



LEON EIFLER¹, AMBROS GLEIXNER², JONAD PULAJ

Chvátal's Conjecture Holds for Ground Sets of Seven Elements

¹  0000-0003-0245-9344

²  0000-0003-0391-5903

Zuse Institute Berlin
Takustr. 7
14195 Berlin
Germany

Telephone: +49 30-84185-0
Telefax: +49 30-84185-125

E-mail: bibliothek@zib.de
URL: <http://www.zib.de>

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Chvátal’s Conjecture Holds for Ground Sets of Seven Elements

Leon Eifler^{*}, Ambros Gleixner^{*}, and Jonad Pulaj^{*,†}

^{*}*Department of Mathematical Optimization, Zuse Institute Berlin (ZIB), Takustr. 7, 14195 Berlin, Germany, {eifler,gleixner,pulaj}@zib.de*

[†]*Department of Computer Science, Mathematics and Physics, Faculty of Science and Technology, The University of the West Indies, Cave Hill, St. Michael, Barbados, jonad.pulaj@cavehill.uwi.edu*

September 5, 2018

Abstract

We establish a general computational framework for Chvátal’s conjecture based on exact rational integer programming. As a result we prove Chvátal’s conjecture holds for all downsets whose union of sets contains seven elements or less. The computational proof relies on an exact branch-and-bound certificate that allows for elementary verification and is independent of the integer programming solver used.

1 Introduction

Chvátal’s conjecture is a well-known open problem in extremal set theory from 1974, later earning a spot among Erdős’ favorite combinatorial problems [Erd81]. Despite its popularity, research efforts have yielded limited progress, mostly restricted to special cases and related variants of the original conjecture. Before continuing in more detail we need the following definitions.

Let $[n] := \{1, 2, \dots, n\}$. A *family* \mathcal{F} is a set of subsets of $[n]$. Let $U(\mathcal{F})$ denote the union of all sets in \mathcal{F} . A family \mathcal{F} is a *downset* if and only if $A \in \mathcal{F}$ and $B \subseteq A$ implies $B \in \mathcal{F}$. If \mathcal{F} is a downset then $F \in \mathcal{F}$ is a *base* if and only if no strict supersets of F are contained, i.e., $F \subseteq D$ and $D \in \mathcal{F}$ implies $F = D$. A family \mathcal{F} is called *intersecting* if and only if the intersection of any pairwise sets in \mathcal{F} is nonempty. A family \mathcal{F} is a *star* if and only there exists an element in $U(\mathcal{F})$ contained in all sets of \mathcal{F} . A family \mathcal{F} has the *star property* if and only if some maximum-sized intersecting family in \mathcal{F} is a star. We are ready to state Chvátal’s conjecture as follows:

Conjecture 1 (Chvátal [Chv74]). *Every downset has the star property.*

Schönheim [Sch75] showed that Chvátal’s conjecture holds for all downsets \mathcal{D} whose bases have a nonempty intersection. Stein [Ste83] proved the conjecture holds for all downsets in which all but one of the bases is a *simple* star, i.e., a star in which the intersection of all of its sets is equal to the intersection of any of its two sets. Miklós [Mik84] showed that Chvátal’s conjecture holds for any downset \mathcal{D} that contains an intersection family of size $\lfloor \mathcal{D}/2 \rfloor$. Sterboul [Ste74] proved that any downsets whose sets have three or less elements always satisfy Conjecture 1. The last result was recently proven again in different ways by Czabarka, Hurlbert, Kamat [CHK17] and Olarte, Santos, Spreer [OSS18].

The work for this article has been partly conducted within the Research Campus MODAL funded by the German Federal Ministry of Education and Research (BMBF grant number 05M14ZAM).

Furthermore, Chvátal maintains a website¹ dedicated to the conjecture with a substantial list of publications on the topic.

Our main result on Chvátal’s conjecture is the following:

Theorem 1. *Conjecture 1 holds for all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq 7$.*

Theorem 1 is proven via machine-assisted methods which will be described in detail in this paper and whose implementations are freely available to the public.² Furthermore it is very likely that Chvátal’s conjecture holds for $|U(\mathcal{D})| = 8$, as we show in Section 4. To the best of our knowledge there is no known computational methodology in the literature that investigates Conjecture 1 even for small ground sets. The previously known best bound on the cardinality of the ground set was $|U(\mathcal{D})| = 4$, which follows directly from [Ste74] as we show in Proposition 4.

More generally, we establish a new *safe* computational framework for Chvátal’s conjecture based on integer programming (IP), which couples an exact rational solver and verification procedures for the correctness of the input and the branch-and-bound tree output. Notably, this framework features the combined use of an exact IP certificate, VIPR [CGS17], and a formal proof assistant, Coq [Coq18], to ensure the correctness of the certificate’s input data. To the best of our knowledge, this has not been explored in the literature before.

