

Risk averse stochastic programming: time consistency and optimal stopping

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Abstract

This paper addresses time consistency of risk averse optimal stopping in stochastic optimization. It is demonstrated that time consistent optimal stopping entails a specific structure of the functionals describing the transition between consecutive stages. The stopping risk measures capture this structural behavior and they allow natural dynamic equations for risk averse decision making over time. Consequently, associated optimal policies satisfy Bellman's principle of optimality, which characterizes optimal policies for optimization by stating that a decision maker should not reconsider previous decisions retrospectively. We also discuss numerical approaches to solving such problems.

Keywords: Stochastic programming, coherent risk measures, time consistency, dynamic equations, optimal stopping time, Snell envelope.

1 Introduction

Optimal Stopping (OS) is a classical topic of research in statistics and operations research going back to the pioneering work of [Wald \(1947, 1949\)](#) on sequential analysis. For a thorough

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discussion of theoretical foundations of OS we can refer to [Shiryaev \(1978\)](#), for example. The classical formulation of OS assumes availability of the probability distribution of the considered data process. Of course in real life applications the ‘true distribution’ is never known exactly and in itself can be viewed as uncertain. This motivates to formulate a considered optimization problem in terms of a chosen family of probability distributions. In stochastic optimization this is going back to the pioneering paper of [Scarf \(1958\)](#), and is often referred to as the distributionally robust approach to stochastic programming. In a certain sense the distributional robustness can be considered as a dual counterpart of the modern approach to risk measures, cf. [Artzner et al. \(1999\)](#). Recently distributionally robust (risk averse) formulations of OS were considered in several publications, e.g., [Föllmer and Schied \(2004, Section 6.5\)](#), [Cheridito et al. \(2006\)](#), [Bayraktar et al. \(2010\)](#), [Krätschmer and Schoenmakers \(2010\)](#), [Belomestny and Krätschmer \(2016\)](#), [Belomestny and Krätschmer \(2017\)](#), [Goldenshluger and Zeevi \(2017\)](#).

However, a straightforward extension of risk neutral stochastic programs to their distributionally robust (risk averse) counterparts involves a delicate issue of time consistency. That is, decisions (optimal policies) designed at the initial time, before observing any realization of the random data process, should not be reconsidered from a perspective of later stages. This is closely related to Bellman’s principle of optimality and to eventual writing of the corresponding dynamic programming equations which is crucial for an efficient numerical solution of such problems. The question of time (in)consistency has been realized very early in economics and management science. [Weller \(1978\)](#) (cf. also [Hammond \(1989\)](#)) discusses time consistency in combination with expected utilities. Quoting [Weller \(1978, p. 263\)](#): “I prove that under certain assumptions, consistency is equivalent to maximizing expected utility on the set of feasible plans, with a restricted set of utility functions and a tree of subjective probability distributions which satisfy the Bayesian updating rule.” This problem setting puts risk measures in relation to stochastic optimization. In this context, however, it is to mention that [Haviv \(1996\)](#) gives a Markov decision process with constraints, so that the optimal solution does *not* satisfy Bellman’s principle.

More recently the discussion of consistency properties of risk measures has become popular in financial mathematics. Pioneers include [Wang \(1999\)](#) and [Jobert and Rogers \(2008\)](#), who introduce a concept of dynamic consistency of risk measures themselves. Other authors (e.g., [Weber \(2006\)](#), [Cheridito and Kupper \(2011\)](#), [Ruszczyński \(2010\)](#)) take a similar approach by discussing axioms of risk measures. These publications try to identify properties of risk measures themselves, which are relevant in a general, time consistent framework. Time consistency also respects increasing information, which can be investigated as essential feature of conditional risk measures (cf. [Kovacevic and Pflug \(2009\)](#)). [Kupper and Schachermayer \(2009\)](#) show that time consistent and law invariant risk measures have a very specific, entropy like representation which depends on not more than a single parameter.

Optimal stopping together with (risk neutral) policy optimization was considered in [Hordijk \(1974\)](#), but was not developed further. It turns out that optimal stopping problems lead to dynamic equations in the classical (risk neutral) setting. In this paper we extend this in several directions. We combine stopping times and risk measures by introducing *stopping*

risk measures and demonstrate how optimal stopping times combine with optimizing over policies (decision rules). The appropriate nested risk measures corresponding to the problem structure entail transition functionals at each stage. Their specific structure is employed to specify dynamic equations and generalized risk martingales. We associated dynamic equations corresponding to optimal stopping and investigate their specific structure. Optimal policies derived from optimal stopping problems are naturally time consistent.

We also develop a concept of time consistency in the distributionally robust (risk averse) setting which is naturally amenable to Bellman's principle of optimality and to formulation of the dynamic programming equations. These equations are similar to dynamic equations in Föllmer and Schied (2004, Theorem 6.52) but do not require the restrictive assumption of stability used there (see Remark 5.2 below). Finally we discuss computational approaches to solving such problems and give numerical examples.

Outline. The paper is organized as follows. In Section 2 we formulate the classical (risk neutral) OS problem in discrete time. Time and dynamic consistency of a general class of multistage optimization problems is discussed in Section 3. In Section 4 the respective dynamic programming equations, based on decomposability of the nested formulation, are derived. Time consistent formulations of the distributionally robust (risk averse) OS problems are presented in Section 5. Finally Section 6 is devoted to computational methods and numerical examples.

2 Optimal stopping time

We will use the following framework. Let (Ω, \mathcal{F}, P) be a probability space and $\mathfrak{F} := (\mathcal{F}_0, \dots, \mathcal{F}_T)$ be a filtration (a sequence of increasing sigma algebras, $\mathcal{F}_0 \subset \dots \subset \mathcal{F}_T$) with¹ $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{Z}_0 \subset \dots \subset \mathcal{Z}_T$ be a sequence of linear spaces of functions $Z: \Omega \rightarrow \mathbb{R}$. We assume that $\mathcal{Z}_t := L_p(\Omega, \mathcal{F}_t, P)$ for some $p \in [1, \infty]$, although more general settings are possible. We denote by Z_t an element of the space \mathcal{Z}_t . Note that an element $Z_t \in \mathcal{Z}_t$ actually is a class of \mathcal{F}_t -measurable functions which can be different from each other on a set of P -measure zero. Since \mathcal{F}_0 is trivial, the space \mathcal{Z}_0 consists of constant functions and will be identified with \mathbb{R} . Since elements of \mathcal{Z}_t are \mathcal{F}_t -measurable, a process $Z_t \in \mathcal{Z}_t$, $t = 0, \dots, T$, is adapted to the filtration \mathfrak{F} . We use the notation $\mathcal{Z}_{t,u} := \mathcal{Z}_t \times \dots \times \mathcal{Z}_u$, $0 \leq t < u \leq T$, in particular $\mathcal{Z}_{0,T} = \mathcal{Z}_0 \times \dots \times \mathcal{Z}_T$. For elements $Z_{t,u} = (Z_t, \dots, Z_u)$ and $Z'_{t,u} = (Z'_t, \dots, Z'_u)$ of $\mathcal{Z}_{t,u}$ we write $Z_{t,u} \preceq Z'_{t,u}$ to denote that $Z_\tau(\omega) \leq Z'_\tau(\omega)$ for almost every (a.e.), or in other words almost surely (a.s.), (with respect to the reference probability measure P) $\omega \in \Omega$, and write $Z_{t,u} \prec Z'_{t,u}$ to denote that $Z_{t,u} \preceq Z'_{t,u}$ and $Z_{t,u} \neq Z'_{t,u}$. By $\mathbb{E}_{|\mathcal{F}}$ or $\mathbb{E}[\cdot | \mathcal{F}]$ we denote the conditional expectation with respect to sigma field \mathcal{F} . By $\mathbb{1}_A$ we denote the indicator function of set A .

Recall that a *stopping time*, adapted to the filtration \mathfrak{F} , is a random variable $\tau: \Omega \rightarrow \{0, \dots, T\}$ such that $\{\omega \in \Omega: \tau(\omega) = t\} \in \mathcal{F}_t$ for $t = 0, \dots, T$. We denote by \mathfrak{T} the set of

¹The sigma algebra consisting of only two sets, the empty set and the whole space Ω , is called trivial.

stopping times (adapted to the filtration \mathfrak{F}). For a process $Z_t \in \mathcal{Z}_t$, $t = 0, \dots, T$, the (risk neutral) optimal stopping time problem can be written as

$$\min / \max_{\tau \in \mathfrak{T}} \mathbb{E}[Z_\tau]. \quad (2.1)$$

By ‘min/max’ we mean that either the minimization or maximization procedure is applied. Of course, the minimization procedure can be changed to the corresponding maximization simply by changing Z_τ to $-Z_\tau$. Later we will consider combinations of stopping times and policy optimizations where it will be essential whether a maximization or minimization procedure is considered.

We will extend the above stopping time problem in two directions. First, we combine this with cost optimization. Consider objective functions $f_0: \mathbb{R}^{n_0} \rightarrow \mathbb{R}$, $f_t: \mathbb{R}^{n_t} \times \Omega \rightarrow \mathbb{R}$, and feasibility constraints defined by $\mathcal{X}_0 \subset \mathbb{R}^{n_0}$ and multifunctions $\mathcal{X}_t: \mathbb{R}^{n_{t-1}} \times \Omega \rightrightarrows \mathbb{R}^{n_t}$, $t = 1, \dots, T$. It is assumed that $f_t(x_{t-1}, \cdot)$ and $\mathcal{X}_t(x_{t-1}, \cdot)$ are \mathcal{F}_t -measurable. A sequence $\pi = \{x_0, \mathbf{x}_1(\cdot), \dots, \mathbf{x}_T(\cdot)\}$ of mappings $\mathbf{x}_t: \Omega \rightarrow \mathbb{R}^{n_t}$, $t = 0, \dots, T$, adapted to the filtration² \mathfrak{F} is called a *policy* or a *decision rule*. Since \mathcal{F}_0 is trivial, the first decision x_0 is deterministic. A policy $\pi = \{x_0, \mathbf{x}_1, \dots, \mathbf{x}_T\}$ is feasible if it satisfies the feasibility constraints, i.e., $x_0 \in \mathcal{X}_0$ and $\mathbf{x}_t(\omega) \in \mathcal{X}_t(\mathbf{x}_{t-1}(\omega), \omega)$, $t = 1, \dots, T$, for a.e. $\omega \in \Omega$. We denote by Π the set of feasible policies such that $(f_0(x_0), f_1(\mathbf{x}_1(\cdot), \cdot), \dots, f_T(\mathbf{x}_T(\cdot), \cdot)) \in \mathcal{Z}_{0,T}$. We then consider the following problems

$$\min / \max_{\pi \in \Pi} \min / \max_{\tau \in \mathfrak{T}} \mathbb{E}[f_\tau(\mathbf{x}_\tau, \omega)]. \quad (2.2)$$

The min-min, i.e., minimization with respect to $\pi \in \Pi$ and $\tau \in \mathfrak{T}$, problem (2.2) is a natural formulation aimed at the simultaneous minimization with respect to the decision rules and stopping time. Similarly the max-max formulation is aimed at the simultaneous maximization of $\pi \in \Pi$ and $\tau \in \mathfrak{T}$. The max-min (min-max) formulation (2.2) could be interpreted as a certain type of compromise, we will discuss this later.

Second, we consider risk averse counterparts of risk neutral problems of the form (2.1)–(2.2). It is tempting to extend the risk neutral formulations simply by replacing the corresponding expectation operator by an appropriate risk measure (such approach was adopted in some recent publications, e.g., Föllmer and Schied (2004, Section 6.5), Belomestny and Krätschmer (2017), Goldenshluger and Zeevi (2017)). However, this has a delicate issue with time consistency considerations. We will discuss time consistency concepts from a somewhat general point of view in the following Sections 3 and 4 and will come back to discussion of optimal stopping time in Section 5.

Let us finish this section with the following observations. We have that Ω is the union of the disjoint sets

$$\Omega_t^\tau := \{\omega: \tau(\omega) = t\}, \quad t = 0, \dots, T, \quad (2.3)$$

²We use bold notation \mathbf{x}_t for (measurable) mappings in order to distinguish it from deterministic vector $x_t \in \mathbb{R}^{n_t}$. Also by writing $\mathbf{x}_t(\cdot)$ we emphasize that this is a function of $\omega \in \Omega$, i.e., is a random variable, rather than a deterministic vector. It is said that the sequence $(x_0, \mathbf{x}_1, \dots, \mathbf{x}_T)$ is adapted to the filtration if $\mathbf{x}_t(\cdot)$ is \mathcal{F}_t -measurable for every $t = 1, \dots, T$.

and hence $\mathbb{1}_\Omega = \sum_{t=0}^T \mathbb{1}_{\{\tau=t\}}$. Note that $\mathbb{1}_{\{\tau=s\}}$ are \mathcal{F}_t measurable for $0 \leq s \leq t \leq T$ and hence $\mathbb{1}_{\{\tau \leq t\}}$ and $\mathbb{1}_{\{\tau > t\}} = \mathbb{1}_\Omega - \mathbb{1}_{\{\tau \leq t\}}$ are also \mathcal{F}_t measurable for $t = 0, \dots, T$. Moreover $\mathbb{1}_{\{\tau=t\}} Z_\tau = \mathbb{1}_{\{\tau=t\}} Z_t$ and thus for $(Z_0, \dots, Z_T) \in \mathcal{Z}_{0,T}$ it follows that

$$Z_\tau = \sum_{t=0}^T \mathbb{1}_{\{\tau=t\}} Z_\tau = \sum_{t=0}^T \mathbb{1}_{\{\tau=t\}} Z_t. \quad (2.4)$$

By replacing Z_t in (2.4) with $Z'_t := Z_0 + \dots + Z_t$ we obtain

$$\begin{aligned} Z_0 + \dots + Z_\tau &= Z'_\tau = \sum_{t=0}^T \mathbb{1}_{\{\tau=t\}} Z'_t = \sum_{t=0}^T \sum_{i=0}^t \mathbb{1}_{\{\tau=t\}} Z_i \\ &= \sum_{t=0}^T \sum_{i=t}^T \mathbb{1}_{\{\tau=i\}} Z_t = \sum_{t=0}^T \mathbb{1}_{\{\tau \geq t\}} Z_t. \end{aligned} \quad (2.5)$$

The problem $\max_{\tau \in \mathcal{T}} \mathbb{E}[\sum_{i=1}^T Z_i]$ can be considered in the framework of problem (2.1) simply by replacing Z_τ with $Z'_\tau = Z_0 + \dots + Z_\tau$, and similarly for problems (2.2). Finally note that since $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial, we have that $\{\omega : \tau(\omega) = 0\}$ is either Ω or \emptyset , and hence either $\mathbb{1}_{\{\tau=0\}} Z_0 = Z_0$ or $\mathbb{1}_{\{\tau=0\}} Z_0 = 0$, and probability of the event $\{\tau = 0\}$ is either 0 or 1.

Because random variables are defined only up to P -almost sure equivalence, it is in general not meaningful to speak of an ' ω by ω ' infimum $\inf\{X(\omega) : X \in \mathcal{X}\}$. The essential infimum substitutes for this concept. We use the following concept of essential infimum, cf. Karatzas and Shreve (1998, Appendix A).

Definition 2.1 (Essential infimum). Let \mathcal{X} be a nonempty family of random variables defined on (Ω, \mathcal{F}, P) . The essential infimum of \mathcal{X} , denoted by $\text{ess inf } \mathcal{X}$, is a random variable X^* satisfying: (i) if $X \in \mathcal{X}$, then $X \geq X^*$ a.s., and (ii) if Y is a random variable satisfying $X \geq Y$ a.s. for all $X \in \mathcal{X}$, then $X^* \geq Y$ a.s. The essential supremum is defined in the similar way.

3 Time consistency

In this section we discuss time consistency of multistage optimization problems from a general point of view. Consider a functional $\mathcal{R} : \mathcal{Z}_{0,T} \rightarrow \mathbb{R}$ and the optimization problem

$$\min_{\pi \in \Pi} \mathcal{R}[f_0(x_0), f_1(\mathbf{x}_1(\cdot), \cdot), \dots, f_T(\mathbf{x}_T(\cdot), \cdot)], \quad (3.1)$$

where Π is the set of feasible policies as defined in Section 2. We refer to \mathcal{R} as a *preference functional* and to (3.1) as the *reference problem*.

