On the NP-hardness of deciding emptiness of the split closure of a rational polytope in the 0,1 hypercube

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Abstract

Split cuts are prominent general-purpose cutting planes in integer programming. The split closure of a rational polyhedron is what is obtained after intersecting the half-spaces defined by all the split cuts for the polyhedron. In this paper, we prove that deciding whether the split closure of a rational polytope is empty is NP-hard, even when the polytope is contained in the unit hypercube. As a direct corollary, we prove that optimization and separation over the split closure of a rational polytope in the unit hypercube are NP-hard, extending an earlier result of Caprara and Letchford.

Keywords: Integer linear programming; Cutting Planes; Split cuts; Split closure; Separation; NP-hardness

1 Introduction

Consider a mixed integer linear program (MILP) defined over a rational polyhedron

$$P = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p : \; Ax + Gy \leq b\},$$

where we denote by x and y the vectors of n integer and p continuous variables, respectively. The objective is to optimize a linear function over $P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Let P_I denote the integer hull of P, namely $P_I := \mathrm{conv}(P \cap (\mathbb{Z}^n \times \mathbb{R}^p))$, the convex hull of the points in $P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Starting with Chvátal-Gomroy cuts proposed by Chvátal [4] and Gomory [12], general-purpose cutting-planes were developed for solving integer programming problems. In particular, Cook, Kannan, and Schrijver [5] proposed the *split cuts or split inequalities*. These cuts are a special case of Balas' disjunctive cuts [2] which can be obtained from a *split disjunction*. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, any point (x,y) in $\mathbb{Z}^n \times \mathbb{R}^p$ satisfies either $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$. An inequality is a split cut if it is valid for both

$$\Pi_1 := P \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p : \; \pi x \leq \pi_0\} \text{ and}$$
$$\Pi_2 := P \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p : \; \pi x \geq \pi_0 + 1\}$$

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for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. We call the set $S(\pi, \pi_0) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : \ \pi x \le \pi_0 \text{ or } \pi x \ge \pi_0 + 1\}$ the *split* or the *split disjunction* derived from $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Clearly, $P_I \subseteq \text{conv}(P \cap S(\pi, \pi_0)) \subseteq P$ and an inequality is a split cut if and only if it is valid for $\text{conv}(P \cap S(\pi, \pi_0))$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. It is straightforward that $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : \ \pi_0 \le \pi x \le \pi_0 + 1\}$ does not contain any integer point in its interior, so split cuts are a type of intersection cuts introduced by Balas [1]. It is also known that Gomory's mixed integer cuts [13] are split cuts.

Cook, Kannan, and Schrijver [5] introduced a notion of closure as follows.

$$P' := \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} \text{conv}\left(P \cap S(\pi, \pi_0)\right)$$

is the *split closure* of P. By its definition, $P_I \subseteq P' \subseteq P$. Cook, Kannan, and Schrijver [5] proved that the split closure of a rational polyhedron is, again, a rational polyhedron, meaning that it can be described by finitely many split inequalities. Although the split closure of a rational polyhedron is finitely generated, it is known that both the optimization and separation problems over the split closure of a rational polyhedron are NP-hard, due to Caprara and Letchford [3]. In addition, Mahajan and Ralphs [15] showed that it is NP-complete to decide whether there exists a split $S(\pi, \pi_0)$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $P \cap S(\pi, \pi_0)$ is empty, which implies that selecting an optimal split in terms of the gap closed is NP-hard. In this paper, we prove the following hardness result:

Theorem 1.1. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polyhedron. It is NP-complete to decide whether the split closure of P is empty, even when P is contained in the unit hypercube* $[0, 1]^n$.

The proof of Theorem 1.1 is given in Section 3. In Section 4, we will argue that our reduction for proving this NP-hardness result extends the result of Caprara and Letchford [3]. The reduction also generalizes the result of Mahajan and Ralphs [15] to an arbitrary number of split disjunctions. Section 4 contains more precise statements.

