On Robust Fractional 0-1 Programming

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Abstract. We study single- and multiple-ratio robust fractional 0-1 programming problems (RFPs). In particular, this work considers RFPs under a wide-range of disjoint and joint uncertainty sets, where the former implies separate uncertainty sets for each numerator and denominator, and the latter accounts for different forms of inter-relatedness between them. First, we demonstrate that, unlike the deterministic case, single-ratio RFP is NP-hard under general polyhedral uncertainty sets. However, if the uncertainty sets are imbued with a certain structure - variants of the well-known budgeted uncertainty - the disjoint and joint single-ratio RFPs are polynomially-solvable when the deterministic counterpart is. We also propose mixed-integer linear programming (MILP) formulations for multiple-ratio RFPs. We conduct extensive computational experiments using test instances based on real and synthetic datasets to evaluate the performance of our MILP reformulations, as well as to compare the disjoint and joint uncertainty sets. Finally, we demonstrate the value of the robust approach by examining the robust solution in a deterministic setting and vice versa.

Keywords. Fractional 0-1 programming, Robust optimization, Nonlinear integer programming, Robust assortment optimization

1 Introduction

Fractional objective functions arise naturally in a broad-range of problems that involve optimization of efficiency measures, averages, probabilities and percentages, among others. A comprehensive overview of continuous fractional programming papers is given by Schaible (1982) and Stancu-Minasian (2017). In this paper, we focus on fractional 0-1 programming (also referred to as hyperbolic 0-1 programming). Formally, the deterministic fractional 0-1 program is defined as

$$\max_{x\in X}\sum_{i\in I}\frac{a_{i0}+\sum\limits_{j\in J}a_{ij}x_j}{b_{i0}+\sum\limits_{j\in J}b_{ij}x_j}, \tag{FP}$$

where $I=\{1,...,m\},\ J=\{1,...,n\}$ and $X\subseteq\mathbb{B}^n$ for $\mathbb{B}:=\{0,1\}$. If $m=1$, then the problem is referred to as single-ratio, else it is multiple-ratio.

Many diverse applications can be readily formulated as FP. Such applications encompass scheduling (Saipe 1975), retail assortment (Subramanian and Sherali 2010, Méndez-Díaz et al. 2014, Davis et al. 2014), set covering (Amaldi et al. 2011, 2012), facility location (Tawarmalani et al. 2002), stochastic service systems (Elhedhli 2005, Han et al. 2013), biclustering (Busygin et al. 2005, Trapp et al. 2010), finding alternative solutions to binary linear programs (Trapp and Konrad 2015), cell formation problem (Bychkov et al. 2014, Pinheiro et al. 2018), and clinical trials (Bertsimas et al. 2019). Additionally, specialized techniques have been proposed for special cases of FP, including the minimum fractional spanning tree problem (Ursulenko et al. 2013), the minimum cost-to-time cycle problem (Dasdan et al. 1999), the maximum mean-cut problem (McCormick and Ervolina 1994), the minimum fractional assignment problem (Shigeno et al. 1995), and the maximum clique ratio problem (Moeini 2015, Sethuraman and Butenko 2015). We refer the reader to a recent survey by Borrero et al. (2017) and the references therein for an overview of the literature, applications and solution methods for fractional 0-1 programming.

In practice, the parameters of an optimization problem are often subject to uncertainty, and existing techniques for FP may not be adequate for problems with unknown parameters. Our approach to uncertain fractional 0-1 programming falls within the framework of robust optimization. Specifically, we assume that some or all of the coefficients a_{ij} and b_{ij} may not be known exactly, but are modeled as bounded random variables $\widetilde{a}_{ij}$ and $\widetilde{b}_{ij}$, respectively. These coefficients are presumed to lie in some uncertainty set $\mathcal{U}$; that is, $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$. Then the robust counterpart of FP with respect to the uncertainty set $\mathcal{U}$ optimizes against the worst-case scenario:

$$Z^{\star}_{\mathcal{U}} = \max_{x \in X} \min_{(\widetilde{a},\widetilde{b}) \in \mathcal{U}} \sum_{i \in I} \frac{a_{i0} + \sum\limits_{j \in J} \widetilde{a}_{ij} x_j}{b_{i0} + \sum\limits_{j \in J} \widetilde{b}_{ij} x_j}. \qquad \text{(RFP}[\mathcal{U}])$$

Throughout the paper, we assume that the data satisfy the following assumption:

Assumption 1 *For all $x \in X$, $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$ and $i \in I$, $a_{i0} + \sum_{j \in J} \widetilde{a}_{ij} x_j \geqslant 0$ and $b_{i0} + \sum_{j \in J} \widetilde{b}_{ij} x_j > 0$.*

Most fractional programming problems typically have non-negative data, since such data represent probabilities, prices, weights, utilities, etc. - see, e.g., Borrero et al. (2017) and the applications described therein. The portion of Assumption 1 related to a strictly positive denominator is a commonly made assumption for the deterministic version, see, e.g., Hansen et al. (1991) and Boros and Hammer (2002). Moreover, the non-negative numerator assumption is not restrictive, since by adding a sufficiently large constant value to each ratio we can transform its numerator into the one which takes only non-negative values for any $(\widetilde{a}, \widetilde{b}) \in \mathcal{U}$ and $x \in X$.

In the following, we define $(t)^+ = \max\{0, t\}$ for any $t \in \mathbb{R}$, and let $A \times B$ denote the Cartesian product of sets A and B.

1.1 Relevant literature

Fractional programming. The single-ratio FP with a strictly positive denominator can be solved to optimality by repeatedly solving a sequence of optimization problems with a linear objective function over X via parametrization algorithms, such as Newton's method (Dinkelbach 1967) and binary-search (Lawler 1976, Ahuja et al. 1993, Radzik 2013). Moreover, if solving such a linear optimization problem over X can be done in polynomial time, then single-ratio FP can be solved in polynomial time. Furthermore, Megiddo (1979) shows that if a nominal binary-linear problem admits a polynomial-time algorithm, then so does single-ratio FP.

Multiple-ratio FP is NP-hard even for two ratios and strictly positive denominators, see, e.g., Hansen et al. (1990) and Prokopyev et al. (2005a, 2005b). For solving multiple-ratio FP, there exist several mixed-integer linear programming (MILP) reformulations of FP. An early formulation was given by Williams (1974) and later generalized by Tawarmalani et al. (2002). A different formulation was suggested by Li (1994), and further discussed by Wu (1997) and Tawarmalani et al. (2002). Additionally, Tawarmalani et al. (2002) present six other formulations. A more recent formulation approach based on performing a base-2 expansion of certain integer-valued expressions was given by Borrero et al. (2016), which can reduce the number of bilinear terms that require linearization and may substantially reduce the number of variables with respect to the previous MILP reformulations.

Robust optimization. The robust optimization approach of El Ghaoui and Lebret (1997), El Ghaoui et al. (1998), and Ben-Tal and Nemirovski (1998, 1999, 2000) presumes that the unknown data lie within some uncertainty set. A critical modeling decision is selecting an appropriate uncertainty set that is close to the true representation of uncertainty in the data, and also ensures tractability of the resulting robust counterpart. In particular, the budgeted uncertainty proposed in Bertsimas and Sim (2003, 2004) is widely used, as it achieves a good balance between accuracy and tractability.

Continuous robust single-ratio fractional programming was introduced by Gorissen (2015). The author shows that, under some convexity assumptions, the robust counterparts can be solved in polynomial time. Additionally, the Lagrange dual of a (continuous) robust single-ratio fractional problem was studied in Jeyakumar and Li (2011) and Jeyakumar et al. (2013) for disjoint uncertainty sets, i.e., the uncertainty set $\mathcal{U}$ can be naturally decomposed as the Cartesian product of simpler sets.

In the case of fractional 0-1 programming, the work of Rusmevichientong and Topaloglu (2012) is the only study that explores robust formulations. The authors study a single-ratio assortment optimization problem under the multinomial logit choice model, where only customer preferences are uncertain. However, their results cannot be directly extended for more general classes of fractional problems including the cases when the revenues are subject to uncertainty or the choice model is mixed-multinomial logit.

1.2 Structure and contributions of the paper

To the best of our knowledge, this study is the first work that addresses the robust fractional 0-1 programming in its general structure. We perform a comprehensive study of RFP[$\mathcal{U}$] that includes several types of the budgeted uncertainty sets, and also encompasses single- and multiple-ratio cases. We also briefly explore the complexity of RFP[$\mathcal{U}$] for general polyhedral $\mathcal{U}$. The structure of the paper can be summarized as follows.

- In §2, we introduce the (disjoint and joint) generalizations of the budgeted uncertainty set for fractional 0–1 programs and discuss computational complexity of RFP.
- In §3, we propose an approach to find an optimal solution of single-ratio RFP by solving a polynomial number of linear optimization problems over X; in particular, if linear optimization over X is polynomial-time solvable, then so is RFP[$\mathcal{U}$].
- In §4, we extend classical MILP formulations for FP to tackle multiple-ratio RFP[$\mathcal{U}$], and exploit the binary-expansion technique to improve the efficacy of the MILPs. We also provide some insights on the selection of the appropriate level of uncertainty.
- In §5, we present computations with real and synthetic data. Additionally, we examine the price of robustness and evaluate the performance of the proposed MILPs via extensive computational experiments.

2 Model of data uncertainty

The selection of an appropriate uncertainty set can affect the tractability of a robust optimization problem. In this section, we describe the budgeted uncertainty set, and several variations thereof, for fractional 0-1 programming as considered in this paper, which lead to tractable (polynomial-time) methods for single-ratio RFP[$\mathcal{U}$] in §3. On the other hand, we also demonstrate that the robust counterpart of a polynomially-solvable unconstrained single-ratio FP (with strictly positive denominator) is NP-hard for a general polyhedral uncertainty set $\mathcal{U}$.

In particular, following the convention introduced by Bertsimas and Sim (2003, 2004), each unknown coefficient $\widetilde{a}_{ij}$ and $\widetilde{b}_{ij}$ lies in a symmetric interval centered on the nominal value, i.e., $\widetilde{a}_{ij} \in [a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}]$ and $\widetilde{b}_{ij} \in [b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}]$ with $d^a_{ij}, d^b_{ij} \geqslant 0$. The coefficients d^a_{ij} and d^b_{ij} denote the potential deviation from nominal values a_{ij} and b_{ij}, respectively, for each $i \in I, j \in J$.

Additionally, it is unlikely for all of the coefficients to simultaneously change to their worst-case values. Hence, only a predetermined number of the unknown coefficients take values different from their nominal value. Given a ratio $i \in I$ and vectors $\widetilde{a}_i, \widetilde{b}_i \in \mathbb{R}^n$, let $S_i(\widetilde{a}_i) = \{j \in J \mid \widetilde{a}_{ij} \neq a_{ij}\}$ and $S_i(\widetilde{b}_i) = \left\{j \in J \mid \widetilde{b}_{ij} \neq b_{ij}\right\}$ be the set of indices of the uncertain parameters whose values are different from the nominal in the numerator and the denominator, respectively.

Uncertainty pertaining to linear 0-1 constraints is covered in literature (Bertsimas and Sim 2003), thus we assume that the constraint coefficients are fixed. Furthermore, we assume without loss of generality that the data is integral (otherwise, the rational coefficients can be scaled to satisfy this assumption). Hence:

Assumption 2 *All data is integer, i.e.,* $a_{i0}, b_{i0}, a_{ij}, b_{ij} \in \mathbb{Z}$*, and* $d^a_{ij}, d^b_{ij} \in \mathbb{Z}_+$ *for all* $i \in I, j \in J$*.*

Disjoint uncertainty set. Given $\Gamma^a_i, \Gamma^b_i \in \{0, 1, \ldots, n\}$ as the budget of uncertainty or the level of conservatism, for each $i \in I$ we define

$$\mathcal{U}^a_i = \Big\{ \widetilde{a}_i \in \mathbb{R}^n \mid \widetilde{a}_{ij} \in [a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}] \text{ for } j \in J,\ |S_i(\widetilde{a}_i)| \leqslant \Gamma^a_i \Big\}, \text{ and} \tag{1}$$

$$\mathcal{U}^b_i = \Big\{ \widetilde{b}_i \in \mathbb{R}^n \mid \widetilde{b}_{ij} \in [b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}] \text{ for } j \in J,\ |S_i(\widetilde{b}_i)| \leqslant \Gamma^b_i \Big\}. \tag{2}$$

Note that $\mathcal{U}^a_i$ and $\mathcal{U}^b_i$ correspond to the budgeted uncertainty sets studied in Bertsimas and Sim (2003, 2004), and Γ^a_i and Γ^b_i are the number of coefficients allowed to vary from their nominal value in the numerator and the denominator of the i-th ratio, respectively. Then the *disjoint uncertainty set* for fractional programming is

$$\mathcal{U}^{ab} = \Big\{ (\widetilde{a}, \widetilde{b}) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \mid (\widetilde{a}_i, \widetilde{b}_i) \in \mathcal{U}^a_i \times \mathcal{U}^b_i, \text{ for all } i \in I \Big\}.$$

We refer to $\mathcal{U}^{ab}$ as disjoint since uncertainty of the coefficients of each numerator/denominator is independent from the rest of the data. Also, observe that in the i-th ratio by setting $\Gamma^a_i = 0$ ($\Gamma^b_i = 0$) we can restrict the uncertainty only to the denominator (numerator) of the ratio. Therefore, set $\mathcal{U}^{ab}$ includes sub-cases in which some ratios are subject to uncertainty either only in their denominators or numerators.

Joint uncertainty sets. We now describe four joint uncertainty sets. In contrast with the disjoint uncertainty set above, there is some dependence between the uncertainties related to different numerators and denominators.

- *Shared ratio budget* - Given $\Gamma_i \in \{0, 1, \ldots, 2n\}$, for each $i \in I$ let

$$\mathcal{U}_i = \Big\{ (\widetilde{a}_i, \widetilde{b}_i) \in \mathbb{R}^n \times \mathbb{R}^n \mid \widetilde{a}_{ij} \in [a_{ij} - d^a_{ij}, a_{ij} + d^a_{ij}],\ \widetilde{b}_{ij} \in [b_{ij} - d^b_{ij}, b_{ij} + d^b_{ij}],\ |S_i(\widetilde{a}_i)| + |S_i(\widetilde{b}_i)| \leqslant \Gamma_i \Big\}.$$

The *shared ratio budget uncertainty set* is

$$\mathcal{U}^{\overline{ab}} = \Big\{ (\widetilde{a}, \widetilde{b}) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \mid (\widetilde{a}_i, \widetilde{b}_i) \in \mathcal{U}_i, \text{ for all } i \in I \Big\}.$$

Under the shared ratio budget uncertainty set, uncertainty for the i-th ratio is independent of other ratios, but the uncertainties of its numerator and denominator are connected by a common budget, Γ_i. Specifically, at most Γ_i of coefficients in the i-th ratio's numerator and denominator can change.

The uncertainty sets $\mathcal{U}^{ab}$ and $\mathcal{U}^{\overline{ab}}$ above arise naturally when there is uncertainty concerning individual coefficients of FP. In some applications, however, the uncertainty of the original problem may have a specific structure which requires a specialized uncertainty set. We now describe three such sets.

- *Matched sets* - Consider the problem of maximizing return on investment or productivity, where "a" corresponds to the return of executing a given project (e.g., dollar amount), and "b" corresponds to the investment costs for the project (e.g., time). Additionally, suppose that undesirable events may occur (e.g., strikes, natural disasters), resulting in a *simultaneous* decrease in the returns and increase in the costs of a given project. Such uncertainty is modeled by the *matched sets uncertainty set*

$$\mathcal{U}_{=}^{\overline{ab}} = \Big\{(\widetilde{a}, \widetilde{b}) \in \mathcal{U}^{\overline{ab}} \mid S_i(\widetilde{a}_i) = S_i(\widetilde{b}_i), \text{ for all } i \in I\Big\}.$$

- *Matched effects* - Consider the assortment optimization problem under the mixed multinomial logit model (see, e.g., Boyd and Mellman 1980, Méndez-Díaz et al. 2014),

$$\max_{x \in X} \sum_{i \in I} \frac{\sum_{j \in J} r_{ij} \rho_{ij} x_j}{1 + \sum_{j \in J} \rho_{ij} x_j}, \tag{3}$$

where r_{ij} and ρ_{ij} are the revenue and customer preference weight associated with selling product j to customer class i, respectively. Note that if the revenues are known, but the preferences are uncertain, then changes with respect to the nominal values of numerator/denominator coefficients that correspond to the same variable are proportional and of the same sign. The *matched effects uncertainty set*

$$\mathcal{U}_{\propto}^{\overline{ab}} = \Big\{(\widetilde{a}, \widetilde{b}) \in \mathcal{U}_{=}^{\overline{ab}} \mid \frac{a_{ij} - \widetilde{a}_{ij}}{d_{ij}^a} = \frac{b_{ij} - \widetilde{b}_{ij}}{d_{ij}^b}, \text{ for all } i \in I,\ j \in J\Big\}$$

captures this effect.

- *Single budget* - In all of the uncertainty sets defined above, we assume each ratio has its own budget(s) of uncertainty. On the other hand, one may consider an uncertainty set in which a single budget controls the degree of conservatism over all ratios. Specifically, the *single budget uncertainty set* for numerators also arises in the assortment problem (3) when the preferences are known, but the revenues are unknown, and is given by

$$\mathcal{U}^{\overline{a}} = \Big\{\widetilde{a} \in \mathbb{R}^{m \times n} \mid \widetilde{a}_{ij} \in [a_{ij} - d_{ij}^a, a_{ij} + d_{ij}^a] \text{ for all } i \in I, j \in J,\ \sum_{i \in I} |S_i(\widetilde{a}_i)| \leqslant \Gamma\},$$

where the budget $\Gamma \in \{0, 1, \ldots, m \cdot n\}$ is shared by all ratios. In words, only numerators are subject to uncertainty and at most Γ of the numerators coefficients are different from their nominal values.

The five uncertainty sets defined above, i.e., $\mathcal{U}^{ab}$, $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_{=}^{\overline{ab}}$, $\mathcal{U}_{\propto}^{\overline{ab}}$, and $\mathcal{U}^{\overline{a}}$, aim at modeling a broad-range of situations arising in practice; moreover, none is a special case of another. Furthermore, it can be verified that RFP[$\mathcal{U}$], in general, is neither quasi-convex nor quasi-concave.

We show in §3 that for a polynomial-time solvable FP the considered uncertainty sets lead to polynomial-time solvable robust counterparts RFP[$\mathcal{U}$]. In contrast, note that the robust counterparts corresponding to general polyhedral uncertainty are NP-hard.

RFP[$\mathcal{U}$] for general polyhedral uncertainty is NP-hard. Consider an unconstrained ($X = \mathbb{B}^n$) single-ratio problem with uncertainty limited to the numerator

$$\max_{x \in \mathbb{B}^n} \frac{a_0 + a^T x - \max_{\gamma \in \mathcal{U}} \left\{(A\gamma)^T x\right\}}{b_0 + b^T x}, \tag{4}$$

where $\mathcal{U} = \{\gamma : D\gamma \leqslant d, \gamma \geqslant 0\}$ is a general polyhedral uncertainty set and Assumption 1 holds. Note that, without uncertainty, the deterministic unconstrained single-ratio problem can be solved in polynomial time via a linear-time median-finding algorithm (Hansen et al. 1991). However, this property does not follow through to the robust counterpart.

Proposition 1 *Problem* (4) *is* NP*-hard.*

Proof. Let $b_0 = 1$ and $b_j = 0$ for $j \in J$, then we have a linear objective with a polyhedral uncertainty set. By Theorem 4 of Buchheim and Kurtz (2018), the resulting problem is NP-hard. □

Similarly, consider the problem with uncertainty restricted to the denominator

$$\max_{x \in \mathbb{B}^n} \frac{a_0 + a^T x}{b_0 + b^T x + \max_{\gamma \in \mathcal{U}} \left\{(A\gamma)^T x\right\}}. \tag{5}$$

Proposition 2 *Problem* (5) *is* NP*-hard.*

Proof. Follows directly from noting that (5) is equivalent to $\min_{x \in \mathbb{B}^n} \frac{b_0 + b^T x + \max_{\gamma \in \mathcal{U}} \left\{(A\gamma)^T x\right\}}{a_0 + a^T x}$, and using an argument similar to the one in Proposition 1. □

In light of these results, in the remainder of this paper we restrict $\mathcal{U}$ to any disjoint or joint uncertainty sets defined in this section, i.e., $\mathcal{U} \in \{\mathcal{U}^{ab}, \mathcal{U}^{\overline{ab}}, \mathcal{U}^{\overline{ab}}_{=}, \mathcal{U}^{\overline{ab}}_{\propto}, \mathcal{U}^{\overline{a}}\}$, and RFP[$\mathcal{U}$] as the corresponding representation of the robust problem.

3 Single-ratio RFP[$\mathcal{U}$]

When the uncertain coefficients of the objective function are in the form of a budgeted uncertainty set, Bertsimas and Sim (2003) prove that the solution of the robust counterpart of the nominal binary-linear problem

$$\min_{x \in X} c_0 + \sum_{j \in J} c_j x_j, \tag{6}$$

can be found by solving n instances of (6). Therefore, if (6) is polynomially-solvable, so is its robust counterpart. Similarly, parametrization algorithms such as Newton's method (Dinkelbach 1967) and binary-search algorithm (Lawler 1976, Radzik 2013, Ahuja et al. 1993) can find an optimal solution for the constrained single-ratio FPs by solving a sequence of problems in the form of (6).

In this section, we combine and extend the ideas from robust linear programming and deterministic fractional optimization, to propose a solution method for single-ratio RFP[$\mathcal{U}$]. In particular, we show that if there exists a polynomial-time algorithm for linear optimization over X, then RFP[$\mathcal{U}$] is polynomial-time solvable when $\mathcal{U}$ is one of the uncertainty sets described in §2. We first consider the disjoint uncertainty set $\mathcal{U}^{ab}$ in §3.1, and then we tackle the joint uncertainty sets in §3.2.

3.1 Disjoint single-ratio case

Herein, we demonstrate how to solve single-ratio RFP[$\mathcal{U}^{ab}$] by solving at most $(n+1)^2$ nominal FPs.