Already Fishburn [Fis88] highlighted the connection between combinatorial optimization and Chvátal’s conjecture and investigated related problems. Thus modeling the conjecture as an IP and using solvers that safeguard against numerical issues is a natural step to further investigate Conjecture 1. Furthermore, relying on an IP framework to investigate Chvátal’s conjecture for small ground sets has other advantages. First, the rich and well-developed theory of polyhedral combinatorics, that is inherent in an IP approach, may lead to new insights on Conjecture 1. Second, as we see in Section 2.3 and Section 4, known partial results on Chvátal’s conjecture can be encoded as “cuts” in our framework. This improves the performance of the exact rational solver and may allow strengthening of Theorem 1 in the future. In contrast, our initial experiments with propositional satisfiability (SAT) formulations that used MiniSat+ [ES06] to obtain a SAT encoding for the IP formulation $P_{inf}(n)$ in Section 2.1 failed to obtain competitive results. Even when employing current SAT solvers such as Lingeling [Bie16], *any* instance for $|U(\mathcal{D})| > 5$ failed to solve within a 24 hour time limit.

The rest of this paper is organized as follows. Section 2 describes the IP formulations that we use to model Conjecture 1 together with valid inequalities for the underlying polytopes and “cuts” from the literature that reduce the number of integral solutions to the IP formulations. Section 3 focuses on exact rational integer programming together with input/output verification as a recent methodology for machine-assisted theorem proving. Section 4 contains a detailed description of our experimental results. Section 5 concludes with an outlook on future work.

2 A Polyhedral Approach to Chvátal’s Conjecture

In this section we present integer programming formulations for Chvátal’s conjecture over fixed-size ground sets. The formulations are based on decision variables that index members of the power set. Due to the exponential nature of power sets, the size of the formulations is bound to grow quickly with the size of the ground set. However, even for small ground sets little is known and the results in Section 4 show that we can make significant progress beyond the current best bound of $|U(\mathcal{F})| = 4$.

¹<http://users.encs.concordia.ca/~chvatal/conjecture.html>.

²<https://github.com/leoneifler/chvatalip>

2.1 An Infeasibility-Based Formulation

The first IP model is formulated such that Chvátal's conjecture holds for the considered ground set if and only if the IP has no solution. In other words, a feasible solution to the IP formulation for any fixed n would yield a counterexample to Chvátal's conjecture. Let $2^{[n]}$ denote the power set of $[n]$, then we consider the integer program $P_{inf}(n)$,

$$\max \sum_{S \in 2^{[n]}} x_S \quad (1a)$$

$$x_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subset T, \quad (1b)$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \quad (1c)$$

$$y_S \leq x_S \quad \forall S \in 2^{[n]}, \quad (1d)$$

$$\sum_{S \in 2^{[n]} : i \in S} x_S + 1 \leq \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \quad \forall i \in [n], \quad (1e)$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]}.$$

Here, x encodes the set family $\mathcal{S}(x) := \{S \subseteq [n] : x_S = 1\}$ and y encodes the subfamily $\mathcal{S}(y) := \{S \subseteq [n] : y_S = 1\}$. The first class of *downset* inequalities (1b) ensures that $\mathcal{S}(x)$ is a downset. The second class of *intersecting* inequalities (1c) ensures that $\mathcal{S}(y) \setminus \{\emptyset\}$ is an intersecting family. The third class of *containment* inequalities (1d) ensures that the intersecting family is contained in the chosen downset, $\mathcal{S}(y) \subseteq \mathcal{S}(x)$. Finally, the fourth class of *star* inequalities (1e) requires that the intersecting family has greater cardinality than any star in the downset.

Theorem 2. *Let n be a positive integer. All downsets \mathcal{F} such that $|U(\mathcal{F})| \leq n$ satisfy Chvátal's conjecture if and only if $P_{inf}(n)$ is infeasible.*

Proof. Fix n . Suppose that Chvátal's conjecture does not hold, i.e., there exists a downset \mathcal{D} and an intersecting family $\mathcal{Y} \subseteq \mathcal{D}$ such that $|\mathcal{Y}|$ is larger than the size of every star in \mathcal{D} . W.l.o.g. assume $\mathcal{D} \subseteq 2^{[n]}$. Let x and y be their incidence vectors, i.e., $\mathcal{D} = \mathcal{S}(x)$ and $\mathcal{Y} = \mathcal{S}(y)$. By construction, x and y satisfy constraints (1b–1d). Furthermore, for each element $i \in [n]$, $|\mathcal{Y}|$ is larger than the size of all stars that have i as common element, hence (1e) is equally satisfied. In total, x and y constitute a feasible solution to $P_{inf}(n)$.

Conversely, suppose all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n$ satisfy Chvátal's conjecture. Suppose x and y are feasible solutions to $P_{inf}(n)$. By (1b), $\mathcal{S}(x)$ forms a downset and $|U(\mathcal{S}(x))| \leq n$. By (1c) and (1d), $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$ forms an intersecting family contained in $\mathcal{S}(x)$. Hence, $|\mathcal{Y}|$ can be at most the size of the largest star contained in $\mathcal{S}(x)$,

$$|\mathcal{Y}| = \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \leq \max_{i \in [n]} \sum_{S \in 2^{[n]} : i \in S} x_S. \quad (2)$$

But then constraint (1e) is violated for $i_0 \in \arg \max_{i \in [n]} \sum_{S \in 2^{[n]} : i \in S} x_S$. \square

Note that the objective function of $P_{inf}(n)$ is, in some sense, arbitrary since we only need to decide whether the integer program has a feasible solution or not. The objective function encodes the cardinality of $\mathcal{S}(x)$, hence solving $P_{inf}(n)$ amounts to searching for a largest counterexample to Conjecture 1. In the following we present a more advanced formulation that uses the optimal value as an essential component.