2 Related work

As we mentioned earlier, Mahajan and Ralphs [15] considered the problem of deciding whether there exists a single split disjunction that can certify that the split closure of a rational polytope is empty, and they proved that the problem is NP-complete. Cornuéjols and Li [7, 8] in their recent papers considered the problem of deciding whether the Chvátal-Gomory closure of a rational polytope is empty. Their technique to show that the problem is NP-complete is similar to Mahajan and Ralphs [15]'s approach. More recently, Cornuéjols, Lee, and Li [6] improved this result, by proving that the problem remains NP-complete even when the input polytope is contained in the unit hypercube. Notice that the Chvátal-Gomory closure of a rational polyhedron $P \subseteq \mathbb{R}^n$ is obtained by applying a special type of split disjunctions $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that either $P \cap \{x : \pi x \le \pi_0\} = \emptyset$ or $P \cap \{x : \pi x \ge \pi_0 + 1\} = \emptyset$.

All the above hardness results were obtained by providing polynomial reductions from either the Partition Problem or the Equality Knapsack Problem (see [11]):

Partition Problem. Given n positive integer weights $a_1, \cdots, a_n$, either find a set of binary integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = \frac{1}{2}\sum_{i=1}^n a_i$ or show that none exists.

Equality Knapsack Problem. Given n positive integer weights $a_1, \ldots, a_n$ and a capacity b, either find a set of nonnegative integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = b$ or show that none exists.

The reductions are basically as follows. Given n positive weights $a_1, \ldots, a_n$ and a positive capacity b for either a partition problem instance ($b = \frac{1}{2}\sum_{i=1}^n a_i$ in this case) or an equality knapsack instance, one can construct a rational polytope as the convex hull of $n + c_1$ points in $\mathbb{R}^{n+c_2}$, where c_1 and c_2 are fixed constants, so that its linear description can be computed in polynomial time.

One might wonder whether there is a similar construction to prove Theorem 1.1 that is about the split closure. Given an equality knapsack instance with n weights, we construct a rational polytope in $[0, 1]^{n+4}$. Although our construction includes $\Omega(2^n)$ extreme points, we can still find its linear description in polynomial time. We provide our construction in the next section.

3 Reduction from Equality Knapsack

In this section, we show two lemmas to prove Theorem 1.1.

Lemma 3.1. *The problem of deciding whether the split closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ given by its linear description is empty is in complexity class NP.*

Proof. Theorem 13 in [9] by Dash, Günlük, and Lodi implies that the split closure of P can be described by finitely many split inequalities whose encoding sizes are polynomially bounded by the encoding size of P. When the split closure is empty, then the intersection of the half-spaces defined by finitely many split inequalities is empty. Then by Helly's theorem, for some $k \leq n + 1$, there are k split inequalities of polynomially bounded encoding size that certify that the split closure of P is empty. Therefore, we have a polynomial size NP certificate for the problem. □

Now that we know the problem is in NP, what remains is to show that the problem is NP-hard, even when the input polytope is contained in the unit hypercube.

Lemma 3.2. *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational polytope $P \subseteq [0, 1]^{n+4}$ contained in the unit hypercube that satisfies the following:*

(a) $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ *is contained in P, but P contains no integer point.*

(b) There exists a solution to the equality knapsack instance if and only if there exists a split cut for P that separates $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

(c) There exists a solution to the equality knapsack instance if and only if the split closure of P is empty and there is a single split disjunction to certify this.

Proof. We may assume that b is sufficiently large so that $b > n+2$, while the knapsack problem still remains NP-hard. We may also assume that $0 < a_1, \dots, a_n < b$. Consider the following $n+6$ points $v^1, \dots, v^{n+6}$ in $[0,1]^{n+4}$.

