	20	0.81	10.13	0.13	1.81
st-m1	20	0.26	0.18	0.08	0.24
ex2-1-8	24	0.19	0.08	0.01	0.19
st-m2	30	0.41	0.35	0.18	0.65
st-rv7	30	0.49	0.87	0.12	0.41
st-rv8	40	0.58	0.77	0.13	1.05
st-rv9	50	2.1	3.69	0.43	1.8

tolerance and running time limit, to terminate the solvers. The absolute gap is defined as a difference between the given lower and upper bounds.

For the first group, we selected twenty concave instances from Globallib folder in [12]. This folder contains all non-convex instances of Globallib test problems [17]. The dimension of problems range from five to fifty.

We set the absolute gap tolerance and the maximum running time to 10^{-4} and 100 seconds, respectively. Since all methods could give us global optimum with the prescribed gap, we just report the running time. The performance of all solvers are summarized in Table 1, which n denotes the dimension of instances and the rest columns denote the execution time for solvers.

The second group of examples involves twenty concave QPs with dense data. The test problems were generated as follows. The feasible set, X, was given by the following linear system

$$Ax \leq 10b, \sum_{i=1}^{n} x_i \leq 100, x \geq 0,$$

where square matrix A and and vector b were generated by MATLAB's function randn and rand, respectively. The randn function generates a sample of a Gaussian random variable, with mean 0 and standard deviation 1, while rand generates a uniformly distributed random number between 0 and 1. We also generated the vector c via randn function. We generated the square matrix Q with the formula $Q = -U^T DU$, where U is an orthogonal matrix obtained form the singular value decomposition of some random matrix and D is a diagonal matrix whose components are chosen by rand function. We generated twenty concave QPs in $\mathbf{R}^{40}$ and $\mathbf{R}^{45}$.

We set the absolute gap tolerance and the maximum running time to 10^{-3} and 1000 seconds, respectively. We report the generated lower bound and running time. If for some instance the running time is less than 1000 seconds, the

Table 2 Dense instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	*time*	*lb*	*time*	*lb*	*time*	*lb*	*time*
Ex1-40	-2286.1	-7421.5	1000	-2842.8	1000	-2286.1	36	-2286.1	7
Ex2-40	-3821.4	-13627	1000	-6390.5	1000	-3821.4	323	-3821.4	204
Ex3-40	-2756.6	-10617	1000	-4522.3	1000	-2756.6	100	-2756.6	11
Ex4-40	-2341.6	-5714	1000	-4287	1000	-2341.6	39	-2341.6	9
Ex5-40	-2808.2	-5660	1000	-4195.5	1000	-2808.2	124	-2808.2	11
Ex6-40	-4341.8	-25538	1000	-5319.4	1000	-4341.8	30	-4341.8	4
Ex7-40	-2465.4	-5916.9	1000	-3326.3	1000	-2465.4	91	-2465.4	3
Ex8-40	-2554.6	-6570.7	1000	-5246.1	1000	-2554.6	564	-2554.6	424
Ex9-40	-4599.6	-16653	1000	-5381.7	1000	-4599.6	26	-4599.6	3
Ex10-40	-3446.6	-8798.8	1000	-4835.9	1000	-3446.6	40	-3446.6	4
Ex1-45	-4493.2	-6463.3	1000	-2842.8	1000	-4493.2	59	-4493.2	13
Ex2-45	-2705.9	-13214	1000	-6039.8	1000	-2722.9	1000	-2705.9	94
Ex3-45	-3057.8	-19086	1000	-6271.7	1000	-3057.8	461	-3057.8	196
Ex4-45	-2714.1	-8607.3	1000	-6092.4	1000	-2714.1	689	-2714.1	698
Ex5-45	-3028.2	-13511	1000	-6822.5	1000	-3075.1	1000	-3028.2	888
Ex6-45	-2354.4	-10756	1000	-6215.4	1000	-2549.2	1000	-2354.4	657
Ex7-45	-3391.4	-15958	1000	-6561	1000	-3391.4	197	-3391.4	96
Ex8-45	-1948.2	-6923.2	1000	-3675.8	1000	-1948.2	341	-1948.2	9
Ex9-45	-2710.2	-8781.2	1000	-5014	1000	-2710.2	172	-2710.2	35
Ex10-45	-3099	-8431.3	1000	-6239.7	1000	-3099	193	-3099	91

Table 3 Dense instances

BARON	Couenne	CPLEX	Algorithm 1
2.1	180	50	0

solver succeeded in solving with the prescribed gap. Table 2 reports computational performances of both methods. In this table, $q^\star$ denotes the optimal value and columns *lb* and *time* show the lower bound and the spent CPU time, respectively. To evaluate the quality of the generated upper bound for the case that a solver exceeded the time bound, we measured the difference between the upper bound and the optimal value for all examples. Table 3 reports the maximum of the differences for all examples corresponding to the solvers.