The principle of optimality, going back to Bellman (1957), postulates that an optimal policy computed at the initial stage of the decision process, before any realization of the uncertainty data became available, remains optimal at the later stages. This formulation is quite vague since it is not clearly specified what optimality at the later stages does mean. In some situations this comes naturally and implicitly assumed. However, in more complex cases

this could lead to a confusion and misunderstandings. Therefore, in order to proceed, we consider a class of preference relations between possible realizations of random data defined by a family of mappings

$$\mathcal{R}_{t,u}: \mathcal{Z}_{t,u} \rightarrow \mathbb{R}, \quad 0 \leq t < u \leq T.$$

We refer to each $\mathcal{R}_{t,u}$ as a *preference mapping* and to the family $\mathfrak{R} = \{\mathcal{R}_{t,u}\}_{0 \leq t < u \leq T}$ as a *preference system*. Since \mathcal{Z}_0 is identified with \mathbb{R} , we view $\mathcal{R}_{0,T}(Z_0, \dots, Z_T)$ as a real number for any $(Z_0, \dots, Z_T) \in \mathcal{Z}_{0,T}$. We assume that $\mathcal{R}_{0,T}$ coincides with the preference functional \mathcal{R} of the reference problem (3.1), i.e., $\mathcal{R} = \mathcal{R}_{0,T}$.

Remark 3.1. The considered preference system does not necessarily represents a risk averse approach. As we will discuss later, indeed in some settings it can be interpreted as conservative, and hence as risk averse, while in other cases it can be viewed as somewhat optimistic (see, in particular, Remark 5.3 below).

Definition 3.2 (Time consistent policies). We say that an optimal policy $\bar{\pi} = \{\bar{x}_0, \dots, \bar{x}_T\}$, solving the reference problem (3.1), is *time consistent* with respect to the preference system \mathfrak{R} , if at stages $t = 1, \dots, T-1$, the policy $\{\bar{x}_t, \dots, \bar{x}_T\}$ is optimal for

$$\begin{aligned} \text{ess inf} \quad & \mathcal{R}_{t,T} \left[f_t(\mathbf{x}_t(\cdot), \cdot), \dots, f_T(\mathbf{x}_T(\cdot), \cdot) \right], \\ \text{s.t.} \quad & \mathbf{x}_u(\cdot) \in \mathcal{X}_u(\mathbf{x}_{u-1}(\cdot), \cdot) \text{ a.s., } u = t, \dots, T, \end{aligned} \quad (3.2)$$

given $\mathbf{x}_{t-1} = \bar{\mathbf{x}}_{t-1}$.

Remark 3.3. At stage $t = 1, \dots, T$, we already observed a realization of $\bar{\mathbf{x}}_{t-1}$ of the policy $\bar{\pi}$. The next decision \mathbf{x}_t should satisfy the feasibility constraint $\mathbf{x}_t(\cdot) \in \mathcal{X}_t(\bar{\mathbf{x}}_{t-1}(\cdot), \cdot)$, and so on at the later stages. In that sense the optimization in (3.2) is performed over policies $(\mathbf{x}_t, \dots, \mathbf{x}_T)$, satisfying the respective feasibility constraints, given $\bar{\mathbf{x}}_{t-1}$. For definition of the concept of essential infimum, used in (3.2), see Definition 2.1.

It could be noted that we allow for the preference mapping $\mathcal{R}_{t,T}$, $t = 1, \dots, T-1$, to depend on realizations of the data process up to time t , i.e., we have that $\mathcal{R}_{t,T}(Z_t, \dots, Z_T)$ is \mathcal{F}_t -measurable. However, we do not allow $\mathcal{R}_{t,T}$ to depend on the decisions. Definition 3.2 formalizes the meaning of optimality of a solution of the reference problem at the later stages of the decision process. Clearly this framework depends on a choice of the preference system $\mathfrak{R} = \{\mathcal{R}_{t,u}\}_{1 \leq t < u \leq T}$. This suggests the following basic questions: (i) what would be a ‘natural’ choice of mappings $\mathcal{R}_{t,u}$, (ii) what properties of mappings $\mathcal{R}_{t,u}$ are sufficient/necessary to ensure that every (at least one) optimal solution of the reference problem is time consistent, (iii) how time consistency is related to dynamic programming equations. As we shall see the last question is closely related to decomposability of $\mathcal{R}_{t,T}$ in terms of one-step mappings $\mathcal{R}_{t,t+1}$.

The minimal property that is required for the preference mappings is monotonicity.³

³In some publications the monotonicity property is understood in the reverse sense, i.e., $Z_{t,u} \preceq Z'_{t,u}$ implies that $\mathcal{R}(Z_{t,u}) \succeq \mathcal{R}(Z'_{t,u})$. In this paper we consider the monotonicity only in the sense of Definition 3.4.

Definition 3.4 (Monotonicity). We say that preference mapping $\mathcal{R}_{t,u}$ is *monotone* if for any $Z_{t,u}, Z'_{t,u} \in \mathcal{Z}_{t,u}$ such that $Z_{t,u} \preceq Z'_{t,u}$ it follows that $\mathcal{R}(Z_{t,u}) \preceq \mathcal{R}(Z'_{t,u})$. We say that preference mapping $\mathcal{R}_{t,u}$ is *strictly monotone* if for any $Z_{t,u}, Z'_{t,u} \in \mathcal{Z}_{t,u}$ such that $Z_{t,u} \prec Z'_{t,u}$ it follows that $\mathcal{R}(Z_{t,u}) \prec \mathcal{R}(Z'_{t,u})$. The preference system $\mathfrak{R} = \{\mathcal{R}_{t,u}\}_{0 \leq t < u \leq T}$ is said to be monotone (strictly monotone) if every preference mapping $\mathcal{R}_{t,u}$ is monotone (strictly monotone).

3.1 Additive case

The above framework (3.1)–(3.2) is very general. The common case considered in the recent literature on risk averse stochastic optimization is when the reference problem is a function of the total cost. That is, when each $\mathcal{R}_{t,u}$ is representable as a function of $Z_t + \dots + Z_u$, i.e.,

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) := \rho_{t,T}(Z_t + \dots + Z_u), \quad 0 \leq t < u \leq T, \quad (3.3)$$

for some⁴ $\rho_{t,T}: \mathcal{Z}_T \rightarrow \mathcal{Z}_t$, $t = 0, \dots, T$. We will refer to such framework as the *additive case*. We will also consider examples of natural and important preference systems which cannot be considered in the additive framework (3.3).

Consider the following properties of the mapping $\rho_{t,T}$:

- (i) $\rho_{t,T}(\lambda Z + (1 - \lambda)Z') \preceq \lambda \rho_{t,T}(Z) + (1 - \lambda)\rho_{t,T}(Z')$, $Z, Z' \in \mathcal{Z}_T$, $\lambda \in [0, 1]$,
- (ii) $\rho_{t,T}(Z + Z') \preceq \rho_{t,T}(Z) + \rho_{t,T}(Z')$, $Z, Z' \in \mathcal{Z}_T$,
- (iii) $\rho_{t,T}(Z + Z') \succeq \rho_{t,T}(Z) + \rho_{t,T}(Z')$, $Z, Z' \in \mathcal{Z}_T$,
- (iv) $\rho_{t,T}(\lambda Z) = \lambda \rho_{t,T}(Z)$, $Z \in \mathcal{Z}_T$, $\lambda \geq 0$ and
- (v) $\rho_{t,T}(Z + Z_t) = \rho_{t,T}(Z) + Z_t$, $Z \in \mathcal{Z}_T$, $Z_t \in \mathcal{Z}_t$.

We refer to these properties as *convexity*, *subadditivity*, *superadditivity*, *positive homogeneity* and *translation equivariance*, respectively. Note that if $\rho_{t,T}$ is positively homogeneous, then $\rho_{t,T}$ is convex iff it is subadditive.

With convex mapping $\rho_{t,T}$ is associated its concave counterpart

$$\nu_{t,T}(Z) := -\rho_{t,T}(-Z). \quad (3.4)$$

Note that $\nu_{t,T}(Z)$ inherits monotonicity, positive homogeneity and translation equivariance properties of $\rho_{t,T}$, and subadditivity of $\rho_{t,T}$ implies superadditivity of $\nu_{t,T}$. Following Artzner et al. (1999) we refer to $\rho_{t,T}$ as (convex) *coherent* if it is subadditive, monotone, positively homogeneous and translation equivariant.

The preference system (3.3) is monotone (strictly monotone) if $\rho_{t,T}$ are monotone (strictly monotone). In particular, let $\varrho: L_p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be a law invariant coherent risk measure and $\varrho_{|\mathcal{F}_t}$ be its conditional analogue. Then $\rho_{t,T} := \varrho_{|\mathcal{F}_t}$ is the corresponding coherent mapping.

⁴Note that since $\mathcal{Z}_u \subset \mathcal{Z}_T$, $u = t + 1, \dots, T$, the corresponding mapping $\rho_{t,T}: \mathcal{Z}_u \rightarrow \mathcal{Z}_t$ is defined as restriction of $\rho_{t,T}$ to \mathcal{Z}_u .

When $\varrho := \mathbb{E}$ is the expectation operator, the corresponding preference system is given by conditional expectations

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) = \mathbb{E}_{|\mathcal{F}_t}[Z_t + \dots + Z_u], \quad 0 \leq t < u \leq T,$$

which corresponds to the risk neutral setting. As another example take $\varrho := \text{AV@R}_\alpha$, where the Average Value-at-Risk measure can be defined as

$$\text{AV@R}_\alpha(Z) := \inf_{u \in \mathbb{R}} \left\{ u + (1 - \alpha)^{-1} \mathbb{E}[Z - u]_+ \right\}, \quad \alpha \in [0, 1). \quad (3.5)$$

For $\alpha = 0$ the AV@R_0 is the expectation operator and for $\alpha = 1$ it becomes $\varrho = \text{ess sup}$, the essential supremum operator. The risk measure AV@R_α is monotone but is not strictly monotone for $\alpha \neq 0$. The conditional analogue of the Average Value-at-Risk (3.5) is

$$\text{AV@R}_{\alpha_t|\mathcal{F}_t}(Z) = \text{ess inf}_{u_t \in L_\infty(\mathcal{F}_t)} \left\{ u_t + (1 - \alpha_t)^{-1} \mathbb{E}_{|\mathcal{F}_t}[Z - u_t]_+ \right\}, \quad \alpha_t \in [0, 1).$$

Example 3.5. Let $\mathcal{Z}_t := L_\infty(\Omega, \mathcal{F}_t, P)$ and⁵

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) := \text{ess sup}_{\mathcal{F}_t} \{Z_t, \dots, Z_u\}, \quad 0 \leq t < u \leq T.$$

The objective of the corresponding reference problem (3.1) is then given by the maximum of the essential supremum of the cost functions in the periods $t = 0, \dots, T$.

In the additive case, the value $\mathcal{R}_{t,u}(Z_t, \dots, Z_u)$ is a function of the sum $Z_t + \dots + Z_u$ and is the same as value of $\mathcal{R}_{t,T}$ applied to $Z_t + \dots + Z_u$ for any $u = t + 1, \dots, T$. That is, in that framework there is no point of considering preference mappings $\mathcal{R}_{t,u}$ for u different from T . On the other hand, the preference system of Example 3.5 is not additive and it is essential there to consider preference mappings $\mathcal{R}_{t,u}$ for $u < T$.

3.2 Two stage setting

It is informative at this point to discuss the two stage case, $T = 2$, since time inconsistency could already happen there. Consider the following two stage stochastic program

$$\begin{aligned} \min_{x_0, \mathbf{x}_1 \in \mathfrak{X}} \quad & \mathcal{R}(f_0(x_0), f_1(\mathbf{x}_1(\cdot), \omega)), \\ \text{s.t.} \quad & x_0 \in \mathcal{X}_0, \quad \mathbf{x}_1(\omega) \in \mathcal{X}_1(x_0, \omega), \end{aligned} \quad (3.6)$$

where $\mathfrak{X} := \{\mathbf{x}_1: \Omega \rightarrow \mathbb{R}^{n_1} \mid f_1(\mathbf{x}_1(\cdot), \cdot) \in \mathcal{Z}\}$ with \mathcal{Z} being a linear space of measurable functions $Z: \Omega \rightarrow \mathbb{R}$. The preference functional $\mathcal{R}: \mathcal{Z}_{0,1} \rightarrow \mathbb{R}$ is defined on the space $\mathcal{Z}_{0,1} = \mathbb{R} \times \mathcal{Z}$. In order to deal with duality issues we consider the following two frameworks for defining the space \mathcal{Z} . In one framework we use, as in the previous section, $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$,

⁵The conditional essential supremum is the smallest \mathcal{F}_t -random variable (X_t , say), so that $Z_\tau \preceq X_t$ for all τ with $t \leq \tau \leq u$ (see Definition 2.1).

$p \in [1, \infty]$, with P viewed as the reference probability measure (distribution). The space \mathcal{Z} is paired with the space $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$, where $q \in [1, \infty]$ is such that $1/p + 1/q = 1$, with the respective bilinear form

$$\langle \zeta, Z \rangle = \int_{\Omega} \zeta(\omega) Z(\omega) P(d\omega), \quad \zeta \in \mathcal{Z}^*, \quad Z \in \mathcal{Z}.$$

This framework became standard in the theory of coherent risk measures. However, it is not applicable to the distributionally robust approach when the set of ambiguous distributions is defined by moment constraints and there is no reference probability measure. In that case we assume that Ω is a compact metric space and use the space $\mathcal{Z} := C(\Omega)$ of continuous functions $Z: \Omega \rightarrow \mathbb{R}$. The dual \mathcal{Z}^* of this space is the space of finite signed Borel measures with the respective bilinear form

$$\langle \mu, Z \rangle = \int_{\Omega} Z(\omega) \mu(d\omega), \quad \mu \in \mathcal{Z}^*, \quad Z \in \mathcal{Z}.$$

Note that in the first framework of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, an element $Z \in \mathcal{Z}$ actually is a class of functions which can differ from each other on a set of P -measure zero.

The optimization in (3.6) is performed over $x_0 \in \mathcal{X}_0 \subset \mathbb{R}^{n_0}$ and $\mathbf{x}_1: \Omega \rightarrow \mathbb{R}^{n_1}$ such that $f_1(\mathbf{x}_1(\cdot), \cdot) \in \mathcal{Z}$. The feasibility constraints $\mathbf{x}_1(\omega) \in \mathcal{X}_1(x_0, \omega)$ in problem (3.6) should be satisfied for almost every $\omega \in \Omega$ in the first framework, and for *all* $\omega \in \Omega$ in the second framework of $\mathcal{Z} = C(\Omega)$. For $Z, Z' \in \mathcal{Z}$ we use the notation $Z \preceq Z'$ to denote that $Z(\omega) \leq Z'(\omega)$ for a.e. $\omega \in \Omega$ in the first framework, and for all $\omega \in \Omega$ in the second framework. As before we write $Z \prec Z'$ to denote that $Z \preceq Z'$ and $Z \neq Z'$.

The common setting considered in the stochastic programming literature is to define $\mathcal{R}(Z_0, Z_1) := \rho(Z_0 + Z_1)$, $(Z_0, Z_1) \in \mathbb{R} \times \mathcal{Z}$, where $\rho: \mathcal{Z} \rightarrow \mathbb{R}$ is a specified risk functional. This is the additive case discussed in Section 3.1. In particular, when ρ is the expectation operator this becomes the risk neutral formulation. However, there are many other possibilities to define preference functionals \mathcal{R} which are useful in various situations. For example, consider⁶ $\mathcal{R}(Z_0, Z_1) := Z_0 \vee \rho(Z_1)$. If moreover, in the framework of $\mathcal{Z} = C(\Omega)$, we take $\rho(Z) := \sup_{\omega \in \Omega} Z(\omega)$, then the corresponding problem (3.6) can be viewed as a robust type problem with minimization of the worst possible outcome of the two stages. That is, if the second stage cost is bigger than the first stage cost for some scenarios, then the worst second stage cost is minimized. On the other hand, if the first stage cost is bigger for all scenarios, then the second stage problem is not considered. Similarly in the framework of $\mathcal{Z} = L_{\infty}(\Omega, \mathcal{F}, P)$, we can take $\rho(Z) := \text{ess sup}(Z)$. This is the case of Example 3.5. As we shall discuss later this is closely related to the problem of optimal stopping time.

In order to proceed we will need the following interchangeability result for a functional $\varrho: \mathcal{Z} \rightarrow \mathbb{R}$. Consider a function $\psi: \mathbb{R}^n \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$. Let

$$\Psi(\omega) := \inf_{y \in \mathbb{R}^n} \psi(y, \omega)$$

⁶We use notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$.

and

$$\mathfrak{Y} := \{\eta: \Omega \rightarrow \mathbb{R}^n \mid \psi_\eta(\cdot) \in \mathcal{Z}\},$$

where $\psi_\eta(\cdot) := \psi(\eta(\cdot), \cdot)$.