$$
\begin{array}{rlllllllllll}
v^1 := (& \frac{a_1}{4b}, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} &) \\
v^2 := (& 0, & \frac{a_2}{4b}, & \cdots, & 0, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} &) \\
 & & & & \vdots & & & & & & \\
v^{n-1} := (& 0, & 0, & \cdots, & \frac{a_{n-1}}{4b}, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} &) \\
v^n := (& 0, & 0, & \cdots, & 0, & \frac{a_n}{4b}, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} &) \\
v^{n+1} := (& \frac{a_1}{4b}, & \frac{a_2}{4b}, & \cdots, & \frac{a_{n-1}}{4b}, & \frac{a_n}{4b}, & 0, & 0, & \frac{1}{4} - \frac{1}{8b}, & 0 &) \\
v^{n+2} := (& 1 - \frac{a_1}{2b}, & 1 - \frac{a_2}{2b}, & \cdots, & 1 - \frac{a_{n-1}}{2b}, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{4} + \frac{1}{8b}, & 0, & \frac{1}{4} + \frac{1}{8b} &) \\
v^{n+3} := (& 0, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{2} + \frac{1}{8b}, & 0, & 0 &) \\
v^{n+4} := (& 0, & 0, & \cdots, & 0, & 0, & 0, & 0, & \frac{1}{2}, & 0 &) \\
v^{n+5} := (& 0, & 0, & \cdots, & 0, & 0, & 0, & 0, & 0, & \frac{1}{2} + \frac{1}{8b} &) \\
v^{n+6} := (& 0, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{4} - \frac{n+2}{8b}, & \frac{1}{4} + \frac{1}{8b}, & \frac{1}{4} - \frac{n+2}{8b} &)
\end{array}
$$

Let P be a rational polytope defined as follows:

$$
P := \left\{ x = \sum_{i=1}^{n+6} v^i y_i : \begin{array}{rcccl} \frac{4b}{4b+1} & \leq & \sum_{i=1}^{n+6} y_i & \leq & n+6-\frac{4b}{4b+1} \\ y_{n+3}+y_{n+5}-1 & \leq & y_{n+4} & \leq & y_{n+3}+y_{n+5} \\ 0 & \leq & y_i & \leq & 1, \quad \forall i \in [n] \end{array} \right\}
$$

Claim 1. *The linear description of P that involves only x variables can be obtained in polynomial time.*

Proof of Claim. We can rewrite P as $P = \{x \in \mathbb{R}^{n+4} : x = Vy,\ Ay \leq b\}$ where V is the matrix whose columns are $v^1, \dots, v^{n+6}$ and $Ay \leq b$ is the system of the other constraints in P. Notice that $v^1, \dots, v^n$, v^{n+2}, v^{n+3}, v^{n+4}, and v^{n+5} are linearly independent, and let B denote the column submatrix of V that consists of these vectors. Let N denote the column submatrix of the remaining columns. Then $x = Vy$ is equivalent to $y_B = B^{-1}x - B^{-1}Ny_N$, where y_B and y_N consist of the components of y that correspond to B and N, respectively. Let A be decomposed into its two column submatrices C and D so that $Ay = Cy_B + Dy_N$. Then, P can be written as $P = \{x \in \mathbb{R}^{n+4} : CB^{-1}x + (D - CB^{-1}N)y_N \leq b\}$. y_N consists of only two variables y_{n+1} and y_{n+6}, so projecting away y_N from P can be done in polynomial time by the Fourier-Motzkin elimination method. Therefore, we can find a linear system describing P that involves x variables only in polynomial time. $\Diamond$

To complete the proof, we show that P satisfies properties (a), (b), and (c). Let u denote $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. To show that (a) is satisfied, we need the following two claims.

Claim 2. *$u \in P$ and P is centrally symmetric with respect to u.*

Proof of Claim. Notice that $\sum_{i=1}^{n+6} v^i = (1, \dots, 1)$. Then $u = \sum_{i=1}^{n+6} \frac{1}{2} v^i \in P$, because $y_i = \frac{1}{2}$ for $i \in [n+6]$ satisfy the constraints. In addition, given $x = \sum_{i=1}^{n+6} v^i y_i$, observe that $2u - x = \sum_{i=1}^{n+6} v^i (1 - y_i)$ as $2u =$

$\sum_{i=1}^{n+6} v^i$. Therefore, $x \in P$ if and only if $2u - x \in P$, so P is centrally symmetric with respect to u, as required. $\diamondsuit$

Claim 3. $P \subseteq [0,1]^{n+4}$ *and* $P \cap \{0,1\}^{n+4} = \emptyset$.