2.2 An Optimality-Based Formulation

As already noted in [OSS18] it is sufficient to only consider downsets generated by an intersecting family. We use this insight in the following, advanced formulation $P_{opt}(n)$,

$$\max \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S - z \quad (3a)$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \quad (3b)$$

$$\sum_{S \in 2^{[n]} : i \in S} x_S \leq z \quad \forall i \in [n], \quad (3c)$$

$$y_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subseteq T, \quad (3d)$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]},$$

$$z \in \mathbb{Z}_{\geq 0}.$$

The first class of *intersecting* inequalities (3b) is the same as (1c), whereas the second class of *star* inequalities (3c) differs from (1e). It ensures that the largest star is bounded above by the positive integer variable z . Finally, the third class of *generation* inequalities (3d) ensures, as will be made clearer in the proof of Theorem 3, that an optimal solution of $P_{opt}(n)$ considers only downsets generated by the intersecting family. We note that the generation inequalities (3d) can also be included in $P_{inf}(n)$ instead of (1b) and (1d) by the same argument. Before we formally state and prove the correctness of $P_{opt}(n)$ with regards to Conjecture 1, we need the following observation.

Observation 1. *Let n be a positive integer. An optimal solution of $P_{opt}(n)$ satisfies at least one star inequality (3c) with equality.*

Observation 1 follows from the objective function of $P_{opt}(n)$. Variable z is restricted only from below by its lower bound zero and the left-hand sides of constraints (3c). Since $P_{opt}(n)$ is a maximization problem and the objective coefficient of z is negative, in an optimal solution the variable z will be as small as possible. This implies that at least one star inequality (3c) is tight.

Theorem 3. *Let n be a positive integer. All downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n$ satisfy Chvátal's conjecture if and only if the objective function value of an optimal solution of $P_{opt}(n)$ is zero.*

Proof. Fix $n \in \mathbb{N}$. First note that $P_{opt}(n)$ is feasible since any star is also an intersecting family. Thus for any downset $\mathcal{D} \subseteq 2^{[n]}$, choosing x as the indicator vector of \mathcal{D} and setting the y -variables such that they represent a maximum-cardinality star in \mathcal{D} yields a feasible solution. Choosing z to be the maximum star cardinality, i.e., the smallest value such that constraints (3c) are satisfied, also proves a lower bound of zero on the objective value.

Now suppose all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n$ satisfy Chvátal's conjecture and let x, y, z be an optimal solution for $P_{opt}(n)$. Then it suffices to show that $\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S \leq z$. Constraints (3b) ensure that $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$ forms an intersecting family. Furthermore, the x -variables do not appear in the objective function and are bounded below only by constraints (3d). Hence, w.l.o.g we may assume that $x_S = \max_{T \supseteq S} y_T$. Then, by constraints (3d), $\mathcal{D} := \mathcal{S}(x)$ is a downset and $\mathcal{Y} \subseteq \mathcal{D}$. Assuming Chvátal's conjecture ensures that $|\mathcal{Y}| = \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S$ is at most the size of the largest star in \mathcal{D} , which by constraints (3c) is less than or equal to z .

Conversely, suppose there exists a counterexample to Chvátal's conjecture, i.e., a downset \mathcal{D} , $|U(\mathcal{D})| \leq n$, and an intersecting family $\mathcal{Y} \subseteq \mathcal{D}$ such that $|\mathcal{Y}|$ is larger than the size of any star in \mathcal{D} . W.l.o.g. assume $\mathcal{D} \subseteq 2^{[n]}$ and let x and y be the incidence

vectors of \mathcal{D} and \mathcal{Y} , respectively, i.e., $\mathcal{D} = \mathcal{S}(x)$ and $\mathcal{Y} = \mathcal{S}(y)$. Let z be the size of the largest star in \mathcal{D} , i.e., $z = \max \sum_{S \in 2^{[n]}; i \in S} x_S$. Then by construction, x, y, z is a feasible solution for $P_{opt}(n)$. Because we consider a counterexample, the objective function value is at least one. \square

2.3 Valid Inequalities and Model Reductions

As mentioned in Section 1, one of the advantages of an IP approach is that $P_{inf}(n)$ and $P_{opt}(n)$ can be studied in greater depth through polyhedral combinatorial techniques. Furthermore known results from the literature can be expressed as valid inequalities and problem reductions for $P_{inf}(n)$ and $P_{opt}(n)$, in the sense that Theorems 2 and 3 still hold with these additional constraints, and the number of feasible solutions is less than or equal to the number of current solutions. This is demonstrated in the following section and may help to increase the size of n for which the models can be solved.

First, consider the intersecting inequalities of form (1c) and (3b). When $T = [n] \setminus S$, these can be interpreted as a special case of the following partition inequalities.

Proposition 1. *Let n be a positive integer and let \mathcal{P} be a partition of $[n]$. Then the inequality*

$$\sum_{S \in \mathcal{P}} y_S \leq 1 \tag{4}$$

is a valid for $P_{inf}(n)$ and $P_{opt}(n)$.

Proof. Suppose $\sum_{S \in \mathcal{P}} y_S \geq 2$ for an integer feasible solution y , then there exist $S, T \in \mathcal{P}$ with $y_S = y_T = 1$, violating (1c) and (3b). \square

As a consequence, intersection inequalities that do not cover the whole ground set can be strengthened.