Assumption 3.1. In the framework of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, suppose that the function $\psi(y, \omega)$ is random lower semicontinuous, i.e., its epigraphical mapping is closed valued and measurable (such functions are called normal integrands in [Rockafellar and Wets \(1997\)](#)). In the framework of $\mathcal{Z} = C(\Omega)$ suppose that the minimum of $\psi(y, \omega)$ over $y \in \mathbb{R}^n$ is attained for all $\omega \in \Omega$.

We have the following result about interchangeability of the minimum and preference functionals (cf. [Shapiro \(2017\)](#)).

Proposition 3.6 (Interchangeability principle). *Suppose that Assumption 3.1 holds, $\Psi \in \mathcal{Z}$ and $\varrho: \mathcal{Z} \rightarrow \mathbb{R}$ is monotone. Then*

$$\varrho(\Psi) = \inf_{\eta \in \mathfrak{Y}} \varrho(\psi_\eta) \quad (3.7)$$

and

$$\bar{\eta}(\cdot) \in \arg \min_{y \in \mathbb{R}^n} \psi(y, \cdot) \text{ implies } \bar{\eta} \in \arg \min_{\eta \in \mathfrak{Y}} \varrho(\psi_\eta). \quad (3.8)$$

If moreover ϱ is strictly monotone, then the converse of (3.8) holds true as well, i.e.,

$$\bar{\eta} \in \arg \min_{\eta \in \mathfrak{Y}} \varrho(\psi_\eta) \text{ implies } \bar{\eta}(\cdot) \in \arg \min_{y \in \mathbb{R}^n} \psi(y, \cdot). \quad (3.9)$$

In the framework of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, it is assumed that $\psi(y, \omega)$ is random lower semicontinuous. It follows that the optimal value function $\Psi(\cdot)$ and the multifunction $\mathfrak{G}(\cdot) := \arg \min_{y \in \mathbb{R}^n} \psi(y, \cdot)$ are measurable ([Rockafellar and Wets, 1997](#), Chapter 14). In that framework, the meaning of the left hand side of (3.8) and right hand side of (3.9) is that $\bar{\eta}(\cdot)$ is a measurable selection of $\mathfrak{G}(\cdot)$.

Equation (3.7) means that the minimization and preference functionals can be interchanged, provided that the preference functional is monotone. Moreover, the pointwise minimizer in the left hand side of (3.8), if it exists, solves the corresponding minimization problem in the right hand side. In order to conclude the inverse implication (3.9), that the corresponding optimal functional solution is also the pointwise minimizer, the stronger condition of strict monotonicity is needed.

Consider now the problem (3.6), that depends on ω , and let

$$V(x_0, \cdot) := \inf_{x_1 \in \mathcal{X}_1(x_0, \cdot)} \mathcal{R}(f_0(x_0), f_1(x_1, \cdot)),$$

which can be viewed as value of the second stage problem. By Proposition 3.6 we have the following.

Theorem 3.7. Suppose that: (i) the functional $\varrho(\cdot) := \mathcal{R}(Z_0, \cdot)$ is monotone for any $Z_0 \in \mathbb{R}$, (ii) $V(x_0, \cdot) \in \mathcal{Z}$ for all $Z_0 \in \mathbb{R}$, and (iii) Assumption 3.1 holds for

$$\psi(x_1, \omega) := \begin{cases} f_2(x_1, \omega) & \text{if } x_1 \in \mathcal{X}_1(x_0, \omega), \\ +\infty & \text{if } x_1 \notin \mathcal{X}_1(x_0, \omega). \end{cases}$$

Then the optimal value of the problem (3.6) is the same as the optimal value of

$$\min_{x_0 \in \mathcal{X}_0} \mathcal{R}(f_1(x_0), V(x_0, \cdot)). \quad (3.10)$$

Further, (\bar{x}_0, \bar{x}_1) is an optimal solution of the two stage problem (3.6) if \bar{x}_0 is an optimal solution of the first stage problem (3.10) and⁷

$$\bar{x}_1(\cdot) \in \arg \min_{x_1 \in \mathcal{X}_1(\bar{x}_0, \cdot)} \mathcal{R}(f_0(\bar{x}_0), f_1(x_1(\cdot), \cdot)). \quad (3.11)$$

Moreover, if $\mathcal{R}(Z_0, \cdot)$ is strictly monotone, then (\bar{x}_0, \bar{x}_1) is an optimal solution of problem (3.6) if and only if \bar{x}_0 is an optimal solution of the first stage problem and (3.11) holds.

Here, time consistency of a policy (\bar{x}_0, \bar{x}_1) , solving the two stage problem (3.6), means that \bar{x}_1 solves the respective second stage problem given $x_0 = \bar{x}_0$, i.e., condition (3.11) holds. That is, if (\bar{x}_0, \bar{x}_1) is not time consistent, then there exists another feasible solution \tilde{x}_1 such that $f_2(\tilde{x}_1(\cdot), \cdot) \prec f_2(\bar{x}_1(\cdot), \cdot)$. Without *strict* monotonicity of \mathcal{R} it could happen that problem (3.6) has optimal solutions which do not satisfy condition (3.11) and hence are not time consistent. That is, condition (3.11) is a sufficient but without strict monotonicity is not necessary for optimality. Such examples can be found, e.g., in Shapiro (2017) and for robust optimization were given in Delage and Iancu (2015). For example, for $\mathcal{R}(Z_0, Z_1) = Z_0 \vee \rho(Z_1)$ we have that if $Z'_1 \prec Z_1$ are such that $\rho(Z'_1) < \rho(Z_1) < Z_0$, then $\mathcal{R}(Z_0, Z_1) = \mathcal{R}(Z_0, Z'_1)$. That is, $\mathcal{R}(Z_0, \cdot)$ is not strictly monotone. It could happen then that a second stage decision does not satisfy (3.11) and is not time consistent.

Dual representation

The space $\mathcal{Z}_{0,1} = \mathbb{R} \times \mathcal{Z}$ can be equipped, for example, with the norm $\|(Z_0, Z_1)\|_{0,1} := |Z_0| + \|Z_1\|$, where $\|\cdot\|$ is the respective norm of the space \mathcal{Z} , and can be paired with the space $\mathcal{Z}_{0,1}^* = \mathbb{R} \times \mathcal{Z}^*$ with the bilinear form

$$\langle (\zeta_0, \zeta_1), (Z_0, Z_1) \rangle := \zeta_0 Z_0 + \langle \zeta_1, Z_1 \rangle, \quad (\zeta_0, \zeta_1) \in \mathcal{Z}_{0,1}^*, \quad (Z_0, Z_1) \in \mathcal{Z}_{0,1}.$$

Suppose that the functional $\mathcal{R}: \mathcal{Z}_{0,1} \rightarrow \mathbb{R}$ is *convex* and *monotone*. Then by the Klee-Nachbin-Namioka Theorem the functional \mathcal{R} is continuous in the norm topology of $\mathcal{Z}_{0,1}$ (cf. Ruszczyński and Shapiro (2006b, Proposition 3.1)). Suppose further that \mathcal{R} is *positively homogeneous*, i.e.,

⁷By writing ' $\bar{x}_1(\cdot) \in \dots$ ' we mean that $\bar{x}_1(\cdot)$ is a measurable selection and such inclusion holds for a.e. $\omega \in \Omega$ in the setting of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, and for all $\omega \in \Omega$ in the setting of $\mathcal{Z} = C(\Omega)$.

$\mathcal{R}(t Z_{0,1}) = t \mathcal{R}(Z_{0,1})$ for any $t \geq 0$ and $Z_{0,1} \in \mathcal{Z}_{0,1}$. Then by the Fenchel-Moreau Theorem, \mathcal{R} has the dual representation

$$\mathcal{R}(Z_0, Z_1) = \sup_{(\zeta_0, \zeta_1) \in \mathfrak{A}_{0,1}} \langle (\zeta_0, \zeta_1), (Z_0, Z_1) \rangle \quad (3.12)$$

for some convex, bounded and weakly* closed set

$$\mathfrak{A}_{0,1} \subset \{(\zeta_0, \zeta_1) \in \mathcal{Z}_{0,1}^* : \zeta_0 \geq 0, \zeta_1 \succeq 0\}$$

(cf. [Ruszczyński and Shapiro \(2006b\)](#), Theorem 2.2)). The subdifferential of \mathcal{R} is then given by

$$\partial \mathcal{R}(Z_0, Z_1) = \arg \max_{(\zeta_0, \zeta_1) \in \mathfrak{A}_{0,1}} \langle (\zeta_0, \zeta_1), (Z_0, Z_1) \rangle. \quad (3.13)$$

In particular $\partial \mathcal{R}(0, 0) = \mathfrak{A}_{0,1}$.

Example 3.8. Consider preference functional of the form $\mathcal{R}(Z_0, Z_1) := \varphi(Z_0, \rho(Z_1))$, where $\rho: \mathcal{Z} \rightarrow \mathbb{R}$ is a coherent risk measure and $\varphi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a convex monotone positively homogeneous function. It follows then that $\mathcal{R}(\cdot, \cdot)$ is convex monotone and positively homogeneous. Let $\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle$ be the dual representation of ρ , where \mathfrak{A} is a convex bounded weakly* closed subset of \mathcal{Z}^* . Then $\partial \varphi(x_0, x_1)$ consists of vectors (subgradients) (γ_0, γ_1) such that

$$\varphi(y_0, y_1) - \varphi(x_0, x_1) \geq \gamma_0(y_0 - x_0) + \gamma_1(y_1 - x_1)$$

for all $(y_0, y_1) \in \mathbb{R}^2$. Since φ is monotone, it follows that $\partial \varphi(x_0, x_1) \subset \mathbb{R}_+^2$. Consequently the representation (3.12) holds with

$$\mathfrak{A}_{0,1} = \partial \mathcal{R}(0, 0) = \{(\zeta_0, \zeta_1) \in \mathbb{R} \times \mathcal{Z}^* : \zeta_1 = \gamma_1 \zeta, \zeta \in \mathfrak{A}, (\zeta_0, \gamma_1) \in \partial \varphi(0, 0)\}.$$

For example let $\varphi(x_0, x_1) := x_0 \vee x_1$, and hence $\mathcal{R}(Z_0, Z_1) = Z_0 \vee \rho(Z_1)$. Then $\partial \varphi(0, 0) = \{(t, 1-t) : t \in [0, 1]\}$ and

$$\mathfrak{A}_{0,1} = \{(t, (1-t)\zeta) : t \in [0, 1], \zeta \in \mathfrak{A}\}.$$

3.3 Dynamic consistency of preference systems

Many authors investigate time consistency by addressing special properties on the preference system itself. This section recalls these concepts and relates these properties to time consistency of optimal policies. The following concept of dynamic consistency (also called time consistency by some authors), applied to the preference systems rather than considered policies, in slightly different forms was used by several authors (cf., [Kreps and Porteus \(1978\)](#); [Wang \(1999\)](#); [Epstein and Schneider \(2003\)](#); [Riedel \(2004\)](#); [Cheridito et al. \(2006\)](#); [Artzner et al. \(2007\)](#); [Ruszczyński \(2010\)](#)).

Definition 3.9 (Dynamical consistency). The preference system $\{\mathcal{R}_{t,u}\}_{1 \leq t < u \leq T}$ is said to be *dynamically consistent* if for $1 \leq s < t < u \leq T$ and $(Z_s, \dots, Z_u), (Z'_s, \dots, Z'_u) \in \mathcal{Z}_{t,u}$ such that $Z_\tau = Z'_\tau$, $\tau = s, \dots, t-1$, the following ‘forward’ implication holds:

$$\text{if } \mathcal{R}_{t,u}(Z_t, \dots, Z_u) \preceq \mathcal{R}_{t,u}(Z'_t, \dots, Z'_u) \text{ then } \mathcal{R}_{s,u}(Z_s, \dots, Z_u) \preceq \mathcal{R}_{s,u}(Z'_s, \dots, Z'_u). \quad (3.14)$$

It turns out that the above ‘forward’ property of dynamic consistency is not always sufficient to ensure that every optimal policy is time consistent. For that we need a stronger notion of dynamic consistency (cf. [Shapiro et al. \(2014, Section 6.8.5\)](#)).

Definition 3.10 (Strict dynamical consistency). A dynamically consistent preference system $\{\mathcal{R}_{t,u}\}_{1 \leq t < u \leq T}$ is said to be *strictly dynamically consistent* if in addition to (3.14) the following implication holds:

$$\text{if } \mathcal{R}_{t,u}(Z_t, \dots, Z_u) \prec \mathcal{R}_{t,u}(Z'_t, \dots, Z'_u) \text{ then } \mathcal{R}_{s,u}(Z_s, \dots, Z_u) \prec \mathcal{R}_{s,u}(Z'_s, \dots, Z'_u) \quad (3.15)$$

for all $1 \leq s < t < u \leq T$.

Note that it follows from (3.14) that

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) = \mathcal{R}_{t,u}(Z'_t, \dots, Z'_u) \text{ implies } \mathcal{R}_{s,u}(Z_s, \dots, Z_u) = \mathcal{R}_{s,u}(Z'_s, \dots, Z'_u).$$

Recall that in the additive case, $\mathcal{R}_{t,T}(Z_t, \dots, Z_T)$ is given by $\rho_{t,T}(Z_t + \dots + Z_T)$. In that case condition (3.14) implies that

$$\text{if } Z, Z' \in \mathcal{Z}_T \text{ and } \rho_{t,T}(Z) \preceq \rho_{t,T}(Z') \text{ then } \rho_{s,T}(Z) \preceq \rho_{s,T}(Z'), \quad 1 \leq s < t \leq T-1. \quad (3.16)$$

Conversely, if moreover $\rho_{s,T}$ is translation equivariant, then we can write for $1 \leq s < t \leq T$,

$$\rho_{t,T}(Z_s + \dots + Z_T) = Z_s + \dots + Z_{t-1} + \rho_{t,T}(Z_t + \dots + Z_T),$$

and hence condition (3.16) implies (3.14) for $u = T$. If $\rho_{t,T} := \mathbb{E}_{|\mathcal{F}_t}$, then for $s < t$ we have that $\rho_{s,T}(Z) = \mathbb{E}_{|\mathcal{F}_s}[\mathbb{E}_{|\mathcal{F}_t}(Z)]$ and hence this preference system is dynamically consistent. In fact it is not difficult to see that this preference system is strictly dynamically consistent.

Similar to the additive case we have the following result (cf. [Shapiro et al. \(2014, Propositions 6.80\)](#)).

Proposition 3.11. *The following holds true: (i) If the preference system is dynamically consistent and $\bar{\pi} \in \Pi$ is the unique optimal solution of the reference problem (3.1), then $\bar{\pi}$ is time consistent. (ii) If the preference system is strictly dynamically consistent, then every optimal solution of the reference problem is time consistent.*

4 Decomposability and dynamic equations

Let us start with definition of the following basic decomposability concept.

Definition 4.1 (Recursivity). The preference system $\{\mathcal{R}_{t,u}\}_{0 \leq t < u \leq T}$ is said to be *recursive*, if

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) = \mathcal{R}_{t,v}(Z_t, \dots, Z_{v-1}, \mathcal{R}_{v,u}(Z_v, \dots, Z_u)) \quad (4.1)$$

for any $0 \leq t < v < u \leq T$ and $(Z_t, \dots, Z_u) \in \mathcal{Z}_{t,u}$.

We have the following relation between recursivity and dynamic consistency (discussed in Section 3.3).

Proposition 4.2. *Suppose that preference mappings $\mathcal{R}_{t,u}$, $1 \leq t < u \leq T$, are monotone (strictly monotone) and recursive. Then $\{\mathcal{R}_{t,u}\}_{1 \leq t < u \leq T}$ is dynamically consistent (strictly dynamically consistent).*

Proof. We need to verify the implication (3.14). By recursivity, for $1 \leq s < t < u \leq T$, we have

$$\begin{aligned}\mathcal{R}_{s,u}(Z_s, \dots, Z_u) &= \mathcal{R}_{s,t}(Z_s, \dots, Z_{t-1}, \mathcal{R}_{t,u}(Z_t, \dots, Z_u)) \text{ and} \\ \mathcal{R}_{s,u}(Z'_s, \dots, Z'_u) &= \mathcal{R}_{s,t}(Z'_s, \dots, Z'_{t-1}, \mathcal{R}_{t,u}(Z'_t, \dots, Z'_u)).\end{aligned}$$

Assuming $Z_\tau = Z'_\tau$, $\tau = s, \dots, t-1$ the implication (3.14) (the implication (3.15)) follows by monotonicity (strict monotonicity) of $\mathcal{R}_{s,t}$. \blacksquare

It follows then by Proposition 3.11(ii) that if the system is recursive and the preference mappings are strictly monotone, then every optimal solution of the reference problem is time consistent. As already mentioned, without *strict* monotonicity the recursivity property does not necessarily imply time consistency of *every* optimal solution.