Proposition 3 *Problem* RFP[$\mathcal{U}^{ab}$] *is equivalent to*

$$Z^\star_{\mathcal{U}^{ab}} = \max_{\substack{x\in X,\\ \alpha\in\{0,d_1^a,d_2^a,\ldots,d_n^a\},\\ \beta\in\{0,d_1^b,d_2^b,\ldots,d_n^b\}}} \frac{a_0 - \Gamma^a\alpha + \sum_{j\in J}\big(a_j - (d_j^a-\alpha)^+\big)x_j}{b_0 + \Gamma^b\beta + \sum_{j\in J}\big(b_j + (d_j^b-\beta)^+\big)x_j}. \tag{7}$$

Proof. Observe that single-ratio RFP[$\mathcal{U}^{ab}$] is equivalent to $\max_{x\in X}\frac{a_0+\min_{\widetilde{a}\in\mathcal{U}^a}\widetilde{a}^Tx}{b_0+\max_{\widetilde{b}\in\mathcal{U}^b}\widetilde{b}^Tx}$, where $\mathcal{U}^a$ and $\mathcal{U}^b$ are the sets given in (1) and (2). Letting u and v be the indicator vectors of sets $S(\widetilde{a})$ and $S(\widetilde{b})$ respectively, we reformulate RFP[$\mathcal{U}^{ab}$] as

$$\begin{aligned}
\max_{x\in X}\ & \frac{a_0 + \sum_{j\in J} a_jx_j - \max_u\Big\{\sum_{j\in J} d_j^a x_j u_j\Big\}}{b_0 + \sum_{j\in J} b_jx_j + \max_v\Big\{\sum_{j\in J} d_j^b x_j v_j\Big\}} && (8)\\
\text{s.t. } & \sum_{j\in J} u_j \leqslant \Gamma^a, \quad \sum_{j\in J} v_j \leqslant \Gamma^b && (\alpha,\beta)\\
& 0\leqslant u_j\leqslant 1, \quad 0\leqslant v_j\leqslant 1, && \forall j\in J.\ (p_j,q_j)
\end{aligned}$$

Note that there exist integral optimal solutions u^* and v^* to the inner optimization problems in (8), since the polytope defined by cardinality and bounding constraints is integral – thus, the formulation above is indeed correct. By taking the dual of (independent) inner optimization problems in the numerator and the denominator of (8) with respect to dual variables α,β and p,q, we obtain

$$\begin{aligned}
\max_{\substack{x\in X,\\ \alpha,\beta,p,q\geqslant 0}}\ & \frac{a_0 + \sum_{j\in J} a_jx_j - \Big(\Gamma^a\alpha + \sum_{j\in J} p_j\Big)}{b_0 + \sum_{j\in J} b_jx_j + \Big(\Gamma^b\beta + \sum_{j\in J} q_j\Big)} && (9)\\
\text{s.t. } & p_j + \alpha \geqslant d_j^a x_j, \quad q_j + \beta \geqslant d_j^b x_j, && \forall j\in J.
\end{aligned}$$

Clearly, in an optimal solution we have $p_j^* = \left(d_j^a x_j^* - \alpha^*\right)^+ = \left(d_j^a - \alpha^*\right)^+ x_j^*$ and $q_j = \left(d_j^b x_j^* - \beta^*\right)^+ = \left(d_j^a - \alpha^*\right)^+ x_j^*$. Otherwise, we can decrease p_j or q_j and find a solution with a better objective function value.

Additionally, let $E = \left\{ j \in J \mid \left(d_j^a - \alpha^*\right)^+ x_j^* > 0 \right\}$ and observe that if $\alpha^* > 0$ and $\alpha^* \neq d_j^a$ for all $j \in J$ then

$$\Gamma^a(\alpha^* \pm \epsilon) + \sum_{j \in J} \left(d_j^a - (\alpha^* \pm \epsilon)\right)^+ x_j^* = \Gamma^a(\alpha^*) + \sum_{j \in J} \left(d_j^a - \alpha^*\right)^+ x_j^* \pm \epsilon(\Gamma^a - |E|)$$

for sufficiently small $\epsilon > 0$. In particular, depending on the sign of $\Gamma^a - |E|$, we can increase or decrease α^* and find solutions with greater or equal objective function values. Thus, we conclude that there exists an optimal solution where $\alpha^* \in \{0, d_1^a, \dots, d_n^a\}$ and, similarly, we can conclude that there exists an optimal solution where $\beta^* \in \{0, d_1^b, \dots, d_n^b\}$. Replacing α, β, p, q in (9) by their corresponding optimal values, we find formulation (7). □

Hence, RFP[$\mathcal{U}^{ab}$] can be tackled by solving (7) for each candidate pair $(\alpha, \beta) \in \{0, d_1^a, d_2^a, \dots, d_n^a\} \times \{0, d_1^b, d_2^b, \dots, d_n^b\}$ independently.

Theorem 1 *Single-ratio* RFP[$\mathcal{U}^{ab}$] *can be solved with* $(k^a + 1)(k^b + 1)$ *calls to an oracle for* FP*, where* k^a *and* k^b *are the numbers of distinct values of* d_j^a *and* d_j^b*,* $j \in J$*, respectively.*

Theorem 1 implies that if single-ratio FP over X is solvable in strongly polynomial time, then so is its robust counterpart RFP[$\mathcal{U}^{ab}$]. Note that in the worst case $(k^a + 1)(k^b + 1) = (n + 1)^2$, and FP is polynomial-time solvable when linear optimization over X is polynomial-time solvable.

3.2 Joint single-ratio case

It can be observed that the method of Proposition 3 cannot handle single-ratio RFP under joint uncertainty sets, due to interaction between uncertainties in the numerator and the denominator of each ratio. To solve single-ratio RFP under joint uncertainty sets we first show that RFP[$\mathcal{U}^{\overline{ab}}$], RFP[$\mathcal{U}^{\overline{ab}}_{=}$], and RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$] can be formulated as mixed-integer nonlinear programs (MINLPs) with a similar structure (Propositions 4, 5 and 6). Then by exploring some properties of the resulting reformulations (Propositions 7 and 8) we propose a specialized algorithm for solving them (Proposition 9).

Proposition 4 *Problem* RFP[$\mathcal{U}^{\overline{ab}}$] *is equivalent to*

$$\begin{aligned} Z^\star_{\mathcal{U}^{\overline{ab}}} = \max_{\substack{x \in X, \\ \mu, \alpha, \beta, \gamma \geqslant 0}} \quad & \mu \qquad (10) \\ \text{s.t.} \quad & (b_0 + \sum_{j \in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j \in J} \beta_j + \sum_{j \in J} \gamma_j \leqslant a_0 + \sum_{j \in J} a_j x_j \\ & \alpha + \beta_j \geqslant d_j^a x_j, \quad \alpha + \gamma_j \geqslant d_j^b x_j \mu \qquad \forall j \in J. \end{aligned}$$

Proof. Let u and v be the indicator variables of the sets $S(\widetilde{a})$ and $S(\widetilde{b})$, respectively. Note that RFP[$\mathcal{U}^{\overline{ab}}$] can be written as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}} = \max_{x \in X} \min_{u,v \in \mathbb{R}^n} \frac{a_0 + \sum_{j \in J} a_j x_j - \sum_{j \in J} d^a_j x_j u_j}{b_0 + \sum_{j \in J} b_j x_j + \sum_{j \in J} d^b_j x_j v_j} \tag{11a}$$

$$\text{s.t. } \sum_{j \in J} u_j + \sum_{j \in J} v_j \leqslant \Gamma \tag{11b}$$

$$0 \leqslant u_j \leqslant 1, \quad 0 \leqslant v_j \leqslant 1, \qquad \forall j \in J. \tag{11c}$$

Observe that we relaxed the binary constraints $u_j \in \mathbb{B}$ and $v_j \in \mathbb{B}$ to convex bound constraints. Since the inner minimization problem is quasi-concave for any $x \in X$ (Dinkelbach 1967), the nonlinear problem has an optimal solution that is an extreme point of the polytope induced by (11b)–(11c); in particular, there exists an optimal binary solution.

We now reformulate the inner minimization problem using the transformation proposed in Charnes and Cooper (1962): letting $y = 1/\big(b_0 + \sum_{j \in J} b_j x_j + \sum_{j \in J} d^b_j x_j v_j\big)$, $z^u_j = u_j y$, and $z^v_j = v_j y$ for all $j \in J$, we can write (11a)–(11c) as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}} = \max_{x \in X} \min_{z^u, z^v, y} (a_0 + \sum_{j \in J} a_j x_j) y - \sum_{j \in J} d^a_j x_j z^u_j \tag{12a}$$

$$\text{s.t. } (b_0 + \sum_{j \in J} b_j x_j) y + \sum_{j \in J} d^b_j x_j z^v_j = 1 \qquad (\mu) \tag{12b}$$

$$\sum_{j \in J} z^u_j + \sum_{j \in J} z^v_j \leqslant \Gamma y \qquad (\alpha) \tag{12c}$$

$$0 \leqslant z^u_j \leqslant y \qquad \forall j \in J \quad (\beta_j) \tag{12d}$$

$$0 \leqslant z^v_j \leqslant y \qquad \forall j \in J. \quad (\gamma_j) \tag{12e}$$

It is seen that for any fixed $x \in X$, the inner minimization problem is a linear program (LP). Thus, using standard LP duality, we obtain formulation (10) where μ, α, β_j, and γ_j are corresponding dual variables to constraints (12b) to (12e). □

Furthermore, in Propositions 5 and 6 we show that RFP[$\mathcal{U}^{\overline{ab}}_{=}$] and RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$], respectively, can also be formulated as equivalent MINLPs. The proofs are relegated to Appendix A.

Proposition 5 *Problem* RFP[$\mathcal{U}^{\overline{ab}}_{=}$] *is equivalent to*

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{\substack{x \in X, \\ \mu, \alpha, \beta \geqslant 0}} \mu \tag{13}$$

$$\text{s.t. } (b_0 + \sum_{j \in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j \in J} \beta_j \leqslant a_0 + \sum_{j \in J} a_j x_j$$

$$\alpha + \beta_j \geqslant d_j^a x_j + d_j^b x_j \mu \qquad \forall j \in J.$$

Proposition 6 *Problem* RFP$[\mathcal{U}_{\propto}^{\overline{ab}}]$ *is equivalent to*

$$Z^{\star}_{\mathcal{U}_{\propto}^{\overline{ab}}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geqslant 0}} \mu \tag{14}$$

$$\text{s.t. } (b_0 + \sum_{j\in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j \leqslant a_0 + \sum_{j\in J} a_j x_j$$

$$\alpha + \beta_j \geqslant -d_j^a x_j + d_j^b x_j \mu, \quad \alpha + \gamma_j \geqslant d_j^a x_j - d_j^b x_j \mu \qquad \forall j \in J.$$

Example 1 *Consider a trivariate ($n = 3$) single-ratio RFP$[\mathcal{U}_{\propto}^{\overline{ab}}]$, i.e., $Z^{\star}_{\mathcal{U}_{\propto}^{\overline{ab}}} = \frac{a_0+\widetilde{a}_1x_1+\widetilde{a}_2x_2+\widetilde{a}_3x_3}{b_0+\widetilde{b}_1x_1+\widetilde{b}_2x_2+\widetilde{b}_3x_3}$, wherein $a_0 = 6$, $\widetilde{a}_1 \in [-3, 13]$, $\widetilde{a}_2 \in [1, 31]$, $\widetilde{a}_3 \in [1, 5]$, and $b_0 = 3$, $\widetilde{b}_1 \in [0, 4]$, $\widetilde{b}_2 \in [0, 16]$, $\widetilde{b}_3 \in [1, 3]$ for $\Gamma = 2$. Thus, the nominal values are: $a_1 = 5, a_2 = 16, a_3 = 3, b_1 = 2, b_2 = 8, b_3 = 2$, and the deviation values are: $d_1^a = 8, d_2^a = 15, d_3^a = 2, d_1^b = 2, d_2^b = 8, d_3^b = 1$.*

Then by Proposition 6, the equivalent reformulation of this RFP$[\mathcal{U}_{\propto}^{\overline{ab}}]$ is given by

$$Z^{\star}_{\mathcal{U}_{\propto}^{\overline{ab}}} = \max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geqslant 0}} \mu$$

$$s.t. \quad (3 + 2x_1 + 8x_2 + 2x_3)\mu + 2\alpha + \sum_{j\in\{1,2,3\}}\beta_j + \sum_{j\in\{1,2,3\}}\gamma_j \leqslant 6 + 5x_1 + 16x_2 + 3x_3$$

$$\alpha + \beta_1 \geqslant -8x_1 + 2x_1\mu, \qquad \alpha + \gamma_1 \geqslant 8x_1 - 2x_1\mu$$

$$\alpha + \beta_2 \geqslant -15x_2 + 8x_2\mu, \qquad \alpha + \gamma_2 \geqslant 15x_2 - 8x_2\mu$$

$$\alpha + \beta_3 \geqslant -2x_3 + x_3\mu, \qquad \alpha + \gamma_3 \geqslant 2x_3 - x_3\mu. \qquad \square$$

Based on Propositions 4, 5, and 6 we see that, in all cases, single-ratio RFP under the joint uncertainty sets $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_{=}^{\overline{ab}}$, and $\mathcal{U}_{\propto}^{\overline{ab}}$ can be formulated as

$$\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geqslant 0}} \mu \tag{15a}$$

$$\text{s.t. } (b_0 + \sum_{j\in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j\in J}\beta_j + \sum_{j\in J}\gamma_j \leqslant a_0 + \sum_{j\in J} a_j x_j \tag{15b}$$

$$\alpha + \beta_j \geqslant (d_j^1 + d_j^2\mu)x_j, \quad \alpha + \gamma_j \geqslant (d_j^3 + d_j^4\mu)x_j \qquad \forall j \in J, \tag{15c}$$

for some $d^1, d^2, d^3, d^4 \in \mathbb{Z}^n$, where $d_j^1 \cdot d_j^3 \leqslant 0$ and $d_j^2 \cdot d_j^4 \leqslant 0$ for all $j \in J$. In particular, if $d_j^1 = d_j^a, d_j^2 = d_j^3 = 0$, and $d_j^4 = d_j^b$ for all $j \in J$, then problem (15) is equivalent to the reformulation of RFP$[\mathcal{U}^{\overline{ab}}]$ given by (10). Similarly, letting $d_j^1 = d_j^a, d_j^2 = d_j^b, d_j^3 = d_j^4 = 0$ and $d_j^1 = -d_j^3 = -d_j^a, d_j^2 = -d_j^4 = d_j^b$ for all $j \in J$ in (15), lead to equivalent reformulation of RFP$[\mathcal{U}_{=}^{\overline{ab}}]$ and RFP$[\mathcal{U}_{\propto}^{\overline{ab}}]$, respectively, provided in (13) and (14).

Problem (15) is a mixed-integer nonlinear program. Note that for a fixed value of μ, problem (15) reduces to an MILP feasibility problem or equivalently checking whether the following MILP

$$\psi(\mu) = \min_{x \in X, \alpha, \beta, \gamma} \Big\{ (b_0 + \sum_{j \in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j \in J} \beta_j + \sum_{j \in J} \gamma_j - (a_0 + \sum_{j \in J} a_j x_j) \;\Big|\; (15c) \Big\} \tag{16}$$

has a non-positive optimal objective function value (i.e., $\psi(\mu) \leqslant 0$). Proposition 7 below shows that $\psi(\mu)$ is a monotone function of μ. Thus, we can solve (15) by applying the binary-search algorithm on μ, where at each iteration of the algorithm we solve (16) for a fixed value of μ. That is if $\psi(\mu) > 0$ we must decrease μ, otherwise, we can increase μ.

Proposition 7 *For given vectors d^1, d^2, d^3, and d^4 such that $d_j^2 \cdot d_j^4 \leqslant 0$ and $|d_j^2|, |d_j^4| \leqslant d_j^b$ for all $j \in J$, if $\psi(\mu) \leqslant 0$ for a fixed $\mu \geqslant 0$, then $\psi(\mu') \leqslant 0$ for any $0 \leqslant \mu' < \mu$.*

Proof. For fixed $\mu \geqslant 0$, let $(\alpha, \beta, \gamma, x)$ denote a feasible solution of (16) for which the objective function value of (16) is non-positive. Then we show that for $\mu' = \mu - \epsilon$, $\epsilon > 0$, there exist $\beta', \gamma' \geqslant 0$ such that $(\alpha, \beta', \gamma', x)$ is a feasible solution of (16) with non-positive objective function value. Toward this goal, define $J^2 = \{j \in J \mid d_j^2 < 0\}$ and $J^4 = \{j \in J \mid d_j^4 < 0\}$; note that $J^2 \cap J^4 = \emptyset$ since $d_j^2 \cdot d_j^4 \leqslant 0$ for all $j \in J$. Then let $\beta'_j = \beta_j$ for $j \in J \setminus J^2$ and $\beta'_j = \beta_j - \epsilon d_j^2 x_j \geqslant 0$ for $j \in J^2$. Similarly, let $\gamma'_j = \gamma_j$ for $j \in J \setminus J^4$ and $\gamma'_j = \gamma_j - \epsilon d_j^4 x_j \geqslant 0$ for $j \in J^4$. Hence, $(\mu', \alpha, \beta', \gamma', x)$ satisfies the constraints of (15c).

Next, we show that for $(\mu', \alpha, \beta', \gamma', x)$ the objective function value of (16) is non-positive.

$$\begin{aligned}
&(b_0 + \sum_{j \in J} b_j x_j)\mu' + \Gamma\alpha + \sum_{j \in J} \beta'_j + \sum_{j \in J} \gamma'_j - (a_0 + \sum_{j \in J} a_j x_j) \\
&= (b_0 + \sum_{j \in J} b_j x_j)(\mu - \epsilon) + \Gamma\alpha + \sum_{j \in J \setminus J^2} \beta_j + \sum_{j \in J^2} (\beta_j - \epsilon d_j^2 x_j) \\
&\qquad\qquad + \sum_{j \in J \setminus J^4} \gamma_j + \sum_{j \in J^4} (\gamma_j - \epsilon d_j^4 x_j) - (a_0 + \sum_{j \in J} a_j x_j) \\
&= (b_0 + \sum_{j \in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j \in J} \beta_j + \sum_{j \in J} \gamma_j - (a_0 + \sum_{j \in J} a_j x_j) - \epsilon(b_0 + \sum_{j \in J} b_j x_j + \sum_{j \in J^2} d_j^2 x_j + \sum_{j \in J^4} d_j^4 x_j) \leqslant 0.
\end{aligned}$$

The last inequality holds because the objective function value of (16) is non-positive for $(\mu, \alpha, \beta, \gamma, x)$; moreover, since $J^2 \cap J^4 = \emptyset$ and $|d_j^2|, |d_j^4| \leqslant d_j^b$, for all $j \in J$, by Assumption 1 we have $(b_0 + \sum_{j \in J} b_j x_j + \sum_{j \in J^2} d_j^2 x_j + \sum_{j \in J^4} d_j^4 x_j) > 0$. □

In order to solve (16) efficiently at each iteration of the binary-search algorithm, we further simplify it by using an argument similar to the one used for proving Proposition 3.

Proposition 8 *Problem* (16) *can be reformulated as*

$$\begin{aligned}
\psi(\mu) = \min_{x \in X, \alpha \in \mathcal{F}} \Big\{ &(b_0 + \sum_{j \in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j \in J} \left(d_j^1 + d_j^2 \mu - \alpha\right)^+ x_j \\
&+ \sum_{j \in J} \left(d_j^3 + d_j^4 \mu - \alpha\right)^+ x_j - (a_0 + \sum_{j \in J} a_j x_j) \Big\},
\end{aligned} \tag{17}$$

where

$$\mathcal{F} = \Big\{0, (d_1^1 + d_1^2\mu)^+, (d_2^1 + d_2^2\mu)^+, \ldots, (d_n^1 + d_n^2\mu)^+, (d_1^3 + d_1^4\mu)^+, (d_2^3 + d_2^4\mu)^+, \ldots, (d_n^3 + d_n^4\mu)^+\Big\}.$$

Proof. In an optimal solution of (16), we have that, for all $j \in J$, $\beta_j^\star = \big((d_j^1 + d_j^2\mu)x_j^\star - \alpha^\star\big)^+ = (d_j^1 + d_j^2\mu - \alpha^\star)^+ x_j^\star$ and $\gamma_j^\star = \big((d_j^3 + d_j^4\mu)x_j^\star - \alpha^\star\big)^+ = (d_j^3 + d_j^4\mu - \alpha^\star)^+ x_j^\star$. Thus, (16) reduces to

$$\psi(\mu) = \min_{x\in X, \alpha\geqslant 0} (b_0 + \sum_{j\in J} b_j x_j)\mu + \Gamma\alpha + \sum_{j\in J}(d_j^1 + d_j^2\mu - \alpha)^+ x_j + \sum_{j\in J}(d_j^3 + d_j^4\mu - \alpha)^+ x_j - (a_0 + \sum_{j\in J} a_j x_j).$$

Additionally, similar to the proof of Proposition 3 observe that if $\alpha^\star > 0$, $\alpha^\star \neq d_j^1 + d_j^2\mu$ and $\alpha^\star \neq d_j^3 + d_j^4\mu$ for all $j \in J$, then it can be verified that either $\alpha^\star + \epsilon$ or $\alpha^\star - \epsilon$ is also feasible for sufficiently small $\epsilon > 0$. Thus, we may assume without loss of generality that $\alpha^\star \in \{0\} \cup \{(d_j^1 + d_j^2\mu)^+\}_{j\in J} \cup \{(d_j^3 + d_j^4\mu)^+\}_{j\in J}$, which completes the proof. □

Example 2 *According to Proposition 8, the corresponding formulation* (17) *for RFP[$\mathcal{U}_\propto^{\overline{ab}}$] in Example 1 is*

$$\begin{aligned}\psi(\mu) = \min_{x\in X, \alpha\in\mathcal{F}} \Big\{ &(3 + 2x_1 + 8x_2 + 2x_3)\mu + 2\alpha \\ &+ (-8 + 2\mu - \alpha)^+ x_1 + (-15 + 8\mu - \alpha)^+ x_2 + (-2 + \mu - \alpha)^+ x_3 \\ &+ (8 - 2\mu - \alpha)^+ x_1 + (15 - 8\mu - \alpha)^+ x_2 + (2 - \mu - \alpha)^+ x_3 - (6 + 5x_1 + 16x_2 + 3x_3)\Big\},\end{aligned}$$

where $\mathcal{F} = \Big\{0, (-8 + 2\mu)^+, (-15 + 8\mu)^+, (-2 + \mu)^+, (8 - 2\mu)^+, (15 - 8\mu)^+, (2 - \mu)^+\Big\}$. □

In the following, we focus our efforts on obtaining the optimal objective function value of (17). To this end, define $T = \{1, 2, \ldots, |\mathcal{F}|\}$, $|T| \leqslant 2n + 1$, and for each $t \in T$ define binary-linear problem

$$\psi_t(\mu) = \min_{x\in X} g_t(x, \mu) \tag{18}$$

where

$$\begin{aligned} g_t(x, \mu) = (b_0 + \sum_{j\in J} b_j x_j)\mu + \Gamma\left(\bar{c}_t + \bar{d}_t\mu\right)^+ + \sum_{j\in J}\Big(d_j^1 + d_j^2\mu - \left(\bar{c}_t + \bar{d}_t\mu\right)^+\Big)^+ x_j \\ + \sum_{j\in J}\Big(d_j^3 + d_j^4\mu - \left(\bar{c}_t + \bar{d}_t\mu\right)^+\Big)^+ x_j - a_0 - \sum_{j\in J} a_j x_j, \end{aligned}$$

and $(\bar{c}_t, \bar{d}_t) \in \{(0,0)\} \cup \{(d_j^1, d_j^2)\}_{j\in J} \cup \{(d_j^3, d_j^4)\}_{j\in J}$.