Proposition 2. *Let n be a positive integer and $S, T \in 2^{[n]} \setminus \{\emptyset\}$ such that $T \cap S = \emptyset$. Suppose $S \cup T \neq [n]$. Then the intersecting inequality*

$$y_T + y_S \leq 1 \tag{5}$$

is dominated by a partition inequality.

Proof. S, T can be completed to a partition by their complement $[n] \setminus (S \cup T)$. Inequality (5) is trivially dominated by the corresponding partition inequality $y_T + y_S + y_{[n] \setminus (S \cup T)} \leq 1$. \square

The large number of partitions prohibits the static addition of these inequalities to the formulation. However, modern IP solvers automatically extract the conflicting y -assignments from constraints (1c) and (3b) and add partition inequalities dynamically. Next, consider the following central result.

Theorem 4 (Berge [Ber75]). *If \mathcal{D} is a downset then \mathcal{D} is a disjoint union of pairs of disjoint sets, together with \emptyset if $|\mathcal{D}|$ is odd.*

This yields the following result, that can easily be expressed as a valid inequality for $P_{inf}(n)$ and $P_{opt}(n)$, as in Corollary 2.

Corollary 1 (Anderson [And02] p.105). *Let \mathcal{D} be a downset and \mathcal{Y} an intersecting family such that $\mathcal{Y} \subseteq \mathcal{D}$. Then $2|\mathcal{Y}| \leq |\mathcal{D}|$.*

Corollary 2. *Let n be any positive integer. Suppose the following inequality*

$$\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} 2y_S \leq \sum_{S \in 2^{[n]}} x_S \quad (6)$$

is added to $P_{inf}(n)$ and $P_{opt}(n)$. Then Theorems 2 and 3 hold for the modified formulations of $P_{inf}(n)$ and $P_{opt}(n)$, respectively.

The following result is used in [OSS18] to give a simple proof that Chvátal's conjecture holds for all downsets whose sets have three elements or less.

Theorem 5 (Kleitman, Magnanti [KM72]). *Any intersecting family that is contained in the union of two stars generates a downset that satisfies Conjecture 1.*

As a consequence, y -variables for sets with one or two elements can be fixed to zero in $P_{inf}(n)$ and $P_{opt}(n)$.

Corollary 3. *Let n be any positive integer. For $P_{inf}(n)$ and $P_{opt}(n)$ fix $y_S = 0$ for all $S \in 2^{[n]}$ such that $1 \leq |S| \leq 2$. Then Theorems 2 and 3 hold for $P_{inf}(n)$ and $P_{opt}(n)$, respectively, with the given fixings.*

Proof. Suppose y stems from a solution of $P_{inf}(n)$ or $P_{opt}(n)$, then it encodes an intersecting family $\mathcal{Y} := \mathcal{S}(y) \setminus \{\emptyset\}$. If $\{i\} \in \mathcal{Y}$, then \mathcal{Y} is a star centered around i . If $\{i, j\} \in \mathcal{Y}$, then \mathcal{Y} is the union of two stars centered around i and j , respectively. By Theorem 5, these do not amount to counterexamples to Chvátal's conjecture and can safely be excluded from the formulations without changing the feasibility status of $P_{inf}(n)$ and the optimal objective value of $P_{opt}(n)$, respectively. \square

Variable fixings are certainly the most effective improvements to the problem formulation, since they directly reduce the problem size as opposed to general valid inequalities that increase the number of constraints. If we know that Conjecture 1 holds for all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n$ for some fixed n , then we can use a simple variable fixing scheme for the case when $|U(\mathcal{D})| = n + 1$, as follows.

Proposition 3. *Let n be a fixed positive integer. Suppose $P_{inf}(n)$ is infeasible and the objective function value of an optimal solution of $P_{opt}(n)$ is zero for all positive integers $n_0 < n$. Fix $x_S = 1$ for all $S \in 2^{[n]}$ such that $|S| = 1$. Then Theorems 2 and 3 hold for $P_{inf}(n)$ and $P_{opt}(n)$, respectively, with the given fixings.*

Proof. Consider the x -vector from a solution of $P_{inf}(n)$ or $P_{opt}(n)$ and suppose that $x_{\{i\}} = 0$ for some element $i \in [n]$. As in the proofs of Theorems 2 and 3, we may assume that x encodes a downset $\mathcal{D} := \mathcal{S}(x)$ and $\mathcal{Y} := \mathcal{S}(y) \subseteq \mathcal{D}$ is an intersecting family. By the downset property, $x_S = 0$ for all $S \ni i$. But then $|U(\mathcal{D})| < n$ and by assumption the solution is not a counterexample to Conjecture 1. Hence, we may fix $x_S = 1$ for all $S \in 2^{[n]}$ such that $|S| = 1$. \square

This can be exploited for $n = 5$ due to the following known fact.

Proposition 4. *Conjecture 1 holds for all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq 4$.*

Proof. Consider a downset \mathcal{D} such that $|U(\mathcal{D})| \leq 4$, w.l.o.g. $\mathcal{D} \subseteq 2^{[4]}$. If $[4] \notin \mathcal{D}$, then Chvátal's conjecture holds by [Ste74]. Otherwise, $\mathcal{D} = 2^{[4]}$, and maximum-size stars have cardinality 8. By Corollary 1 intersecting families cannot be larger, and Chvátal's conjecture holds. \square

For larger n , Proposition 3 can be used incrementally. Finally, we discuss how to exploit the fundamental result of [Ste74] as an additional fixing scheme.