In the additive case (discussed in Section 3.1), when $\mathcal{R}_{t,T}(Z_t, \dots, Z_T) = \rho_{t,T}(Z_t + \dots + Z_T)$, $t = 0, \dots, T-1$, the recursive property can be written as

$$\rho_{t,T}(\rho_{v,T}(Z)) = \rho_{t,T}(Z), \quad Z \in \mathcal{Z}_T, \quad 0 \leq t < v \leq T-1. \quad (4.2)$$

Note that since $\rho_{v,T}(Z) \in \mathcal{Z}_v$, we have that $\rho_{t,T}(\rho_{v,T}(Z)) = \rho_{t,v}(\rho_{v,T}(Z))$. By applying (4.2) recursively for $v = t+1, \dots, T-1$, this means that $\rho_{t,T}$ can be decomposed as

$$\rho_{t,T}(Z) = \rho_{t,T}(\rho_{t+1,T}(\dots \rho_{T-1,T}(Z))), \quad Z \in \mathcal{Z}_T. \quad (4.3)$$

If moreover $\rho_{t,T}$ is translation equivariant, this becomes

$$\rho_{t,T}(Z_t + \dots + Z_T) = Z_t + \rho_{t,T}(Z_{t+1} + \rho_{t+1,T}(Z_{t+2}) + \dots + \rho_{T-1,T}(Z_T)). \quad (4.4)$$

For a law invariant convex coherent measure ϱ and $\rho_{t,T} := \varrho|_{\mathcal{F}_t}$, the recursive property (4.2) can hold only in two cases – for the ‘expectation’ and the ‘ess sup’ operators (cf. Kupper and Schachermayer (2009)). For example the Average Value-at-Risk preference system, $\rho_{t,T} := \text{AV@R}_{\alpha|_{\mathcal{F}_t}}$, is not recursive for $\alpha \in (0, 1)$. Recursive preference system, in the additive case, can be constructed in the nested form

$$\rho_{t,T}(Z) := \phi_t(\phi_{t+1}(\dots \phi_{T-1}(Z))), \quad Z \in \mathcal{Z}_T, \quad (4.5)$$

where $\phi_s: \mathcal{Z}_{s+1} \rightarrow \mathcal{Z}_s$, $s = 1, \dots, T-1$, are one-step mappings. For example, taking $\varrho := \text{AV@R}_{\alpha}$ this becomes

$$\rho_{t,T}(\cdot) = \text{AV@R}_{\alpha|_{\mathcal{F}_t}} \left(\text{AV@R}_{\alpha|_{\mathcal{F}_{t+1}}}(\dots \text{AV@R}_{\alpha|_{\mathcal{F}_{T-1}}}(\cdot)) \right),$$

the so-called *nested Average Value-at-Risk* mappings. As it was pointed out above, for $\alpha \in (0, 1)$ these nested Average Value-at-Risk mappings are different from the $\text{AV@R}_{\alpha|\mathcal{F}_t}$.

Consider now the general case of the preference system $\{\mathcal{R}_{t,u}\}_{1 \leq t < u \leq T}$. The recursive property (4.1) implies that $\mathcal{R}_{t,u}$ can be decomposed in terms of one step mappings $\mathcal{R}_{s,s+1}$, $s = t, \dots, u - 1$, as

$$\mathcal{R}_{t,u}(Z_t, \dots, Z_u) = \mathcal{R}_{t,t+1}\left(Z_t, \mathcal{R}_{t+1,t+2}\left(Z_{t+1}, \dots, \mathcal{R}_{u-1,u}(Z_{u-1}, Z_u)\right)\right). \quad (4.6)$$

Conversely recursive preference mappings can be constructed in the form (4.6) by choosing one step mappings $\mathcal{R}_{s,s+1}: \mathcal{Z}_s \times \mathcal{Z}_{s+1} \rightarrow \mathcal{Z}_s$, $s = 1, \dots, T - 1$.

4.1 Dynamic programming equations

The recursive property (4.1) and monotonicity of the preference system allow to write the following dynamic programming equations for the reference problem (3.1), derivations are similar to the two stage case discussed in Section 3.2. At the terminal stage T the cost-to-go function is defined as

$$V_T(x_{T-1}, \omega) := \operatorname{ess\,inf}_{x_T \in \mathcal{X}_T(x_{T-1}, \omega)} f_T(x_T, \omega). \quad (4.7)$$

Suppose that $x_0, \dots, \mathbf{x}_{T-1}$ are given. Since $\mathcal{R} = \mathcal{R}_{0,T}$ we have by the interchangeability principle (Proposition 3.6) and recursivity that

$$\begin{aligned} \inf_{\mathbf{x}_T \in \mathcal{X}_T(\mathbf{x}_{T-1}, \cdot)} \mathcal{R}[f_0(x_0), \dots, f_T(\mathbf{x}_T(\cdot), \cdot)] \\ &= \mathcal{R}[f_0(x_0), \dots, f_{T-1}(\mathbf{x}_{T-1}, \omega), \inf_{\mathbf{x}_T \in \mathcal{X}_T(\mathbf{x}_{T-1}, \omega)} f_T(\mathbf{x}_T, \omega)] \\ &= \mathcal{R}_{0,T}[f_0(x_0), \dots, f_{T-2}(\mathbf{x}_{T-2}, \omega), f_{T-1}(\mathbf{x}_{T-1}, \omega), V_T(x_{T-1}, \omega)] \\ &= \mathcal{R}_{0,T}[f_0(x_0), \dots, f_{T-2}(\mathbf{x}_{T-2}, \omega), \mathcal{R}_{T-1,T}[f_{T-1}(\mathbf{x}_{T-1}, \omega), V_T(x_{T-1}, \omega)]], \end{aligned} \quad (4.8)$$

assuming that $V_T(\mathbf{x}_{T-1}, \cdot) \in \mathcal{Z}_T$. Continuing this backward in time we obtain at stages $t = T - 1, \dots, 1$, the cost-to-go functions

$$V_t(x_{t-1}, \omega) := \operatorname{ess\,inf}_{x_t \in \mathcal{X}_t(x_{t-1}, \omega)} \mathcal{R}_{t,t+1}(f_t(x_t, \omega), V_{t+1}(x_t, \omega)), \quad (4.9)$$

representing the corresponding dynamic programming equations. Finally, at the first stage, the problem

$$\min_{x_0 \in \mathcal{X}_0} \mathcal{R}_{0,1}(f_0(x_0), V_1(x_0, \cdot))$$

should be solved. We conclude with (4.8) that

$$V_0 := \min_{\pi \in \Pi} \mathcal{R}[f_0(x_0), f_1(\mathbf{x}_1(\cdot), \cdot), \dots, f_T(\mathbf{x}_T(\cdot), \cdot)] \quad (4.10)$$

for the recursive preference system \mathcal{R} .

In a rudimentary form such approach to writing dynamic equations with relation to time consistency was outlined in Shapiro (2009).

Definition 4.3 (Dynamic programming equations). We say that a policy $\pi = (x_0, \mathbf{x}_1, \dots, \mathbf{x}_T)$ satisfies the *dynamic programming equations* (4.7), (4.9) and (4.10) if

$$\mathbf{x}_T(\cdot) \in \arg \min_{x_T \in \mathcal{X}_T(x_{T-1}, \cdot)} f_T(x_T, \cdot), \quad (4.11)$$

$$\mathbf{x}_t(\cdot) \in \arg \min_{x_t \in \mathcal{X}_t(x_{t-1}, \cdot)} \mathcal{R}_{t,t+1}(f_t(x_t, \cdot), V_{t+1}(x_t, \cdot)), \quad t = 1, \dots, T-1, \quad (4.12)$$

$$x_0 \in \arg \min_{x_0 \in \mathcal{X}_0} \mathcal{R}_{0,1}(f_0(x_0), V_1(x_0, \cdot)). \quad (4.13)$$

If a policy satisfies the dynamic programming equations, then it is optimal for the reference multistage problem (3.1) and is time consistent. Without strict monotonicity it could happen that a policy, which is optimal for the reference problem (3.1), does not satisfy the dynamic programming equations and is not time consistent. As it was discussed in Section 3.2 this could happen even in the two stage case and a finite number of scenarios.

Remark 4.4. Consider the additive case where $\mathcal{R}_{t,t+1}(Z_t, Z_{t+1}) = \rho_{t,T}(Z_t + Z_{t+1})$. If moreover mappings ρ_t are translation equivariant, this becomes

$$\mathcal{R}_{t,t+1}(Z_t, Z_{t+1}) = Z_t + \rho_{t,T}(Z_{t+1}). \quad (4.14)$$

Suppose further that mappings $\rho_{t,T}$ are decomposable via a family of one-step coherent mappings ϕ_t , as in (4.5). In that case equations (4.7)–(4.10) coincide with the respective equations of the additive case (cf. Ruszczyński and Shapiro (2006a)).

Example 4.5. Let us define one step mappings as

$$\mathcal{R}_{s,s+1}(Z_s, Z_{s+1}) := Z_s \vee \varrho|_{\mathcal{F}_s}(Z_{s+1}), \quad s = 0, \dots, T-1, \quad (4.15)$$

and the corresponding preference mappings of the form (4.6), where ϱ is a law invariant coherent measure. In particular for $\varrho := \text{ess sup}$ we obtain the preference system of Example 3.5. Here the dynamic equations (4.9) take the form

$$V_t(x_{t-1}, \omega) = \text{ess inf}_{x_t \in \mathcal{X}_t(x_{t-1}, \omega)} \left\{ f_t(x_t, \omega) \vee \varrho|_{\mathcal{F}_t}(V_{t+1}(x_t, \omega)) \right\}, \quad (4.16)$$

and the reference problem can be viewed as minimization of the worst possible outcome over the considered period of time measured in terms of the measure ϱ .

As we shall see in Section 5.2, this example is closely related to the stopping time risk averse formulation of multistage programs.

5 Time consistent optimal stopping

In this section we discuss a combination of the optimal stopping time and time consistent formulations of multiperiod preference measures.

Definition 5.1. Let $\varrho_{t|\mathcal{F}_t}: \mathcal{Z}_{t+1} \rightarrow \mathcal{Z}_t$, $t = 0, \dots, T-1$, be monotone, translation equivariant mappings and consider the corresponding mappings $\rho_{s,t}: \mathcal{Z}_t \rightarrow \mathcal{Z}_s$ represented in the nested form

$$\rho_{s,t}(\cdot) := \varrho_{s|\mathcal{F}_s} \left(\varrho_{s+1|\mathcal{F}_{s+1}} \left(\dots \varrho_{t-1|\mathcal{F}_{t-1}}(\cdot) \right) \right), \quad 0 \leq s < t \leq T. \quad (5.1)$$

The *stopping risk measure* is

$$\rho_{0,T}(Z_\tau) = \mathbb{1}_{\{\tau=0\}} Z_0 + \varrho_{0|\mathcal{F}_0} \left(\mathbb{1}_{\{\tau=1\}} Z_1 + \dots + \varrho_{T-1|\mathcal{F}_{T-1}}(\mathbb{1}_{\{\tau=T\}} Z_T) \right). \quad (5.2)$$

The stopping risk measure is well-defined by virtue of (2.4) and the translation equivariance of the mappings $\varrho_{t|\mathcal{F}_t}$. Since \mathcal{F}_0 is trivial, the corresponding functional $\rho_{0,T}: \mathcal{Z}_T \rightarrow \mathbb{R}$ is real valued.

In the risk neutral case when $\varrho_{t|\mathcal{F}_t} := \mathbb{E}_{|\mathcal{F}_t}$, we have that $\rho_{s,t} = \mathbb{E}_{|\mathcal{F}_s}$ for $0 \leq s < t \leq T$, in particular $\rho_{0,T} = \mathbb{E}_{|\mathcal{F}_0} = \mathbb{E}$, hence

$$\mathbb{E}(Z_\tau) = \mathbb{E} \left[\sum_{t=0}^T \mathbb{1}_{\{\tau=t\}} Z_t \right] = \mathbb{1}_{\{\tau=0\}} Z_0 + \mathbb{E}_{|\mathcal{F}_0} \left(\mathbb{1}_{\{\tau=1\}} Z_1 + \dots + \mathbb{E}_{|\mathcal{F}_{T-1}}(\mathbb{1}_{\{\tau=T\}} Z_T) \right). \quad (5.3)$$

The stopping time risk measure suggests the following counterpart of the risk neutral stopping time problem (2.1) (recall that by ‘min/max’ we mean that either the minimization or maximization procedure is applied):

$$\min / \max_{\tau \in \mathfrak{T}} \rho_{0,T}(Z_\tau). \quad (5.4)$$

As it was argued in the previous sections, the above formulation (5.4) can be viewed as time consistent and is amenable to writing the dynamic programming equations.

5.1 Distributionally robust approach

It is possible to view the stopping time formulation (5.4) from the following distributionally robust point of view. Consider a *convex* coherent functional $\varrho: L_p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$. It can be represented in the dual form

$$\varrho(Z) = \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z], \quad (5.5)$$

where \mathfrak{M} is a set of probability measures absolutely continuous with respect to the reference probability measure P and such that the densities dQ/dP , $Q \in \mathfrak{M}$, form a bounded convex weakly* closed set $\mathfrak{A} \subset L_q(\Omega, \mathcal{F}, P)$ in the dual space $L_q(\Omega, \mathcal{F}, P) = L_p(\Omega, \mathcal{F}, P)^*$. For the concave counterpart $-\varrho(-Z)$ of ϱ (see (3.4)), the corresponding dual representation is obtained by replacing ‘sup’ in (5.5) with ‘inf’, that is

$$-\varrho(-Z) = \inf_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z].$$

Conversely, given a set \mathfrak{M} of probability measures absolutely continuous with respect to the reference probability measure P and such that the densities dQ/dP , $Q \in \mathfrak{M}$, form a

bounded convex weakly* closed set $\mathfrak{A} \subset L_q(\Omega, \mathcal{F}, P)$, one can use the righthand side of (5.5) as a definition of the corresponding functional ϱ . The so defined functional ϱ is convex coherent. It is law invariant iff the set \mathfrak{A} is law invariant in the sense that if $\zeta \in \mathfrak{A}$ and ζ' is a density distributionally equivalent⁸ to ζ , then $\zeta' \in \mathfrak{A}$. This holds even if the reference probability measure P has atoms (cf. Shapiro (2017, Theorem 2.3)).

Since $\mathbb{E}_Q[\cdot] = \mathbb{E}_Q[\mathbb{E}_Q[\cdot | \mathcal{F}_t]]$, for $Q \in \mathfrak{M}$, we have that

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[\cdot] = \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[\mathbb{E}_Q[\cdot | \mathcal{F}_t]] \leq \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q \left[\operatorname{ess\,sup}_{Q \in \mathfrak{M}} \mathbb{E}_Q[\cdot | \mathcal{F}_t] \right]. \quad (5.6)$$

The functional $\varrho_{|\mathcal{F}_t}(\cdot) := \operatorname{ess\,sup}_{Q \in \mathfrak{M}} \mathbb{E}_Q[\cdot | \mathcal{F}_t]$ can be viewed as the conditional counterpart of the corresponding functional ϱ . Equality in (5.6) would mean the recursive property $\varrho(\cdot) = \varrho(\varrho_{|\mathcal{F}_t}(\cdot))$. In the law invariant case and when the reference probability measure is nonatomic, the functional ϱ has such recursive property only when the set \mathfrak{M} is a singleton or consists of all probability measures absolutely continuous with respect to the reference measure (cf. Kupper and Schachermayer (2009)).

The nested functional $\rho_{0,T}$, defined in (5.1), is decomposable, i.e., has the recursive property (4.2). For not decomposable (law invariant) risk measure ϱ the corresponding *nested* stopping objective $\rho_{0,T}(Z_\tau)$, of the form (5.2), is different from $\varrho(Z_\tau)$. As we shall see below the *nested* formulation of stopping time is amenable for writing dynamic programming equations, and in the sense of nested decomposition is time consistent.

Remark 5.2. In some recent publications it was suggested to consider the following formulation of distributionally robust (risk averse) optimal stopping time problems

$$\max_{\tau \in \mathfrak{T}} \left\{ \varrho(Z_\tau) := \inf_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z_\tau] \right\}. \quad (5.7)$$

As it is pointed above, unless the set \mathfrak{M} is a singleton or consists of all probability measures, this is not the same as the corresponding nested formulation. In order to deal with this, and eventually to write the associated dynamic equations, it was assumed in Föllmer and Schied (2004, Section 6.5) that the set \mathfrak{M} possesses a certain property, called stability. By the above discussion it appears that such stability property will hold only in rather exceptional cases.