Evidently, $\psi(\mu) = \min_{t\in T} \psi_t(\mu)$. Thus, for μ fixed, checking whether $\psi(\mu) \leqslant 0$ can be done by verifying whether there exists $t \in T$ such that $\psi_t(\mu) \leqslant 0$. Thereby, in the following result we conclude that problem (15) can be solved efficiently using the binary-search method.

Proposition 9 *Problem* (15) *can be solved with* $O\big(n\log(U/\epsilon)\big)$ *calls to an oracle for* (18)*, where* $U = |a_0| + \sum_{j\in J}|a_j|$ *and* $\epsilon > 0$ *is a precision parameter.*

Proof. The binary search requires $O\big(\log(\frac{U}{\epsilon})\big)$ iterations and each iteration requires solving at most $|\mathcal{F}| = |T| = 2n+1$ problems of the form (18). Moreover, let $\tau(n)$ denote the complexity of solving binary-linear problem (18). Then the binary-search algorithm to solve problem (15) has the worst-case complexity $O\big(n\log(U/\epsilon)\tau(n)\big)$. $\square$

As a direct consequence of Propositions 7 to 9, we get the main result of this subsection, i.e.,

Theorem 2 *Single-ratio* RFP$[\mathcal{U}^{\overline{ab}}]$, RFP$[\mathcal{U}^{\overline{ab}}_{=}]$ *and* RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ *can be solved in* $O\big(n\log(U/\epsilon)\tau(n)\big)$*, where* $\tau(n)$ *is the complexity of solving problem* (18)*. In particular, if linear optimization over* X *is polynomial-time solvable, then so is single-ratio* RFP *under the joint uncertainty sets.*

Notably, when $X = \mathbb{B}^n$ the complexity of solving problem (18) is $O(n)$, i.e., $\tau(n) = n$, resulting in the overall complexity $O\big(n^2\log(U/\epsilon)\big)$ to solve RFP$[\mathcal{U}^{\overline{ab}}]$, RFP$[\mathcal{U}^{\overline{ab}}_{=}]$ and RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$. Additionally, if $X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j \leqslant k\}$ or $X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j = k\}$ we have $\tau(n) = n\log(n)$, resulting in the overall complexity $O\big(n^2\log(n)\log(U/\epsilon)\big)$. Therefore,

Corollary 1 *The unconstrained and cardinality constrained single-ratio* RFP$[\mathcal{U}]$ *under joint uncertainty sets* $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}^{\overline{ab}}_{=}$ *and* $\mathcal{U}^{\overline{ab}}_{\propto}$ *can be solved in polynomial time.*

It is worth mentioning that the cardinality-constrained ($X = \{x \in \mathbb{B}^n \mid \sum_{j\in J} x_j \leqslant k\}$) single-ratio assortment problem (3) when customer preferences (ρ_j) are subject to rectangular uncertainty $\mathcal{U} = \prod_{j=0}^{n}[l_j, u_j] \subset \mathbb{R}^{n+1}_{++}$, where ℓ_j and u_j are lower and upper bounds on ρ_j, can be solved in $O(n^2)$, see Rusmevichientong and Topaloglu (2012). This problem is a special case of RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ when $\Gamma = n$, and $\widetilde{a}_j, \widetilde{b}_j > 0$. However, the aforementioned result cannot be extended, e.g., when revenues (r_j) are uncertain or, more importantly, for generally structured single-ratio RFP$[\mathcal{U}]$ (such as other choice models) under other types of the budgeted uncertainty sets or (weaker) Assumption 1.

We conclude the discussion on single-ratio RFP$[\mathcal{U}]$ with the following remarks.

Remark 1 The solutions methods outlined in this section are particularly efficient for unconstrained problems. Additionally, they are useful when there exist specialized algorithms to solve the corresponding constrained linear binary problem, e.g., those that exploit the constraint structure of the underlying combinatorial optimization problem. If these algorithms are polynomial time (for example, such as those for the linear assignment, the shortest path and the minimum spanning tree problems, see Ahuja et al. 1993), then the single-ratio RFP$[\mathcal{U}]$ is also polynomial-time solvable. $\square$

Remark 2 In the case of single-ratio RFPs under the disjoint uncertainty set, the approach of Theorem 1 is superior to the binary search approach developed in §3.2 since the former is strongly polynomial, $O(n^2)$, while the latter involves the binary search algorithm with the number of iterations $O\big(\log(\frac{U}{\epsilon})\big)$. $\square$

4 Multiple-ratio RFP[$\mathcal{U}$]

In this section, we present MILP formulations for multiple-ratio RFP[$\mathcal{U}$]. First, for the disjoint uncertainty set, we reformulate RFP[$\mathcal{U}^{ab}$] as robust linear problems. Then with these reformulations in hand, we adapt the methods of Bertsimas and Sim (2004) to transform them into MILPs, see §4.1. For the joint uncertainty sets (except $\mathcal{U}^{\overline{a}}$) we use the results from §3.2, see §4.2.1; for $\mathcal{U}^{\overline{a}}$ we use a same approach provided in §4.1, see §4.2.2. Then, in §4.3 we discuss the sizes (numbers of variables and constraints) of the obtained MILP reformulations. Finally, in §4.4 we show that the optimal value of the robust formulations provided in this paper with high probability are not overestimator of the true value of the fractional problems with symmetrical and bounded random coefficients.

4.1 Disjoint uncertainty set

For the present discussion, we consider the uncertainty set $\mathcal{U}^{ab}$, and present three MILP reformulations of RFP[$\mathcal{U}^{ab}$]. For the first two formulations presented in §4.1.1 and §4.1.2 we exploit the ideas from fractional programming literature, see Li (1994) and Williams (1974). The third formulation, presented in §4.1.3 corresponds to a binary expansion reformulation proposed by Borrero et al. (2016).

4.1.1 Reformulation 1 $\big($MILP$_1$[$\mathcal{U}^{ab}$]$\big)$.

Note that RFP[$\mathcal{U}^{ab}$] can be written as

$$\max_{x\in X} \min_{(\widetilde{a},\widetilde{b})\in\mathcal{U}^{ab}} \sum_{i\in I}\left(a_{i0}+\sum_{j\in J}\widetilde{a}_{ij}x_j\right)\left(\frac{1}{b_{i0}+\sum_{j\in J}\widetilde{b}_{ij}x_j}\right).$$

Using the substitutions

$$\omega_i \leqslant \frac{1}{b_{i0}+\sum_{j\in J}\widetilde{b}_{ij}x_j},$$

for all $\widetilde{b}_i\in\mathcal{U}^b_i$ and $i\in I$, and exploiting the fact that $\mathcal{U}^{ab}$ is disjoint, we find the equivalent formulation

$$\begin{aligned}
\max_{\substack{x\in X,\\ \omega\geqslant 0}} \min_{\widetilde{a}\in\mathcal{U}^a} \; & \sum_{i\in I}(a_{i0}+\sum_{j\in J}\widetilde{a}_{ij}x_j)\omega_i \\
\text{s.t. } & (b_{i0}+\sum_{j\in J}\widetilde{b}_{ij}x_j)\omega_i \leqslant 1 && \forall \widetilde{b}_i\in\mathcal{U}^b_i,\ \forall i\in I,
\end{aligned}$$

where $\mathcal{U}^a := \{\widetilde{a}\in\mathbb{R}^{m\times n} \mid \widetilde{a}_i\in\mathcal{U}^a_i \text{ for all } i\in I\}$. Similarly, defining new variables μ_i such that $\mu_i \leqslant (a_{i0}+\sum_{j\in J}\widetilde{a}_{ij}x_j)\omega_i$ for all $\widetilde{a}_i\in\mathcal{U}^a_i$ and $i\in I$ yields the robust optimization problem

$$\begin{aligned}
\max_{\substack{x\in X,\\ \mu,\omega\geqslant 0}} \; & \sum_{i\in I}\mu_i && (\text{RFP}_1[\mathcal{U}^{ab}]) \\
\text{s.t. } & \mu_i \leqslant (a_{i0}+\sum_{j\in J}\widetilde{a}_{ij}x_j)\omega_i && \forall \widetilde{a}_i\in\mathcal{U}^a_i,\ \forall i\in I
\end{aligned}$$

$$(b_{i0} + \sum_{j \in J} \widetilde{b}_{ij} x_j)\omega_i \leqslant 1 \qquad \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I.$$

Note that the directions of the inequalities ($\leqslant$) rely on the sense of the objective function and Assumption 1. Since $x \in X \subseteq \mathbb{B}^n$, we linearize the bilinear terms $x_j \omega_i$ using standard techniques (e.g., Wu 1997, Adams and Forrester 2005) as follows

$$\Delta_{ij} := \{(x_j, \omega_i, z_{ij}) \in \mathbb{B} \times \mathbb{R}_+^2 \mid z_{ij} \leqslant \omega_i^U x_j, z_{ij} \geqslant \omega_i^L x_j, z_{ij} \leqslant \omega_i + \omega_i^L(x_j - 1), z_{ij} \geqslant \omega_i + \omega_i^U(x_j - 1)\},$$

where ω_i^U and ω_i^L are an upper bound and a lower bound on ω_i, respectively, and note that $(x_j, w_i, z_{ij}) \in \Delta_{ij} \Leftrightarrow z_{ij} = w_i x_j$. Hence, $\text{RFP}_1[\mathcal{U}^{ab}]$ is equivalent to the robust linear problem

$$\max_{\substack{x \in X \\ \omega, \mu, z \geqslant 0}} \sum_{i \in I} \mu_i \tag{19}$$

$$\text{s.t. } \mu_i \leqslant a_{i0}\omega_i + \sum_{j \in J} \widetilde{a}_{ij} z_{ij} \qquad \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall i \in I$$

$$b_{i0}\omega_i + \sum_{j \in J} \widetilde{b}_{ij} z_{ij} \leqslant 1 \qquad \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I$$

$$(x_j, \omega_i, z_{ij}) \in \Delta_{ij} \qquad \forall i \in I, j \in J.$$

Following the approach of Bertsimas and Sim (2004), the robust linear problem (19) can be transformed into an MILP reformulation of $\text{RFP}[\mathcal{U}^{ab}]$ as follows.

$$\max \sum_{i \in I} \mu_i \tag{$\text{MILP}_1[\mathcal{U}^{ab}]$}$$

$$\text{s.t. } \mu_i - \sum_{j \in J} a_{ij} z_{ij} + \sum_{j \in J} \beta_{ij} + \Gamma_i^a \alpha_i \leqslant a_{i0}\omega_i \qquad \forall i \in I$$

$$b_{i0}\omega_i + \sum_{j \in J} b_{ij} z_{ij} + \sum_{j \in J} \gamma_{ij} + \Gamma_i^b \lambda_i \leqslant 1 \qquad \forall i \in I$$

$$\alpha_i + \beta_{ij} \geqslant d_{ij}^a z_{ij} \qquad \forall i \in I, \forall j \in J$$

$$\lambda_i + \gamma_{ij} \geqslant d_{ij}^b z_{ij} \qquad \forall i \in I, \forall j \in J$$

$$x \in X,\ (x_j, \omega_i, z_{ij}) \in \Delta_{ij},\ \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \geqslant 0 \qquad \forall i \in I, \forall j \in J.$$

4.1.2 Reformulation 2 $(\textbf{MILP}_2[\mathcal{U}^{ab}])$.

As an alternative to the approach of §4.1.1, one could instead replace each ratio with an auxiliary variable. Let

$$\mu_i \leqslant \frac{a_{i0} + \sum_{j \in J} \widetilde{a}_{ij} x_j}{b_{i0} + \sum_{j \in J} \widetilde{b}_{ij} x_j} \qquad \forall i \in I,\ (\widetilde{a}_i, \widetilde{b}_i) \in \mathcal{U}_i^a \times \mathcal{U}_i^b.$$

Then we can write RFP$[\mathcal{U}^{ab}]$ as

$$\max_{\substack{x\in X,\\ \mu\geqslant 0}} \sum_{i\in I} \mu_i \qquad (\text{RFP}_2[\mathcal{U}^{ab}])$$

$$\text{s.t. } (b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j)\mu_i \leqslant a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j \qquad \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall \widetilde{b}_i \in \mathcal{U}_i^b, \forall i \in I.$$

Finally, after linearization of $x_j\mu_i$ using a variant of the set Δ_{ij} and applying the transformation of a robust linear problem to an MILP similar to the one used in §4.1.1, we find the MILP reformulation of RFP$_2[\mathcal{U}^{ab}]$.

$$\max \sum_{i\in I} \mu_i \qquad (\text{MILP}_2[\mathcal{U}^{ab}])$$

$$\text{s.t. } b_{i0}\mu_i - \sum_{j\in J} a_{ij}x_j + \sum_{j\in J} b_{ij}z_{ij} + \Gamma_i^a\alpha_i + \Gamma_i^b\lambda_i + \sum_{j\in J}\beta_{ij} + \sum_{j\in J}\gamma_{ij} \leqslant a_{i0} \qquad \forall i \in I$$

$$\alpha_i + \beta_{ij} \geqslant d_{ij}^a x_j \qquad \forall i \in I, \forall j \in J$$

$$\lambda_i + \gamma_{ij} \geqslant d_{ij}^b z_{ij} \qquad \forall i \in I, \forall j \in J$$

$$x \in X, (x_j, \mu_i, z_{ij}) \in \Delta_{ij}, \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \geqslant 0 \qquad \forall i \in I, \forall j \in J.$$

4.1.3 Binary-expansion reformulation $\big(\text{MILP}_2^{\log}[\mathcal{U}^{ab}]\big)$.

The third formulation considered uses a base-2 expansion (Borrero et al. 2016) to reduce the number of bilinear terms that require linearization. In the context of RFP, we employ this idea to reformulate RFP$_2[\mathcal{U}^{ab}]$.

Observe that for any $x \in X$ and worst-case realization $\widetilde{b}_i \in \mathcal{U}_i^b$, the term $\sum_{j\in J} \widetilde{b}_{ij}x_j$ is integer since the data are integral (Assumption 2). To ascertain the (logarithmic) number of additional variables needed, let $\max^r(H_i)$ return the r-th largest element in the set $H_i = \{d_{ij}^b \mid j \in J\}$. Then for all $i \in I$, we define π_i as follows

$$\pi_i := \Big\lfloor \log_2 \Big(\sum_{j\in J} |b_{ij}| + \sum_{r\leqslant \Gamma_i^b} \max{}^r(H_i) \Big) \Big\rfloor + 1. \qquad (20)$$

We then define the binarization variables $w_{ik} \in \mathbb{B}$ for all $k \in K_i := \{1, 2, \dots, \pi_i\},\ i \in I$. We also define $\bar{B}_i := \sum_{j\in J, b_{ij}<0} |b_{ij}|$. Observe that $\sum_{j\in J} \widetilde{b}_{ij}x_j + \bar{B}_i \geqslant 0$ for any $x \in X$ and $\widetilde{b}_i \in \mathcal{U}_i^b$. Replacing the terms $\sum_{j\in J} \widetilde{b}_{ij}x_j$ with $-\bar{B}_i + \sum_{k=1}^{\pi_i} 2^{k-1}w_{ik}$ for all $i \in I$ in RFP$_2[\mathcal{U}^{ab}]$, yields

$$\max \sum_{i\in I} \mu_i \qquad (\text{RFP}_2^{\log}[\mathcal{U}^{ab}])$$

$$\text{s.t. } (b_{i0} - \bar{B}_i + \sum_{k\in K_i} 2^{k-1}w_{ik})\mu_i \leqslant a_{i0} + \sum_{j\in J} \widetilde{a}_{ij}x_j \qquad \forall \widetilde{a}_i \in \mathcal{U}_i^a,\ \forall i \in I$$

$$\sum_{j\in J} \widetilde{b}_{ij} x_j + \bar{B}_i \leqslant \sum_{k\in K_i} 2^{k-1} w_{ik} \qquad \forall \widetilde{b}_i \in \mathcal{U}_i^b,\ \forall i \in I$$

$$x \in X, w_{ik} \in \mathbb{B}, \mu_i \geqslant 0 \qquad \forall k \in K_i, \forall i \in I.$$

Let $z_{ik} = w_{ik}\mu_i$. By using a variant of the set Δ_{ij} to linearize bilinear terms of model $\text{RFP}_2^{\log}[\mathcal{U}^{ab}]$, i.e., $w_{ik}\mu_i$, and applying the transformation of a robust linear problem to an MILP similar to the one used in §4.1.1, $\text{RFP}_2^{\log}[\mathcal{U}^{ab}]$ can be reformulated as the following MILP.

$$\max \sum_{i\in I} \mu_i \qquad (\text{MILP}_2^{\log}[\mathcal{U}^{ab}])$$

$$\text{s.t. } (b_{i0} - \bar{B}_i)\mu_i + \sum_{k\in K_i} 2^{k-1} z_{ik} - \sum_{j\in J} a_{ij} x_j + \sum_{j\in J} \beta_{ij} + \Gamma_i^a \alpha_i \leqslant a_{i0} \qquad \forall i \in I$$

$$-\sum_{k\in K_i} 2^{k-1} w_{ik} + \sum_{j\in J} b_{ij} x_j + \bar{B}_i + \sum_{j\in J} \gamma_{ij} + \Gamma_i^b \lambda_i \leqslant 0 \qquad \forall i \in I$$

$$\alpha_i + \beta_{ij} \geqslant d_{ij}^a x_j \qquad \forall i \in I, \forall j \in J$$

$$\lambda_i + \gamma_{ij} \geqslant d_{ij} x_j \qquad \forall i \in I, \forall j \in J$$

$$x \in X,\ w_{ik} \in \mathbb{B},\ (w_{ik}, \mu_i, z_{ik}) \in \Delta_{ik},\ \beta_{ij}, \gamma_{ij}, \alpha_i, \lambda_i, \mu_i \geqslant 0 \qquad \forall i \in I, \forall j \in J, \forall k \in K_i.$$

Remark 3 It is also possible to develop a binary-expansion reformulation of $\text{RFP}_1[\mathcal{U}^{ab}]$. However, based on our experiments such a formulation performs poorly in computations; also, refer to Borrero et al. (2016) and Mehmanchi et al. (2019) for an analogous comparison regarding deterministic FP. Hence, we omit this formulation for brevity. □

4.2 Joint uncertainty sets

We now present MILP formulations of RFP[$\mathcal{U}$] for the joint uncertainty sets $\mathcal{U} \in \{\mathcal{U}^{\overline{ab}}, \mathcal{U}_{=}^{\overline{ab}}, \mathcal{U}_{\propto}^{\overline{ab}}, \mathcal{U}^{\overline{a}}\}$. Toward this goal, we use the results of §3.2 to develop MILPs for multiple-ratio RFP[$\mathcal{U}^{\overline{ab}}$], RFP[$\mathcal{U}_{=}^{\overline{ab}}$], and RFP[$\mathcal{U}_{\propto}^{\overline{ab}}$]; see §4.2.1. For RFP[$\mathcal{U}^{\overline{a}}$] we use a similar approach to the one used in §4.1.1, see §4.2.2. Note that, for the joint uncertainty sets we cannot take the advantage of the binary-expansion technique, either due to dependencies in the uncertainty sets, or because it does not reduce the number of bilinear terms for the joint cases.

4.2.1 Reformulation for RFP[$\mathcal{U}$] when $\mathcal{U} \in \{\mathcal{U}^{\overline{ab}}, \mathcal{U}_{=}^{\overline{ab}}, \mathcal{U}_{\propto}^{\overline{ab}}\}$ (MILP[$\mathcal{U}^{\overline{ab}}$], MILP[$\mathcal{U}_{=}^{\overline{ab}}$], and MILP[$\mathcal{U}_{\propto}^{\overline{ab}}$]).

By Propositions 4, 5, and 6 it can be verified that multiple-ratio RFP[$\mathcal{U}$] under joint uncertainties $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_{=}^{\overline{ab}}$, and $\mathcal{U}_{\propto}^{\overline{ab}}$ can be represented as the following problem.

$$\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geqslant 0}} \sum_{i\in I} \mu_i \qquad (21)$$

$$\text{s.t. } (b_{i0} + \sum_{j\in J} b_{ij}x_j)\mu_i + \Gamma_i\alpha_i + \sum_{j\in J}\beta_{ij} + \sum_{j\in J}\gamma_{ij} \leqslant a_{i0} + \sum_{j\in J} a_{ij}x_j \qquad \forall i \in I$$

$$\alpha_i + \beta_{ij} \geqslant (d^1_{ij} + d^2_{ij}\mu_i)x_j, \quad \alpha_i + \gamma_{ij} \geqslant (d^3_{ij} + d^4_{ij}\mu_i)x_j \qquad \forall i \in I, \forall j \in J,$$

for some $d^1, d^2, d^3, d^4 \in \mathbb{Z}^{m\times n}$. By linearizing the bilinear terms $x_j\mu_i$, problem (21) can be reformulated as an equivalent MILP.

$$\max_{\substack{x\in X,\\ \mu,\alpha,\beta,\gamma\geqslant 0}} \sum_{i\in I}\mu_i \tag{22}$$

$$\text{s.t. } b_{i0}\mu_i - \sum_{j\in J} a_{ij}x_j + \sum_{j\in J} b_{ij}z_{ij} + \Gamma_i\alpha_i + \sum_{j\in J}\beta_{ij} + \sum_{j\in J}\gamma_{ij} \leqslant a_{i0} \qquad \forall i \in I$$

$$\alpha_i + \beta_{ij} \geqslant d^1_{ij}x_j + d^2_{ij}z_{ij}, \quad \alpha_i + \gamma_{ij} \geqslant d^3_{ij}x_j + d^4_{ij}z_{ij} \qquad \forall i \in I, \forall j \in J$$

$$x \in X, (x_j, \mu_i, z_{ij}) \in \Delta_{ij}, \beta_{ij}, \gamma_{ij}, \alpha_i, \mu_i \geqslant 0 \qquad \forall i \in I, \forall j \in J.$$

Specifically, if we let $d^1_j = d^a_j, d^2_j = d^3_j = 0$, and $d^4_j = d^b_j$ for all $j \in J$, then problem (22) is an equivalent MILP reformulation of RFP[$\mathcal{U}^{\overline{ab}}$] denoted by MILP[$\mathcal{U}^{\overline{ab}}$]. Similarly, letting $d^1_j = d^a_j, d^2_j = d^b_j, d^3_j = d^4_j = 0$ and $d^1_j = -d^3_j = -d^a_j, d^2_j = -d^4_j = d^b_j$ for all $j \in J$ in (22), lead to equivalent MILP reformulations of RFP[$\mathcal{U}^{\overline{ab}}_{=}$] and RFP[$\mathcal{U}^{\overline{ab}}_{\propto}$] indicated by MILP[$\mathcal{U}^{\overline{ab}}_{=}$] and MILP[$\mathcal{U}^{\overline{ab}}_{\propto}$], respectively. Finally, note that in MILP[$\mathcal{U}^{\overline{ab}}_{=}$] since $d^3_j = d^4_j = 0$ variable γ_{ij} and constraint $\alpha_i + \gamma_{ij} \geqslant d^3_{ij}x_j + d^4_{ij}z_{ij}$ can be removed for all $i \in I, j \in J$; see Table 1 for the size of formulations.