Proposition 5. *Let $n \geq 4$ be a fixed integer. In $P_{inf}(n)$ and $P_{opt}(n)$ fix $x_S = 1$ for all $S \in 2^{[4]}$. Then Theorems 2 and 3 hold, respectively, with the given fixings.*

Proof. According to [Ste74], counterexamples to Chvátal's conjecture must feature a downset that contains at least one set of size four. By permuting the elements in $[n]$ suitably, we can always ensure that this set is $\{1,2,3,4\}$. \square

To summarize, we arrive at the following improved formulation $P_{red}(n)$,

$$\max \sum_{S \in 2^{[n]} \setminus \{\emptyset\}} y_S - z \tag{7a}$$

$$y_T + y_S \leq 1 \quad \forall T \in 2^{[n]} \setminus \{\emptyset\}, \forall S \in 2^{[n]} \setminus \{\emptyset\} : T \cap S = \emptyset, \tag{7b}$$

$$\sum_{S \in 2^{[n]} : i \in S} x_S \leq z \quad \forall i \in [n], \tag{7c}$$

$$y_T \leq x_S \quad \forall T \in 2^{[n]}, \forall S \in 2^{[n]} : S \subseteq T, \tag{7d}$$

$$\sum_{S \in 2^{[n]} \setminus \{\emptyset\}} 2y_S \leq \sum_{S \in 2^{[n]}} x_S, \tag{7e}$$

$$y_S = 0 \quad \forall S \in 2^{[n]} : 1 \leq |S| \leq 2, \tag{7f}$$

$$x_S = 1 \quad \forall S \in 2^{[n]} : |S| = 1, \tag{7g}$$

$$x_S = 1 \quad \forall S \subseteq [4], \tag{7h}$$

$$x_S, y_S \in \{0, 1\} \quad \forall S \in 2^{[n]},$$

$$z \in \mathbb{Z}_{\geq 0}.$$

This formulation serves as the basis for our proof of Theorem 1 that uses the following equivalence incrementally for $n = 5, 6$, and 7.

Theorem 6. *Let n be a positive integer and suppose Chvátal's conjecture holds for all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n - 1$. Then all downsets \mathcal{D} such that $|U(\mathcal{D})| \leq n$ satisfy Chvátal's conjecture if and only if the objective function value of an optimal solution of $P_{red}(n)$ is zero.*

Proof. As follows from Corollary 2, constraint (7e) is a valid inequality for $P_{opt}(n)$. Corollary 3 shows that constraints (7f) do not exclude any counterexamples that may have objective function value greater than zero. According to Proposition 3, the same holds for constraints (7g) under the assumption that Chvátal's conjecture is correct for smaller ground sets. According to Proposition 5, constraints (7h) may exclude counterexamples, but only as long as at least one symmetric counterexample remains feasible. \square

2.4 Partitioning the Solution Space of $P_{red}(n)$

Fixing x_S variables to one when $|S|$ is large in relation to n implies the fixing of all the variables that index the power set of S and thus may bring about significant computational savings. In particular, if we are able to account for the natural action of the symmetric group on families of sets, the exact version of SCIP may be guided in such a way as to avoid redundant work when possible. First, we need the following definition and observation.

We say that two families of sets contained in $2^{[n]}$ are *isomorphic*, if and only if there exists a permutation of $[n]$ that transforms one into the other.

Observation 2. *Any two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ of k sets of size $n - 1$ are isomorphic to each other, for each $1 \leq k \leq n$.*

This is trivial when $k = 1$ and follows immediately for all other k by noting that there are k elements of the ground set such that each element is contained in all k sets but one.

Our fixing scheme runs as follows. For each k such that $0 \leq k \leq n$, denote by $P_{red}(n)^k$ the integer program $P_{red}(n)$ with additional fixings such that $n - k$ variables x_S with $|S| = n - 1$ are set to zero while the rest of the k variables x_S such that $|S| = n - 1$ are set to one. Furthermore for each k such that $1 \leq k \leq n$, $P_{red}(n)^k$ does not contain (7h).

Proposition 6. *Let n be any positive integer. Suppose that the objective function value of an optimal solutions of $P_{red}(n)^k$ is zero, for each k such $0 \leq k \leq n$. Then the objective function value of an optimal solution of $P_{red}(n)$ is zero.*

Proof. Consider an optimal solution of $P_{red}(n)$ and its corresponding downset $\mathcal{S}(x)$. Suppose $\mathcal{S}(x)$ contains k sets of size $n - 1$, where $0 \leq k \leq n$. If $k = 0$ then an optimal solution of $P_{red}(n)$ is isomorphic to an optimal solution of $P_{red}(n)^0$. By Observation 2 an optimal solution of $P_{red}(n)^k$ for some k such that $1 \leq k \leq n$ is isomorphic to an optimal solution of $P_{red}(n)$. \square

More generally, consider families of k sets of size m , such $4 \leq m \leq n - 1$ and $0 \leq k \leq \binom{n}{m}$. Let $N(k, m)$ denote the collection of all families of k sets of size m such that no two distinct families are isomorphic.