Remark 5.3. Consider the maximization variant of problem (5.4). By the dual representation (5.5), such formulation with *convex* preference functional can be viewed as somewhat optimistic; hoping that the uncertainty of the probability distribution, represented by the respective set \mathfrak{M} , works in our favor potentially giving a larger value of the return Z_τ . From the risk averse (pessimistic) point of view it makes more sense either to use the corresponding concave counterpart $\nu_{0,T}$ (defined in (3.4)), or to work with the minimization variant while employing the convex preference functional. As we shall see later these considerations raise delicate issues of preserving convexity of considered problems which is crucial for efficient numerical procedures.

⁸It is said that ζ and ζ' are distributionally equivalent if $P(\zeta \leq z) = P(\zeta' \leq z)$ for all $z \in \mathbb{R}$.

5.2 Multistage risk averse optimal stopping time

By employing nested functionals⁹ $\rho_{0,T}$ of the form (5.1), and by using (2.4), the corresponding optimal stopping time counterparts of problems (2.2) can be written as

$$\min_{\pi \in \Pi} / \max_{\tau \in \mathfrak{T}} \min_{\pi \in \Pi} / \max_{\tau \in \mathfrak{T}} \rho_{0,T}(f_\tau(\mathbf{x}_\tau, \omega)), \quad (5.8)$$

with

$$\begin{aligned} \rho_{0,T}(f_\tau(\mathbf{x}_\tau, \omega)) = & \mathbb{1}_{\{\tau=0\}} f_0(x_0) + \varrho_{0|\mathcal{F}_0} \left(\mathbb{1}_{\{\tau=1\}} f_1(\mathbf{x}_1, \omega) + \right. \\ & \left. \cdots + \varrho_{T-1|\mathcal{F}_{T-1}} (\mathbb{1}_{\{\tau=T\}} f_T(\mathbf{x}_T, \omega)) \right). \end{aligned} \quad (5.9)$$

Remark 5.4. It is also possible to consider optimization of $\rho_{0,T}(f_0(x_0) + \cdots + f_\tau(\mathbf{x}_\tau, \omega))$ by using (2.5) rather than (2.4), i.e., by replacing the cost function $f_\tau(\mathbf{x}_\tau, \omega)$ with the cumulative cost $f_0(x_0) + \cdots + f_\tau(\mathbf{x}_\tau, \omega)$ (for the risk neutral case and fixed stopping time τ this type of problems were considered recently in Guigues (2018)). These formulations are equivalent and we concentrate below on the formulations (5.8).

The problems (5.8) are twofold, they consist in finding simultaneously an optimal policy $\pi^* = (x_0^*, \mathbf{x}_1^*, \dots, \mathbf{x}_T^*)$ and an optimal stopping time $\tau^* \in \mathfrak{T}$. In the risk neutral case when $\varrho_{t|\mathcal{F}_t} = \mathbb{E}_{|\mathcal{F}_t}$, for a given (fixed) policy $\pi \in \Pi$, these problems become the classical problem (2.1) of stopping time for the process

$$Z_t(\omega) := f_t(\mathbf{x}_t(\omega), \omega). \quad (5.10)$$

For a given stopping time $\tau \in \mathfrak{T}$ we can write the corresponding dynamic programming equations, of the form (4.9), for the minimization (with respect to $\pi \in \Pi$) problem (5.8) (cf. Ruszczyński and Shapiro (2006a))

$$V_T^\tau(x_{T-1}, \omega) := \operatorname{ess\,inf}_{x_T \in \mathcal{X}_T(x_{T-1}, \omega)} \mathbb{1}_{\{\tau=T\}} f_T(x_T, \omega), \quad (5.11)$$

$$V_t^\tau(x_{t-1}, \omega) := \operatorname{ess\,inf}_{x_t \in \mathcal{X}_t(x_{t-1}, \omega)} \mathbb{1}_{\{\tau=t\}} f_t(x_t, \omega) + \varrho_{t|\mathcal{F}_t}(V_{t+1}^\tau(x_t, \omega)), \quad (5.12)$$

$t = 1, \dots, T-1$, and the first stage problem at $t = 0$ is (note that $\varrho_{0|\mathcal{F}_0} = \varrho_0$)

$$\min_{x_0 \in \mathcal{X}_0} f_0(x_0) + \varrho_0(V_1^\tau(x_0, \omega)).$$

5.2.1 The min-max problem

Let us consider the min-max (minimization with respect to $\pi \in \Pi$ – maximization with respect to $\tau \in \mathfrak{T}$) variant of problem (5.8). For a fixed policy $\pi = (x_0, \mathbf{x}_1, \dots, \mathbf{x}_T) \in \Pi$ we need to solve the optimal stopping time problem

$$\max_{\tau \in \mathfrak{T}} \left\{ \rho_{0,T}(Z_\tau) = \mathbb{1}_{\{\tau=0\}} Z_0 + \varrho_{0|\mathcal{F}_0} \left(\mathbb{1}_{\{\tau=1\}} Z_1 + \cdots + \varrho_{T-1|\mathcal{F}_{T-1}} (\mathbb{1}_{\{\tau=T\}} Z_T) \right) \right\}, \quad (5.13)$$

⁹Recall that the considered nested functionals are assumed to be monotone and translation equivariant, while can be convex or concave.

with Z_t given in (5.10). The following is a natural extension of the classical results in the risk neutral case to the considered risk averse setting (5.13), e.g., Shiryaev (1978, Section 2.2).

Definition 5.5 (Snell envelope). Let $(Z_0, \dots, Z_T) \in \mathcal{Z}_{0,T}$ be a stochastic process. The Snell envelope (associated with functional $\rho_{0,T}$) is the stochastic process

$$\begin{aligned} E_T &:= Z_T, \\ E_t &:= Z_t \vee \varrho_{t|\mathcal{F}_t}(E_{t+1}), \quad t = 0, \dots, T-1, \end{aligned} \quad (5.14)$$

defined in backwards recursive way.

For $m = 0, \dots, T$, consider $\mathfrak{T}_m := \{\tau \in \mathfrak{T} : \tau \geq m\}$, the optimization problem

$$\max_{\tau \in \mathfrak{T}_m} \rho_{0,T}(Z_\tau), \quad (5.15)$$

and

$$\tau_m^*(\omega) := \min\{t : E_t(\omega) = Z_t(\omega), \ m \leq t \leq T\}, \ \omega \in \Omega. \quad (5.16)$$

Denote by \mathbf{v}_m the optimal value of the problem (5.15). Of course, for $m = 0$, the problem (5.15) coincides with problem (5.13) and \mathbf{v}_0 is the optimal value of problem (5.13). Note that by the recursive property (4.2) we have that¹⁰ $\rho_{0,T}(Z_\tau) = \rho_{0,m}(\rho_{m,T}(Z_\tau))$, $m = 1, \dots, T$.

The following assumption was used by several authors under different names (see, for example, Cheridito and Kupper (2009), where it is called *local property* and references therein):

$$\varrho_{t|\mathcal{F}_t}(\mathbb{1}_A \cdot Z) = \mathbb{1}_A \cdot \varrho_{t|\mathcal{F}_t}(Z), \quad \text{for all } A \in \mathcal{F}_t, \ t = 0, \dots, T-1. \quad (5.17)$$

For coherent mappings $\varrho_{t|\mathcal{F}_t}$ it always holds (cf. Shapiro et al. (2014, Theorem 6.70)).

The following can be compared with classical results in the risk neutral case (e.g., Bingham and Peskir (2008, Theorem 1)).

Theorem 5.6. *Let $\varrho_{t|\mathcal{F}_t} : \mathcal{Z}_{t+1} \rightarrow \mathcal{Z}_t$, $t = 0, \dots, T-1$, be monotone translation equivariant mappings possessing property (5.17) and $\rho_{s,t}$, $0 \leq s < t \leq T$, be the corresponding nested functionals defined in (5.1). Then for $(Z_0, \dots, Z_T) \in \mathcal{Z}_{0,T}$ the following holds:*

(i) *for $m = 0, \dots, T$, and τ_m^* defined in (5.16),*

$$\begin{aligned} E_m &\succeq \rho_{m,T}(Z_\tau), \ \forall \tau \in \mathfrak{T}_m, \\ E_m &= \rho_{m,T}(Z_{\tau_m^*}), \end{aligned}$$

(ii) *the stopping time τ_m^* is optimal for the problem (5.15),*

(iii) *if $\hat{\tau}_m$ is an optimal stopping time for the problem (5.15), then $\hat{\tau}_m \succeq \tau_m^*$,*

(iv) $\mathbf{v}_m = \varrho_{0,m}(E_m)$, $m = 1, \dots, T$, and $\mathbf{v}_0 = E_0$.

¹⁰By the definition $\rho_{T,T}(Z_T) \equiv Z_T$.

Proof. We use induction in m going backwards in time. Recall that $E_T = Z_T$ and hence the assertions follow for $m = T$. Now let $m = T - 1$ and $\tau \in \mathfrak{T}_{T-1}$. Since $\rho_{T-1,T} = \varrho_{T-1|\mathcal{F}_{T-1}}$, by using the translation equivariance and property (5.17), we can write

$$\begin{aligned}\rho_{T-1,T}(Z_\tau) &= \varrho_{T-1|\mathcal{F}_{T-1}} \left(\mathbb{1}_{\{\tau=T-1\}} Z_{T-1} + \mathbb{1}_{\{\tau=T\}} Z_T \right) \\ &= \mathbb{1}_{\{\tau=T-1\}} Z_{T-1} + \mathbb{1}_{\{\tau=T\}} \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T).\end{aligned}$$

We have that Ω is the union of the disjoint sets Ω_{T-1}^τ and Ω_T^τ (defined in (2.3)), and hence (recall that $E_T = Z_T$)

$$\mathbb{1}_{\{\tau=T-1\}} Z_{T-1} + \mathbb{1}_{\{\tau=T\}} \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T) \preceq \max\{Z_{T-1}, \varrho_{T-1|\mathcal{F}_{T-1}}(E_T)\} = E_{T-1}. \quad (5.18)$$

It follows that $\rho_{T-1,T}(Z_\tau) \preceq E_{T-1}$.

Conditional on the event $\{Z_{T-1} \geq \varrho_{T-1|\mathcal{F}_{T-1}}(E_T)\}$ we have: $E_{T-1} = Z_{T-1}$ and $\tau_{T-1}^* = T - 1$, and

$$\rho_{T-1,T}(Z_{\tau_{T-1}^*}) = \varrho_{T-1|\mathcal{F}_{T-1}}(Z_{T-1}) = Z_{T-1} = E_{T-1},$$

and τ_{T-1}^* is optimal for the corresponding problem (5.15). Otherwise conditional on the event $\{Z_{T-1} < \varrho_{T-1|\mathcal{F}_{T-1}}(E_T)\}$, we have that $E_{T-1} = \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T)$, and $\tau_{T-1}^* = T$ is optimal for the corresponding problem (5.15). In both cases the assertion (iii) also holds.

Now let $m = T - 2$ and $\tau \in \mathfrak{T}_{T-2}$. We have that $\rho_{T-2,T}(\cdot) = \varrho_{T-2|\mathcal{F}_{T-2}}(\varrho_{T-1|\mathcal{F}_{T-1}}(\cdot))$ and

$$\begin{aligned}\rho_{T-2,T}(Z_\tau) &= \varrho_{T-2|\mathcal{F}_{T-2}} \left(\varrho_{T-1|\mathcal{F}_{T-1}} \left(\mathbb{1}_{\{\tau=T-2\}} Z_{T-2} + \mathbb{1}_{\{\tau \geq T-1\}} Z_\tau \right) \right) \\ &= \mathbb{1}_{\{\tau=T-2\}} Z_{T-2} + \varrho_{T-2|\mathcal{F}_{T-2}} \left(\mathbb{1}_{\{\tau=T-1\}} Z_{T-1} + \mathbb{1}_{\{\tau=T\}} \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T) \right) \\ &= \mathbb{1}_{\{\tau=T-2\}} Z_{T-2} + \mathbb{1}_{\{\tau > T-2\}} \varrho_{T-2|\mathcal{F}_{T-2}} \left(\mathbb{1}_{\{\tau=T-1\}} Z_{T-1} + \mathbb{1}_{\{\tau=T\}} \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T) \right),\end{aligned}$$

where the last equation holds since $\mathbb{1}_{\{\tau > T-2\}} \mathbb{1}_{\{\tau=T-1\}} = \mathbb{1}_{\{\tau=T-1\}}$ and $\mathbb{1}_{\{\tau > T-2\}} \mathbb{1}_{\{\tau=T\}} = \mathbb{1}_{\{\tau=T\}}$ and by (5.17). Then by (5.18) and monotonicity of $\varrho_{T-2|\mathcal{F}_{T-2}}$ we obtain

$$\begin{aligned}\rho_{T-2,T}(Z_\tau) &\preceq \mathbb{1}_{\{\tau=T-2\}} Z_{T-2} + \mathbb{1}_{\{\tau > T-2\}} \varrho_{T-2|\mathcal{F}_{T-2}}(E_{T-1}) \\ &\preceq \max\{Z_{T-2}, \varrho_{T-2|\mathcal{F}_{T-2}}(E_{T-1})\} = E_{T-2}.\end{aligned}$$

Conditional on $\{Z_{T-2} \geq \varrho_{T-2|\mathcal{F}_{T-2}}(E_{T-1})\}$, we have that $E_{T-2} = Z_{T-2}$ and $\tau_{T-2}^* = T - 2$, and

$$\rho_{T-2,T}(Z_{\tau_{T-2}^*}) = \varrho_{T-2|\mathcal{F}_{T-2}}(\varrho_{T-1|\mathcal{F}_{T-1}}(Z_{T-2})) = Z_{T-2} = E_{T-2},$$

and τ_{T-2}^* is optimal for the corresponding problem (5.15). Otherwise conditional on the event $\{Z_{T-2} < \varrho_{T-2|\mathcal{F}_{T-2}}(E_{T-1})\}$, we have $E_{T-2} = \varrho_{T-2|\mathcal{F}_{T-2}}(E_{T-1})$ and $\tau_{T-2}^* \geq T - 1$. Conditioning further on $\{Z_{T-1} < \varrho_{T-1|\mathcal{F}_{T-1}}(E_T)\}$ we have that $E_{T-1} = \varrho_{T-1|\mathcal{F}_{T-1}}(Z_T)$, and $\tau_{T-2}^* = T$ is optimal for the corresponding problem (5.15). Otherwise conditional further on $\{Z_{T-1} \geq \varrho_{T-1|\mathcal{F}_{T-1}}(E_T)\}$ we have that $\tau_{T-2}^* = T - 1$ and the assertions are verified.

The assertion follows by going backwards in time for $m = T - 3, \dots$ ■

Note that it follows by (5.14) that $Z_t(\omega) \leq E_t(\omega)$ for all $t = 0, \dots, T$ and a.e. $\omega \in \Omega$. We have that

$$\tau_0^*(\omega) = \min\{t: Z_t(\omega) \geq E_t(\omega), t = 0, \dots, T\} \quad (5.19)$$

is an optimal solution of problem (5.13), and the optimal value of problem (5.13) is equal to E_0 . That is, going forward the optimal stopping time τ_0^* stops at the first time $Z_t = E_t$. In particular it stops at $t = 0$ if $Z_0 \geq E_0$ (recall that Z_0 and E_0 are deterministic), i.e., iff $Z_0 \geq \varrho_{0|\mathcal{F}_0}(E_1)$; and it stops at $t = T$ iff $Z_t < E_t$ for $t = 0, \dots, T-1$. As in the risk neutral case the time consistency (Bellman's principle) is ensured here by the decomposable structure of the considered nested risk measure. That is, if it was not optimal to stop within the time set $\{0, \dots, m-1\}$, then starting the observation at time $t = m$ and being based on the information \mathcal{F}_m (i.e., conditional on \mathcal{F}_m), the same stopping rule is still optimal for the problem (5.15).

Consider (compare with (4.15))

$$\mathcal{R}_{s,s+1}(Z_s, Z_{s+1}) := Z_s \vee \varrho_{s|\mathcal{F}_s}(Z_{s+1}), \quad s = 0, \dots, T-1. \quad (5.20)$$

Then we can write E_t in the following recursive form (compare with (4.6))

$$E_t = \mathcal{R}_{t,t+1}\left(Z_t, \mathcal{R}_{t+1,t+2}\left(Z_{t+1}, \dots, \mathcal{R}_{T-1,T}(Z_{T-1}, Z_T)\right)\right), \quad t = 0, \dots, T-1.$$

Consequently (recall that $\mathbf{v}_0 = E_0$) problem (5.8) can be written in the form (3.1) with $\mathcal{R} := \mathcal{R}_{0,T}$, where $\mathcal{R}_{0,T}$ is given in the nested form discussed in Section 4, $\mathcal{R}_{s,s+1}$ defined in (5.20), and with the respective dynamic programming equations of the form (4.16). For an optimal policy $\bar{\pi} = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_T\}$, the Snell envelop of the corresponding stopping time problem is $E_t(\omega) = V_t(\bar{x}_{t-1}, \omega)$, $t = 1, \dots, T$, where $V_t(\cdot, \omega)$ is the value function defined by the dynamic equations (4.16).