4.2.2 Reformulation for RFP[$\mathcal{U}^{\overline{a}}$] (MILP[$\mathcal{U}^{\overline{a}}$]).

Let ω as in §4.1.1, define a new variable $\mu \leqslant \sum_{i\in I}(a_{i0} + \sum_{j\in J}\widetilde{a}_{ij}x_j)\omega_i$ for all $\widetilde{a} \in \mathcal{U}^{\overline{a}}$, and write RFP[$\mathcal{U}^{\overline{a}}$] as

$$\max_{x\in X,\omega,\mu\geqslant 0} \mu$$

$$\text{s.t. } \mu \leqslant \sum_{i\in I} a_{i0}\omega_i + \sum_{i\in I}\sum_{j\in J}\widetilde{a}_{ij}x_j\omega_i \qquad \forall \widetilde{a} \in \mathcal{U}^{\overline{a}}$$

$$b_{i0}\omega_i + \sum_{j\in J} b_{ij}x_j\omega_i \leqslant 1 \qquad \forall i \in I.$$

Letting $z_{ij} = x_j\omega_i$ and u be the indicator variables of set $S_i(\widetilde{a})$, we obtain

$$\max_{\substack{x\in X,\\ \mu,\omega\geqslant 0}} \mu$$

$$\text{s.t. } \mu - \sum_{i\in I} a_{i0}\omega_i - \sum_{i\in I}\sum_{j\in J} a_{ij}z_{ij} + \max_{u\in V}\left\{\sum_{j\in J} d^a_{ij}z_{ij}u_{ij}\right\} \leqslant 0 \qquad \forall i \in I$$

$$b_{i0}\omega_i + \sum_{j\in J} b_{ij}z_{ij} \leqslant 1 \qquad \forall i \in I$$

$$(x_j, \omega_i, z_{ij}) \in \Delta_{ij} \qquad \forall i \in I,\ j \in J,$$

where V is the polytope defined by the constraints

$$\sum_{i \in I} \sum_{j \in J} u_{ij} \leqslant \Gamma \qquad (\alpha)$$

$$0 \leqslant u_{ij} \leqslant 1 \qquad \forall i \in I, j \in J.\ (\beta_{ij})$$

Using LP-duality for the inner maximization problem, we obtain the MILP formulation:

$$\max \mu \qquad (\text{MILP}[\mathcal{U}^{\overline{a}}])$$

$$\text{s.t. } \mu - \sum_{i \in I} a_{i0}\omega_i - \sum_{i \in I} \sum_{j \in J} a_{ij} z_{ij} + \Gamma\alpha + \sum_{i \in I} \sum_{j \in J} \beta_{ij} \leqslant 0$$

$$b_{i0}\omega_i + \sum_{j \in J} b_{ij} z_{ij} \leqslant 1 \qquad \forall i \in I$$

$$\alpha + \beta_{ij} \geqslant d^a_{ij} z_{ij} \qquad \forall i \in I, \forall j \in J$$

$$x \in X,\ (x_j, \omega_i, z_{ij}) \in \Delta_{ij},\ \beta_{ij}, \alpha, \mu, \omega_i \geqslant 0 \qquad \forall i \in I, \forall j \in J.$$

4.3 Problems sizes and MILP enhancement $\left(\textbf{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]\right)$

Table 1 shows the number of variables and constraints for all MILP reformulations provided in §4.1 and §4.2. This table also includes the sizes of the well-known MILPs for FP, denoted by FP_1 (Li 1994, Tawarmalani et al. 2002, Wu 1997) and FP_2 (Tawarmalani et al. 2002), as well as their respective binary-expansion versions (Borrero et al. 2016), denoted by FP_3 and FP_4.

Later in §5.2.3 we observe that, among the MILPs for the disjoint uncertainty, $\text{MILP}_1[\mathcal{U}^{ab}]$ typically has the best LP relaxation and $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$ often has the best performance due to a reduced number of variables and constraints - see Table 1. Hence, we enhance $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$ by adding the valid inequality $\sum_{i \in I} \mu_i \leqslant z_{LP}^{MILP_1[\mathcal{U}^{ab}]}$ to $\text{MILP}^{\log}_2[\mathcal{U}^{ab}]$ where $z_{LP}^{MILP_1[\mathcal{U}^{ab}]}$ is the objective function value of the LP relaxation of $\text{MILP}_1[\mathcal{U}^{ab}]$, and we call the new formulation $\text{MILP}^{\log}_{2'}[\mathcal{U}^{ab}]$. In the deterministic fractional programming, Borrero et al. (2016) made a similar observation regarding FP_1 and FP_4. They call the new formulation $\text{FP}_{4'}$, and we compare its performance versus the performances of the developed MILPs for the disjoint uncertainty in the next section.

Table 1: Sizes of the MILPs for nominal problems FP_1 to FP_4, and the robust problems, where n and m are defined as in FP, c is the number of constraints defining X, and π_i is defined as in (20). Moreover, $\theta_i^a := \lfloor \log_2(\sum_{j\in J}|a_{ij}|) \rfloor + 1$ and $\theta_i^b := \lfloor \log_2(\sum_{j\in J}|b_{ij}|) \rfloor + 1$.

MILP reformulation	No. of continuous variables	No. of binary variables	No. of linear constraints
	Nominal reformulations		
FP_1	$m(n+1)$	n	$m(4n+1)+c$
FP_2	$m(n+1)$	n	$m(4n+1)+c$
FP_3	$m+\sum_{i\in I}(\theta_i^a+\theta_i^b)$	$n+\sum_{i\in I}(\theta_i^a+\theta_i^b)$	$3m+4\sum_{i\in I}(\theta_i^a+\theta_i^b)+c$
FP_4	$m+\sum_{i\in I}\theta_i^b$	$n+\sum_{i\in I}\theta_i^b$	$2m+4\sum_{i\in I}\theta_i^b+c$
	Robust reformulations (Disjoint)		
$MILP_1[\mathcal{U}^{ab}]$	$m(3n+4)$	n	$m(6n+2)+c$
$MILP_2[\mathcal{U}^{ab}]$	$m(3n+3)$	n	$m(6n+1)+c$
$MILP_2^{\log}[\mathcal{U}^{ab}]$	$m(2n+3)+\sum_{i\in I}\pi_i$	$n+\sum_{i\in I}\pi_i$	$m(2n+2)+4\sum_{i\in I}\pi_i+c$
	Robust reformulations (Joint)		
$MILP_2[\mathcal{U}^{\overline{ab}}]$ & $MILP_2[\mathcal{U}^{\overline{ab}}_{\propto}]$	$m(3n+2)$	n	$m(6n+1)+c$
$MILP_2[\mathcal{U}^{\overline{ab}}_{=}]$	$m(2n+2)$	n	$m(5n+1)+c$
$MILP[\mathcal{U}^{\overline{a}}]$	$m(2n+1)+2$	n	$m(5n+1)+c+1$

4.4 Insights on the price of robustness

In robust linear optimization when uncertain coefficients are symmetric, bounded and independent random variables, Bertsimas and Sim (2004) provide a probabilistic guarantee for each constraint violation. Next, we exploit their approach to establish somewhat similar results for RFPs under dis/joint uncertainty sets.

Let $x^\star$ and $\mu_i^\star$ denote a robust optimal solution and the robust value of the i-th ratio in $RFP[\mathcal{U}]$, respectively. By using the binomial distribution

$$B(r,P) = \frac{1}{2^r}\left\{(1-\nu+\lfloor\nu\rfloor)\binom{r}{\lfloor\nu\rfloor} + \sum_{j=\lfloor\nu\rfloor+1}^{r}\binom{r}{j}\right\},$$

for $\nu = (P+r)/2$, and $r, P \in \mathbb{Z}_+$, we show the probability that $\mu_i^\star$ overestimates the true value of the i-th ratio for random variables $\widetilde{a}$ and $\widetilde{b}$ is bounded above.

Proposition 10 *Let $\widetilde{a}$ and $\widetilde{b}$ be symmetric, bounded, and independent random variables, i.e., $\widetilde{a}_{ij} = a_{ij} + \eta_{ij} d_{ij}^a$ and $\widetilde{b}_{ij} = b_{ij} + \eta_{i,j+n} d_{ij}^b$, where $\eta_{ij}, \eta_{i,j+n} \in [-1,1]$, for all $i \in I, j \in J$, are independently* *distributed random variables. For each $i \in I$, in* RFP[$\mathcal{U}$]

$$(i)\ \textit{if}\ \mathcal{U} = \mathcal{U}^{ab},\ \textit{then}\ Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star}\right) \leqslant B\big(2n, \Gamma_i^a + \Gamma_i^b\big), \qquad \Gamma_i^a, \Gamma_i^b \in \{0,\ldots,n\};$$

$$(ii)\ \textit{if}\ \mathcal{U} = \mathcal{U}^{\overline{ab}},\ \textit{then}\ Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star}\right) \leqslant B\big(2n, \Gamma_i\big), \qquad \Gamma_i \in \{0,\ldots,2n\};$$

additionally,

$$(iii)\ \textit{if}\ \mathcal{U} = \mathcal{U}^{\overline{a}},\ \textit{then}\ Pr\left(\sum_{i\in I} \mu_i^\star > \sum_{i\in I} \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} b_{ij} x_j^\star}\right) \leqslant B\big(m\cdot n, \Gamma\big), \qquad \Gamma \in \{0,\ldots,m\cdot n\}.$$

Proposition 11 *Let $\widetilde{a}$ and $\widetilde{b}$ be symmetric and bounded random variables, i.e., $\widetilde{a}_{ij} = a_{ij} + \eta_{ij} d_{ij}^a$ and $\widetilde{b}_{ij} = b_{ij} + \eta_{ij} d_{ij}^b$, where $\eta_{ij} \in [-1,1]$, for all $i \in I, j \in J$, are independently distributed random variables. For each $i \in I$, in* RFP[$\mathcal{U}$]

$$\textit{if}\ \mathcal{U} = \mathcal{U}_{\propto}^{\overline{ab}},\ \textit{then}\ Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star}\right) \leqslant B\big(n, \Gamma_i\big), \qquad \Gamma_i \in \{0,\ldots,n\}.$$

See Appendix A for the proofs of Propositions 10 and 11. Evidently, as the decision-maker is more conservative and selects larger level of uncertainty (Γ), the probability that $\mu_i^\star$ is larger than the value of the i-th ratio for $x^\star$ and random variables $\widetilde{a}$ and $\widetilde{b}$ is smaller. Note that we do not derive a similar upper-bound when $\mathcal{U} = \mathcal{U}_{=}^{\overline{ab}}$ since we cannot satisfy the key assumption that random variables η are independently distributed.

5 Computational results

The computational experiments in this section encompass a case study of a particular assortment problem (see §5.1), as well as experiments on instances with synthetic data to evaluate the performance of our MILP reformulations (see §5.2). In both of the following subsections, we describe the relevant test instances, compare the robust and nominal solutions, and discuss relevant aspects of the solutions. Our experiments were performed using CPLEX 12.7.1 IBM on an 8-core CPU (3.7 GHz) with 32 GB of RAM.

5.1 Case study: robust assortment optimization for frozen pizza

Assortment optimization problems arise in many applications such as retailing, revenue management problems, and online advertising. Assortment optimization with uncertainty considerations is a growing area of research; in addition to Rusmevichientong and Topaloglu (2012), discussed in §1.1 and §3.2, Bertsimas and Mišić (2016) and Désir et al. (2019) have proposed robust optimization approaches for different classes of assortment optimization problems.

Our case study, outlined next, optimizes an assortment problem for a real retailer of frozen pizza studied by Keane and Wada (2012); the data is available at `http://cblib.zib.de`. The objective of the assortment problem is to maximize revenue for a company, given a large number of potential product offerings, associated revenues for those offerings, and estimations of customer preferences between those offerings. Additionally, the customers are divided into several different classes, thus the mixed-multinomial logit choice model is a natural fit for the problem.

5.1.1 Test instances

The test instances comprise customer preference data on frozen pizzas from Keane and Wada (2012). In particular, there are 130 potential product offerings divided into 5 tiers of revenue (\$1.49, \$1.75, \$1.79, \$1.89, and \$2.75), and there are 3 classes of customers. Thus, the problem is an instance of (3) with $m = 3$ ratios and $n = 130$ variables. The same data was used for each test, with variations in the type of uncertainty set, as well as the level of uncertainty Γ. We fixed $d^a_{ij} = 0.5a_{ij}$, and $d^b_{ij} = 0.5b_{ij}$ (where relevant) for all uncertainty sets.

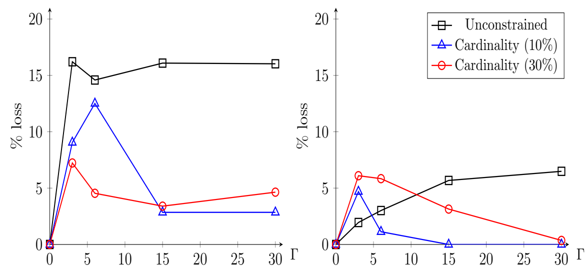
For the case study we consider four robust problems; specifically, we consider unconstrained ($X = \mathbb{B}^n$) and cardinality-constrained ($X = \{x \in \mathbb{B}^n \mid \sum_{j \in J} x_j \leqslant k\}$) versions of RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ and RFP$[\mathcal{U}^{\overline{a}}]$ which are a natural fit for this application. Uncertainty in customer preferences (ρ_{ij}) and revenues (r_{ij}) can be captured by the matched effects, $\mathcal{U}^{\overline{ab}}_{\propto}$, and the single budget, $\mathcal{U}^{\overline{a}}$, uncertainty sets respectively; see §2. With respect to the feasible region, we test both the unconstrained case - for an online retailer with the ability to market many options - as well as two sizes of cardinality constraint: $k = 13$ and $k = 39$, corresponding to 10% and 30% of the 130 variables, respectively. The latter problem classes correspond to a small and large retailer, respectively, where there is a physical limitation on the number of products which can be offered to customers.

5.1.2 The price of robustness

The value in the robust approach is demonstrated by checking the performance of the nominal (optimal) solution in the uncertain environment, and vice versa. These results are shown in Figures 1 and 2. Figure 1 shows the relative decrease in the robust objective function value when the optimal nominal solution is used in the uncertain setting instead of the optimal robust solution (at the given uncertainty level) as "% loss". Figure 2 depicts the opposite case - the loss of using the robust optimal solution when the unknown coefficient take their nominal values. Thus, higher "% loss" in these two figures implies worse results.

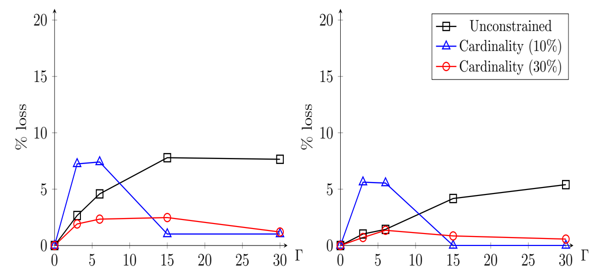
The results for the unconstrained case show that the nominal optimal solution performs worse in the robust setting than the robust solution does in the deterministic environment. Additionally, we observe that, as the level of uncertainty increases for both uncertainty sets, the percentage loss ("% loss") of using both nominal and robust solutions in the opposite setting increases.

The cardinality results exhibit a somewhat different pattern of behavior, although we continue to see that the robust solution performs better in the nominal setting than vice versa. For the cardinality feasible regions, in both uncertainty sets, the nominal and robust solutions are different for small to moderate values of Γ_i, but for the larger values of Γ_i the nominal and robust solutions become similar again. The reason for this behavior is that, as Γ_i grows, all (or almost all) of the variable coefficients in the optimal robust solution are reduced by uncertainty; that is, Γ_i is close to or larger than the size of the cardinality k. Since each uncertain coefficient is reduced by 50% (see above), the most favorable products without uncertainty reduction remain the most favorable products when everything (within the limited cardinality size k) is reduced 50% by uncertainty.



(a) $\mathcal{U}_{\propto}^{\overline{ab}}$ (b) $\mathcal{U}^{\overline{a}}$

Figure 1: Decrease in the robust optimal objective function value by plugging a nominal optimal solution into the robust problem for frozen pizza. Specifically, let $Z^{\star}_{\mathcal{U}}$ denote the optimal objective function value of RFP[$\mathcal{U}$]. Additionally, let $\hat{Z}_{\mathcal{U}} = \min_{(\widetilde{a},\widetilde{b})\in\mathcal{U}} \sum_{i\in I} \frac{a_{i0}+\widetilde{a}_i^T x^{\star}}{b_{i0}+\widetilde{b}_i^T x^{\star}}$ where $x^{\star}$ is a nominal optimal solution. Then % loss for each Γ is $\frac{Z^{\star}_{\mathcal{U}}-\hat{Z}_{\mathcal{U}}}{Z^{\star}_{\mathcal{U}}} \times 100\%$.



(a) $\mathcal{U}_{\propto}^{\overline{ab}}$ (b) $\mathcal{U}^{\overline{a}}$

Figure 2: Decrease in the nominal optimal objective function value by plugging a robust optimal solution into the nominal problem for frozen pizza. Specifically, let $Z^{\star}$ denote the optimal objective function value of FP. Additionally, let $\hat{Z} = \sum_{i\in I} \frac{a_{i0}+a_i^T x^{\star}_{\mathcal{U}}}{b_{i0}+b_i^T x^{\star}_{\mathcal{U}}}$ where $x^{\star}_{\mathcal{U}}$ is an optimal solution of RFP[$\mathcal{U}$]. Then % loss for each Γ is $\frac{Z^{\star}-\hat{Z}}{Z^{\star}} \times 100\%$.

5.1.3 Solution Analysis

A salient feature of the unconstrained robust solutions in our case study is that, under both uncertainty sets $\mathcal{U}^{\overline{a}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$, the robust optimal solution contains more variables with $x_j = 1$ as Γ_i increases, see Figure 3. For example, under $\mathcal{U}^{\overline{a}}$, each increase in Γ_i results in roughly 10 more variables included in the optimal solution. With $\Gamma_i = 0$, the optimal solution contains more variables from the highest 2 revenue classes, and as uncertainty increases, more choices from lower revenue classes become part of the solution. This can be explained by observing that, with increasing uncertainty, the Γ_i most favorable products are the ones with their coefficients changed by uncertainty. Hence, the reduction in preference and/or revenue brings these products more in line with the lesser revenue products, which then become part of the optimal solution.

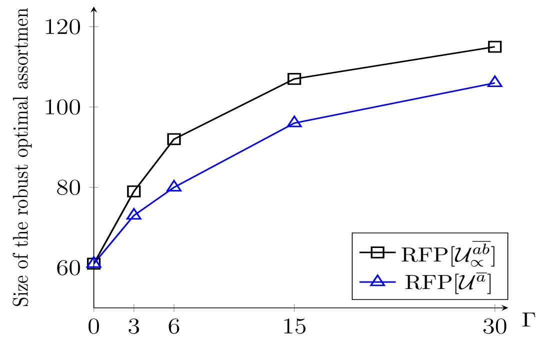


Figure 3: Size of the unconstrained robust optimal assortment versus the the level of uncertainty (Γ).

However, somewhat counter-intuitively, given a cardinality size of 13, the optimal solutions (both nominal and robust) consist of variables mostly from the second-highest revenue tier, \$1.89. When the cardinality size is expanded to 39, more variables from both the first and second highest revenue tiers become part of the optimal solutions. An examination of the data shows that the highest revenue tier items are generally (significantly) less-preferred (they have smaller values of preference ρ) than the more reasonably priced second tier items, hence the second tier items show themselves to be superior generators of revenue.

The outlined observations for (either constrained or unconstrained) multi-class deterministic and robust assortment optimizations can be compared to the previous results in the literature for unconstrained single-class deterministic and robust assortment optimizations. For example, assuming (without loss of generality) that the revenues are ordered such that $r_1 \geqslant r_2 \geqslant \ldots r_n$, Talluri and Van Ryzin (2004) show that the unconstrained single-class nominal assortment optimization problems under multi-nominal logit choice model are "revenue-ordered assortments", i.e., there exists a set of optimal solutions of the form $\{1, 2, \ldots, j\}$, for some index j. Rusmevichientong and Topaloglu (2012) derive a similar result for the robust case, where uncertainty is limited to customer preferences.

5.2 Synthetic instances

We now conduct extensive computational experiments on randomly generated instances to gain insights into the performance of the disjoint and joint MILP reformulations provided in §4. Additionally, we evaluate the nominal solution in a robust setting, and vice versa, to determine the "price of robustness." In §5.2.1, we outline the structure and parameters of the computational experiments. The price of robustness is studied in §5.2.2. We describe the results for the disjoint and joint uncertainty sets in §5.2.3 and §5.2.4, respectively.

5.2.1 Test instances

We chose combinations of $m \in \{1, 3, 5\}$ and $n \in \{50, 100, 150\}$. The uncertainty parameters Γ_i^a, Γ_i^b were chosen based on m, n, and the relevant uncertainty set $\mathcal{U}$, and these choices are given in the appropriate section below. For each choice of m, n, Γ and a particular constraint type (detailed below), five instances were sampled and the results averaged. The instances were each given a time limit of 1 hour (3600 seconds).