For the third group of the instances, we regarded concave QPs with sparsity. We selected twenty examples form RandQP folder in [12]. In most of the instances, Q were indefinite. We shifted eigenvalues such that Q was transformed to a negative semi-definite matrix. Moreover, we considered the instances without equality constraints. As there were box constraints in all instances, the feasible set was bounded. Table 4 summarizes the computational performances. For this group of test problems, all solvers gave the upper bound equal to the optimal value.

Table 4 Sparse instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	*time*	*lb*	*time*	*lb*	*time*	*lb*	*time*
qp40-20-2-1	-286.31	-286.31	3	-286.31	10	-286.31	1	-286.31	14
qp40-20-2-2	-169.572	-169.572	21	-169.572	23	-169.572	1	-169.572	556
qp40-20-2-3	-152.31	-152.31	91	-152.31	51	-152.31	4	-152.31	500
qp40-20-3-1	-219.664	-219.664	35	-219.665	1000	-219.664	1	-219.664	28
qp40-20-3-2	-171.255	-171.255	35	-171.255	43	-171.255	2	-171.255	501
qp40-20-3-3	-101.248	-101.248	25	-101.248	39	-101.248	2	-101.248	255
qp40-20-3-4	-118.119	-118.119	136	-118.12	1000	-118.119	5	-118.119	53
qp40-20-4-1	-240.464	-240.464	188	-240.464	772	-240.464	6	-240.464	104
qp40-20-4-2	-168.813	-168.813	117	-168.813	90	-168.813	5	-168.813	28
qp40-20-4-3	-93.511	-93.511	662	-93.511	337	-93.511	38	-93.511	561
qp50-25-1-1	-430.892	-430.892	49	-430.892	518	-430.892	3	-430.892	557
qp50-25-1-2	-131.88	-131.88	71	-131.88	189	-131.88	8	-131.88	619
qp50-25-1-3	-137.567	-137.567	167	-137.569	1000	-137.567	7	-137.567	799
qp50-25-1-4	-133.52	-134.151	1000	-133.521	1000	-133.52	10	-133.52	716
qp50-25-2-1	-269.924	-269.924	84	-269.924	136	-269.924	7	-269.924	648
qp50-25-2-2	-204.733	-204.733	654	-204.733	369	-204.733	28	-204.733	612
qp50-25-2-3	-167.34	-167.341	1000	-167.34	934	-167.34	8	-167.34	8
qp50-25-2-4	-129.209	-129.21	1000	-129.209	107	-129.209	7	-129.209	615
qp50-25-3-1	-393.76	-393.76	40	-393.761	1000	-393.76	2	-393.76	40
qp50-25-3-2	-224.266	-224.268	1000	-224.267	1000	-224.266	14	-224.266	23

For last group of instances, we considered the norm maximization problem. This problem can be formulated as concave QP

$$\begin{aligned} &\min\ -x^T x \\ &s.t.\ Ax \leq b, \end{aligned}$$

where $X = \{x : Ax \leq b\}$ is a polytope. Unlike norm maximization problem, the above problem is NP-hard. To evaluate the performance of the solvers, we considered the polytopes which were generated for the second group. Tables 5 and 6 give computational performances.

On average, CPLEX outperformed other solvers in most instances. After CPLEX, Algorithm 1 had better performance compared to BARON and Couenne in most instances. Especially, it had the best performance on the second group of instances, but its performances on the third group of instances was not satisfactory.