Remark 5.7. As it was already mentioned in Remark 5.3, there is a delicate issue of preserving convexity of the considered optimization problems. Suppose that the functionals $\varrho_{t|\mathcal{F}_t}$ are convex. Together with the assumed monotonicity of $\varrho_{t|\mathcal{F}_t}$ and since maximum of two convex functions is convex, convexity of the respective value functions $V_t(\cdot, \omega)$ is implied by convexity of the objective functions $f_t(\cdot, \omega)$ if, for example, the feasibility constraints are linear of the form

$$\mathcal{X}_t(x_{t-1}, \omega) := \{x_t \geq 0: B_t(\omega)x_{t-1} + A_t(\omega)x_t = b_t(\omega)\}. \quad (5.21)$$

On the other hand if $\varrho_{t|\mathcal{F}_t}$ are concave, then convexity of $V_t(\cdot, \omega)$ is not guaranteed.

5.2.2 The min-min problem

Consider the min-min (minimization with respect to $\pi \in \Pi$ and $\tau \in \mathfrak{T}$) variant of problem (5.8). In that case for a fixed policy $\pi \in \Pi$ we need to solve the optimal stopping time problem

$$\min_{\tau \in \mathfrak{T}} \rho_{0,T}(Z_\tau). \quad (5.22)$$

Here the one step mappings $\mathcal{R}_{s,s+1}$, used in construction of the corresponding preference system (discussed in Section 4), take the form

$$\mathcal{R}_{s,s+1}(Z_s, Z_{s+1}) := Z_s \wedge \varrho_{s|\mathcal{F}_s}(Z_{s+1}), \quad s = 0, \dots, T-1. \quad (5.23)$$

In contrast to (5.20) considered above, the mappings $\mathcal{R}_{s,s+1}$ defined in (5.23) do not preserve convexity of $\varrho_{s|\mathcal{F}_s}$. The corresponding dynamic programming equations (4.9)–(4.10) (and (4.11)–(4.13)) apply here as well and are somewhat simpler as the essential infimum and the minimum can be interchanged.

5.2.3 Supermartingales and delayed stopping

Recall that a sequence of random variables $\{X_t\}_{0 \leq t \leq T}$ is said to be *supermartingale* relative to the filtration \mathfrak{F} , if $X_t \succeq \mathbb{E}_{|\mathcal{F}_t}(X_{t+1})$, $t = 0, \dots, T-1$. By analogy we say that the sequence $X_t \in \mathcal{Z}_t$ is \mathfrak{P} -supermartingale, with respect to the collection of mappings $\mathfrak{P} = \{\varrho_{t|\mathcal{F}_t}\}_{t=0, \dots, T-1}$, if

$$X_t \succeq \varrho_{t|\mathcal{F}_t}(X_{t+1}), \quad t = 0, \dots, T-1. \quad (5.24)$$

It follows by the definition (5.14) that the Snell envelope sequence $\{E_t\}_{0 \leq t \leq T}$ is \mathfrak{P} -supermartingale. It also follows from (5.14) that $E_t \succeq Z_t$, i.e., E_t dominates Z_t . We have that $\{E_t\}_{0 \leq t \leq T}$ is the smallest \mathfrak{P} -supermartingale which dominates the corresponding sequence $\{Z_t\}_{0 \leq t \leq T}$.

Remark 5.8. Consider two collections $\varrho_{t|\mathcal{F}_t}: \mathcal{Z}_{t+1} \rightarrow \mathcal{Z}_t$ and $\varrho'_{t|\mathcal{F}_t}: \mathcal{Z}_{t+1} \rightarrow \mathcal{Z}_t$, $t = 0, \dots, T-1$, of monotone translation equivariant mappings possessing property (5.17), with the respective Snell envelope sequences E_t and E'_t and stopping times $\tau'_0 = \inf\{t: E'_t = Z_t\}$ and $\tau_0^* = \inf\{t: E_t = Z_t\}$, defined for $(Z_0, \dots, Z_T) \in \mathcal{Z}_{0,T}$. Suppose that $\varrho_{t|\mathcal{F}_t}(\cdot) \preceq \varrho'_{t|\mathcal{F}_t}(\cdot)$, $t = 0, \dots, T-1$. It follows then that $E_t \preceq E'_t$ for $t = 0, \dots, T$, and hence $\tau_0^* \leq \tau'_0$. That is, for larger risk mappings the optimal (maximization) stopping time, defined in (5.16), is delayed.

For convex law invariant risk functionals it holds that $\mathbb{E}[X] \leq \rho(X)$ (e.g., Shapiro et al. (2014, Corollary 6.52)). In that case it follows together with (5.24) that every \mathfrak{P} -supermartingale X_t , $t = 0, \dots, T-1$, is also a martingale in the usual sense, i.e., with respect to the expectation. For the concave counterpart $\nu(X) = -\rho(-X)$ the converse inequality $\nu(X) \leq \mathbb{E}[X]$ follows of course.

6 Numerical illustration

This section discusses computational approaches to solving stopping time problems with preference systems of the general form (5.8). We illustrate two different approaches based on the pricing of American put options. This stopping time problem is well-known in mathematical finance.

We start by solving the dynamical equations explicitly in the following section and then elaborate on Stochastic Dual Dynamic Programming (SDDP) type algorithm in Section 6.2. We give numerical examples for optimal stopping in univariate, as well as for multivariate

problems (basket options). Further, the examples cover both, the convex and the concave nested (stopping) preference systems.

6.1 Optimal stopping by solving the dynamic equations

Like a simple stock, the option is an investment which comes with risk. The risk averse investor is not willing to pay the fair price for risk-prone investments, instead, the potential buyer expects a discount corresponding to his general decline of risk. As the American put option with strike price $K > 0$ can be exercised any time the investor considers the optimal stopping problem

$$\sup_{\tau \in \mathfrak{T}} \rho_{0,T} \left(e^{-r\tau} \cdot [K - S_\tau]_+ \right), \quad (6.1)$$

where $r > 0$ is a fixed discount rate and S_t is the price of the underlying at time t .

The risk neutral investor chooses the expectation, $\rho_{0,T} = \mathbb{E}$, in (6.1), this problem is the well-known optimal stopping problem in mathematical finance. The risk averse investor, in contrast, chooses a preference measure $\rho_{0,T}$ reflecting his personal risk profile.

Note that the situation reverses for the *owner* of a risky asset. This market participant is willing to accept a fee (a commission, like a premium in insurance) to get rid of his risky asset, i.e., to sell the risk. As well, he will choose a preference functional when computing his personal price of the option, but his personal preference functional reveals opposite preferences than the buyer's functional.

Following the Black-Scholes scheme we consider the geometric random walk process

$$S_t = S_{t-1} \cdot \exp \left(r - \sigma^2/2 + \varepsilon_t \right), \quad t = 1, \dots, T, \quad (6.2)$$

in discrete time with ε_t being an i.i.d. Gaussian white noise process, $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$.

By considering S_t as the state variable the corresponding dynamic equations can be written recursively as

$$\mathcal{V}_T(S_T) := [K - S_T]_+ \quad (6.3)$$

and

$$\mathcal{V}_t(S_t) = [K - S_t]_+ \vee e^{-r} \cdot \varrho_{t|S_t} \left(\mathcal{V}_{t+1}(S_{t+1}) \right), \quad (6.4)$$

for $t = T - 1, \dots, 1$, where ϱ is a chosen law invariant coherent risk measure. In particular we can use the Entropic Value-at-Risk, $\varrho := \text{EV@R}^\beta$, where for $\beta > 0$,

$$\text{EV@R}^\beta(Z) := \inf_{u > 0} \left\{ u^{-1} \left(\beta + \log \mathbb{E}[e^{uZ}] \right) \right\}. \quad (6.5)$$

Note that this is a *convex* law invariant coherent risk measure, it is the homogeneous version of the risk measure studied in [Kupper and Schachermayer \(2009\)](#). We refer to [Ahmadi-Javid and Pichler \(2017\)](#) for details on this particular risk measure, which allows explicit evaluations for Gaussian random variables.

Figure 1 displays the decision rules for $\varrho = \text{EV@R}^\beta$ and varying levels β of risk aversion. The option is not exercised, as long as the price of the stock stays in the continuation region.

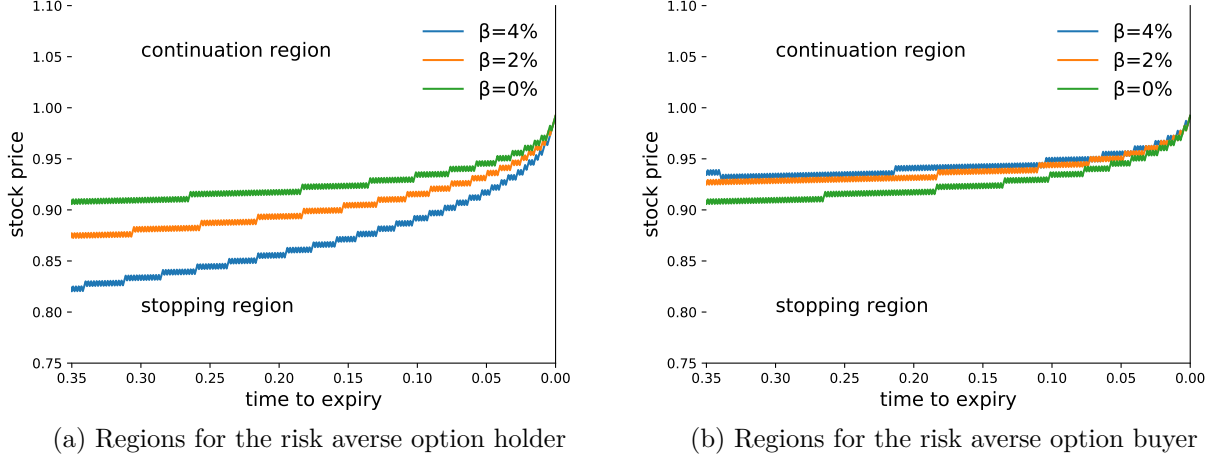


Figure 1: Stopping and continuation regions for risk averse option traders

Once the price of the stock drops out of the continuation region into the stopping region, then it is optimal to exercise the put option. The optimal stopping time thus is

$$\tau = \inf \left\{ i \in \{0, \dots, n\} : (t_i, S_{t_i}) \notin \text{continuation region} \right\}.$$

In view of (6.4) the stopping rule can be restated as

$$S_t + \mathcal{V}_t(S_t) \leq K, \tag{6.6}$$

where S_t is the current price of the stock and $\mathcal{V}_t(S_t)$ the actual and updated price of the corresponding put option. The rule (6.6) is known as *fugit* in mathematical finance.

It holds as well that $-\varrho(-Y) \leq \varrho(Y)$ ¹¹ and Figure 1b displays the continuation and stopping region for the respective concave functional $\nu(Y) := -\text{EV@R}^\beta(-Y)$, the solution of the problem

$$\sup_{\tau \in \mathfrak{T}} \nu_{0,T} \left(e^{-r\tau} \cdot [K - S_\tau]_+ \right);$$

this plot describes the trading regions for the risk averse option buyer. In the risk neutral case, $\beta = 0$, the regions are notably identical as the expectation is linear and thus $\mathbb{E}[Y] = -\mathbb{E}[-Y]$.

6.2 Stochastic Dual Dynamic Programming

The SDDP algorithm was introduced in [Pereira and Pinto \(1991\)](#) and extended to a risk averse setting in [Shapiro \(2011\)](#). For a discussion of the SDDP method we can refer to [Shapiro et al. \(2014, Section 5.10\)](#) and references therein.

¹¹Indeed, $0 = \varrho(0) \leq \varrho(Y) + \varrho(-Y)$.

6.2.1 Univariate SDDP

To study the performance of the SDDP approach to solving optimal stopping problems, we remove nonlinear influences and consider the arithmetic version of the reference problem (6.1), which is

$$\sup_{\tau \in \mathfrak{T}} \rho_{0,T}([K - S_\tau]_+ - r\tau)$$

(Schachermayer and Teichmann (2007) discuss the differences of these models). We use risk measure $\varrho := (1 - \lambda)\mathbb{E} + \lambda \text{AV@R}_\alpha$ for some $\lambda \in [0, 1]$ and $\alpha \in (0, 1)$ and consider $T = 25$ stages. The SDDP algorithm first discretizes the random variables ε_t . If ε_t takes the values $\pm\sigma S_0$ with probability $1/2$ each, then the dynamic programming equations reduce to the binomial option pricing model. Nevertheless, the SDDP algorithm can handle the general situation where ε_t follows an arbitrary distribution, as long as it is possible to obtain samples from such distribution. In the following experiments we let $\varepsilon_t \sim \mathcal{N}(0, \sigma^2 S_0^2)$. In order to solve the problem numerically we discretize the (continuous) distribution by randomly generating N realizations at every stage. We refer to the obtained approximation as the *discretized problem*. The following numerical experiments use $N = 100$ discretized points for each time period.

We assess the quality of the SDDP algorithm on two aspects, namely the statistical properties of the approximations of dynamic programming equations formed under discretization, and the efficiency of the algorithm in solving the discretized problem. Since the problem is univariate (i.e., a single stock price S_t), it is possible to solve the discretized problem quite accurately simply by discretization of the state variables. So the main purpose of the following exercise is to verify efficiency of the SDDP algorithm. After that we investigate a multivariate setting where simple discretization of state variables is not possible. We first run the algorithm, applied to the discretized problem, for 1000 iterations and record the lower bounds generated in each iteration.

The upper bound of the optimal value is constructed by piecewise linear approximations of the value functions $V_t(\cdot)$ (this could be inefficient in the multivariate setting). Note that the SDDP algorithm, applied to the discretized problem, also generates a policy for the original problem with continuous distributions of ε_t . Its value for the original problem can be estimated by generating random sample paths (scenarios) from the original distributions and averaging the obtained values. We sample 2000 scenarios from the distributions of ε_t (either the original or the discretized one) and plot the distributions of the corresponding stopping time and the interest discounted profit. To assess how well the discretization approximates the true dynamic programming equations, we run the SDDP algorithm 30 times and compare the distributions of stopping time and interest discounted profit.

Formulation 1: $\lambda = 0$ (the risk neutral case). Figure 2 exhibits a typical convergence of lower bounds generated by the SDDP algorithm. In particular, the gap between the upper and the lower bounds is already small after, say, 500 iterations. The algorithm solves the discretized problem quite well. Moreover values of the constructed policies for the discretized and original problems are very similar.

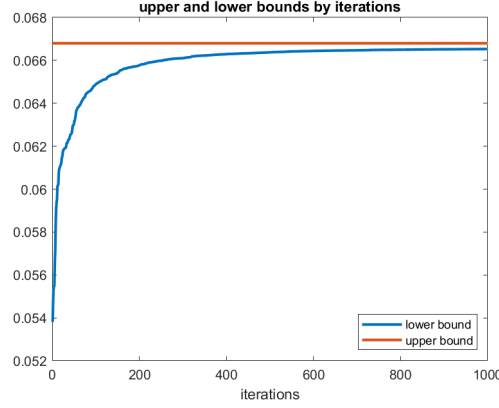


Figure 2: Upper and lower bounds of the optimal objective values (of formulation 1) generated by the SDDP algorithm in 1000 iterations.

We run 30 trials of the SDDP algorithm for randomly generated discretizations of the original continuous distributions of ε_t . We sample 2000 scenarios from both the true and the discretized distributions of ε_t and plot the distributions of the corresponding stopping time and the interest discounted profit. We refer to the discretized distribution as the empirical distribution.

Figures 3 and 4 contain plots for typical distributions of interest discounted profits and stopping times generated by the discretized problems in 30 trials, respectively. In particular, the subplot on the left of each plot corresponds to the scenarios sampled from the discretized distribution of ε_t , whereas the subplot on the right corresponds to the original normal distribution. Although figure 4 indicates that the distributions of the stopping times generated in different trials could be different, the distribution of the profits share a common shape as shown in Figure 3. This indicates that the optimal objective value of (6.1) generated by the discretization is reasonably accurate when $N = 100$, while optimal stopping times are unstable.