The LP relaxation quality, denoted by R in the following tables, is computed by $\frac{Z^*_{LP}}{Z^*}$, where Z^*_{LP} is the optimal solution of the LP continuous relaxation, and Z^* is the optimal integer solution (if Z^* cannot be found within the time limit by any solution approach, then the best-known integer solution is used in place of Z^*). Moreover, the optimality gap is denoted by G and is computed by $\frac{UB-LB}{LB}$, where UB and LB are the upper- and the lower-bound on the optimal objective function value, respectively.

Coefficients sampling. The coefficients a_{ij} and b_{ij} were each sampled from a (discrete) $U[0, 20]$ distribution, except for b_{i0} which was sampled from a $U[1, 20]$. Subsequently, each d^a_{ij} and d^b_{ij} were sampled from $U[0, \lfloor \frac{1}{2} a_{ij} \rfloor]$ or $U[0, \lfloor \frac{1}{2} b_{ij} \rfloor]$, respectively. Note that these parameter choices satisfy Assumptions 1 and 2.

Constraints. Three different constraint types were used: unconstrained (denoted by U in the following tables), cardinality-constrained (C), and knapsack-constrained (K). The cardinality constraint is of the *equality* type so that $\sum_{j \in J} x_j = k$, where $k = \frac{2}{5} n$. The knapsack constraint was of the *inequality* type, structured so that $\sum_{j \in J} k_j x_j \leqslant k$, where k_j was sampled from a $U[1, 10]$ distribution, and $k = 2n$.

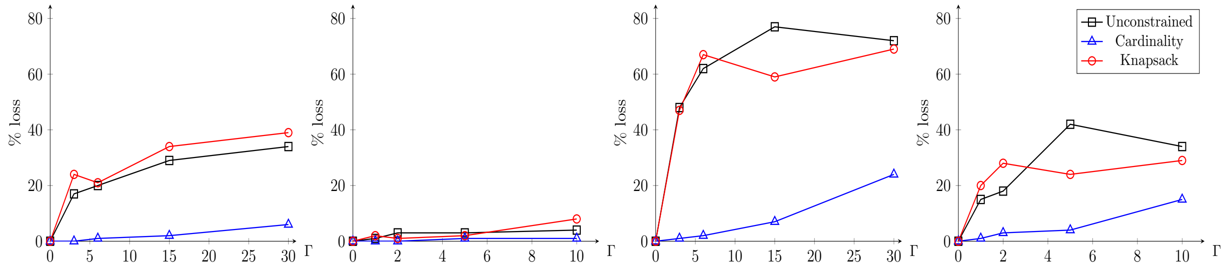
Linearization Bounds. For $\text{MILP}_1[\mathcal{U}^{ab}]$, note that $\omega^L_i = 0$ and $\omega^U_i = 1$ are valid bounds. Similar (not necessarily tight) lower and upper bound computations were performed for the other linearization procedures.

5.2.2 The price of robustness

Herein, we demonstrate the value of the robust approach; that is, we show that ignoring the possibility of uncertain data can be more costly than being conservative. In Figures 4 and 5, the "small" d^a_{ij} and d^b_{ij} were sampled using the procedure described in §5.2.1. The "large" d^a_{ij} and d^b_{ij} in these two figures were sampled by instead letting d^a_{ij} and d^b_{ij} be distributed as $U[\lfloor \frac{1}{2} a_{ij} \rfloor, a_{ij}]$ and $U[\lfloor \frac{1}{2} b_{ij} \rfloor, b_{ij}]$, respectively (that is, a higher level of uncertainty). Each sub-figure is comparable to the one directly above/below it.

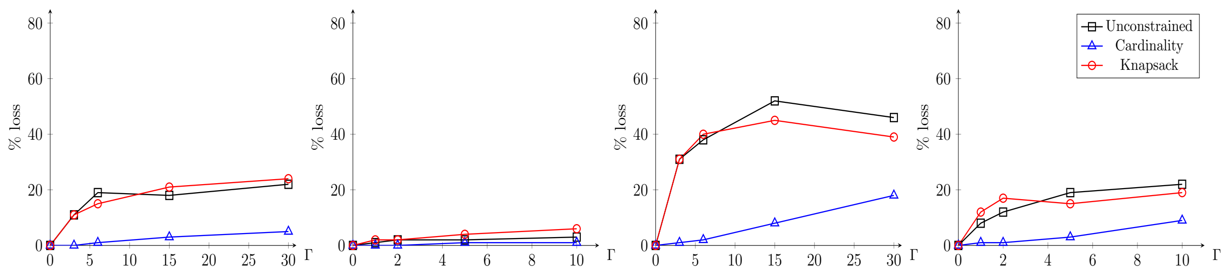
Figure 4 exhibits the benefit from applying the robust approach. It shows that under the worst-case scenario in the robust setting the objective function value attained by an optimal nominal solution can be rather poor and thus, illustrates how much the decision-maker can gain by taking into account the data uncertainty. More precisely, Figure 4 depicts the average decrease in the robust objective function value for $m \in \{1, 3, 5\}$, by inserting optimal x from the associated nominal problem into the robust problem. We observe that in case of large d, especially for the unconstrained and knapsack-constrained cases, inserting the nominal solution into the robust problem can cause a loss of up to 80%. This observation holds, albeit with scaled-down percentages, for the smaller d values as well.

Therefore, we conclude that the decision-maker has more to lose by failing to account for uncertainty than she does by being over-conservative. Simply speaking, if the decision-maker is overly conservative (chooses the Γ_i, for all $i \in I$, too large), then the loss on the objective function is outweighed by the amount she would lose by incorrectly ignoring the uncertainty (i.e., assuming Γ_i=0 for all $i \in I$). These results are similar to those of robust linear problems - see, e.g., Bertsimas and Sim (2003).



(a) "small" d, $\mathcal{U}^{ab}$ (b) "small" d, $\mathcal{U}^{\overline{a}}$ (c) "large" d, $\mathcal{U}^{ab}$ (d) "large" d, $\mathcal{U}^{\overline{a}}$

Figure 4: Average decrease in the robust optimal objective function value by plugging a nominal optimal solution into the robust problem for the synthetic data and $n = 150$. Specifically, let $Z^\star_{\mathcal{U}}$ denote the optimal objective function value of RFP[$\mathcal{U}$]. Additionally, let $\hat{Z}_{\mathcal{U}} = \min_{(\widetilde{a},\widetilde{b})\in\mathcal{U}} \sum_{i\in I} \frac{a_{i0}+\widetilde{a}_i^T x^\star}{b_{i0}+\widetilde{b}_i^T x^\star}$ where $x^\star$ is a nominal optimal solution. Then % loss for each Γ is the average of $\frac{Z^\star_{\mathcal{U}}-\hat{Z}_{\mathcal{U}}}{Z^\star_{\mathcal{U}}} \cdot 100$ over five test instances and three ratio sizes $m \in \{1, 3, 5\}$.



(a) "small" d, $\mathcal{U}^{ab}$ (b) "small" d, $\mathcal{U}^{\overline{a}}$ (c) "large" d, $\mathcal{U}^{ab}$ (d) "large" d, $\mathcal{U}^{\overline{a}}$

Figure 5: Average decrease in the nominal optimal objective function value by plugging a robust optimal solution into the nominal problem for the synthetic data and $n = 150$. Specifically, let $Z^\star$ denote the optimal objective function value of FP. Additionally, let $\hat{Z} = \sum_{i\in I} \frac{a_{i0}+a_i^T x^\star_{\mathcal{U}}}{b_{i0}+b_i^T x^\star_{\mathcal{U}}}$ where $x^\star_{\mathcal{U}}$ is an optimal solution of RFP[$\mathcal{U}$]. Then % loss for each Γ is the average of $\frac{Z^\star-\hat{Z}}{Z^\star} \cdot 100$ over five test instances and three ratio sizes $m \in \{1, 3, 5\}$.

Figure 5 illustrates the opposite situation. That is, it shows how much the decision-maker can gain by having precise information about the problem data parameters. Specifically, Figure 5 depicts the average decrease in the nominal objective function value for $m \in \{1, 3, 5\}$, by inserting robust optimal solution x into the nominal problem. This insertion causes a loss of up to 50% in the objective function value of the nominal problem for large d in case of unconstrained and knapsack-constrained problems.

5.2.3 Disjoint reformulations

The results for the disjoint uncertainty set $\mathcal{U}^{ab}$, for single-ratio ($m = 1$) and multiple-ratio ($m \in \{3, 5\}$) as well as small ($n = 50$) and large size ($n = 150$) problems are presented in Tables 2 and 3; see also

Table 6 in Appendix B for medium size problems ($n = 100$). The uncertainty parameters were chosen so that $\Gamma_i^a = \Gamma_i^b$ for all $i \in I$, as stated in the tables. Observe that, in general, single-ratio problem is easy to solve for any of the constraint types. In particular, the binary reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ (recall §4.3) can handle the single-ratio setting, in that its average solution times for $m = 1$ in Tables 2 and 3 are the same as those for the nominal problem $\text{FP}_{4'}$.

As one would expect, increasing either m or n increases the difficulty of the fractional problem under disjoint uncertainty. In the nominal case (see, e.g., Tawarmalani et al. 2002), FP_1 generally outperforms the FP_2 across all constraint types for the multiple-ratio problem, and we find that this result carries over into the robust case. Specifically, for $m = 3$ and $m = 5$ in Table 3, $\text{MILP}_1[\mathcal{U}^{ab}]$ solves more than half of unconstrained and knapsack instances to optimality, while $\text{MILP}_2[\mathcal{U}^{ab}]$ solves almost none.

However, the *binarized* $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ outperforms both $\text{MILP}_1[\mathcal{U}^{ab}]$ and $\text{MILP}_2[\mathcal{U}^{ab}]$. In Table 3, note that when $m = 5$, $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ solves all except one of the unconstrained and knapsack instances to optimality, while $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ all solves the cardinality-constrained instances to optimality.

For the multiple-ratio problem, the cardinality-constrained problems seem to be the most computationally difficult (when the best solution approach is chosen for each constraint type), although this observation holds for the nominal case as well - see, for example the $m = 5$ case under constraint C in Table 3. On the other hand, the unconstrained problem is sometimes more difficult than the knapsack-constrained problem (as when $\Gamma_i = 1, m = 5$ in Table 2), though not universally so (e.g., $\Gamma_i = 2, m = 5$ in Table 2). Finally, we note that there appears to be no particular pattern or relationship between the level of uncertainty Γ_i^a, Γ_i^b and the computational difficulty for any of the parameter settings.

To summarize these results, we observe that $\text{MILP}_1[\mathcal{U}^{ab}]$ tends to have the best continuous relaxation bound. This observation is consistent with the earlier observations in the literature that the corresponding nominal reformulation FP_1 typically has the best relaxation quality; see, Borrero et al. (2016) and Mehmanchi et al. (2019). Nonetheless, this does not always (or even often) lead to superior solution times mainly due to the large size of the reformulation. In particular, for a small number of variables (Table 2), it appears that $\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$ is the best choice for disjoint cardinality-constrained problems, while $\text{MILP}_1[\mathcal{U}^{ab}]$ is usually better for unconstrained or knapsack-constrained models. However, as the number of variables increases (Table 3), the logarithmic reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ is a better choice for unconstrained and knapsack-constrained problems, although it appears that the binarized reformulations have weaker relaxation qualities than the corresponding original MILPs.

5.2.4 Joint reformulations

Results for joint uncertainty sets $\mathcal{U}^{\overline{ab}}$, $\mathcal{U}_=^{\overline{ab}}$ and $\mathcal{U}_\propto^{\overline{ab}}$ are given in Tables 4 and 5 for $n \in \{50, 150\}$ (see also Table 7 in Appendix B for $n = 100$). These tables also include the respective results of the most efficient reformulation for the disjoint uncertainty, i.e., $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ provided in Tables 2 and 3, to compare the difficulty of solving $\text{RFP}[\mathcal{U}]$ under disjoint versus joint uncertainty sets.

The uncertainty parameters were chosen based upon those chosen for the disjoint case. With Γ_i^a, Γ_i^b as the relevant disjoint uncertainty parameters, we have: for $\mathcal{U}^{\overline{ab}}$ that $\Gamma_i = 2\ \Gamma_i^a$, for $\mathcal{U}_=^{\overline{ab}}$ and $\mathcal{U}_\propto^{\overline{ab}}$ that

$\Gamma_i = \Gamma_i^a$, and for $\mathcal{U}^{\overline{a}}$ that $\Gamma = m\,\Gamma_i^a$ for problems with similar m, n.

Observe that MILP$_2[\mathcal{U}]$ performs similarly (with respect to solution times/optimality gap) on both the disjoint and joint uncertainty sets, by comparing the MILP$_2[\mathcal{U}^{ab}]$ of Table 2 with the relevant columns of Table 4, and conducting similar comparisons for columns of the 100 and 150 variable tables. However, for the disjoint uncertainty case we were able to use a binary reformulation (MILP$_{2'}^{\log}[\mathcal{U}^{ab}]$) to obtain superior solution times. Thus, the joint problems are generally more computationally difficult than the disjoint due to the absence of such a binary reformulation for them, which can be seen by comparing the first column of Tables 4 and 5 with the other columns.

Though the multiple-ratio problem utilized the entire hour of solution time allowed for most joint uncertainty sets, the single-ratio problem was solved quickly in most cases. Additionally, for the multiple-ratio problem, $\mathcal{U}^{\overline{a}}$ remains tractable for unconstrained and knapsack-constrained problems. In these two special cases, MILP$[\mathcal{U}^{\overline{a}}]$ typically solved the joint problem to optimality in a similar time as MILP$_2^{\log}[\mathcal{U}^{ab}]$ solved the disjoint instance. Finally, we observe that the cardinality constraint is universally difficult (as in the disjoint case) for all multiple-ratio instances with the joint uncertainty sets.

6 Conclusion

This paper addresses single- and multiple-ratio RFPs defined as the robust counterparts of the fractional 0-1 programming problems (FPs) under various disjoint and joint uncertainty sets. We demonstrate that single-ratio RFP, contrary to its deterministic counterpart, is NP-hard for a general polyhedral uncertainty set. However, if the uncertainties are in the form of the budgeted uncertainty sets, then we develop polynomial-time solution methods for single-ratio RFP provided that the nominal problem is polynomial-time solvable.

In particular, for the disjoint uncertainty set we propose an approach to solve single-ratio RFP by calling at most $(n+1)^2$ instances of FP. Moreover, in the case of joint uncertainty sets we show that single-ratio RFP can be solved by solving a polynomial number of instances of a linear binary problem. Therefore, if the latter admits a polynomial-time solution algorithm, then single-ratio RFP under dis/joint uncertainty sets is polynomial-time solvable, as well.

In case of multiple-ratio RFPs, we exploit the structure of the budgeted dis/joint uncertainty sets in order to propose various MILPs to solve them. Particularly, based on our extensive computational experiments it is noted that RFPs are more challenging to solve under the joint sets than the disjoint one, as the former cannot take advantage of the binary-expansion technique. Indeed, it appears that as the size of the problem increases, the binarized formulations are often a better choice for the robust problem under the disjoint uncertainty set.

We also explore the value of the robust optimal solution for instances with both the real and synthetic data and find that ignoring the data uncertainty can lead to poor decisions. These results coupled with the insights on the selection of budget(s) of uncertainties can provide guidance to consider the suitable solution method and level of uncertainty in practice.

Finally, it is worth mentioning that conic quadratic programming based reformulations can be used as an alternative solution approach to tackle deterministic fractional 0-1 programs, see, e.g., Şen et al. (2018), Atamtürk and Gómez (2018), and Mehmanchi et al. (2019) as they lead to strong convex relaxations. However, solving large-scale mixed-integer conic quadratic programs is still challenging in practice despite the recent advances in off-the-shelf commercial optimization software. Nevertheless, such approaches can be pursued as a promising future research direction for solving robust fractional 0-1 programs.

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Table 2: Results for disjoint reformulations. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 50$. In each row, among the solution methods that solve the most number of instances to optimality, the best average time and the best average gap (if #< 5) are in **bold**.

$n = 50$	Cons.	$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{\log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$			
$m = 1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.1	5	0.00	1.0	**0.0**	5	0.00	1.0	0.3	5	0.00	10.8	0.1	5	0.00	12.0	0.1	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 0$	C	0.2	5	0.00	1.0	1.6	5	0.00	1.9	0.3	5	0.00	16.8	**0.1**	5	0.00	17.1	**0.1**	5	0.00	1.9
	K	0.1	5	0.00	1.0	**0.0**	5	0.00	1.0	0.3	5	0.00	9.3	0.1	5	0.00	9.9	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.2	**0.1**	5	0.00	1.0	0.9	5	0.00	16.0	0.2	5	0.00	17.2	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 1$	C	0.2	5	0.00	1.2	5.1	5	0.00	1.7	0.7	5	0.00	26.0	0.2	5	0.00	27.6	**0.1**	5	0.00	1.7
	K	0.0	5	0.00	1.4	**0.1**	5	0.00	1.0	0.4	5	0.00	23.0	**0.1**	5	0.00	27.2	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.5	**0.1**	5	0.00	1.0	0.8	5	0.00	19.4	0.2	5	0.00	21.7	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 2$	C	0.2	5	0.00	1.2	0.8	5	0.00	1.4	0.7	5	0.00	16.4	0.2	5	0.00	16.8	**0.1**	5	0.00	1.4
	K	0.1	5	0.00	1.4	**0.1**	5	0.00	1.0	0.4	5	0.00	13.8	0.2	5	0.00	14.8	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	0.7	5	0.00	21.2	0.2	5	0.00	24.7	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 5$	C	0.2	5	0.00	1.5	2.1	5	0.00	1.4	0.6	5	0.00	19.6	0.2	5	0.00	20.0	**0.1**	5	0.00	1.4
	K	0.1	5	0.00	1.9	**0.1**	5	0.00	1.0	0.4	5	0.00	14.8	**0.1**	5	0.00	15.5	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	0.5	5	0.00	23.3	0.2	5	0.00	28.0	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 10$	C	0.2	5	0.00	1.7	4.9	5	0.00	2.2	0.7	5	0.00	32.6	0.2	5	0.00	34.6	**0.1**	5	0.00	2.2
	K	0.0	5	0.00	1.5	**0.0**	5	0.00	1.0	0.5	5	0.00	10.4	0.2	5	0.00	10.6	**0.0**	5	0.00	1.0
Average		0.1	5.0	0.00	1.4	1.0	5.0	0.00	1.3	0.5	5.0	0.00	18.2	0.2	5.0	0.00	19.8	**0.1**	5.0	0.00	1.3
$m = 3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.6	5	0.00	1.5	**0.3**	5	0.00	1.5	2,223.4	3	0.23	18.7	0.5	5	0.00	23.4	0.4	5	0.00	1.5
$\Gamma_i^a = \Gamma_i^b = 0$	C	1.0	5	0.00	1.2	1,798.0	4	0.02	3.1	3,600.0	0	1.07	26.4	1.0	5	0.00	27.8	**0.9**	5	0.00	3.1
	K	0.4	5	0.00	1.5	**0.2**	5	0.00	1.5	1,324.4	4	0.07	16.6	**0.2**	5	0.00	19.9	0.3	5	0.00	1.5
	U	0.4	5	0.00	2.2	**1.1**	5	0.00	1.8	3,600.0	0	0.52	28.5	2.0	5	0.00	34.4	1.2	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 1$	C	0.4	5	0.00	1.2	143.6	5	0.00	2.6	3,600.0	0	0.77	41.4	0.7	5	0.00	48.7	**0.9**	5	0.00	2.6
	K	0.4	5	0.00	1.9	2.0	5	0.00	1.6	2,171.2	2	0.41	19.6	2.0	5	0.00	22.7	**1.3**	5	0.00	1.6
	U	0.3	5	0.00	2.3	**0.6**	5	0.00	1.6	2,972.0	1	0.56	34.7	2.4	5	0.00	44.5	1.4	5	0.00	1.6
$\Gamma_i^a = \Gamma_i^b = 2$	C	0.8	5	0.00	1.3	529.6	5	0.00	2.2	3,600.0	0	0.84	19.4	**1.8**	5	0.00	19.6	2.0	5	0.00	2.2
	K	0.3	5	0.00	2.1	2.7	5	0.00	1.5	1,170.7	4	0.15	19.6	2.9	5	0.00	21.2	**2.3**	5	0.00	1.5
	U	0.4	5	0.00	2.2	7.7	5	0.00	1.5	2,218.2	2	0.43	27.3	2.2	5	0.00	33.5	**1.1**	5	0.00	1.5
$\Gamma_i^a = \Gamma_i^b = 5$	C	0.7	5	0.00	1.6	848.0	4	0.01	2.6	3,600.0	0	0.91	44.1	**3.1**	5	0.00	48.3	12.6	5	0.00	2.6
	K	0.6	5	0.00	2.6	13.7	5	0.00	1.6	2,980.0	2	0.24	26.7	3.9	5	0.00	31.4	**2.3**	5	0.00	1.6
	U	0.4	5	0.00	2.7	**0.5**	5	0.00	1.7	2,920.0	1	0.47	32.8	2.8	5	0.00	40.8	1.3	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 10$	C	0.8	5	0.00	1.8	632.0	5	0.00	2.4	3,600.0	0	1.00	28.2	**6.7**	5	0.00	28.7	29.6	5	0.00	2.4
	K	0.6	5	0.00	2.3	**0.7**	5	0.00	1.5	2,340.3	3	0.18	16.0	2.2	5	0.00	17.2	0.8	5	0.00	1.5
Average		0.5	5.0	0.00	1.9	265.4	4.9	0.00	1.9	2,794.7	1.5	0.52	26.7	**2.3**	5.0	0.00	30.8	3.9	5.0	0.00	1.9
$m = 5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	3.3	5	0.00	1.9	**1.0**	5	0.00	1.9	2,883.6	1	0.67	24.0	7.7	5	0.00	30.5	8.0	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 0$	C	57.8	5	0.00	1.2	3,600.0	0	0.16	3.9	3,600.0	0	1.56	46.5	**12.5**	5	0.00	51.8	20.5	5	0.00	3.9
	K	4.8	5	0.00	1.8	**1.2**	5	0.00	1.8	2,884.2	1	0.55	18.7	10.7	5	0.00	20.9	10.9	5	0.00	1.8
	U	8.4	5	0.00	2.4	**76.3**	5	0.00	1.9	2,360.0	2	0.77	34.7	816.3	4	0.00	47.2	307.0	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 1$	C	26.1	5	0.00	1.4	3,080.0	1	0.09	2.6	3,600.0	0	1.11	26.5	**14.4**	5	0.00	27.2	24.0	5	0.00	2.6
	K	9.2	5	0.00	2.5	342.2	5	0.00	1.9	2,948.0	1	0.55	22.3	216.8	5	0.00	25.4	**132.2**	5	0.00	1.9
	U	7.8	5	0.00	2.4	**74.4**	5	0.00	1.8	2,922.0	1	0.86	29.2	645.0	5	0.00	37.6	111.6	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 2$	C	18.7	5	0.00	1.4	3,600.0	0	0.08	3.1	3,600.0	0	1.20	33.6	**22.4**	5	0.00	36.8	67.3	5	0.00	3.1
	K	16.7	5	0.00	2.7	906.8	4	0.01	1.9	3,600.0	0	0.98	26.0	1,629.0	3	0.01	27.9	**297.0**	5	0.00	1.9
	U	4.7	5	0.00	2.9	**9.3**	5	0.00	1.8	3,600.0	0	0.91	30.4	273.6	5	0.00	37.8	52.2	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 5$	C	25.5	5	0.00	1.6	3,600.0	0	0.08	2.6	3,600.0	0	1.22	34.3	**74.8**	5	0.00	37.3	513.0	5	0.00	2.6
	K	2.9	5	0.00	2.1	**0.7**	5	0.00	1.5	1,521.6	3	0.21	23.6	42.9	5	0.00	28.0	25.4	5	0.00	1.5
	U	4.2	5	0.00	2.6	**22.4**	5	0.00	1.7	2,244.0	2	0.46	28.5	264.6	5	0.00	36.6	59.6	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 10$	C	17.7	5	0.00	1.8	3,600.0	0	0.11	3.5	3,600.0	0	1.36	40.3	**94.4**	5	0.00	43.5	751.6	5	0.00	3.5
	K	3.3	5	0.00	2.8	**0.6**	5	0.00	1.8	3,000.0	2	0.30	22.6	176.2	5	0.00	24.1	51.0	5	0.00	1.8
Average		14.1	5.0	0.00	2.1	1,261.0	3.3	0.03	2.2	3,064.2	0.9	0.85	29.4	286.8	4.8	0.00	34.2	**162.1**	5.0	0.00	2.2

Table 3: Results for disjoint reformulations. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 150$. In each row, among the solution methods that solve the most number of instances to optimality, the best average time and the best average gap (if #< 5) are in **bold**.

