TIGHT MIP FORMULATIONS FOR BOUNDED LENGTH CYCLIC SEQUENCES

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ABSTRACT. We study cyclic binary strings with bounds on the lengths of the intervals of consecutive ones and zeros. This is motivated by scheduling problems where such binary strings can be used to represent the state (on/off) of a machine. In this context the bounds correspond to minimum and maximum lengths of on- or off-intervals, and cyclic strings can be used to model periodic schedules. Extending results for non-cyclic strings is not straight forward. We present a non-trivial tight compact extended network flow formulation, as well as valid inequalities in the space of the state and start-up variables some of which are shown to be facet-defining. Applying a result from disjunctive programming, we also convert the extended network flow formulation into an extended formulation over the space of the state and start-up variables.

1. INTRODUCTION

In scheduling problems it is often natural to use time-indexed binary variables to model the availability of resources, such as the state of machines (on/off) or roster patterns for the workforce. In these contexts there are often bounds on the lengths of on- and off-intervals, and there is a significant literature on mixed integer programming formulations for this [4, 5, 6, 7, 9, 11, 13]. In particular, Malkin [11] showed that for lower bounds on the lengths of on- and off-intervals, the valid inequalities that can be found in [14] are sufficient to describe the convex hull in the space of the state and start-up variables. Pochet and Wolsey [12] give the convex hull for the case of constant upper and lower bounds, and this was generalized by Queyranne and Wolsey [13] who considered upper and lower bounds, and allowed these bounds to vary over time. They present a tight extended network formulation, and obtain the convex hull in the space of the state and start-up variables via a projection from a different path formulation.

Our work is motivated by applications in the scheduling of railway maintenance [10], where it is required in some situations that schedules are cyclic. For this reason, we let the sequence of state variables "wrap around" the time horizon and apply the bounds on the lengths of on- or off-intervals also to intervals that start in the end of the time horizon and continue in the beginning. A more formal problem description is provided in Section 2. In Section 3, we follow the approach from [13] to derive a compact extended network flow formulation. It turns out that the straightforward cyclic variant of the network formulation from[13] does not lead to an integral polytope in the space of the flow variables, but we can obtain an integral network flow formulation by considering a larger network that arises from exploiting a simple disjunction. In Section 4 we study a cyclic variant of the Queyranne/Wolsey formulation in the space of the state and start-up variables. We prove that it is a valid formulation, but in contrast to the non-cyclic case the polytope is not integral. For the case that the bounds on the interval lengths are constant over time we provide some valid inequalities,

and give sufficient conditions for them to be facet-defining. We also use a result from disjunctive programming to derive an extended formulation for the convex hull in the space of the state and start-up variables. Finally, in Section 5 we describe some directions for further investigations.

2. Problem description

Throughout this paper, we denote the set $\{a, a+1, \ldots, b\}$ for integers $a \leqslant b$ by $[a, b]$. Let the time horizon be indexed by $[0, n-1]$ with the convention that time is added modulo n, that is, 0 is the time period after $n-1$. For integers a and b with $0 \leqslant b < a < n$ representing time periods we let the interval wrap around in the natural way, that is, $[a, b] = \{a, a+1, \ldots, n-1, 0, 1, \ldots, b\}$.

As in [13], we consider parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \in \mathbb{Z}^{4n}$ that impose bounds on the length of on- and off-intervals in the following way:

- $\alpha_t \in [1, n-1]$ is a lower bound on the length of an on-interval starting in period t,
- $\beta_t \in [\alpha_t, n-1]$ is an upper bound on the length of an on-interval starting in period t,
- $\gamma_t \in [1, n-1]$ is a lower bound on the length of an off-interval starting in period t,
- $\delta_t \in [\gamma_t, n-1]$ is an upper bound on the length of an off-interval starting in period t.

In particular, we require that there are at least one on-period and at least one off-period (otherwise there is an on-interval of length n or an off-interval of length n, and no matter where we let this start the upper bound on the length of the corresponding interval will be violated). We define binary state variables y_t for $t \in [0, n-1]$ to be

$$
y_t = \begin{cases} 1 & \text{if period } t \text{ is an on-period,} \\ 0 & \text{if period } t \text{ is an off-period.} \end{cases}
$$

The set of feasible state sequences $(y_0, \ldots, y_{n-1}) \in \{0,1\}^n$ is characterized by the following implications:

$$
\begin{array}{lll}
y_t - y_{t-1} = 1 \implies y_{t+i} = 1 \text{ for all } i \in [0, \alpha_t - 1] & t \in [0, n-1], & (1) \\
y_t - y_{t-1} = 1 \implies y_{t+i} = 0 \text{ for some } i \in [\alpha_t, \beta_t] & t \in [0, n-1], & (2) \\
y_{t-1} - y_t = 1 \implies y_{t+i} = 0 \text{ for all } i \in [0, \gamma_t - 1] & t \in [0, n-1], & (3) \\
y_{t-1} - y_t = 1 \implies y_{t+i} = 1 \text{ for some } i \in [\gamma_t, \delta_t] & t \in [0, n-1]. & (4)
\end{array}
$$