Denote by $P_{red}(n)^{0,m}$, where $4 \leq m \leq n - 1$, the integer program $P_{red}(n)$ with the additional fixings that all variables x_S such that $|S| = m_0$ for all $m \leq m_0 \leq n - 1$ are set to zero. Furthermore denote by $P_{red}(n)_S^{k,m}$ the integer program $P_{red}(n)$ with additional fixings such that $n - k$ variables x_S with $|S| = m$ and $S \in \mathcal{S}$ are set to zero while the rest of the k variables x_S such that $|S| = m$ are set to one. Additionally all variables x_S such that $|S| = m_0$ for all $m < m_0 \leq n - 1$ are set to zero. Finally for each k such that $1 \leq k \leq \binom{n}{m}$, $P_{red}(n)_S^{k,m}$ does not contain (7h). For $k = 0$ we define $P_{red}(n)_S^{k,m} := P_{red}(n)^{0,m}$.

Corollary 4. *Let n be any positive integer. Let \mathcal{O}_m denote an optimal solution of $P_{red}(n)^{0,m}$ for each m such that $5 \leq m \leq n - 1$. Then \mathcal{O}_m is isomorphic to an optimal solution of some $P_{red}(n)_S^{k,m-1}$ such that $\mathcal{S} \in N(k, m - 1)$.*

Corollary 5. *Let n be any positive integer. Let \mathcal{O}_m denote an optimal solution of $P_{red}(n)^{0,m}$ for each m such that $5 \leq m \leq n - 1$. To determine \mathcal{O}_m it suffices to determine optimal solutions of $P_{red}(n)_S^{k,m-1}$ such that $\mathcal{S} \in N(k, m - 1)$ for each $0 \leq k \leq \binom{n}{m-1}$.*

Before detailing the computational results, we give a general description of how to *safely* use computational integer programming to prove suitable statements of interests, with an overview of the available tools used and developed in this paper.

3 Verifiable Proofs for Integer Programming Results

There are two main computational difficulties in using the solution of integer programs for investigating mathematical conjectures. First, virtually all state-of-the-art IP solvers rely on fast floating-point arithmetic, hence their results are compromised by roundoff errors. Second, compact certificates for integer programming results are not available and most solvers do not even provide output that would allow to check and verify the correctness of their result. Thus in recent years researchers have been turning to SAT solvers to investigate suitable questions of interest with considerable success, as the solution of the boolean Pythagorean triples problem [HKM16] demonstrates. However, recent progress in the direction of exact rational IP has already enabled the use of computational IP frameworks to settle several open questions related to Frankl's conjecture [Pul17]. In the following, we outline our computational methodology used to solve the integer programs presented in Section 2 such that the results can be trusted and both input and output can be verified independently of the IP solver used. Figure 1 illustrates this framework.

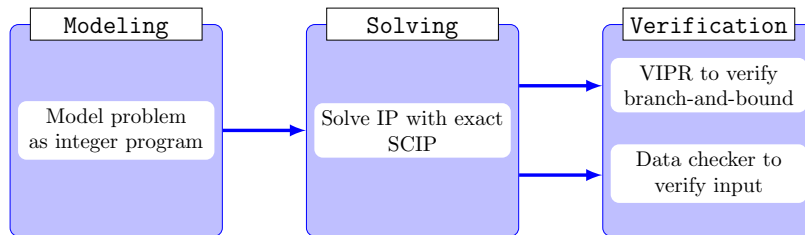


Figure 1: General framework for modeling, solving, and verifying the results of integer programs.

Modeling. As the first step, we use the modeling language ZIMPL [Koc04] to formulate the integer program. ZIMPL employs exact rational arithmetic when instantiating the model in order to ensure that no roundoff errors are introduced before passing the model to a solver.

Solving. Next, we solve the IP using the exact rational variant of the MIP solver SCIP [Coo+13]. Exact SCIP implements a hybrid branch-and-bound algorithm that combines floating-point and exact rational arithmetic in a safe manner. Several methods are used in order to obtain safe dual bounds by correcting relaxation solutions from fast floating-point linear programming (LP) solvers. An exact rational LP solver, QSopt_ex [App+07], is used as sparingly as possible. Although exact SCIP still lacks many more sophisticated techniques implemented in state-of-the-art floating-point solvers such as presolving reductions, cutting planes, or symmetry handling, its design helps to yield superior performance compared to a naïve branch-and-bound method solely relying on rational LP solves.

Output Verification. Although exact SCIP is designed to provide safe results, the correctness of the algorithm and implementation cannot easily be verified externally. To address this issue, we use VIPR [CGS17], a recently developed certificate format that consists of the problem definition followed by an encoding of the branch-and-bound proof as a list of valid inequalities. It rests on three simple inference steps that allow for elementary, stepwise verification: aggregation of inequalities, rounding of right-hand sides, and resolution of a binary disjunction. In this sense, a VIPR certificate can, in theory, be checked by hand, although in practice this may be prohibitive for larger certificates. Hence, the VIPR project comes with an automatic, standalone checker, but the simplicity of the format allows for the implementation of alternative checkers.

Exact SCIP can be configured to generate VIPR certificates during the solving process such that its result must not be trusted blindly. Its correctness can be verified completely independently of the solving process.