$n = 150$	Cons.	$\text{FP}_{4'}$				$\text{MILP}_1[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{ab}]$				$\text{MILP}_2^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$			
$m = 1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.1	5	0.00	1.0	**0.1**	5	0.00	1.0	0.7	5	0.00	29.6	0.2	5	0.00	37.5	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 0$	C	0.2	5	0.00	1.0	3,600.0	0	0.31	2.4	32.5	5	0.00	45.6	0.3	5	0.00	46.7	**0.1**	5	0.00	2.4
	K	0.1	5	0.00	1.0	**0.1**	5	0.00	1.0	1.0	5	0.00	20.7	0.2	5	0.00	22.5	**0.1**	5	0.00	1.0
	U	0.1	5	0.00	1.3	**0.2**	5	0.00	1.0	2.3	5	0.00	34.7	**0.2**	5	0.00	42.3	**0.2**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 3$	C	0.2	5	0.00	1.1	3,600.0	0	0.30	2.0	80.0	5	0.00	31.3	0.3	5	0.00	31.8	**0.2**	5	0.00	2.0
	K	0.1	5	0.00	1.3	**0.1**	5	0.00	1.0	1.3	5	0.00	26.0	0.3	5	0.00	27.4	**0.1**	5	0.00	1.0
	U	0.2	5	0.00	1.4	**0.1**	5	0.00	1.1	6.0	5	0.00	38.8	0.3	5	0.00	47.2	0.2	5	0.00	1.1
$\Gamma_i^a = \Gamma_i^b = 6$	C	0.2	5	0.00	1.1	3,600.0	0	0.31	2.4	1,112.9	4	0.48	88.8	0.3	5	0.00	109.5	**0.2**	5	0.00	2.4
	K	0.1	5	0.00	1.4	**0.2**	5	0.00	1.1	1.5	5	0.00	18.6	0.3	5	0.00	18.9	**0.2**	5	0.00	1.1
	U	0.2	5	0.00	1.6	**0.2**	5	0.00	1.0	2.5	5	0.00	47.3	0.3	5	0.00	57.3	**0.2**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 15$	C	0.2	5	0.00	1.2	3,600.0	0	0.25	1.8	473.3	5	0.00	46.6	**0.3**	5	0.00	47.3	**0.3**	5	0.00	1.8
	K	0.1	5	0.00	1.9	**0.1**	5	0.00	1.0	1.9	5	0.00	43.8	0.4	5	0.00	46.9	0.2	5	0.00	1.0
	U	0.2	5	0.00	1.9	**0.1**	5	0.00	1.0	1.9	5	0.00	39.8	0.5	5	0.00	43.0	0.2	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 30$	C	0.2	5	0.00	1.4	3,600.0	0	0.28	2.1	931.9	4	0.12	72.2	**0.4**	5	0.00	74.5	**0.4**	5	0.00	2.1
	K	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	1.6	5	0.00	26.3	0.3	5	0.00	27.4	0.2	5	0.00	1.0
Average		0.2	5.0	0.00	1.4	1,200.1	3.3	0.10	1.4	176.7	4.9	0.04	40.7	0.3	5.0	0.00	45.3	**0.2**	5.0	0.00	1.4
$m = 3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.7	5	0.00	1.8	721.0	4	0.01	1.8	3,600.0	0	3.62	46.0	0.8	5	0.00	62.3	**0.7**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 0$	C	4.9	5	0.00	1.2	3,600.0	0	0.99	5.1	3,600.0	0	9.66	118.6	4.2	5	0.00	141.2	**3.2**	5	0.00	5.1
	K	0.5	5	0.00	1.7	3.1	5	0.00	1.7	3,600.0	0	2.86	38.9	0.9	5	0.00	46.4	**0.8**	5	0.00	1.7
	U	0.5	5	0.00	2.3	450.2	5	0.00	1.8	3,600.0	0	5.54	69.9	7.0	5	0.00	92.7	**4.1**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 3$	C	3.1	5	0.00	1.2	3,600.0	0	0.81	3.9	3,600.0	0	9.36	109.8	**7.4**	5	0.00	122.7	30.8	5	0.00	3.9
	K	0.5	5	0.00	2.4	929.3	4	0.02	1.9	3,600.0	0	4.94	48.0	7.4	5	0.00	52.2	**5.2**	5	0.00	1.9
	U	0.7	5	0.00	2.8	1,660.5	3	0.04	2.0	3,600.0	0	6.62	70.6	11.5	5	0.00	89.2	**5.4**	5	0.00	2.0
$\Gamma_i^a = \Gamma_i^b = 6$	C	8.8	5	0.00	1.3	3,600.0	0	0.68	3.1	3,600.0	0	8.40	56.2	**6.1**	5	0.00	57.4	29.6	5	0.00	3.1
	K	0.9	5	0.00	2.4	1,472.3	3	0.04	1.8	3,600.0	0	4.28	38.5	12.5	5	0.00	43.0	**6.9**	5	0.00	1.8
	U	0.5	5	0.00	2.9	2,164.4	2	0.04	1.8	3,600.0	0	7.84	91.2	20.0	5	0.00	122.4	**13.7**	5	0.00	1.8
$\Gamma_i^a = \Gamma_i^b = 15$	C	6.3	5	0.00	1.4	3,600.0	0	0.59	2.9	3,600.0	0	9.58	62.3	**13.0**	5	0.00	64.0	49.9	5	0.00	2.9
	K	0.8	5	0.00	2.8	2,160.5	2	0.10	1.9	3,600.0	0	5.98	45.1	30.0	5	0.00	47.7	**16.5**	5	0.00	1.9
	U	0.9	5	0.00	2.7	721.5	4	0.05	1.7	3,600.0	0	6.72	58.7	119.8	5	0.00	69.0	**30.0**	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 30$	C	3.7	5	0.00	1.5	3,600.0	0	0.58	3.1	3,600.0	0	11.48	65.1	**22.8**	5	0.00	66.2	1,448.6	5	0.00	3.1
	K	0.8	5	0.00	2.7	730.0	4	0.06	1.6	3,600.0	0	4.68	47.2	**55.4**	5	0.00	53.8	204.1	5	0.00	1.6
Average		2.2	5.0	0.00	2.1	1,934.2	2.4	0.27	2.4	3,600.0	0.0	6.77	64.4	**21.3**	5.0	0.00	75.3	123.3	5.0	0.00	2.4
$m = 5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	16.9	5	0.00	2.4	2,004.4	3	0.06	2.4	3,600.0	0	7.18	57.3	46.0	5	0.00	73.3	**32.6**	5	0.00	2.4
$\Gamma_i^a = \Gamma_i^b = 0$	C	2,210.0	4	0.00	1.3	3,600.0	0	1.18	4.5	3,600.0	0	11.58	63.6	**234.0**	5	0.00	65.5	666.6	5	0.00	4.5
	K	20.2	5	0.00	2.3	2,164.8	2	0.09	2.3	3,600.0	0	5.06	40.6	39.1	5	0.00	46.3	**34.7**	5	0.00	2.3
	U	30.6	5	0.00	3.2	2,888.8	1	0.23	2.5	3,600.0	0	10.22	91.4	3,012.0	1	0.11	113.7	**960.0**	5	0.00	2.5
$\Gamma_i^a = \Gamma_i^b = 3$	C	2,250.0	5	0.00	1.3	3,600.0	0	1.03	4.0	3,600.0	0	12.40	105.5	**370.0**	5	0.00	114.4	2,302.0	5	0.00	4.0
	K	15.3	5	0.00	3.0	2,884.2	1	0.25	2.4	3,600.0	0	7.28	58.2	1,726.0	4	0.02	67.5	**734.0**	5	0.00	2.4
	U	24.2	5	0.00	3.0	2,160.7	2	0.25	2.2	3,600.0	0	8.84	84.6	1,574.6	3	0.05	110.6	**1,578.6**	4	**0.00**	2.2
$\Gamma_i^a = \Gamma_i^b = 6$	C	1,067.8	5	0.00	1.4	3,600.0	0	1.04	4.3	3,600.0	0	13.80	159.8	**1,047.2**	5	0.00	185.9	1,608.8	5	0.00	4.3
	K	7.8	5	0.00	2.6	1,442.0	3	0.16	2.0	3,600.0	0	5.48	55.5	1,505.0	3	0.03	64.1	**417.0**	5	0.00	2.0
	U	17.9	5	0.00	3.0	2,161.8	2	0.16	1.9	3,600.0	0	9.22	82.3	1,478.0	4	0.03	105.7	**1,018.0**	4	**0.02**	1.9
$\Gamma_i^a = \Gamma_i^b = 15$	C	1,356.6	5	0.00	1.5	3,600.0	0	0.81	3.5	3,600.0	0	13.80	102.7	**1,568.0**	5	0.00	111.1	3,600.0	4	0.01	3.5
	K	55.0	5	0.00	3.4	2,166.2	2	0.31	2.0	3,600.0	0	9.84	76.5	2,278.0	2	0.14	85.9	**1,806.0**	3	0.03	2.0
	U	9.8	5	0.00	3.1	741.4	4	0.05	1.9	3,600.0	0	8.46	81.9	1,790.0	3	0.26	100.9	**306.0**	5	0.00	1.9
$\Gamma_i^a = \Gamma_i^b = 30$	C	3,160.0	5	0.00	1.6	3,600.0	0	0.71	3.5	3,600.0	0	14.60	107.0	**3,440.0**	5	0.00	111.5	3,600.0	1	0.01	3.5
	K	20.3	5	0.00	3.3	**782.4**	4	0.13	1.9	3,600.0	0	7.74	75.8	1,308.0	4	0.05	94.9	1,056.0	4	**0.02**	1.9
Average		684.1	4.9	0.00	2.4	2,493.1	1.6	0.43	2.8	3,600.0	0.0	9.70	82.8	1,427.7	3.9	0.05	96.8	**1,314.7**	4.3	**0.01**	2.8

Table 4: Comparison of results for the best disjoint reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ versus joint reformulations $\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$, and $\text{MILP}[\mathcal{U}^{\overline{a}}]$. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 50$. We have: for $\mathcal{U}^{\overline{ab}}$ that $\Gamma_i = 2\,\Gamma_i^a$, for $\mathcal{U}_{=}^{\overline{ab}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$ that $\Gamma_i = \Gamma_i^a$, and for $\mathcal{U}^{\overline{a}}$ that $\Gamma = m\,\Gamma_i^a$.

$n = 50$	Cons.	$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$				$\text{MILP}[\mathcal{U}^{\overline{a}}]$			
$m = 1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.1	5	0.00	1.0	0.2	5	0.00	10.8	0.1	5	0.00	10.8	0.1	5	0.00	10.8	0.0	5	0.00	1.0
$\Gamma_i^a = 0$	C	0.1	5	0.00	1.9	0.3	5	0.00	16.8	0.3	5	0.00	16.8	0.3	5	0.00	16.8	1.6	5	0.00	1.9
	K	0.0	5	0.00	1.0	0.3	5	0.00	9.3	0.2	5	0.00	9.3	0.3	5	0.00	9.3	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.0	0.7	5	0.00	16.2	0.6	5	0.00	15.7	0.7	5	0.00	14.9	0.1	5	0.00	1.0
$\Gamma_i^a = 1$	C	0.1	5	0.00	1.7	0.6	5	0.00	25.9	0.7	5	0.00	25.7	0.8	5	0.00	24.8	2.3	5	0.00	2.0
	K	0.1	5	0.00	1.0	0.3	5	0.00	22.9	0.4	5	0.00	22.3	0.3	5	0.00	21.1	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.7	5	0.00	19.2	0.6	5	0.00	19.0	0.7	5	0.00	16.2	0.1	5	0.00	1.0
$\Gamma_i^a = 2$	C	0.1	5	0.00	1.4	0.7	5	0.00	16.2	0.6	5	0.00	15.7	0.7	5	0.00	15.1	0.5	5	0.00	1.6
	K	0.1	5	0.00	1.0	0.4	5	0.00	13.7	0.4	5	0.00	13.6	0.4	5	0.00	12.6	0.0	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.7	5	0.00	20.4	0.7	5	0.00	21.2	0.7	5	0.00	18.5	0.1	5	0.00	1.0
$\Gamma_i^a = 5$	C	0.1	5	0.00	1.4	0.8	5	0.00	19.0	0.7	5	0.00	18.6	0.5	5	0.00	16.7	1.7	5	0.00	1.5
	K	0.1	5	0.00	1.0	0.3	5	0.00	14.2	0.4	5	0.00	14.6	0.4	5	0.00	11.7	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	0.5	5	0.00	22.0	0.5	5	0.00	23.3	0.6	5	0.00	20.8	0.1	5	0.00	1.0
$\Gamma_i^a = 10$	C	0.1	5	0.00	2.2	0.7	5	0.00	31.6	0.7	5	0.00	30.3	0.9	5	0.00	26.8	2.3	5	0.00	2.2
	K	0.0	5	0.00	1.0	0.4	5	0.00	10.0	0.4	5	0.00	10.4	0.4	5	0.00	8.7	0.0	5	0.00	1.0
Average		0.1	5.0	0.00	1.3	0.5	5.0	0.00	17.9	0.5	5.0	0.00	17.8	0.5	5.0	0.00	16.3	0.6	5.0	0.00	1.3
$m = 3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.4	5	0.00	1.5	2,229.4	3	0.23	18.7	2,249.4	3	0.23	18.7	2,225.4	3	0.23	18.7	0.4	5	0.00	1.5
$\Gamma_i^a = 0$	C	0.9	5	0.00	3.1	3,600.0	0	1.07	26.4	3,600.0	0	1.07	26.4	3,600.0	0	1.07	26.4	1,898.0	4	0.02	3.1
	K	0.3	5	0.00	1.5	1,324.4	4	0.07	16.6	1,324.4	4	0.07	16.6	1,324.3	4	0.07	16.6	0.3	5	0.00	1.5
	U	1.2	5	0.00	1.8	3,600.0	0	0.53	28.6	3,600.0	0	0.77	27.7	3,600.0	0	0.43	26.0	0.4	5	0.00	1.7
$\Gamma_i^a = 1$	C	0.9	5	0.00	2.6	3,600.0	0	0.77	41.8	3,600.0	0	0.88	41.1	3,600.0	0	0.78	40.1	833.2	4	0.01	2.8
	K	1.3	5	0.00	1.6	2,170.8	2	0.38	19.7	2,198.6	2	0.39	19.3	2,169.6	2	0.34	17.8	0.9	5	0.00	1.6
	U	1.4	5	0.00	1.6	2,982.0	1	0.56	34.4	3,600.0	0	0.62	34.4	3,064.0	1	0.47	31.5	0.3	5	0.00	1.6
$\Gamma_i^a = 2$	C	2.0	5	0.00	2.2	3,600.0	0	0.85	18.9	3,600.0	0	0.97	18.8	3,600.0	0	0.85	18.1	1,420.0	5	0.00	2.4
	K	2.3	5	0.00	1.5	1,202.7	4	0.14	19.4	2,194.6	3	0.24	18.9	1,204.8	4	0.11	17.5	0.4	5	0.00	1.4
	U	1.1	5	0.00	1.5	2,214.3	2	0.40	26.1	2,420.3	2	0.52	27.3	2,340.3	2	0.40	24.7	0.7	5	0.00	1.6
$\Gamma_i^a = 5$	C	12.6	5	0.00	2.6	3,600.0	0	0.89	42.3	3,600.0	0	0.86	41.8	3,600.0	0	0.85	38.3	2,190.4	2	0.03	3.0
	K	2.3	5	0.00	1.6	2,800.0	2	0.21	25.5	3,580.0	1	0.48	26.7	2,960.0	1	0.23	21.7	5.9	5	0.00	1.6
	U	1.3	5	0.00	1.7	2,900.0	1	0.41	30.6	3,260.0	1	0.56	32.8	2,948.0	1	0.42	28.4	0.4	5	0.00	1.7
$\Gamma_i^a = 10$	C	29.6	5	0.00	2.4	3,600.0	0	0.97	26.9	3,600.0	0	1.17	27.1	3,600.0	0	0.81	22.7	1,242.0	5	0.00	2.6
	K	0.8	5	0.00	1.5	2,040.2	3	0.16	15.2	2,880.3	1	0.37	16.0	2,360.2	3	0.16	14.3	0.5	5	0.00	1.6
Average		3.9	5.0	0.00	1.9	2,764.3	1.5	0.51	26.1	3,020.5	1.1	0.61	26.2	2,813.1	1.4	0.48	24.2	506.2	4.7	0.00	2.0
$m = 5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	8.0	5	0.00	1.9	2,883.6	1	0.67	24.0	2,883.6	1	0.67	24.0	2,883.6	1	0.67	24.0	1.0	5	0.00	1.9
$\Gamma_i^a = 0$	C	20.5	5	0.00	3.9	3,600.0	0	1.56	46.7	3,600.0	0	1.56	46.7	3,600.0	0	1.56	46.7	3,600.0	0	0.16	3.9
	K	10.9	5	0.00	1.8	2,884.2	1	0.55	18.7	2,884.2	1	0.55	18.7	2,884.2	1	0.55	18.7	1.2	5	0.00	1.8
	U	307.0	5	0.00	1.9	2,390.0	2	0.75	34.7	2,374.0	2	0.74	34.4	2,390.0	2	0.69	32.2	10.6	5	0.00	2.0
$\Gamma_i^a = 1$	C	24.0	5	0.00	2.6	3,600.0	0	1.07	26.3	3,600.0	0	1.10	26.1	3,600.0	0	1.12	25.5	3,180.0	1	0.09	2.9
	K	132.2	5	0.00	1.9	2,916.0	1	0.51	22.1	2,926.0	1	0.56	22.0	2,968.0	1	0.54	21.7	78.6	5	0.00	1.9
	U	111.6	5	0.00	1.8	2,892.4	1	0.85	28.8	2,894.0	1	0.88	29.0	2,902.0	1	0.81	27.5	8.4	5	0.00	1.8
$\Gamma_i^a = 2$	C	67.3	5	0.00	3.1	3,600.0	0	1.18	32.9	3,600.0	0	1.14	32.7	3,600.0	0	1.18	31.0	3,600.0	0	0.10	3.4
	K	297.0	5	0.00	1.9	3,600.0	0	0.97	25.3	3,600.0	0	0.94	25.4	3,600.0	0	0.96	24.1	114.2	5	0.00	2.0
	U	52.2	5	0.00	1.8	3,600.0	0	0.89	29.1	3,600.0	0	0.98	30.4	3,600.0	0	0.82	26.4	2.5	5	0.00	2.0
$\Gamma_i^a = 5$	C	513.0	5	0.00	2.6	3,600.0	0	1.20	33.0	3,600.0	0	1.16	32.7	3,600.0	0	1.15	29.7	3,600.0	0	0.07	2.9
	K	25.4	5	0.00	1.5	1,503.8	3	0.20	22.4	1,559.6	3	0.24	23.6	1,770.8	3	0.27	23.2	1.1	5	0.00	1.8
	U	59.6	5	0.00	1.7	2,207.2	2	0.38	26.7	2,244.0	2	0.48	28.5	2,394.0	2	0.40	25.8	20.4	5	0.00	1.8
$\Gamma_i^a = 10$	C	751.6	5	0.00	3.5	3,600.0	0	1.28	38.6	3,600.0	0	1.22	38.4	3,600.0	0	1.14	32.8	3,600.0	0	0.10	3.5
	K	51.0	5	0.00	1.8	2,880.0	2	0.27	21.1	3,020.0	2	0.31	22.6	3,600.0	0	0.36	20.3	0.8	5	0.00	1.9
Average		162.1	5.0	0.00	2.2	3,050.5	0.9	0.82	28.7	3,065.7	0.9	0.84	29.0	3,132.8	0.7	0.81	27.3	1,187.9	3.4	0.03	2.4

Table 5: Comparison of results for the best disjoint reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ versus joint reformulations $\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$, $\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$, and $\text{MILP}[\mathcal{U}^{\overline{a}}]$. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 150$. We have: for $\mathcal{U}^{\overline{ab}}$ that $\Gamma_i = 2\,\Gamma_i^a$, for $\mathcal{U}^{\overline{ab}}_{=}$ and $\mathcal{U}^{\overline{ab}}_{\propto}$ that $\Gamma_i = \Gamma_i^a$, and for $\mathcal{U}^{\overline{a}}$ that $\Gamma = m\,\Gamma_i^a$.