Proof of Claim. For $x = \sum_{i=1}^{n+6} v^i y_i \in P$, we know that $0 \le \sum_{i=1}^{n+6} v^i y_i \le \sum_{i=1}^{n+6} v^i = (1, \dots, 1)$, because $v^1, \dots, v^{n+6} \ge 0$. That means P is contained in $[0,1]^{n+4}$. Let $z = \sum_{i=1}^{n+6} v^i y_i \in P$. We would like to show that $z \notin \{0,1\}^n$. Suppose otherwise. If $z_j = 1$ for some $1 \le j \le n$, then it must be the case that $y_j = y_{n+1} = y_{n+2} = 1$ because $z_j = \frac{a_j}{4b} y_j + \frac{a_j}{4b} y_{n+1} + \frac{2b - a_j}{2b} y_{n+2} \le 1$ and the equality holds only if $y_j = y_{n+1} = y_{n+2} = 1$. In fact, $y_{n+1} = y_{n+2} = 1$ implies that $z_j > 0$ for each $j \in [n+4]$ and thus $z = (1, \dots, 1)$ and $y_i = 1$ for each $i \in [n+6]$. However, this violates constraint $\sum_{i=1}^{n+6} y_i < n + 6$, a contradiction. Thus, $z_j = 0$ for all $1 \le j \le n$. This implies $y_i = 0$ for $1 \le i \le n+2$, so $z = (0, \dots, 0)$ is the only possibility. However, we observed that $(1, \dots, 1) \notin P$, so $(0, \dots, 0) \notin P$ by Claim 2. This contradicts the assumption that $z \in P$. Therefore, we get that $P \cap \{0,1\}^{n+4} = \emptyset$, as required. $\diamondsuit$

By Claim 2 and Claim 3, we know that P satisfies (a). To prove that P also satisfies (b) and (c), we show the following two claims:

Claim 4. *If there exists a solution to the equality knapsack instance, then the split closure of P is empty and there is a single split disjunction to certify this.*

Proof of Claim. Let $(d_1, \dots, d_n)$ be a solution to the equality knapsack instance, so $\sum_{i=1}^{n} a_i d_i = b$ and $d_i \ge 0$ for $i \in [n]$. Let $\pi := (d_1, \dots, d_n, -\sum_{i=1}^{n} d_i, 1, -1, 1) \in \mathbb{Z}^{n+4}$. Observe that

$$\pi v^i = \frac{a_i d_i}{4b} + \frac{1}{4b} \quad i = 1, \dots, n, \quad \pi v^{n+1} = \frac{1}{8b}, \quad \pi v^{n+2} = \frac{1}{4b},$$

$$\pi v^{n+3} = \frac{1}{2} + \frac{1}{8b}, \quad \pi v^{n+4} = -\frac{1}{2}, \quad \pi v^{n+5} = \frac{1}{2} + \frac{1}{8b}, \quad \pi v^{n+6} = \frac{1}{4} - \frac{n}{4b} - \frac{5}{8b}.$$

Let $x \in P$. Then $x = \sum_{i=1}^{n+6} v^i y_i$ for some y satisfying the constraints for P. Notice that $\sum_{i=n+3}^{n+5} y_i \pi v^i = \frac{1}{8b}(y_{n+3} + y_{n+5}) + \frac{1}{2}(y_{n+3} - y_{n+4} + y_{n+5})$. Then we have

$$0 \le \sum_{i=n+3}^{n+5} y_i \pi v^i \le \frac{1}{4b} + \frac{1}{2} \tag{1}$$

where the first equality holds only if $y_{n+3} = y_{n+4} = y_{n+5} = 0$ and the second equality holds only if $y_{n+3} = y_{n+4} = y_{n+5} = 1$. Now, consider $y_{n+6} \pi v^{n+6} + \sum_{i=1}^{n+2} y_i \pi v^i$. Clearly, $\pi v^i \ge 0$ for $1 \le i \le n+2$ and $\pi v^{n+6} \ge 0$ as we assumed that $b \ge n+3$. This implies