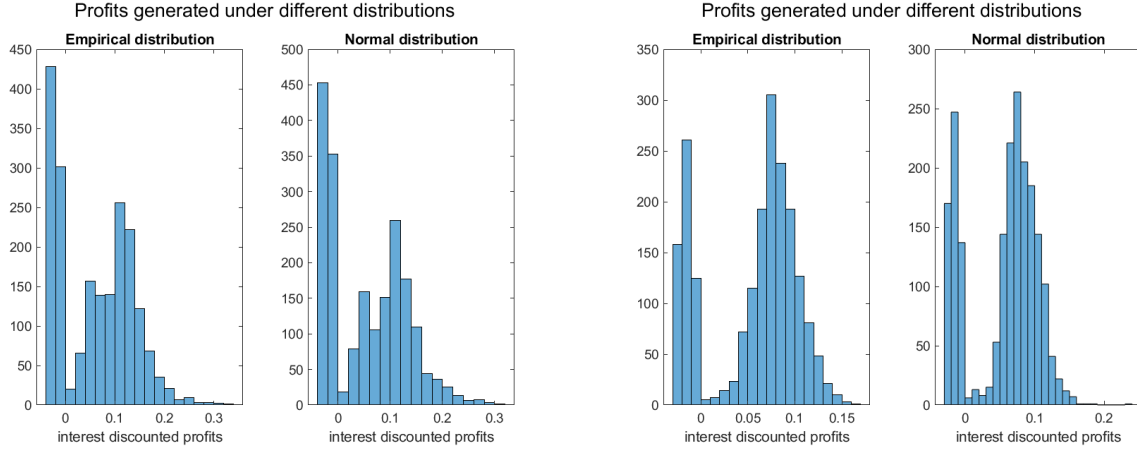


Figure 3: Typical distributions of interest discounted profit (of formulation 1) where the scenarios are sampled from both the empirical and the original normal distribution.

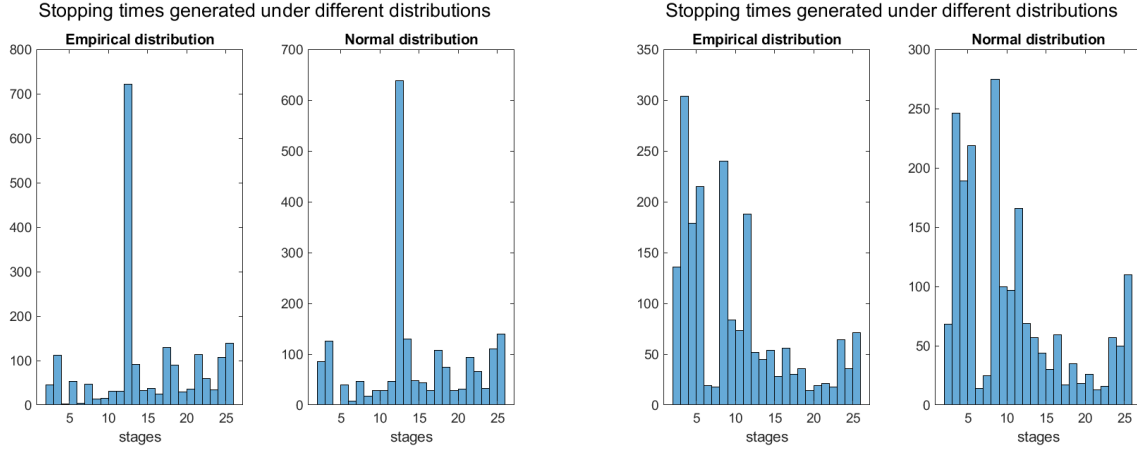


Figure 4: Typical distributions of stopping time (of formulation 1) where the scenarios are sampled from both the empirical and the original normal distribution.

Formulation 2: $\lambda = 0.2$ and $\alpha = 0.05$. We present similar analysis as in the risk neutral case. Figure 5 shows the gap between the upper and the lower bounds which is already small after 500 iterations. Moreover, all distributions of stopping times and profits have the same shape (see Figure 6). As predicted in Remark 5.3, such formulation hopes that the uncertainty of the probability distribution works in our favor potentially giving a larger value of the return, thereby delays the stopping time to late stages.

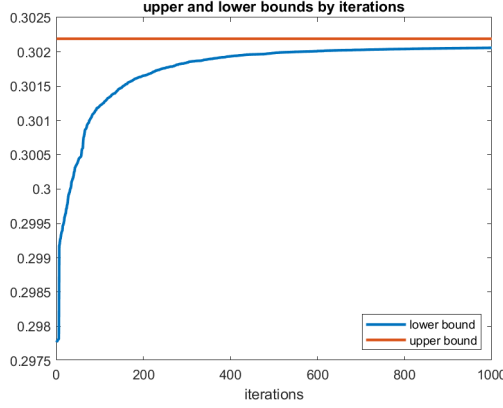


Figure 5: Upper and lower bounds of the optimal objective values (of formulation 2) generated by the SDDP algorithm in 1000 iterations.

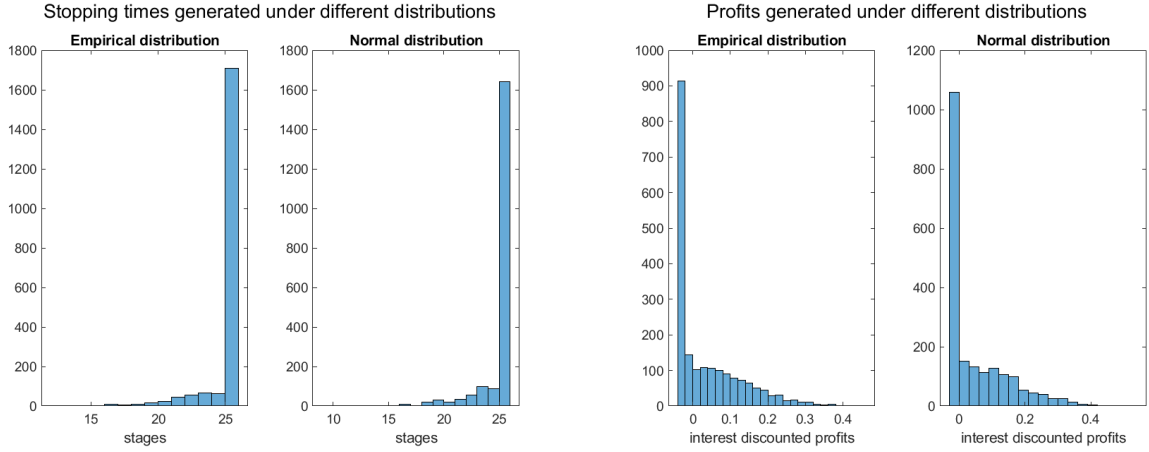


Figure 6: Typical distributions of stopping time and interest discounted profit (of formulation 2) where the scenarios are sampled from both the empirical and the original normal distribution.

6.2.2 Basket options

We also analyze the SDDP algorithm applied to the (multivariate) American basket option pricing problem

$$\sup_{\tau \in \mathfrak{T}} \rho_{0,T} \left(\left[\sum_{j \in J} w_j S_\tau^j - K \right]_+ - r\tau \right). \quad (6.7)$$

Here J is an index set of assets with $|J| = 5$, and $T = 25$. Also $\varrho = (1 - \lambda)\mathbb{E} + \lambda \mathbf{AV}@\mathbf{R}_\alpha$ for some $\lambda \in [0, 1]$ and $\alpha \in (0, 1)$, $w_j \in \mathbb{R}$ are weights, K is the strike price, S_t^j is the price of asset j at time t . In particular, suppose $\mu^j = rS_0^j$ and iid $\epsilon_t \sim N(\mathbf{0}, \Sigma)$ for some covariance

matrix Σ of order $|J| \times |J|$, then S_t^j is a random walk process for each $j \in J$ such that

$$S_t^j = \mu^j + S_{t-1}^j + (\epsilon_t)_j, \quad t = 1, \dots, T. \quad (6.8)$$

The SDDP algorithm for (6.7) proceeds the same way as the one for (6.1), and our analysis of the algorithm is the same as before, i.e., we assess how well the discretization approximates the true dynamic programming equations and how efficient the algorithm in solving the original problem. As mentioned before, the construction of a deterministic upper bound could be inefficient in the multivariate setting, hence we simply let the algorithm run 1000 iterations.

Formulation 3: basket option with $\lambda = 0$ (the risk neutral case). Figure 7 shows that the lower bounds generated by the SDDP algorithm stabilize after about 500 iterations. It appears that the SDDP algorithm solves the discretized problem quite accurately, and we evaluate its performance for the original problem by generating scenarios from the original distribution. As shown in Figures 8 and 9, the distributions of policies, namely interest discounted profits and stopping times, vary across trials.

To further understand the relation between the quality of the discretization and the number of discretized points, we run 30 additional trials with $N = 300$ for each time period, and the results are summarized in Figures 10 and 11. In short, the distributions of profits now look similar, though the distributions of stopping times still have a moderate variation.

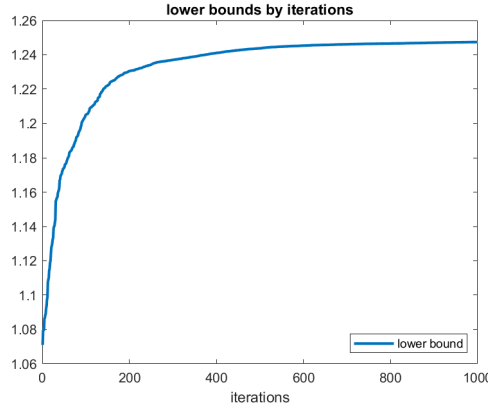


Figure 7: Lower bounds of the optimal objective values (of formulation 3) generated by the SDDP algorithm in 1000 iterations.

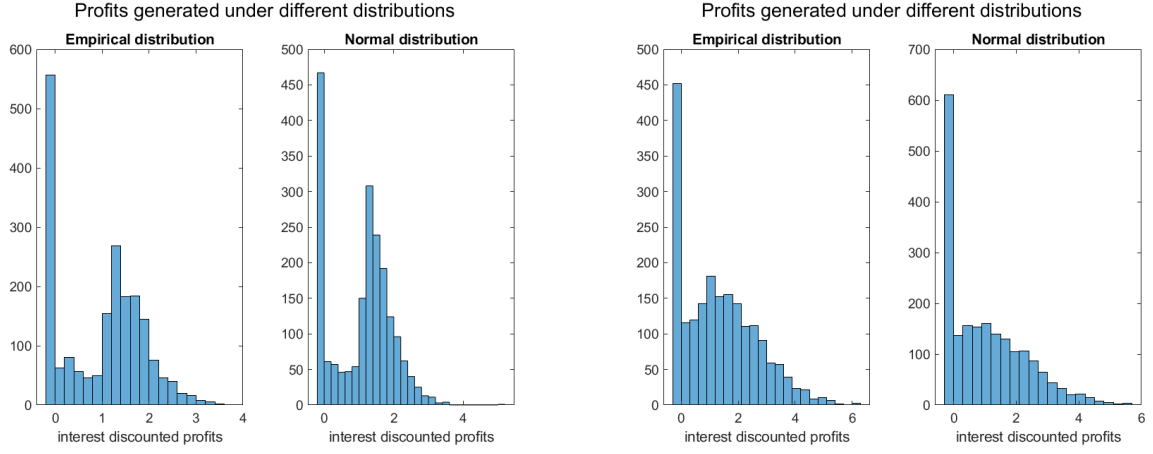


Figure 8: Typical distributions of interest discounted profit (of formulation 3 with $N = 100$) where the scenarios are sampled from both the empirical and the original normal distribution.

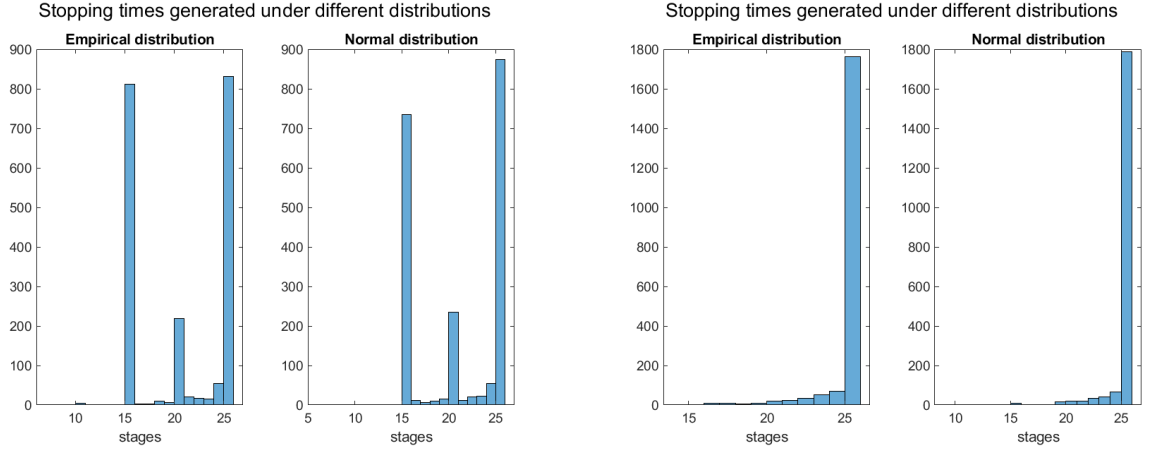


Figure 9: Typical distributions of stopping time (of formulation 3 with $N = 100$) where the scenarios are sampled from both the empirical and the original normal distribution.

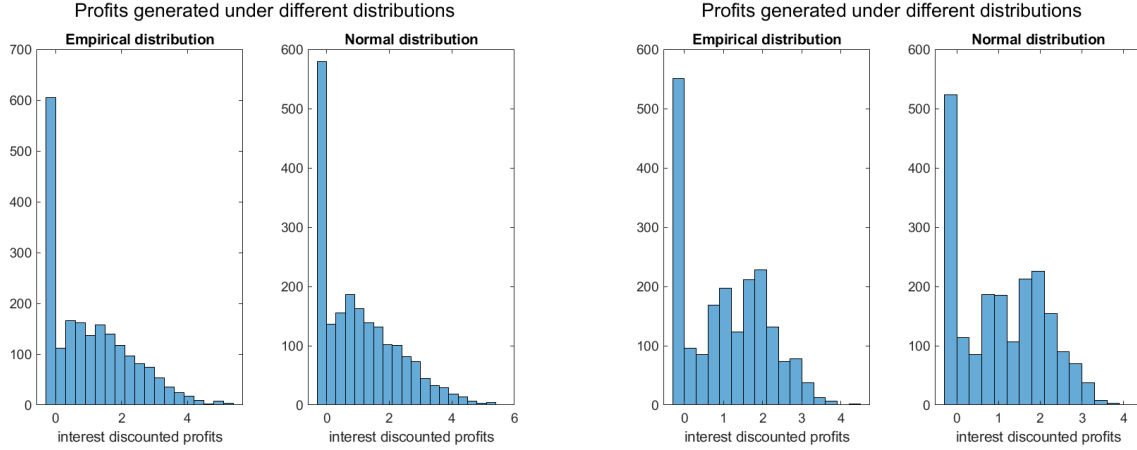


Figure 10: Typical distributions of interest discounted profit (of formulation 3 with $N = 300$) where the scenarios are sampled from both the empirical and the original normal distribution.

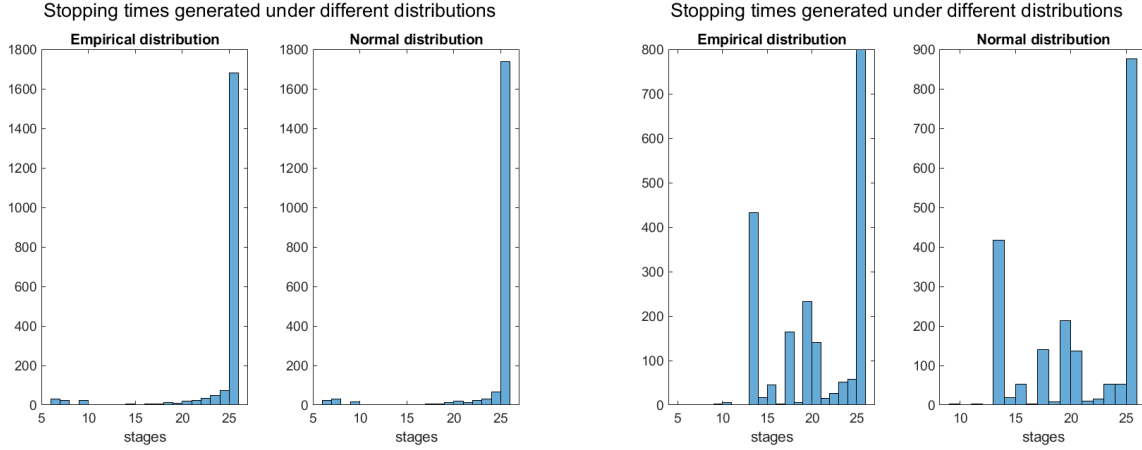


Figure 11: Typical distributions of stopping time (of formulation 3 with $N = 300$) where the scenarios are sampled from both the empirical and the original normal distribution.