We define the binary start-up variables z_t for $t \in [0, n-1]$ to be

$$
z_t = 1 \iff y_{t-1} = 0 \text{ and } y_t = 1, \tag{5}
$$

and define the set

$$
Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = \left\{ (\boldsymbol{y}, \boldsymbol{z}) \in \{0,1\}^{2n} : (1), (2), (3), (4), (5), 1 \leqslant y_0 + \cdots + y_{n-1} \leqslant n-1 \right\}.
$$

We are interested in tight linear formulations for $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, and our approach is to adapt the arguments used in [13]. Before studying the general case we derive a simple feasibility criterion for the constant bound case in which the bounds ε_t for $\boldsymbol{\varepsilon} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$ do not change over time. If $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (\alpha, \beta, \gamma, \delta)$ for all t then the number of start-up periods is an integer between $n/(\beta + \delta)$ and $n/(\alpha + \gamma)$. The following proposition states that for every integer k in this range there exists a feasible solution with k start-up periods.

Proposition 1. *If* $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (\alpha, \beta, \gamma, \delta)$ *all* $t \in [0, n-1]$, *then*

$$
\{z_0 + \cdots + z_{n-1} : (\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})\} = \{k \in \mathbb{Z} : n/(\beta + \delta) \leqslant k \leqslant n/(\alpha + \gamma)\}.
$$

In particular, $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \neq \emptyset$ *if and only if* $k(\alpha + \gamma) \leqslant n \leqslant k(\beta + \delta)$ *for some integer* k.

Proof. Let $K = \{k \in \mathbb{Z} : n / (\beta + \delta) \leqslant k \leqslant n / (\alpha + \gamma)\}$. We have to show that there exists $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ with $z_0 + \cdots + z_{n-1} = k$ if and only if $k \in K$. First, suppose $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, set $k = z_0 + \cdots + z_{T-1}$ and let $0 \leqslant t_1 < t_2 < \cdots < t_k \leqslant n - 1$ denote the indices with $z_{t_i} = 1$ for $i \in [1, k]$. Then, for every $i \in [1, k]$, $t_{i+1} = t_i + p_i + q_i$ with $\alpha \leqslant p_i \leqslant \beta$ and $\gamma \leqslant q_i \leqslant \delta$ for all $i \in [1, k]$. Summing over i, we obtain $n = (p_1 + q_1) + \cdots + (p_k + q_k)$, hence $k(\alpha + \gamma) \leqslant n \leqslant k(\beta + \delta)$, which implies $k \in K$. For the converse, start with any $k \in K$. Then $k(\alpha + \gamma) \leqslant n \leqslant k(\beta + \delta)$, hence $n - (\alpha + \gamma) \geqslant (k - 1)(\alpha + \gamma)$ and $n - (\beta + \delta) \leqslant (k - 1)(\beta + \delta)$. This implies that we can choose $p_k \in [\alpha, \beta]$ and $q_k \in [\gamma, \delta]$ such that $(k - 1)(\alpha + \gamma) \leqslant n - (p_k + q_k) \leqslant (k - 1)(\beta + \delta)$. Continuing this way, we obtain $n = (p_k + q_k) + (p_{k-1} + q_{k-1}) + \cdots + (p_1 + q_1)$ with $p_i \in [\alpha, \beta]$ and $q_i \in [\gamma, \delta]$. Then

$$\boldsymbol{y} = \underbrace{11\ldots1}_{p_1}\underbrace{00\ldots0}_{q_1}\underbrace{11\ldots1}_{p_2}\underbrace{00\ldots0}_{q_2} \cdots \underbrace{11\ldots1}_{p_k}\underbrace{00\ldots0}_{q_k}, \quad \boldsymbol{z} = 1\underbrace{00\ldots0}_{p_1+q_1-1}1\underbrace{00\ldots0}_{p_2+q_2-1} \cdots 1\underbrace{00\ldots0}_{p_k+q_k-1}$$

defines a vector $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ satisfying $z_1 + \cdots + z_{n-1} = k$. $\qquad\square$

3. AN EXTENDED NETWORK FORMULATION

We consider a directed graph (V, A) with node set $V = \{0, 1\} \times [0, n - 1]$, and arc set

$$A = \{((0, t), (1, l)) \ : \ l \in [t + \alpha_t, \, t + \beta_t]\} \cup \{((1, t), (0, l)) \ : \ l \in [t + \gamma_t, \, t + \delta_t]\}.$$