$$0 \le y_{n+6} \pi v^{n+6} + \sum_{i=1}^{n+2} y_i \pi v^i \le \pi v^{n+6} + \sum_{i=1}^{n+2} \pi v^i = \frac{1}{2} - \frac{1}{4b} \tag{2}$$

where the first equality holds only when $y_1 = \dots = y_{n+2} = y_{n+6} = 0$ and the second equality holds only when $y_1 = \dots = y_{n+2} = y_{n+6} = 1$. From (1) and (2), we get that $0 \le \pi x \le 1$ where $\pi x = 0$ only if $y_i = 0$ for all

$i \in [n+6]$ and $\pi x = 1$ only if $y_i = 1$ for all $i \in [n+6]$. As $0 < \sum_{i=1}^{n+6} y_i < n+6$, we know that πx can be neither 0 nor 1. That means $P \subseteq \{x : 0 < \pi x < 1\}$. Therefore, $P \cap S(\pi, 0) = \emptyset$ and thus the split closure of P is empty, as required. $\Diamond$

Claim 4 proves one direction of each of (b) and (c). The other direction of each can be shown by the following claim.

Claim 5. *If there exists a split cut separating* $u = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$*, then there exists a solution to the equality knapsack instance.*

Proof of Claim. Since there is a split cut that separates u, there exist $\pi \in \mathbb{Z}^{n+4}$ and $\pi_0 \in \mathbb{Z}$ such that $u \notin \operatorname{conv}(P \cap S(\pi, \pi_0))$, Then $\pi_0 < \pi u < \pi_0 + 1$. As $S(-\pi, -\pi_0 - 1)$ is identical to $S(\pi, \pi_0)$, we may assume that $\pi_0 \geq 0$ without loss of generality. We will show that π and π_0 satisfy the following five properties.

(1) $\pi_{n+1} = -\sum_{i=1}^{n} \pi_i$.

(2) $\pi_{n+2} = \pi_{n+4} = 1$ and $\pi_{n+3} = -1$.

(3) $\pi_0 = 0$.

(4) $\sum_{i=1}^{n} a_i \pi_i = b$.

(5) $\pi_i \geq 0$ for $i = 1, \ldots, n$.

(1) – (5) imply that $(\pi_1, \ldots, \pi_n)$ is a solution to the equality knapsack instance. Since $\sum_{i=1}^{n+4} \pi_i$ is an integer and $\pi u = \frac{1}{2} \sum_{i=1}^{n+4} \pi_i$ is strictly between two consecutive integers π_0 and $\pi_0 + 1$, we get $\pi u = \pi_0 + \frac{1}{2}$. Let $x \in P$. Then $2u - x \in P$ by Claim 2. If $x, 2u - x \in S(\pi, \pi_0)$, then $u = \frac{1}{2}x + \frac{1}{2}(2u - x) \in \operatorname{conv}(P \cap S(\pi, \pi_0))$, a contradiction. Hence, for every $x \in P$, either $\pi_0 < \pi x < \pi_0 + 1$ or $\pi_0 < \pi(2u - x) < \pi_0 + 1$ holds.

(1): Consider $w^1 := \left(0, \ldots, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{4b}{4b+1} v^{n+3} + v^{n+4} + \frac{4b}{4b+1} v^{n+5} \in P$. Then $\pi w^1 = \pi u - \frac{1}{2} \sum_{i=1}^{n+1} \pi_i$ and $\pi\left(2u - w^1\right) = \pi u + \frac{1}{2} \sum_{i=1}^{n+1} \pi_i$. We know that $\pi u = \pi_0 + \frac{1}{2}$ and that either $\pi_0 < \pi w^1 < \pi_0 + 1$ or $\pi_0 < \pi\left(2u - w^1\right) < \pi_0 + 1$ holds, and we get $-1 < \sum_{i=1}^{n+1} \pi_i < 1$ in each case. Since $\sum_{i=1}^{n+1} \pi_i$ is an integer strictly between -1 and 1, it is equal to 0. Hence, (1) is satisfied.