$n=150$	Cons.	$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{=}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}_{\propto}]$				$\text{MILP}[\mathcal{U}^{\overline{a}}]$			
$m=1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.1	5	0.00	1.0	0.4	5	0.00	29.6	0.4	5	0.00	29.6	0.4	5	0.00	29.6	0.1	5	0.00	1.0
$\Gamma_i^a = 0$	C	0.1	5	0.00	2.4	30.5	5	0.00	45.6	170.6	5	0.00	45.6	30.5	5	0.00	45.6	3,600.0	0	0.32	2.4
	K	0.1	5	0.00	1.0	1.0	5	0.00	20.7	0.9	5	0.00	20.7	0.9	5	0.00	20.7	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.8	5	0.00	35.4	3.7	5	0.00	33.5	3.1	5	0.00	32.7	0.2	5	0.00	1.0
$\Gamma_i^a = 3$	C	0.2	5	0.00	2.0	41.4	5	0.00	31.6	68.0	5	0.00	31.1	740.6	4	0.01	30.9	3,600.0	0	0.34	2.3
	K	0.1	5	0.00	1.0	1.3	5	0.00	27.9	2.9	5	0.00	25.5	1.1	5	0.00	24.2	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.1	2.5	5	0.00	39.6	3.7	5	0.00	37.8	2.7	5	0.00	36.6	0.1	5	0.00	1.0
$\Gamma_i^a = 6$	C	0.2	5	0.00	2.4	790.4	4	0.70	90.2	1,451.6	3	0.07	87.7	751.8	4	0.30	86.1	3,600.0	0	0.34	3.1
	K	0.2	5	0.00	1.1	1.7	5	0.00	19.4	2.3	5	0.00	18.1	1.3	5	0.00	16.9	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.1	5	0.00	46.6	3.0	5	0.00	46.8	1.7	5	0.00	42.0	0.1	5	0.00	1.0
$\Gamma_i^a = 15$	C	0.3	5	0.00	1.8	1,451.7	5	0.00	46.8	24.4	5	0.00	45.5	1,843.6	4	0.00	43.7	3,600.0	0	0.33	2.3
	K	0.2	5	0.00	1.0	2.0	5	0.00	42.7	3.1	5	0.00	43.0	1.9	5	0.00	35.7	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	2.4	5	0.00	38.3	3.5	5	0.00	39.3	2.3	5	0.00	31.2	0.1	5	0.00	1.0
$\Gamma_i^a = 30$	C	0.4	5	0.00	2.1	690.5	5	0.00	71.2	823.7	4	0.46	68.7	1,539.2	3	0.70	63.9	3,600.0	0	0.40	2.4
	K	0.2	5	0.00	1.0	2.1	5	0.00	25.3	3.4	5	0.00	26.2	1.5	5	0.00	23.7	0.1	5	0.00	1.0
Average		0.2	5.0	0.00	1.4	201.5	4.9	0.05	40.7	171.0	4.8	0.04	39.9	328.2	4.7	0.07	37.6	1,200.1	3.3	0.12	1.5
$m=3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.7	5	0.00	1.8	3,600.0	0	3.64	46.0	3,600.0	0	3.64	46.0	3,600.0	0	3.64	46.0	721.0	4	0.01	1.8
$\Gamma_i^a = 0$	C	3.2	5	0.00	5.1	3,600.0	0	9.66	118.7	3,600.0	0	9.74	118.7	3,600.0	0	9.66	118.7	3,600.0	0	1.00	5.1
	K	0.8	5	0.00	1.7	3,600.0	0	2.86	38.9	3,600.0	0	2.86	38.9	3,600.0	0	2.86	38.9	2.9	5	0.00	1.7
	U	4.1	5	0.00	1.8	3,600.0	0	5.26	71.9	3,600.0	0	5.18	67.4	3,600.0	0	4.94	64.8	46.5	5	0.00	1.8
$\Gamma_i^a = 3$	C	30.8	5	0.00	3.9	3,600.0	0	9.42	109.8	3,600.0	0	9.40	109.3	3,600.0	0	10.04	108.1	3,600.0	0	0.91	4.5
	K	5.2	5	0.00	1.9	3,600.0	0	4.88	48.9	3,600.0	0	4.98	48.2	3,600.0	0	4.58	45.2	57.2	5	0.00	1.9
	U	5.4	5	0.00	2.0	3,600.0	0	7.04	71.5	3,600.0	0	6.48	70.5	3,600.0	0	6.18	65.8	322.0	5	0.00	1.9
$\Gamma_i^a = 6$	C	29.6	5	0.00	3.1	3,600.0	0	8.92	56.1	3,600.0	0	8.04	55.5	3,600.0	0	8.44	55.3	3,600.0	0	0.83	3.6
	K	6.9	5	0.00	1.8	3,600.0	0	4.67	38.8	3,600.0	0	4.26	37.7	3,600.0	0	3.90	35.0	375.8	5	0.00	1.8
	U	13.7	5	0.00	1.8	3,600.0	0	7.52	89.6	3,600.0	0	7.40	91.2	3,600.0	0	6.28	73.3	10.9	5	0.00	1.8
$\Gamma_i^a = 15$	C	49.9	5	0.00	2.9	3,600.0	0	9.04	61.4	3,600.0	0	8.56	60.6	3,600.0	0	8.72	58.5	3,600.0	0	0.83	3.7
	K	16.5	5	0.00	1.9	3,600.0	0	5.94	44.0	3,600.0	0	5.72	44.4	3,600.0	0	5.20	40.8	830.6	4	0.01	1.9
	U	30.0	5	0.00	1.7	3,600.0	0	6.54	56.7	3,600.0	0	6.58	59.6	3,600.0	0	6.04	53.2	722.6	4	0.01	1.8
$\Gamma_i^a = 30$	C	1,448.6	5	0.00	3.1	3,600.0	0	9.40	62.6	3,600.0	0	9.50	62.0	3,600.0	0	9.48	58.2	3,600.0	0	0.80	3.8
	K	204.1	5	0.00	1.6	3,600.0	0	5.32	45.3	3,600.0	0	5.02	47.2	3,600.0	0	4.52	41.7	761.3	4	0.00	1.8
Average		123.3	5.0	0.00	2.4	3,600.0	0.0	6.67	64.0	3,600.0	0.0	6.49	63.8	3,600.0	0.0	6.30	60.2	1,456.7	3.1	0.29	2.6
$m=5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	32.6	5	0.00	2.4	3,600.0	0	7.18	58.0	3,600.0	0	7.20	58.0	3,600.0	0	7.18	58.0	1,983.8	3	0.06	2.4
$\Gamma_i^a = 0$	C	666.6	5	0.00	4.5	3,600.0	0	11.58	63.6	3,600.0	0	11.58	63.6	3,600.0	0	11.58	63.7	3,600.0	0	1.18	4.5
	K	34.7	5	0.00	2.3	3,600.0	0	5.08	40.6	3,600.0	0	5.32	40.6	3,600.0	0	5.08	40.6	2,164.6	2	0.09	2.3
	U	960.0	5	0.00	2.5	3,600.0	0	10.08	95.2	3,600.0	0	9.84	91.5	3,600.0	0	9.24	83.4	2,172.4	2	0.14	2.5
$\Gamma_i^a = 3$	C	2,302.0	5	0.00	4.0	3,600.0	0	12.00	105.3	3,600.0	0	12.20	105.1	3,600.0	0	11.80	104.1	3,600.0	0	1.10	4.7
	K	734.0	5	0.00	2.4	3,600.0	0	7.70	61.1	3,600.0	0	6.94	57.5	3,600.0	0	7.06	55.3	2,882.6	1	0.12	2.5
	U	1,578.6	4	0.00	2.2	3,600.0	0	8.94	81.3	3,600.0	0	9.12	79.5	3,600.0	0	7.90	73.4	1,447.0	3	0.13	2.3
$\Gamma_i^a = 6$	C	1,608.8	5	0.00	4.3	3,600.0	0	13.40	159.5	3,600.0	0	12.40	159.0	3,600.0	0	13.60	156.1	3,600.0	0	1.22	5.3
	K	417.0	5	0.00	2.0	3,600.0	0	6.10	55.7	3,600.0	0	5.44	54.8	3,600.0	0	5.16	53.3	943.3	4	0.06	2.1
	U	1,018.0	4	0.02	1.9	3,600.0	0	8.44	77.1	3,600.0	0	8.76	77.6	3,600.0	0	8.44	70.9	809.2	4	0.09	2.1
$\Gamma_i^a = 15$	C	3,600.0	4	0.01	3.5	3,600.0	0	13.80	100.5	3,600.0	0	13.40	100.0	3,600.0	0	13.20	96.1	3,600.0	0	1.08	4.4
	K	1,806.0	3	0.03	2.0	3,600.0	0	9.60	71.1	3,600.0	0	9.84	73.1	3,600.0	0	8.24	62.9	2,129.0	3	0.19	2.6
	U	306.0	5	0.00	1.9	3,600.0	0	8.00	79.1	3,600.0	0	8.74	81.9	3,600.0	0	7.54	74.8	291.4	5	0.00	2.2
$\Gamma_i^a = 30$	C	3,600.0	1	0.01	3.5	3,600.0	0	13.20	102.9	3,600.0	0	13.40	102.8	3,600.0	0	13.00	95.5	3,600.0	0	1.03	4.5
	K	1,056.0	4	0.02	1.9	3,600.0	0	7.36	68.0	3,600.0	0	7.44	70.3	3,600.0	0	7.30	64.1	752.3	4	0.08	2.1
Average		1,314.7	4.3	0.01	2.8	3,600.0	0.0	9.50	81.3	3,600.0	0.0	9.44	81.0	3,600.0	0.0	9.09	76.8	2,238.4	2.1	0.44	3.1

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A Proofs

Proof of Proposition 5. Let u be the indicator variables of the sets $S(\widetilde{a}) = S(\widetilde{b})$. Note that RFP$[\mathcal{U}^{\overline{ab}}_{=}]$ can be written as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{x\in X} \min_{u\in\mathbb{R}^n} \frac{a_0 + \sum_{j\in J} a_j x_j - \sum_{j\in J} d^a_j x_j u_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d^b_j x_j u_j}$$

$$\text{s.t.} \sum_{j\in J} u_j \leqslant \Gamma$$

$$0 \leqslant u_j \leqslant 1, \qquad \forall j \in J.$$

Using the Charnes and Cooper (1962) transformation as in the proof of Proposition 4, we find the equivalent formulation

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{=}} = \max_{x\in X} \min_{z,y} \ (a_0 + \sum_{j\in J} a_j x_j) y - \sum_{j\in J} d^a_j x_j z_j$$

$$\text{s.t.} (b_0 + \sum_{j\in J} b_j x_j) y + \sum_{j\in J} d^b_j x_j z_j = 1 \qquad (\mu)$$

$$\sum_{j\in J} z_j \leqslant \Gamma y \qquad (\alpha)$$

$$0 \leqslant z_j \leqslant y \qquad \forall j \in J. \quad (\beta_j)$$

Using the standard LP duality for the inner minimization problem, we obtain formulation (13). □

Proof of Proposition 6. Let w be the indicator variables of the sets $S(\widetilde{a}) = S(\widetilde{b})$. To model the proportionality conditions, i.e., $\frac{a_j - \widetilde{a}_j}{d^a_j} = \frac{b_j - \widetilde{b}_j}{d^b_j} \in [-1, 1]$ for all $j \in J$, we introduce additional continuous variables $\eta \in [-1, 1]^n$, and write RFP$[\mathcal{U}^{\overline{ab}}_{\propto}]$ as

$$Z^{\star}_{\mathcal{U}^{\overline{ab}}_{\propto}} = \max_{x\in X} \min_{w,\eta} \frac{a_0 + \sum_{j\in J} a_j x_j + \sum_{j\in J} d^a_j x_j \eta_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d^b_j x_j \eta_j}$$

$$\text{s.t.} \ \sum_{j\in J} w_j \leqslant \Gamma$$

$$-w_j \leqslant \eta_j \leqslant w_j, \quad w_j \in \{0, 1\}, \qquad \forall j \in J.$$

Since the inner minimization problem is quasi-concave, it follows that $\eta_j \in \{-w_j, w_j\}$ in an optimal solution. Letting $u_j = 1$ if $\eta_j = w_j > 0$ and 0 otherwise, $v_j = 1$ if $\eta_j = w_j < 0$ and 0 otherwise, we can rewrite RFP[$\mathcal{U}_{\propto}^{\overline{ab}}$] as

$$Z^\star_{\mathcal{U}_{\propto}^{\overline{ab}}} = \max_{x\in X} \min_{u,v\in[0,1]^n} \frac{a_0 + \sum_{j\in J} a_j x_j + \sum_{j\in J} d_j^a x_j u_j - \sum_{j\in J} d_j^a x_j v_j}{b_0 + \sum_{j\in J} b_j x_j + \sum_{j\in J} d_j^b x_j u_j - \sum_{j\in J} d_j^b x_j v_j}$$
$$\text{s.t. } \sum_{j\in J} u_j + \sum_{j\in J} v_j \leqslant \Gamma.$$

Then using the Charnes and Cooper (1962) transformation and linear programming duality as in the proofs of Propositions 4 and 5, we obtain formulation (14). □

Proof of Proposition 10. We prove part (*i*); parts (*ii*) and (*iii*) can be proved in a similar manner. Note that the fractional binary problems subject to uncertain (random variable) coefficients can be represented as

$$\max_{\substack{x\in X,\\ \mu\geqslant 0}} \sum_{i\in I} \mu_i \tag{23a}$$
$$\text{s.t. } \sum_{j\in J} \widetilde{b}_{ij} x_j \mu_i - \sum_{j\in J} \widetilde{a}_{ij} x_j \leqslant a_{i0} - b_{i0}\mu_i \qquad \forall i\in I, \tag{23b}$$

when $b_{i0} + \sum_{j\in J} \widetilde{b}_{ij} x_j^\star > 0$. For given $(x^\star, \mu^\star)$, random variables $\widetilde{a}$ and $\widetilde{b}$, and for each $i \in I$, we aim to compute an upper-bound for the probability that i-th constraint in (23b) is violated, i.e.,

$$Pr\Big(\sum_{j\in J} \widetilde{b}_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} \widetilde{a}_{ij} x_j^\star > a_{i0} - \mu_i^\star b_{i0}\Big) = Pr\left(\mu_i^\star > \frac{a_{i0} + \sum\limits_{j\in J} \widetilde{a}_{ij} x_j^\star}{b_{i0} + \sum\limits_{j\in J} \widetilde{b}_{ij} x_j^\star}\right).$$

Then, for each $i \in I$,

$$\begin{aligned}
&Pr\Big(\sum_{j\in J} \widetilde{b}_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} \widetilde{a}_{ij} x_j^\star > a_{i0} - \mu_i^\star b_{i0}\Big)\\
&= Pr\Big(\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} \eta_{ij} d_{ij}^b \mu_i^\star x_j^\star - \sum_{j\in J} a_{ij} x_j^\star - \sum_{j\in J} \eta_{i,j+n} d_{ij}^a x_j^\star > a_{i0} - \mu_i^\star b_{i0}\Big) && (24)\\
&= Pr\Big(\sum_{j\in J} \eta_{ij} d_{ij}^b \mu_i^\star x_j^\star + \sum_{j\in J} \eta_{i,j+n} d_{ij}^a x_j^\star > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} a_{ij} x_j^\star\Big) && (25)\\
&\leqslant Pr\Big(\sum_{j\in J} \eta_{ij} d_{ij}^b \mu_i^\star x_j^\star + \sum_{j\in J} \eta_{i,j+n} d_{ij}^a x_j^\star > \sum_{j\in S_{i,b}^\star} d_{ij}^b \mu_i^\star x_j^\star + \sum_{j\in S_{i,a}^\star} d_{ij}^a x_j^\star\Big) && (26)\\
&= Pr\Big(\sum_{j\in J\setminus S_{i,b}^\star} \eta_{ij} d_{ij}^b \mu_i^\star x_j^\star + \sum_{j\in J\setminus S_{i,a}^\star} \eta_{i,j+n} d_{ij}^a x_j^\star > \sum_{j\in S_{i,b}^\star} d_{ij}^b \mu_i^\star x_j^\star (1-\eta_{ij}) + \sum_{j\in S_{i,a}^\star} d_{ij}^a x_j^\star (1-\eta_{i,j+n})\Big)\\
&\leqslant Pr\Big(\sum_{j\in J\setminus S_{i,b}^\star} \eta_{ij} d_{ij}^b \mu_i^\star x_j^\star + \sum_{j\in J\setminus S_{i,a}^\star} \eta_{i,j+n} d_{ij}^a x_j^\star > c_i \sum_{j\in S_{i,b}^\star} (1-\eta_{ij}) + c_i \sum_{j\in S_{i,a}^\star} (1-\eta_{i,j+n})\Big) && (27)\\
&= Pr\Big(\sum_{j\in S_{i,b}^\star} \eta_{ij} + \sum_{j\in S_{i,a}^\star} \eta_{i,j+n} + \sum_{j\in J\setminus S_{i,b}^\star} \eta_{ij} \frac{d_{ij}^b \mu_i^\star x_j^\star}{c_i} + \sum_{j\in J\setminus S_{i,a}^\star} \eta_{i,j+n} \frac{d_{ij}^a x_j^\star}{c_i} > \Gamma_i^a + \Gamma_i^b\Big) && (28)
\end{aligned}$$

Probability (24) is correct for independently and symmetrically distributed random variables $\eta_{ij} \in [-1,1]$ for all $j \in \{1,\ldots,2n\}$. Probability (25) is correct since $\eta_{i,j+n} \in [-1,1]$. Let $S^\star_{i,a}$ and $S^\star_{i,b}$ be the sets of indices of parameters that take the robust value in the numerator and the denominator of the i-th ratio, respectively, in a robust optimal solution. Then note that

$$\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in S^\star_{i,b}} d^b_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} a_{ij}x_j^\star + \sum_{j\in S^\star_{i,a}} d^a_{ij}x_j^\star \leqslant a_{i0} - \mu_i^\star b_{i0}$$

is a valid inequality for problem (23) under uncertainty set $\mathcal{U}^{ab}$. Thus, probability (26) is correct. Additionally, probability (27) is correct for $c_i = \min\Big\{\{d^b_{ij}\mu_i^\star x_j^\star\}_{j\in S^\star_{i,b}}, \{d^a_{ij}x_j^\star\}_{j\in S^\star_{i,a}}\Big\}$. Next, for $j \in \{1,2,\ldots,2n\}$ define

$$\gamma_{ij} = \begin{cases} 1, & \text{if } j \in S^\star_{i,b} \text{ or } j-n \in S^\star_{i,a} \\ \frac{d^b_{ij}\mu_i^\star x_j^\star}{c_i}, & \text{if } j \in J\backslash S^\star_{i,b} \\ \frac{d^a_{ij}x_j^\star}{c_i}, & \text{if } j-n \in J\backslash S^\star_{i,a}, \end{cases}$$

(note that $\gamma_{ij} \leqslant 1$ for all $j \in \{1,\ldots,2n\}$, otherwise $S^\star_{i,a}$ or $S^\star_{i,b}$ are not the robust optimal set of indices). Hence, probability (28) is equivalent to

$$Pr\Big(\sum_{j\in\{1,\ldots,2n\}} \gamma_{ij}\eta_{ij} > \Gamma_i^a + \Gamma_i^b\Big) \leqslant Pr\Big(\sum_{j\in\{1,\ldots,2n\}} \gamma_{ij}\eta_{ij} \geqslant \Gamma_i^a + \Gamma_i^b\Big) \leqslant B(2n, \Gamma_i^a + \Gamma_i^b).$$

The last inequality above follows from Theorem 3 part (a) in Bertsimas and Sim (2004) for independent and symmetrically distributed random variables $\eta_j \in [-1,1]$ and $\gamma_{ij} \leqslant 1$, for $j \in J$. □

Proof of Proposition 11. Following the proof of Proposition 10, for uncertainty set $\mathcal{U}^{ab}_\propto$ and each $i \in I$ we have

$$\begin{aligned}
&Pr\left(\mu_i^\star > \frac{a_{i0} + \sum_{j\in J}\widetilde{a}_{ij}x_j^\star}{b_{i0} + \sum_{j\in J}\widetilde{b}_{ij}x_j^\star}\right) \\
&= Pr\Big(\sum_{j\in J}\widetilde{b}_{ij}\mu_i^\star x_j^\star - \sum_{j\in J}\widetilde{a}_{ij}x_j^\star > a_{i0} - \mu_i^\star b_{i0}\Big) \\
&= Pr\Big(\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J}\eta_{ij}d^b_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} a_{ij}x_j^\star - \sum_{j\in J}\eta_{ij}d^a_{ij}x_j^\star > a_{i0} - \mu_i^\star b_{i0}\Big) \\
&= Pr\Big(\sum_{j\in J}\eta_{ij}(d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star) > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} a_{ij}x_j^\star\Big) \\
&= Pr\Big(\sum_{j\in J}\eta_{ij}|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star| > a_{i0} - \mu_i^\star b_{i0} - \sum_{j\in J} b_{ij}\mu_i^\star x_j^\star + \sum_{j\in J} a_{ij}x_j^\star\Big) && (29) \\
&\leqslant Pr\Big(\sum_{j\in J}\eta_{ij}|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star| > \sum_{j\in S_i^\star}|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star|\Big) && (30) \\
&= Pr\Big(\sum_{j\in J\backslash S_i^\star}\eta_{ij}|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star| > \sum_{j\in S_i^\star}|d^b_{ij}\mu_i^\star x_j^\star - d^a_{ij}x_j^\star|(1-\eta_{ij})\Big)
\end{aligned}$$

$$\leqslant Pr\Big(\sum_{j\in J\setminus S_i^\star}\eta_{ij}|d_{ij}^b\mu_i^\star x_j^\star - d_{ij}^a x_j^\star| > \sum_{j\in S_i^\star} c_i(1-\eta_{ij})\Big) \tag{31}$$

$$= Pr\Big(\sum_{j\in S_i^\star}\eta_{ij} + \sum_{j\in J\setminus S_i^\star}\eta_{ij}\frac{|d_{ij}^b\mu_i^\star x_j^\star - d_{ij}^a x_j^\star|}{c_i} > \Gamma_i\Big) \tag{32}$$

Probability (29) is correct for $\eta_{ij} \in [-1,1]$. Let $S_i^\star$ be the set of indices of parameters that take the robust value in a robust optimal solution of the i-th ratio. Then note that

$$\sum_{j\in J} b_{ij}\mu_i^\star x_j^\star - \sum_{j\in J} a_{ij}x_j^\star + \sum_{j\in S_i^\star}|d_{ij}^b\mu_i^\star x_j^\star - d_{ij}^a x_j^\star| \leqslant a_{i0} - \mu_i^\star b_{i0}$$

is a valid inequality for for problem (23) under uncertainty set $\mathcal{U}_\propto^{ab}$. Thus, probability (30) is correct. Additionally, probability (31) is correct for $c_i = \min\Big\{|d_{ij}^b\mu_i^\star x_j^\star - d_{ij}^a x_j^\star|\Big\}_{j\in S_i^\star}$. Next, for $j \in \{1,2,\ldots,n\}$ define

$$\gamma_{ij} = \begin{cases} 1, & \text{if } j \in S_i^\star \\ \frac{|d_{ij}^b\mu_i^\star x_j^\star - d_{ij}^a x_j^\star|}{c_i}, & \text{if } j \in J\setminus S_i^\star, \end{cases}$$

(note that $\gamma_{ij} \leqslant 1$ for all $j \in J$, otherwise $S_i^\star$ is not the robust optimal set of indices). Hence, probability (32) is equivalent to

$$Pr\Big(\sum_{j\in J}\gamma_{ij}\eta_{ij} > \Gamma_i\Big) \leqslant Pr\Big(\sum_{j\in J}\gamma_{ij}\eta_{ij} \geqslant \Gamma_i\Big) \leqslant B\big(n,\Gamma_i\big).$$

The last inequality follows from Theorem 3 part (a) in Bertsimas and Sim (2004) for independent and symmetrically distributed random variables $\eta_j \in [-1,1]$ and $\gamma_{ij} \leqslant 1$, for $j \in J$. □

B Additional computational results

Table 6: Results for disjoint reformulations. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 100$. In each row, among the solution methods that solve the most number of instances to optimality, the best average time and the best average gap (if #< 5) are in **bold**.