Input Verification. VIPR verification only ensures the correctness of the branch-and-bound certificate with respect to the integer program encoded in the problem section of the certificate file. However, due to implementation errors, the problem section of the certificate file may actually not match the integer program of interest. Therefore we implemented a safe input-checker that internally creates its own representation of the constraint matrix for Problem $P_{red}(n)$. It then reads the problem section of the certificate file and checks if the two constraint matrices coincide. This input checker is written using the Coq proof assistant [Coq18], a mathematical proof management system. The matrix creation in this input checker is problem-specific, nevertheless it can easily be adapted to formulations for similar problems.

All in all, we are confident that this framework ensures a high level of trust in the computational proof of Theorem 1. All tools are made publicly available for review, including the certificate files for the computational results presented in the next section.³

4 Computational Results

Using the safe computational framework outlined in Section 3, we could solve $P_{opt}(n)$ and $P_{red}(n)$ for $n = 5, 6$, and 7 , producing a machine-assisted proof of Theorem 1. Furthermore, several floating-point MIP solvers could solve $P_{red}(8)$ to optimality. Although this does not constitute a safe proof, it makes it highly likely that Chvátal’s conjecture holds for $n = 8$ and a search for counterexamples should focus on larger ground sets.

Beyond the plain question of solvability, in this section we provide details regarding the following questions: *What are the times spent for solving the integer programs and how are they affected by the improvements in the formulations? How large are the resulting certificates and how expensive is their verification? How does the performance of the exact framework compare to the performance of standard floating-point MIP solvers?*

The results for the optimality-based formulations are provided in Table 1. All tests were run on a cluster of computing nodes with Intel Xeon Gold 5122 CPUs with 3.6 GHz and 96 GB of main memory. The exact version of SCIP was built with CPLEX 12.6.3 as floating-point LP solver and QSopt_ex 2.5.10 [App+07] for exact rational LP solves. As floating-point MIP solver, we used SCIP 6.0.0 [Gle+18], built with CPLEX 12.8.0 as the underlying LP solver. The time limit was set to 12 hours for all runs and on each computing node only one job was executed at a time.

IP	n	#vars	#ineqs	SCIP 6.0.0	SCIP exact	VIPR	
				time [s]	time [s]	size [MB]	time [s]
$P_{opt}(n)$	5	63	427	0.2	0.5	0.5	0.07
	6	127	1336	2.3	22.6	73	4.6
	7	255	4125	91.0	4024.2	21000	1258.3
	8	511	12618	–	–	–	–
$P_{red}(n)$	5	31	433	0.1	0.1	0.018	0.005
	6	88	1317	0.1	0.2	0.25	0.2
	7	208	4050	13.5	124.5	163	28.9
	8	455	12424	7278.9	–	–	–

Table 1: Computational details for solving the Chvátal IPs for the two formulations $P_{opt}(n)$ and $P_{red}(n)$. The sizes reported for the VIPR certificates are for uncompressed text files. The running time for input verification is negligible and always below 5 seconds.

First, we observe the effectiveness of the additional inequalities and fixings applied in $P_{red}(n)$. The running times of exact SCIP are significantly reduced, as are the sizes and verification times for the VIPR certificates. Furthermore, only $P_{red}(8)$ can be solved by floating-point SCIP, while it times out for $P_{opt}(8)$. Second, note that the size of the IPs is not large compared to what MIP solvers today can often handle easily in many industrial applications. This underlines the difficulty of the underlying combinatorial question. Third, the sizes and running times of checking the VIPR certificate are significant but do not constitute a bottleneck for the current framework. Note that the times for input verification in Coq are not reported, because they are negligible and always below 5 seconds.

³See <https://github.com/leoneifler/chvatalip> and links therein.

5 Conclusion

In this paper we established a safe computational framework for Chvátal’s conjecture based on exact rational integer programming. As a result, we proved that Chvátal’s conjecture holds for all downsets whose union of sets contains seven elements or less. One advantage of the approach is the flexibility of the IP formulations to include partial results from the literature as valid inequalities and variable fixings. Thus they may be strengthened further as new theoretical results become known.

One promising direction for future progress is to combine our method for excluding large finite sets of potential counterexamples with the proof techniques in [CHK17] and [OSS18]. This could aim at a proof that Chvátal’s conjecture holds for downsets of rank four or more. Furthermore, our IP models can be appropriately modified to investigate variations on Chvátal’s conjecture such as proposed by Snevily⁴ or a generalization of Chvátal’s conjecture proposed by Borg [Bor11].

Last, but not least, we hope that the generality of the computational framework developed makes it useful for the investigation of other open questions in extremal combinatorics. The results certainly motivate sustained work on closing the current performance gap between exact and inexact MIP solvers. As safe methods for applying presolving reductions, cutting planes, and symmetry handling become available, this will likely lead to a certificate that Chvátal’s conjecture holds for ground sets with eight elements.