(2) & (3): By (1), we obtain $\frac{1}{2} \sum_{i=n+2}^{n+4} \pi_i = \pi u$. Consider $w^2 := \left(0, \ldots, 0, 0, \frac{1}{2}, 0, 0\right) = \frac{4b}{4b+1} v^{n+3} \in P$. By symmetry, $2u - w^2 = \left(1, \ldots, 1, 1, \frac{1}{2}, 1, 1\right) \in P$. Notice that $\pi w^2 = \pi u - \frac{1}{2}(\pi_{n+3} + \pi_{n+4})$ and $\pi\left(2u - w^2\right) = \pi u + \frac{1}{2}(\pi_{n+3} + \pi_{n+4})$. As we argued before, we get $\pi_{n+3} + \pi_{n+4} = 0$. By considering $w^3 := \left(0, \ldots, 0, 0, 0, 0, \frac{1}{2}\right) = \frac{4b}{4b+1} v^{n+5} \in P$, we can similarly argue that $\pi_{n+2} + \pi_{n+3} = 0$. Next, consider $w^4 := \left(0, \ldots, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{2} w^1 \in P$. Then, $\pi w^4 = \pi u - \frac{1}{4} \sum_{i=n+2}^{n+4} \pi_i$ and $\pi\left(2u - w^4\right) = \pi u + \frac{1}{4} \sum_{i=n+2}^{n+4} \pi_i$. Since we know that $\pi u = \pi_0 + \frac{1}{2}$ and that either $\pi_0 < \pi w^4 < \pi_0 + 1$ or $\pi_0 < \pi\left(2u - w^4\right) < \pi_0 + 1$ holds, we obtain $-1 \leq \sum_{i=n+2}^{n+4} \pi_i \leq 1$. We observed that $\pi u = \frac{1}{2} \sum_{i=n+2}^{n+4} \pi_i = \pi_0 + \frac{1}{2}$ and assumed earlier that $\pi_0 \geq 0$, so we get $\sum_{i=n+2}^{n+4} \pi_i \geq 1$. Then $\sum_{i=n+2}^{n+4} \pi_i = 1$ and this means $\pi_{n+2} = \pi_{n+4} = 1$ and $\pi_{n+3} = -1$, because we already have $\pi_{n+2} + \pi_{n+3} = \pi_{n+3} + \pi_{n+4} = 0$. As a result, $\pi_0 = \pi u - \frac{1}{2} = 0$. Therefore, (2) and (3) are satisfied.

(4): By (3) and $\pi u = \pi_0 + \frac{1}{2}$, we have $\pi u = \frac{1}{2}$. We first consider $v^{n+1} \in P$. We have that $\pi v^{n+1} = -(\frac{1}{4} - \frac{1}{8b}) + \frac{1}{4b}\sum_{i=1}^{n} a_i\pi_i$. As $\pi_0 = 0$, either $0 < \pi v^{n+1} < 1$ or $0 < \pi\left(2u - v^{n+1}\right) < 1$ should hold. Since $\pi\left(2u - v^{n+1}\right) = 1 - \pi v^{n+1}$, we in fact have $0 < \pi v^{n+1} < 1$. In particular, $\pi v^{n+1} > 0$ implies that $\sum_{i=1}^{n} a_i\pi_i > b - \frac{1}{2}$ and thus we obtain $\sum_{i=1}^{n} a_i\pi_i \geq b$. Next, consider $v^{n+2} \in P$. Notice that $\pi v^{n+2} = (\frac{1}{2} + \frac{1}{4b}) - \frac{1}{2b}\sum_{i=1}^{n} a_i\pi_i$ and $\pi\left(2u - v^{n+2}\right) = 1 - \pi v^{n+2}$. Similarly, we get $\pi v^{n+2} > 0$, and this implies $\sum_{i=1}^{n} a_i\pi_i < b + \frac{1}{2}$. Since $\sum_{i=1}^{n} a_i\pi_i$ is an integer, it is indeed at most b. Consequently, $\sum_{i=1}^{n} a_i\pi_i = b$, as required.