Table 5 Max norm instances

Instance	$q^\star$	BARON		Couenne		CPLEX		Algorithm 1	
		lb	*time*	*lb*	*time*	*lb*	*time*	*lb*	*time*
Ex1-40	-1087.4	-1087.4	360	-1087.4	8	-1087.4	45	-1087.4	4
Ex2-40	-1425	-1425	221	-1448.2	1000	-1425	34	-1425	100
Ex3-40	-1514.5	-1514.5	205	-1514.5	618	-1514.5	7	-1514.5	15
Ex4-40	-1324.1	-1324.1	18	-1324.1	126	-1324.1	3	-1324.1	4
Ex5-40	-1206.1	-1206.1	214	1206.1	1000	-1206.1	20	-1206.1	24
Ex6-40	-2104.8	-2104.8	4	-2104.8	288	-2104.8	3	-2104.8	3
Ex7-40	-1150.4	-1150.8	1000	-1204.1	1000	-1150.4	37	-1150.4	36
Ex8-40	-1268.6	-1268.6	305	-1268.6	702	-1268.6	24	-1268.6	112
Ex9-40	-2090.7	-2090.7	10	-2090.7	80	-2090.7	2	-2090.7	12
Ex10-40	-1503.3	-1503.3	62	-1503.3	527	-1503.3	3	-1503.3	3
Ex1-45	-2097.9	-2097.9	12	-2097.9	85	-2097.9	3	-2097.9	40
Ex2-45	-1190.4	-1190.4	578	-1301.6	1000	-1190.4	51	-1190.4	59
Ex3-45	-1636.8	-1636.8	472	-1785.1	1000	-1636.8	19	-1636.8	178
Ex4-45	-1527.3	-1527.3	64	-1527.3	537	-1527.3	7	-1527.3	81
Ex5-45	-1484.8	-1484.8	803	-1557.3	1000	-3.18896	64	-1484.8	531
Ex6-45	-1106.1	-1117.8	1000	-1191.2	1000	-1106.1	65	-1106.1	502
Ex7-45	-1489.6	-1489.6	244	-1519.4	1000	-1489.6	10	-1489.6	37
Ex8-45	-939.6	-1021.7	1000	-985	1000	-939.6	33	-939.6	6
Ex9-45	-1235.2	-1235.2	243	-1277.7	1000	-1235.2	17	-1235.2	47
Ex10-45	-1557.8	-1557.8	102	-1557.8	870	-1557.8	10	-1557.8	38

Table 6 Max norm instances

BARON	Couenne	CPLEX	Algorithm 1
0	0.9	0	0

References

1. Anstreicher, K.M.: Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. Journal of Global Optimization **43**(2-3), 471–484 (2009)
2. Bao, X., Sahinidis, N.V., Tawarmalani, M.: Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons. Mathematical programming **129**(1), 129 (2011)
3. Belotti, P.: Couenne: a users manual. Tech. rep., Lehigh University (2009)
4. Ben-Tal, A., Nemirovski, A.: Lectures on modern convex optimization: analysis, algorithms, and engineering applications, vol. 2. Siam (2001)
5. Bomze, I.M.: Copositive relaxation beats lagrangian dual bounds in quadratically and linearly constrained quadratic optimization problems. SIAM Journal on Optimization **25**(3), 1249–1275 (2015)
6. Bomze, I.M., Locatelli, M., Tardella, F.: New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability. Mathematical Programming **115**(1), 31 (2008)
7. Boyd, S., Vandenberghe, L.: Convex optimization. Cambridge university press (2004)
8. Burer, S.: On the copositive representation of binary and continuous nonconvex quadratic programs. Mathematical Programming **120**(2), 479–495 (2009)
9. Burer, S., Dong, H.: Separation and relaxation for cones of quadratic forms. Mathematical Programming **137**(1-2), 343–370 (2013)
10. Burer, S., Vandenbussche, D.: A finite branch-and-bound algorithm for nonconvex quadratic programming via semidefinite relaxations. Mathematical Programming **113**(2), 259–282 (2008)
11. Burkard, R.E., Cela, E., Pardalos, P.M., Pitsoulis, L.S.: The quadratic assignment problem. In: Handbook of combinatorial optimization, pp. 1713–1809. Springer (1998)
12. Chen, J., Burer, S.: Globally solving nonconvex quadratic programming problems via completely positive programming. Mathematical Programming Computation **4**(1), 33–52 (2012)
13. Chuong, T., Jeyakumar, V.: Generalized lagrangian duality for nonconvex polynomial programs with polynomial multipliers. Journal of Global Optimization pp. 1–24 (2018)
14. Dür, M., Horst, R.: Lagrange duality and partitioning techniques in nonconvex global optimization. Journal of Optimization Theory and Applications **95**(2), 347–369 (1997)
15. Floudas, C.A., Visweswaran, V.: Quadratic optimization. In: Handbook of global optimization, pp. 217–269. Springer (1995)
16. Fourer, R., Gay, D., Kernighan, B.: Ampl (vol. 117). Danvers, MA: Boyd & Fraser (1993)
17. Globallib: Gamsworld. http://www.gamsworld.org/global/globallib.htm (2013)
18. Gorge, A., Lisser, A., Zorgati, R.: Generating cutting planes for the semidefinite relaxation of quadratic programs. Computers & Operations Research **55**, 65–75 (2015)
19. Horst, R., Tuy, H.: Global optimization: Deterministic approaches. Springer Science & Business Media (1996)
20. ILOG, I.: Cplex 9.0 reference manual. ILOG CPLEX Division (2003)
21. Jiang, B., Li, Z., Zhang, S.: On cones of nonnegative quartic forms. Foundations of Computational Mathematics **17**(1), 161–197 (2017)
22. Kim, S., Kojima, M., Waki, H.: Generalized lagrangian duals and sums of squares relaxations of sparse polynomial optimization problems. SIAM Journal on Optimization **15**(3), 697–719 (2005)
23. Konno, H.: Maximization of a convex quadratic function under linear constraints. Mathematical programming **11**(1), 117–127 (1976)
24. Konno, H., Thach, P.T., Tuy, H.: Optimization on low rank nonconvex structures, vol. 15. Springer Science & Business Media (2013)
25. Lasserre, J.B.: An introduction to polynomial and semi-algebraic optimization, vol. 52. Cambridge University Press (2015)
26. Lasserre, J.B., Toh, K.C., Yang, S.: A bounded degree sos hierarchy for polynomial optimization. EURO Journal on Computational Optimization **5**(1-2), 87–117 (2017)