References

- [And02] I. Anderson. *Combinatorics of finite sets*. Courier Corporation, 2002.
- [App+07] D. L. Applegate, W. Cook, S. Dash, and D. G. Espinoza. Exact solutions to linear programming problems. *Operations Research Letters* 35.6 (2007), pp. 693–699. ISSN: 0167-6377. DOI: [10.1016/j.orl.2006.12.010](https://doi.org/10.1016/j.orl.2006.12.010).
- [Ber75] C Berge. A theorem related to the Chvátal conjecture. *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*. 1975, pp. 35–40.
- [Bie16] A. Biere. Splatz, Lingeling, Plingeling, Treengeling, YalSAT Entering the SAT Competition 2016. *Proc. of SAT Competition 2016 – Solver and Benchmark Descriptions*. Ed. by T. Balyo, M. Heule, and M. Jarvisalo. Vol. B-2016-1. Department of Computer Science Series of Publications B. University of Helsinki, 2016, pp. 44–45. URL: <http://hdl.handle.net/10138/164630>.
- [Bor11] P. Borg. On Chvátal’s conjecture and a conjecture on families of signed sets. *European Journal of Combinatorics* 32.1 (2011), pp. 140–145. DOI: [10.1016/j.ejc.2010.07.002](https://doi.org/10.1016/j.ejc.2010.07.002).
- [CGS17] K. K. Cheung, A. Gleixner, and D. E. Steffy. Verifying Integer Programming Results. *International Conference on Integer Programming and Combinatorial Optimization*. Springer, 2017, pp. 148–160.
- [CHK17] E. Czabarka, G. Hurlbert, and V. Kamat. Chvátal’s conjecture for downsets of small rank. *ArXiv e-prints* (Mar. 2017). arXiv: [1703.00494](https://arxiv.org/abs/1703.00494) [math.CO].
- [Chv74] V. Chvátal. Intersecting families of edges in hypergraphs having the hereditary property. *Hypergraph Seminar*. Springer, 1974, pp. 61–66.
- [Coo+13] W. Cook, T. Koch, D. E. Steffy, and K. Wolter. A hybrid branch-and-bound approach for exact rational mixed-integer programming. *Mathematical Programming Computation* 5.3 (2013), pp. 305–344. DOI: [10.1007/s12532-013-0055-6](https://doi.org/10.1007/s12532-013-0055-6).

⁴<https://faculty.math.illinois.edu/~west/regs/chvatal.html>

- [Erd81] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica* 1.1 (1981), pp. 25–42. DOI: [10.1007/BF02579174](https://doi.org/10.1007/BF02579174).
- [ES06] N. Eén and N. Sorensson. Translating pseudo-boolean constraints into SAT. *Journal on Satisfiability, Boolean Modeling and Computation* 2 (2006), pp. 1–26.
- [Fis88] P. Fishburn. Combinatorial Optimization Problems for Systems of Subsets. *SIAM Review* 30.4 (1988), pp. 578–588. DOI: [10.1137/1030133](https://doi.org/10.1137/1030133).
- [Gle+18] A. Gleixner, M. Bastubbe, L. Eifler, T. Gally, G. Gamrath, R. L. Gottwald, G. Hendel, C. Hojny, T. Koch, M. E. Lübbecke, S. J. Maher, M. Miltenberger, B. Müller, M. E. Pfetsch, C. Puchert, D. Rehfeldt, F. Schlösser, C. Schubert, F. Serrano, Y. Shinano, J. M. Viernickel, M. Walter, F. Wegscheider, J. T. Witt, and J. Witzig. The SCIP Optimization Suite 6.0. eng. Tech. rep. 18-26. Takustr. 7, 14195 Berlin: ZIB, 2018.
- [HKM16] M. J. Heule, O. Kullmann, and V. W. Marek. Solving and verifying the boolean pythagorean triples problem via cube-and-conquer. *International Conference on Theory and Applications of Satisfiability Testing*. Springer. 2016, pp. 228–245.
- [KM72] D. J. Kleitman and T. L. Magnanti. On the Number of Latent Subsets of Intersecting Collections. *Operations Research Center Working Papers* (Oct. 1972). URL: <http://hdl.handle.net/1721.1/5343>.
- [Koc04] T. Koch. Rapid Mathematical Prototyping. PhD thesis. Technische Universität Berlin, 2004.
- [Mik84] D Miklós. Great intersecting families of edges in hereditary hypergraphs. *Discrete mathematics* 48.1 (1984), pp. 95–99. DOI: [10.1016/0012-365X\(84\)90135-3](https://doi.org/10.1016/0012-365X(84)90135-3).
- [OSS18] J. Olarte, F. Santos, and J. Spreer. Short proof of two cases of Chvátal’s conjecture. *ArXiv e-prints* (Apr. 2018). arXiv: [1804.03646](https://arxiv.org/abs/1804.03646) [math.CO].
- [Pul17] J. Pulaž. Cutting Planes for Union-Closed Families. Dissertation. Technische Universität Berlin, 2017.
- [Sch75] J Schönheim. Hereditary systems and Chvátal’s conjecture. *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*. 1975, pp. 537–539.
- [Ste74] F Sterboul. Sur une conjecture de V. Chvátal. *Hypergraph Seminar*. Ed. by D. Berge Claude and Ray-Chaudhuri. Springer. 1974, pp. 152–164. DOI: [10.1007/BFb0066190](https://doi.org/10.1007/BFb0066190).
- [Ste83] P. Stein. On Chvátal’s conjecture related to a hereditary system. *Discrete Mathematics* 43.1 (1983), pp. 97–105. DOI: [10.1016/0012-365X\(83\)90025-0](https://doi.org/10.1016/0012-365X(83)90025-0).
- [Coq18] Coq Development Team. The Coq Proof Assistant, version 8.8.0. Apr. 2018. DOI: [10.5281/zenodo.1219885](https://doi.org/10.5281/zenodo.1219885).