$n = 100$	Cons.	$FP_{4'}$				$MILP_1[\mathcal{U}^{ab}]$				$MILP_2[\mathcal{U}^{ab}]$				$MILP_2^{\log}[\mathcal{U}^{ab}]$				$MILP_{2'}^{\log}[\mathcal{U}^{ab}]$			
$m = 1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$\Gamma_i^a = \Gamma_i^b = 0$	U	0.3	5	0.00	1.0	**0.1**	5	0.00	1.0	0.5	5	0.00	20.9	0.2	5	0.00	24.0	**0.1**	5	0.00	1.0
	C	0.2	5	0.00	1.0	3,048.0	1	0.08	2.8	1.0	5	0.00	64.9	0.2	5	0.00	74.4	**0.1**	5	0.00	2.8
	K	0.0	5	0.00	1.0	**0.0**	5	0.00	1.0	0.6	5	0.00	14.4	0.2	5	0.00	15.0	**0.0**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 2$	U	0.1	5	0.00	1.2	**0.1**	5	0.00	1.0	1.4	5	0.00	25.4	0.3	5	0.00	31.8	0.2	5	0.00	1.0
	C	0.2	5	0.00	1.1	2,942.0	1	0.18	2.7	13.1	5	0.00	64.8	0.3	5	0.00	69.5	**0.2**	5	0.00	2.7
	K	0.1	5	0.00	1.2	**0.1**	5	0.00	1.0	1.2	5	0.00	16.1	**0.1**	5	0.00	16.6	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 4$	U	0.1	5	0.00	1.5	**0.1**	5	0.00	1.0	1.2	5	0.00	34.0	0.3	5	0.00	42.5	**0.1**	5	0.00	1.0
	C	0.2	5	0.00	1.2	2,891.2	1	0.13	2.3	70.9	5	0.00	80.1	0.4	5	0.00	87.5	**0.1**	5	0.00	2.3
	K	0.1	5	0.00	1.5	**0.1**	5	0.00	1.1	1.2	5	0.00	23.5	0.2	5	0.00	25.3	0.2	5	0.00	1.1
$\Gamma_i^a = \Gamma_i^b = 10$	U	0.1	5	0.00	1.6	**0.1**	5	0.00	1.0	1.0	5	0.00	39.8	0.3	5	0.00	46.4	0.2	5	0.00	1.0
	C	0.2	5	0.00	1.3	3,600.0	0	0.14	2.4	125.6	5	0.00	96.5	**0.3**	5	0.00	111.1	**0.3**	5	0.00	2.4
	K	0.0	5	0.00	1.4	**0.1**	5	0.00	1.0	1.2	5	0.00	19.2	0.2	5	0.00	19.9	**0.1**	5	0.00	1.0
$\Gamma_i^a = \Gamma_i^b = 20$	U	0.1	5	0.00	1.4	**0.1**	5	0.00	1.0	1.2	5	0.00	31.3	0.3	5	0.00	40.4	**0.1**	5	0.00	1.0
	C	0.2	5	0.00	1.5	3,600.0	0	0.13	2.9	14.4	5	0.00	74.8	0.4	5	0.00	80.8	**0.3**	5	0.00	2.9
	K	0.1	5	0.00	1.6	**0.0**	5	0.00	1.0	1.1	5	0.00	35.2	0.2	5	0.00	39.0	0.1	5	0.00	1.0
Average		0.1	5.0	0.00	1.3	1,072.1	3.5	0.04	1.5	15.7	5.0	0.00	42.7	0.3	5.0	0.00	48.3	**0.1**	5.0	0.00	1.5
$m = 3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$\Gamma_i^a = \Gamma_i^b = 0$	U	1.0	5	0.00	1.9	6.4	5	0.00	1.9	3,600.0	0	2.48	31.5	1.5	5	0.00	35.9	**0.9**	5	0.00	1.9
	C	2.2	5	0.00	1.2	3,600.0	0	0.49	3.3	3,600.0	0	4.22	35.0	1.8	5	0.00	35.7	**1.5**	5	0.00	3.3
	K	0.4	5	0.00	1.7	2.0	5	0.00	1.7	3,160.0	1	1.30	31.5	1.0	5	0.00	38.4	**0.6**	5	0.00	1.7
$\Gamma_i^a = \Gamma_i^b = 2$	U	0.4	5	0.00	2.1	4.6	5	0.00	1.7	3,600.0	0	2.28	47.1	4.1	5	0.00	65.4	**1.3**	5	0.00	1.7
	C	3.4	5	0.00	1.2	3,600.0	0	0.46	3.5	3,600.0	0	4.84	66.3	**4.2**	5	0.00	69.4	5.0	5	0.00	3.5
	K	0.4	5	0.00	2.0	17.5	5	0.00	1.6	3,600.0	0	1.99	32.6	3.3	5	0.00	39.2	**2.3**	5	0.00	1.6
$\Gamma_i^a = \Gamma_i^b = 4$	U	0.7	5	0.00	2.5	981.4	5	0.00	1.8	3,600.0	0	3.62	48.5	9.0	5	0.00	61.3	**4.8**	5	0.00	1.8
	C	1.9	5	0.00	1.3	3,600.0	0	0.36	2.7	3,600.0	0	4.40	60.1	5.3	5	0.00	67.1	**5.1**	5	0.00	2.7
	K	1.0	5	0.00	2.7	1,656.0	5	0.00	2.0	3,600.0	0	3.46	31.6	17.3	5	0.00	32.8	**7.8**	5	0.00	2.0
$\Gamma_i^a = \Gamma_i^b = 10$	U	0.5	5	0.00	2.4	47.4	5	0.00	1.6	3,600.0	0	3.35	54.2	6.7	5	0.00	73.9	**2.4**	5	0.00	1.6
	C	5.2	5	0.00	1.4	3,600.0	0	0.33	2.9	3,600.0	0	4.92	86.6	**11.6**	5	0.00	100.2	407.9	5	0.00	2.9
	K	0.3	5	0.00	2.1	**0.5**	5	0.00	1.4	3,600.0	0	1.42	41.1	2.4	5	0.00	50.4	0.8	5	0.00	1.4
$\Gamma_i^a = \Gamma_i^b = 20$	U	0.9	5	0.00	2.8	11.2	5	0.00	1.7	3,600.0	0	4.42	55.7	7.3	5	0.00	68.3	**3.3**	5	0.00	1.7
	C	1.4	5	0.00	1.6	3,600.0	0	0.32	2.8	3,600.0	0	5.44	63.6	**30.4**	5	0.00	65.8	1,273.2	5	0.00	2.8
	K	0.5	5	0.00	2.3	724.9	4	0.00	1.8	3,600.0	0	2.84	28.9	5.9	5	0.00	31.0	**2.2**	5	0.00	1.8
Average		1.4	5.0	0.00	1.9	1,430.1	3.3	0.13	2.2	3,570.7	0.1	3.40	47.6	**7.4**	5.0	0.00	55.7	114.6	5.0	0.00	2.2
$m = 5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
$\Gamma_i^a = \Gamma_i^b = 0$	U	10.4	5	0.00	2.1	282.6	5	0.00	2.1	3,600.0	0	2.98	41.5	58.4	5	0.00	56.4	**24.0**	5	0.00	2.1
	C	1,676.0	5	0.00	1.3	3,600.0	0	0.67	4.3	3,600.0	0	5.92	70.5	**93.4**	5	0.00	79.9	393.8	5	0.00	4.3
	K	8.3	5	0.00	2.0	169.6	5	0.00	2.0	3,600.0	0	1.21	23.7	**21.6**	5	0.00	26.8	21.8	5	0.00	2.0
$\Gamma_i^a = \Gamma_i^b = 2$	U	7.1	5	0.00	2.6	1,567.5	3	0.07	2.1	3,600.0	0	4.84	50.5	1,312.4	4	0.01	64.8	**206.4**	5	0.00	2.1
	C	1,560.6	5	0.00	1.4	3,600.0	0	0.62	3.4	3,600.0	0	5.94	53.7	**313.6**	5	0.00	55.6	1,299.6	5	0.00	3.4
	K	9.7	5	0.00	2.7	2,198.4	2	0.07	2.2	3,600.0	0	2.28	25.0	400.0	5	0.00	26.3	**167.2**	5	0.00	2.2
$\Gamma_i^a = \Gamma_i^b = 4$	U	4.8	5	0.00	3.3	1,692.8	3	0.07	2.2	3,600.0	0	5.40	66.9	1,139.8	5	0.00	89.8	**480.0**	5	0.00	2.2
	C	1,146.2	5	0.00	1.4	3,600.0	0	0.57	3.3	3,600.0	0	6.16	79.1	**98.0**	5	0.00	86.0	431.4	5	0.00	3.3
	K	5.9	5	0.00	3.1	2,161.2	2	0.12	2.1	3,600.0	0	1.98	33.7	1,023.8	4	0.03	37.2	**378.2**	5	0.00	2.1
$\Gamma_i^a = \Gamma_i^b = 10$	U	13.4	5	0.00	3.1	2,166.1	2	0.20	2.1	3,600.0	0	5.78	70.3	1,870.0	3	0.06	99.3	**1,524.8**	3	**0.02**	2.1
	C	1,764.0	4	0.00	1.5	3,600.0	0	0.50	3.1	3,600.0	0	6.86	70.5	**804.0**	5	0.00	77.1	3,600.0	4	0.01	3.1
	K	11.3	5	0.00	2.7	**725.6**	4	0.07	1.7	3,600.0	0	1.61	30.3	897.4	4	**0.01**	32.6	964.0	4	**0.01**	1.7
$\Gamma_i^a = \Gamma_i^b = 20$	U	14.3	5	0.00	3.2	904.0	4	0.01	1.9	3,600.0	0	5.12	56.2	1,304.0	4	0.02	66.6	**182.0**	5	0.00	1.9
	C	839.0	5	0.00	1.7	3,600.0	0	0.50	3.5	3,600.0	0	7.44	64.7	**1,614.0**	5	0.00	66.2	3,600.0	1	0.01	3.5
	K	16.2	5	0.00	3.3	751.6	4	0.04	2.0	3,600.0	0	2.06	25.1	762.0	5	0.00	25.8	**238.2**	5	0.00	2.0
Average		472.5	4.9	0.00	2.4	2,041.3	2.3	0.23	2.5	3,600.0	0.0	4.37	50.8	**780.8**	4.6	**0.01**	59.4	900.8	4.5	0.00	2.5

Table 7: Comparison of results for the best disjoint reformulation $\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$ versus joint reformulations $\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$, $\text{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$, and $\text{MILP}[\mathcal{U}^{\overline{a}}]$. Average time (T) in seconds with the number (#) of instances solved within default optimality gap 0.01, and the average remaining optimality gap (G) along with the average relaxation quality (R) across instances for $n = 100$. We have: for $\mathcal{U}^{\overline{ab}}$ that $\Gamma_i = 2\,\Gamma_i^a$, for $\mathcal{U}_{=}^{\overline{ab}}$ and $\mathcal{U}_{\propto}^{\overline{ab}}$ that $\Gamma_i = \Gamma_i^a$, and for $\mathcal{U}^{\overline{a}}$ that $\Gamma = m\,\Gamma_i^a$.

$n = 100$	Cons.	$\text{MILP}_{2'}^{\log}[\mathcal{U}^{ab}]$				$\text{MILP}_2[\mathcal{U}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{=}^{\overline{ab}}]$				$\text{MILP}_2[\mathcal{U}_{\propto}^{\overline{ab}}]$				$\text{MILP}[\mathcal{U}^{\overline{a}}]$			
$m = 1$	type	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.1	5	0.00	1.0	0.4	5	0.00	20.9	0.5	5	0.00	20.9	0.4	5	0.00	20.9	0.1	5	0.00	1.0
$\Gamma_i^a = 0$	C	0.1	5	0.00	2.8	1.0	5	0.00	64.9	1.1	5	0.00	64.9	1.1	5	0.00	64.9	3,046.0	1	0.10	2.8
	K	0.0	5	0.00	1.0	0.5	5	0.00	14.4	0.6	5	0.00	14.4	0.6	5	0.00	14.4	0.0	5	0.00	1.0
	U	0.2	5	0.00	1.0	1.3	5	0.00	25.7	1.0	5	0.00	25.3	1.1	5	0.00	24.0	0.1	5	0.00	1.0
$\Gamma_i^a = 2$	C	0.2	5	0.00	2.7	27.8	5	0.00	65.0	8.3	5	0.00	63.8	21.7	5	0.00	63.4	3,600.0	1	0.16	3.0
	K	0.1	5	0.00	1.0	1.1	5	0.00	16.9	1.0	5	0.00	15.9	1.0	5	0.00	15.1	0.1	5	0.00	1.0
	U	0.1	5	0.00	1.0	1.1	5	0.00	34.3	1.3	5	0.00	33.6	1.1	4	0.28	30.9	0.1	5	0.00	1.0
$\Gamma_i^a = 4$	C	0.1	5	0.00	2.3	247.1	5	0.00	80.5	108.3	5	0.00	79.0	56.0	5	0.00	76.8	2,340.0	2	0.14	2.8
	K	0.2	5	0.00	1.1	1.2	5	0.00	23.4	1.0	5	0.00	23.2	1.0	5	0.00	21.8	0.1	5	0.00	1.0
	U	0.2	5	0.00	1.0	1.0	5	0.00	38.9	1.0	5	0.00	39.6	0.9	5	0.00	34.3	0.1	5	0.00	1.0
$\Gamma_i^a = 10$	C	0.3	5	0.00	2.4	128.3	5	0.00	95.7	42.2	5	0.00	93.1	81.9	5	0.00	86.7	3,600.0	0	0.21	3.1
	K	0.1	5	0.00	1.0	1.2	5	0.00	18.6	1.1	5	0.00	18.9	1.1	5	0.00	17.2	0.0	5	0.00	1.0
	U	0.1	5	0.00	1.0	1.3	5	0.00	30.1	1.3	5	0.00	31.3	1.2	5	0.00	29.6	0.1	5	0.00	1.0
$\Gamma_i^a = 20$	C	0.3	5	0.00	2.9	11.8	5	0.00	73.2	12.3	5	0.00	69.4	74.0	5	0.00	64.4	3,440.0	1	0.14	3.0
	K	0.1	5	0.00	1.0	1.0	5	0.00	33.5	0.9	5	0.00	35.2	1.1	5	0.00	30.6	0.0	5	0.00	1.0
Average		0.1	5.0	0.00	1.5	28.4	5.0	0.00	42.4	12.1	5.0	0.00	41.9	16.3	4.9	0.02	39.7	1,068.5	3.7	0.05	1.6
$m = 3$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	0.9	5	0.00	1.9	3,600.0	0	2.48	31.5	3,600.0	0	2.50	31.5	3,600.0	0	2.48	31.5	6.4	5	0.00	1.9
$\Gamma_i^a = 0$	C	1.5	5	0.00	3.3	3,600.0	0	4.22	35.4	3,600.0	0	4.22	35.4	3,600.0	0	4.22	35.4	3,600.0	0	0.49	3.3
	K	0.6	5	0.00	1.7	3,160.0	1	1.30	31.5	3,160.0	1	1.30	31.5	3,160.0	1	1.30	31.5	1.7	5	0.00	1.7
	U	1.3	5	0.00	1.7	3,600.0	0	2.27	47.6	3,600.0	0	2.24	46.8	3,600.0	0	2.22	44.4	1.1	5	0.00	1.7
$\Gamma_i^a = 2$	C	5.0	5	0.00	3.5	3,600.0	0	4.76	66.2	3,600.0	0	4.84	65.8	3,600.0	0	4.94	64.9	3,600.0	0	0.54	4.1
	K	2.3	5	0.00	1.6	3,600.0	0	1.96	33.0	3,600.0	0	1.90	32.1	3,600.0	0	1.89	31.8	1.8	5	0.00	1.6
	U	4.8	5	0.00	1.8	3,600.0	0	3.54	48.3	3,600.0	0	3.60	48.0	3,600.0	0	3.14	44.5	8.9	5	0.00	1.9
$\Gamma_i^a = 4$	C	5.1	5	0.00	2.7	3,600.0	0	4.62	60.3	3,600.0	0	4.40	59.2	3,600.0	0	4.40	57.9	3,600.0	0	0.46	3.5
	K	7.8	5	0.00	2.0	3,600.0	0	3.68	32.2	3,600.0	0	3.44	30.6	3,600.0	0	3.24	28.9	26.3	5	0.00	1.9
	U	2.4	5	0.00	1.6	3,600.0	0	3.38	52.9	3,600.0	0	3.18	54.0	3,600.0	0	2.90	49.9	2.1	5	0.00	1.8
$\Gamma_i^a = 10$	C	407.9	5	0.00	2.9	3,600.0	0	5.02	85.1	3,600.0	0	4.80	83.0	3,600.0	0	4.84	79.7	3,600.0	0	0.46	3.4
	K	0.8	5	0.00	1.4	3,600.0	0	1.43	39.8	3,600.0	0	1.48	41.1	3,600.0	0	1.56	40.3	0.6	5	0.00	1.6
	U	3.3	5	0.00	1.7	3,600.0	0	4.26	53.1	3,600.0	0	4.40	55.7	3,600.0	0	4.00	50.5	3.3	5	0.00	1.9
$\Gamma_i^a = 20$	C	1,273.2	5	0.00	2.8	3,600.0	0	5.10	61.3	3,600.0	0	5.10	60.4	3,600.0	0	4.86	54.4	3,600.0	0	0.47	3.5
	K	2.2	5	0.00	1.8	3,600.0	0	2.90	28.0	3,600.0	0	2.94	28.9	3,600.0	0	2.62	26.5	20.0	5	0.00	1.8
Average		114.6	5.0	0.00	2.2	3,570.7	0.1	3.39	47.1	3,570.7	0.1	3.36	46.9	3,570.7	0.1	3.24	44.8	1,204.8	3.3	0.16	2.4
$m = 5$		T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R	T	#	G	R
	U	24.0	5	0.00	2.1	3,600.0	0	2.98	41.5	3,600.0	0	2.98	41.6	3,600.0	0	2.98	41.6	262.8	5	0.00	2.1
$\Gamma_i^a = 0$	C	393.8	5	0.00	4.3	3,600.0	0	5.92	70.6	3,600.0	0	5.94	70.6	3,600.0	0	5.92	70.7	3,600.0	0	0.67	4.3
	K	21.8	5	0.00	2.0	3,600.0	0	1.21	23.7	3,600.0	0	1.21	23.7	3,600.0	0	1.21	23.7	197.8	5	0.00	2.0
	U	206.4	5	0.00	2.1	3,600.0	0	4.96	52.8	3,600.0	0	4.62	49.9	3,600.0	0	4.38	47.6	159.3	5	0.00	2.1
$\Gamma_i^a = 2$	C	1,299.6	5	0.00	3.4	3,600.0	0	5.64	53.3	3,600.0	0	5.80	53.3	3,600.0	0	5.66	52.6	3,600.0	0	0.68	3.9
	K	167.2	5	0.00	2.2	3,600.0	0	2.12	25.1	3,600.0	0	2.14	24.5	3,600.0	0	2.04	23.5	773.6	4	0.01	2.2
	U	480.0	5	0.00	2.2	3,600.0	0	5.40	67.2	3,600.0	0	5.66	67.7	3,600.0	0	4.84	58.4	10.3	5	0.00	2.1
$\Gamma_i^a = 4$	C	431.4	5	0.00	3.3	3,600.0	0	6.66	78.6	3,600.0	0	5.90	77.8	3,600.0	0	6.60	76.0	3,600.0	0	0.68	4.1
	K	378.2	5	0.00	2.1	3,600.0	0	1.98	33.3	3,600.0	0	2.06	33.4	3,600.0	0	1.94	31.8	786.6	4	0.01	2.2
	U	1,524.8	3	0.02	2.1	3,600.0	0	5.72	67.4	3,600.0	0	5.52	69.3	3,600.0	0	5.16	62.0	779.0	4	0.03	2.3
$\Gamma_i^a = 10$	C	3,600.0	4	0.01	3.1	3,600.0	0	6.16	68.4	3,600.0	0	6.14	68.0	3,600.0	0	6.66	64.6	3,600.0	0	0.65	4.0
	K	964.0	4	0.01	1.7	3,600.0	0	1.56	27.5	3,600.0	0	1.66	28.8	3,600.0	0	1.58	26.7	723.7	4	0.01	1.9
	U	182.0	5	0.00	1.9	3,600.0	0	4.70	53.4	3,600.0	0	4.86	56.2	3,600.0	0	4.36	50.5	180.9	5	0.00	2.2
$\Gamma_i^a = 20$	C	3,600.0	1	0.01	3.5	3,600.0	0	6.88	61.7	3,600.0	0	6.36	61.3	3,600.0	0	6.38	55.6	3,600.0	0	0.67	4.0
	K	238.2	5	0.00	2.0	3,600.0	0	2.20	23.3	3,600.0	0	2.24	25.1	3,600.0	0	1.86	21.3	969.5	5	0.00	2.2
Average		900.8	4.5	0.00	2.5	3,600.0	0.0	4.27	49.9	3,600.0	0.0	4.21	50.1	3,600.0	0.0	4.10	47.1	1,522.9	3.1	0.23	2.8