Formulation 4: basket option with $\lambda = 0.2$ and $\alpha = 0.05$. Like the other formulations, the lower bounds generated by the SDDP algorithm in this case stabilize after about 500 iterations, hence we omit the plot. Similar to formulation 2, all distributions of stopping times and profits have the same shape (see Figure 12), and the stopping time concentrates on late stages. This indicates that the discretization gives is a good approximation of the original problem even when $N = 100$ for each time period.

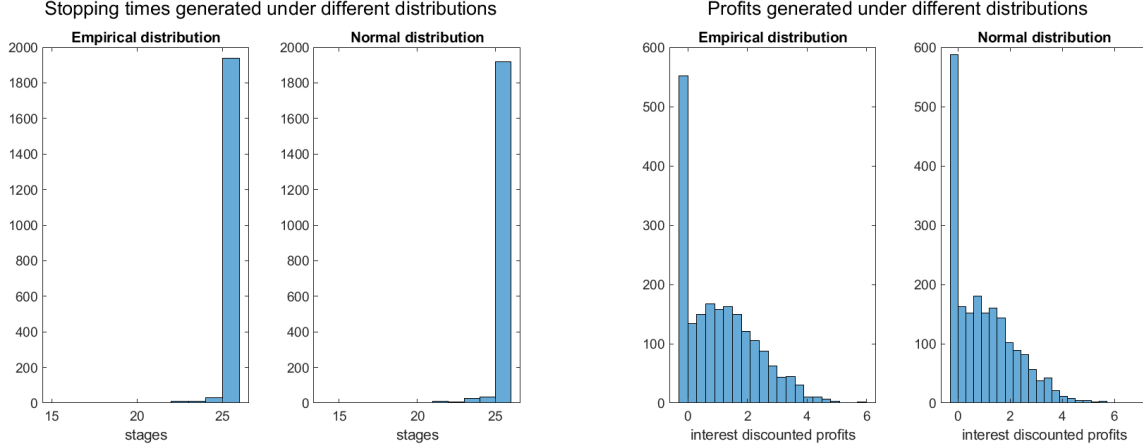


Figure 12: Typical distributions of stopping time and interest discounted profit (of formulation 4) where the scenarios are sampled from both the empirical and the original normal distribution.

We next consider the American basket option pricing problem (6.7) with concave risk measure, with $T = 25$ and $\varrho(Z) = (1 - \lambda)\mathbb{E}(Z) - \lambda \text{AV@R}_\alpha(-Z)$ for some $\lambda \in [0, 1]$ and $\alpha \in (0, 1)$. This is a risk averse formulation since the concave risk measure shifts more weights to scenarios that reduce the profits. As mentioned in Remark 5.3, the convexity is not preserved when a concave risk measure is used.

Without convexity, the SDDP algorithm cannot be applied directly. We circumvent this by approximating the risk averse formulation by solving a sequence of risk neutral problems. The basic idea is to treat every stage of the risk averse formulation as a saddle point problem like (5.2). Starting with the discretized distribution that assigns equal probability to each scenario at every stage, we iteratively solve the risk neutral problem and construct a new discretized distribution by shifting more weights to the “bad” scenarios for the current risk neutral problem. Specifically, in each iteration, we first let the SDDP algorithm solve the discretized risk neutral problem. Then we sample scenarios from the discretized distribution; each scenario corresponds to a sequence of stock prices $\{S_0^j\}_{j \in J}, \{S_1^j\}_{j \in J}, \dots, \{S_T^j\}_{j \in J}$ according to the equation (6.8). For each $t = 1, \dots, T$, we consider $V_t(\{S_{t-1}^j\}_{j \in J}, \hat{\epsilon}_t)$ constructed by the discretized problem, where $\hat{\epsilon}_t$ are the possible realizations of ϵ_t in the discretized distribution. Consequently we arrange these values in the ascending order. For each $\hat{\epsilon}_t$, we then count the frequency (i.e., the number of scenarios) that $\underline{V}_t(S_{t-1}, \hat{\epsilon}_t)$ has one of the $\lceil \alpha N \rceil$ -lowest values. The new distribution is constructed based on the frequency such that the realization $\hat{\epsilon}_t$ is assigned the probability

- $\frac{1-\lambda}{N} + \frac{\lambda}{\alpha N}$, if its frequency is among the top $\lceil \alpha N \rceil$ frequencies for stage t .
- $\frac{1-\lambda}{N} + \frac{\lambda(\alpha N - \lceil \alpha N \rceil)}{\alpha N}$, if its frequency is the $(\lceil \alpha N \rceil + 1)$ -highest frequency for stage t .
- $\frac{1-\lambda}{N}$, otherwise.

Note that those probability weights are exactly the ones assigned by ϱ to different outcomes of the random variable Z when Z has only finitely many outcomes.

The sequence of risk neutral problems is not guaranteed to converge to the original problem in general. Nonetheless, this approach works well on the basket option pricing problem we tested. First of all, the sequence of constructed distributions converges. Besides, the policies generated by the discretization, obtained in the last iteration, exhibit risk aversion. Indeed, compared to Figures 9 and 11, the stopping time shown in Figure 13 has more weights in the early stages, and the distribution of the interest discounted profit (see Figure 13) has a shorter range, because the higher potential returns are turned down in exchange for less risk.

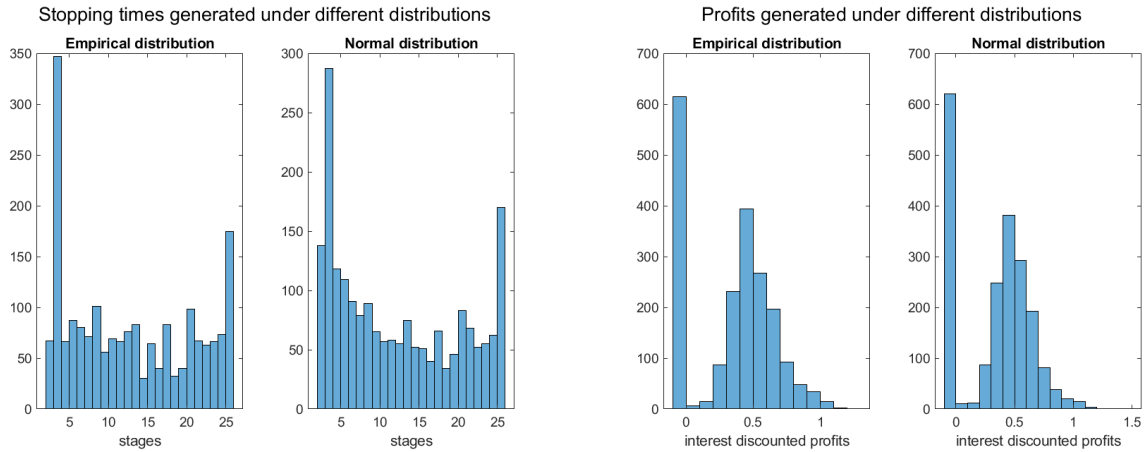


Figure 13: Typical distributions of stopping time and interest discounted profit (of formulation 4) where the scenarios are sampled from both the empirical and the original normal distribution.

References

- A. Ahmadi-Javid and A. Pichler. An analytical study of norms and Banach spaces induced by the entropic value-at-risk. *Mathematics and Financial Economics*, 11(4):527–550, 2017. doi:[10.1007/s11579-017-0197-9](https://doi.org/10.1007/s11579-017-0197-9). 24
- P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent Measures of Risk. *Mathematical Finance*, 9:203–228, 1999. doi:[10.1111/1467-9965.00068](https://doi.org/10.1111/1467-9965.00068). 2, 7
- P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, and H. Ku. Coherent multiperiod risk adjusted values and Bellman’s principle. *Annals of Operations Research*, 152(1):5–22, 2007. doi:[10.1007/s10479-006-0132-6](https://doi.org/10.1007/s10479-006-0132-6). 12
- E. Bayraktar, I. Karatzas, and S. Yao. Optimal stopping for dynamic convex risk measures. *Illinois Journal of Mathematics*, 54(3):1025–1067, 2010. URL <https://projecteuclid.org/euclid.ijm/1336049984>. 2

- R. E. Bellman. *Dynamic Programming*. Princeton University Press, Princeton, NJ, 1957. 5
- D. Belomestny and V. Krätschmer. Optimal stopping under model uncertainty: Randomized stopping times approach. *The Annals of Applied Probability*, 26(2):1260–1295, 4 2016. doi:[10.1214/15-AAP1116](https://doi.org/10.1214/15-AAP1116). 2
- D. Belomestny and V. Krätschmer. Optimal stopping under probability distortions. *Mathematics of Operations Research*, 42:806–833, 2017. 2, 4
- N. H. Bingham and G. Peskir. Optimal stopping and dynamic programming. In *Encyclopedia of Quantitative Risk Assessment and Analysis*, pages 1236–1243. John Wiley and Sons, 2008. 20
- P. Cheridito and M. Kupper. Recursiveness of indifference prices and translation-invariant preferences. *Mathematics and Financial Economics*, 2(3):173–188, 2009. doi:[10.1007/s11579-009-0020-3](https://doi.org/10.1007/s11579-009-0020-3). 20
- P. Cheridito and M. Kupper. Composition of time-consistent dynamic monetary risk measures in discrete time. *International Journal of Theoretical and Applied Finance*, 14(1):137–162, 2011. doi:[10.1142/S0219024911006292](https://doi.org/10.1142/S0219024911006292). 2
- P. Cheridito, F. Delbaen, and M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. *Electronic Journal of Probability*, 11(3):57–106, 2006. doi:[10.1214/EJP.v11-302](https://doi.org/10.1214/EJP.v11-302). 2, 12
- E. Delage and D. Iancu. Robust multistage decision making. *Tutorials in Operations Research, INFORMS*, pages 20–46, 2015. 11
- L. G. Epstein and M. Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003. doi:[10.1016/s0022-0531\(03\)00097-8](https://doi.org/10.1016/s0022-0531(03)00097-8). 12
- H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. de Gruyter Studies in Mathematics 27. Berlin, Boston: De Gruyter, 2004. ISBN 978-3-11-046345-3. doi:[10.1515/9783110218053](https://doi.org/10.1515/9783110218053). URL <http://books.google.com/books?id=cL-bZS0rqWoC>. 2, 3, 4, 18
- A. Goldenshluger and A. Zeevi. Optimal stopping of a random sequence with unknown distribution. 2017. URL <https://www0.gsb.columbia.edu/faculty/azeevi/PAPERS/opstop-4.pdf>. 2, 4
- V. Guigues. Multistage stochastic programs with a random number of stages: dynamic programming equations, solution methods, and application to portfolio selection. Preprint, Cornell University Library, arXiv:1803.06034, 2018. 19
- P. J. Hammond. Consistent plans, consequentialism, and expected utility. *Econometrica*, 57(6):1445–1449, 1989. doi:[10.2307/1913716](https://doi.org/10.2307/1913716). 2

- M. Haviv. On constrained Markov decision processes. *Operations Research Letters*, 19(1): 25–28, 1996. doi:[10.1016/0167-6377\(96\)00003-x](https://doi.org/10.1016/0167-6377(96)00003-x). 2
- A. Hordijk. *Dynamic Programming and Markov Potential Theory*. Mathematical Centre Tracts No. 51, Amsterdam, 1974. 2
- A. Jobert and L. C. G. Rogers. Valuations and dynamic convex risk measures. *Mathematical Finance*, 18(1):1–22, 2008. doi:[10.1111/j.1467-9965.2007.00320.x](https://doi.org/10.1111/j.1467-9965.2007.00320.x). 2
- I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance*. Stochastic Modelling and Applied Probability. Springer, 1998. doi:[10.1007/b98840](https://doi.org/10.1007/b98840). 5
- R. Kovacevic and G. Ch. Pflug. Time consistency and information monotonicity of multiperiod acceptability functionals. In H. Albrecher, W. J. Runggaldier, and W. Schachermayer, editors, *Advanced Financial Modelling*, number 8 in Radon Series on Computational and Applied Mathematics, pages 347–369. de Gruyter, 2009. doi:[10.1515/9783110213140](https://doi.org/10.1515/9783110213140). 2
- V. Krätschmer and J. Schoenmakers. Representations for optimal stopping under dynamic monetary utility functionals. *SIAM Journal on Financial Mathematics*, 1(1):811–832, 2010. doi:[10.1137/090775841](https://doi.org/10.1137/090775841). 2
- D. M. Kreps and E. L. Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46(1):185, 1978. doi:[10.2307/1913656](https://doi.org/10.2307/1913656). 12
- M. Kupper and W. Schachermayer. Representation results for law invariant time consistent functions. *Mathematics and Financial Economics*, 2:189–210, 2009. doi:[10.1007/s11579-009-0019-9](https://doi.org/10.1007/s11579-009-0019-9). 2, 14, 18, 24
- M. V. Pereira and L. M. Pinto. Multi-stage stochastic optimization applied to energy planning. *Mathematical programming*, 52(1-3):359–375, 1991. 25
- F. Riedel. Dynamic coherent risk measures. *Stochastic Processes and their Applications*, 112(2):185–200, 2004. doi:[10.1016/j.spa.2004.03.004](https://doi.org/10.1016/j.spa.2004.03.004). 12
- R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer Verlag, 1997. doi:[10.1007/978-3-642-02431-3](https://doi.org/10.1007/978-3-642-02431-3). URL <https://books.google.com/books?id=w-NdOE5fD8AC>. 10
- A. Ruszczyński. Risk-averse dynamic programming for Markov decision processes. *Math. Program., Ser. B*, 125:235–261, 2010. doi:[10.1007/s10107-010-0393-3](https://doi.org/10.1007/s10107-010-0393-3). 2, 12
- A. Ruszczyński and A. Shapiro. Conditional risk mappings. *Mathematics of Operations Research*, 31(3):544–561, 2006a. doi:[10.1287/moor.1060.0204](https://doi.org/10.1287/moor.1060.0204). 16, 19
- A. Ruszczyński and A. Shapiro. Optimization of convex risk functions. *Mathematics of Operations Research*, 31:433–452, 2006b. doi:[10.1287/moor.1050.0186](https://doi.org/10.1287/moor.1050.0186). 11, 12

- H. Scarf. A min-max solution of an inventory problems. *Studies in the Mathematical Theory of Inventory and Production*, pages 201–209. Stanford University Press, 1958. 2
- W. Schachermayer and J. Teichmann. How close are the option pricing formulas of Bachelier and Black-Merton-Scholes? *Mathematical Finance*, 18(1):155–170, 2007. doi:[10.1111/j.1467-9965.2007.00326.x](https://doi.org/10.1111/j.1467-9965.2007.00326.x). 26
- A. Shapiro. On a time consistency concept in risk averse multistage stochastic programming. *Operations Research Letters*, 37(37):143–147, 2009. doi:[10.1016/j.orl.2009.02.005](https://doi.org/10.1016/j.orl.2009.02.005). 15
- A. Shapiro. Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research*, 209(1):63–72, 2011. 25
- A. Shapiro. Distributionally robust stochastic programming. *SIAM Journal on Optimization*, 27(4):2258–2275, jan 2017. doi:[10.1137/16M1058297](https://doi.org/10.1137/16M1058297). URL http://www.optimization-online.org/DB_FILE/2015/12/5238.pdf. 10, 11, 18
- A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming, Modeling and Theory*. MOS-SIAM Series on Optimization. SIAM, 2nd edition, 2014. 13, 20, 23, 25
- A. N. Shiryaev. *Optimal Stopping Rules*. Springer New York, 1978. doi:[10.1007/978-3-540-74011-7](https://doi.org/10.1007/978-3-540-74011-7). 2, 20
- A. Wald. *Sequential Analysis*. John Wiley & Sons, Inc., 1947. 1
- A. Wald. Statistical decision functions. *Ann. Math. Statist.*, 20(2):165–205, 1949. doi:[10.1214/aoms/1177730030](https://doi.org/10.1214/aoms/1177730030). 1
- T. Wang. A class of dynamic risk measures. Technical report, University of British Columbia, 1999. 2, 12
- S. Weber. Distribution-invariant risk measures, information, and dynamic consistency. *Mathematical Finance*, 16(2):419–441, April 2006. doi:[10.1111/j.1467-9965.2006.00277.x](https://doi.org/10.1111/j.1467-9965.2006.00277.x). 2
- P. A. Weller. Consistent intertemporal decision making under uncertainty. *Review of Economic Studies*, 45:263–266, 1978. doi:[10.2307/2297340](https://doi.org/10.2307/2297340). 2