(5): Let $i \in [n]$. To show that $\pi_i \geq 0$, we consider $v^i \in P$. Notice that $\pi v^i = \frac{1}{4b}a_i\pi_i + \frac{1}{4b}$ and $\pi\left(2u - v^i\right) = 1 - \pi v^i$. As we know that either $0 < \pi v^i < 1$ or $0 < \pi\left(2u - v^i\right) < 1$, we get $0 < \pi v^i < 1$. Then, $\pi v^i > 0$ implies that $a_i\pi_i > -1$. Since $a_i\pi_i$ is an integer, $a_i\pi_i \geq 0$ and thus $\pi_i \geq 0$, as required. ♢

Claim 4 and Claim 5 finally prove that P satisfies (b) and (c), as required. □

4 Implications

In this section, we note some consequences of Theorem 1.1 and Lemma 3.2. The separation problem over the split closure of a rational polyhedron is defined as follows.

Separation Problem. Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and a rational vector $\bar{x} \in \mathbb{Q}^n$, either show that $\bar{x}$ is contained in the split closure of P or a split cut that is violated by $\bar{x}$.

Theorem 4.1 (Separation). *The separation problem over the split closure of a rational polyhedron is NP-hard, even when P is contained in the unit hypercube.*

Proof. Lemma 3.2 implies that, given an equality knapsack instance of $n - 4$ positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time construct a rational polytope $P \subseteq [0,1]^n$ such that there exists a split cut separating $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ from P if and only if the equality knapsack instance has a solution. Therefore, the separation problem over the split closure of a rational polytope in the unit hypercube is NP-hard. □

We remark that Theorem 1.1 also trivially implies Theorem 4.1, as the separation problem over the split closure considers a rational polytope whose split closure is empty as a special case. Furthermore, due to Grötschel, Lovász, and Schrijver [14]'s theorem on the equivalence between optimization and separation, we also get the hardness result for the optimization problem over the split closure.

Corollary 4.2 (Optimization). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron and $c \in \mathbb{Q}^n$ be a rational vector. Optimizing linear function cx over the split closure of P is NP-hard, even when P is contained in the unit hypercube $[0,1]^n$.*

Mahajan and Ralphs [15] proved that selecting a split disjunction certifying that a rational polytope has empty split closure is NP-hard. Lemma 3.2, in particular, part (c) generalizes this result.

Theorem 4.3. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polytope and k be an any arbitrary integer. It is NP-hard to decide whether there exist k split disjunctions $S(\pi^i, \pi_0^i)$ where $(\pi^i, \pi_0^i) \in \mathbb{Z}^n \times \mathbb{Z}$ for $i = 1, \ldots, k$ such that $\bigcap_{i=1}^k \operatorname{conv}\left(P \cap S(\pi^i, \pi_0^i)\right) = \emptyset$.*

When P contains no integer point, deciding emptiness of the split closure of P is the same as checking whether the split closure of P coincides with its integer hull and is the same as checking whether the split rank of P is 1. As a result, we obtain another direct corollary of Theorem 1.1.

Theorem 4.4. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polyhedron. It is NP-hard to decide whether the split rank of P is exactly 1, even when P is contained in the unit hypercube $[0,1]^n$ and P contains no integer point.*

Corollary 4.2 indicates that it is difficult to optimize over the split closure of a rational polyhedron. On the other hand, when we assume that the split closure of a rational polyhedron is identical to its integer hull, optimizing over the split closure seems to become easier. In fact, we can show that

Proposition 4.5. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational poltyope whose split rank is exactly 1 and $c \in \mathbb{Q}^n$. The problem of optimizing linear function cx over $P \cap \mathbb{Z}^n$ is in NP $\cap$ co-NP.*

One might wonder whether there is a polynomial time algorithm to solve integer programming over a rational polytope that has split rank 1. The same question for the Chvátal rank was studied in [6]. The matching problem [10] is an example where there exists a polynomial time algorithm. However, as Theorem 4.4 suggests, it seems hard to use the split rank 1 condition when trying to find an efficient algorithm.

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