27. Laurent, M., Sun, Z.: Handelmans hierarchy for the maximum stable set problem. Journal of Global Optimization **60**(3), 393–423 (2014)
28. Lovász, L., Schrijver, A.: Cones of matrices and set-functions and 0–1 optimization. SIAM journal on optimization **1**(2), 166–190 (1991)
29. Luenberger, D.G.: A double look at duality. IEEE transactions on automatic control **37**(10), 1474–1482 (1992)
30. Luo, Z.Q., Ma, W.K., So, A.M.C., Ye, Y., Zhang, S.: Semidefinite relaxation of quadratic optimization problems. IEEE Signal Processing Magazine **27**(3), 20–34 (2010)
31. Mangasarian, O.L.: Nonlinear Programming. Siam (1974)
32. MOSEK, A.: The mosek optimization toolbox for matlab manual. version 8.1. http://docs.mosek.com/8.1/toolbox/index.html (2017)
33. Nesterov, Y., Wolkowicz, H., Ye, Y.: Semidefinite programming relaxations of nonconvex quadratic optimization. In: Handbook of semidefinite programming, pp. 361–419. Springer (2000)
34. Pardalos, P.M., Vavasis, S.A.: Quadratic programming with one negative eigenvalue is np-hard. Journal of Global Optimization **1**(1), 15–22 (1991)
35. Pólik, I., Terlaky, T.: A survey of the s-lemma. SIAM review **49**(3), 371–418 (2007)
36. Renegar, J.: A mathematical view of interior-point methods in convex optimization, vol. 3. Siam (2001)
37. Sahinidis, N.V., Tawarmalani, M.: Baron 18.11.12: Global optimization of mixed-integer nonlinear programs. Users manual (2017)
38. Sahni, S.: Computationally related problems. SIAM Journal on Computing **3**(4), 262–279 (1974)
39. Sherali, H.D., Tuncbilek, C.H.: A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique. Journal of Global Optimization **2**(1), 101–112 (1992)
40. Sherali, H.D., Tuncbilek, C.H.: A reformulation-convexification approach for solving nonconvex quadratic programming problems. Journal of Global Optimization **7**(1), 1–31 (1995)
41. Shor, N.Z.: Dual quadratic estimates in polynomial and boolean programming. Annals of Operations Research **25**(1), 163–168 (1990)
42. Sturm, J.F., Zhang, S.: On cones of nonnegative quadratic functions. Mathematics of Operations Research **28**(2), 246–267 (2003)
43. Tuy, H.: Convex Analysis and Global Optimization (Springer Optimization and Its Applications Book 110). Springer (2016)
44. Xia, W., Vera, J., Zuluaga, L.F.: Globally solving non-convex quadratic programs via linear integer programming techniques. arXiv preprint arXiv:1511.02423 (2015)
45. Zheng, X., Sun, X., Li, D., Xu, Y.: On zero duality gap in nonconvex quadratic programming problems. Journal of Global Optimization **52**(2), 229–242 (2012)