Figure 1 illustrates this graph for $n = 6$ and $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1, 2, 1, 2)$ for all $t \in [0, n - 1]$. In terms
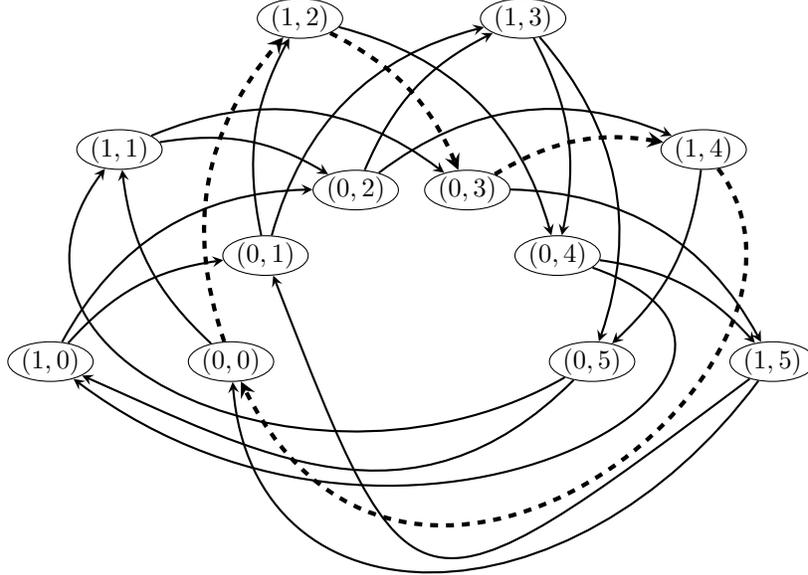

FIGURE 1. The network representation for $n = 6$ and $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1, 2, 1, 2)$ for all $t \in [0, n - 1]$. The dashed cycle corresponds to $\boldsymbol{y} = (1, 1, 0, 1, 0, 0)$ and $\boldsymbol{z} = (1, 0, 0, 1, 0, 0)$.

of switching sequences, an arc $((0, t), (1, l))$ corresponds to switching on in period t and switching off in period l, and an arc $((1, t), (0, l))$ corresponds to switching off in period t and switching on in period l. Feasible switching sequences correspond to directed cycles of length n where the length of an arc $((i, t), (1 - i, t + p))$ for $i \in \{0, 1\}$ is p. As in [13] we can use the flow interpretation to obtain a formulation for $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ in the following way. For every node $v \in V$, let $A_{\text{in}}(v)$ and $A_{\text{out}}(v)$ denote the sets of arcs entering and leaving v, respectively. For convenience, we will omit one pair

of brackets, whenever a node (i, t) appears as an argument, that is, we will write $A_{\text{in}}(i, t)$ instead of $A_{\text{in}}((i, t))$. For $t \in [0, n - 1]$, we define

$$A_{\text{off}}(t) = \{((1, r), (0, l)) \in A : t \in [r, l - 1]\}\}, \quad A_{\text{on}}(t) = \{((0, r), (1, l)) \in A : t \in [r, l - 1]\}\}.$$

If C is a cycle of length n, then for every $t \in [0, n-1]$, C contains exactly one arc from $A_{\text{off}}(t) \cup A_{\text{on}}(t)$, and in the correspondence between cycles C and vectors $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, we have

$$y_t = \begin{cases} 0 & \text{if } C \text{ contains an arc } a \in A_{\text{off}}(t) \\ 1 & \text{if } C \text{ contains an arc } a \in A_{\text{on}}(t) \end{cases} \qquad t \in [0, n-1],$$

$$z_t = 1 \iff C \text{ contains an arc } a \in A_{\text{out}}(0, t) \qquad t \in [0, n-1].$$

Let $Q = Q(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \subseteq \mathbb{R}^{|A|+2n}$ be the polytope defined by the constraints

$$\sum_{a \in A_{\text{off}}(0) \cup A_{\text{on}}(0)} x_a = 1, \tag{6}$$

$$\sum_{a \in A_{\text{in}}(v)} x_a - \sum_{a \in A_{\text{out}}(v)} x_a = 0 \qquad v \in V, \tag{7}$$

$$y_t = \sum_{a \in A_{\text{on}}(t)} x_a \qquad t \in [0, n-1], \tag{8}$$

$$z_t = \sum_{a \in A_{\text{out}}(0,t)} x_a \qquad t \in [0, n-1], \tag{9}$$

$$x_a \geqslant 0 \qquad a \in A. \tag{10}$$

Proposition 2. *The polytope Q is an extended formulation for $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, that is, $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = \text{proj}_{y,z}(Q) \cap \mathbb{Z}^{2n}$.*

Proof. For every $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ we have a corresponding cycle C of length n. Let us define $\boldsymbol{x} \in \{0, 1\}^{|A|}$ as $x_a = 1 \iff a \in C$. This provides a point $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q$, and shows $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \subseteq \text{proj}_{y,z}(Q) \cap \mathbb{Z}^{2n}$. For the converse inclusion we start with an arbitrary $(\boldsymbol{y}, \boldsymbol{z}) \in \text{proj}_{y,z}(Q) \cap \mathbb{Z}^{2n}$, and fix a vector $\boldsymbol{x} \in \mathbb{R}^{|A|}$ with $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q$. We need to verify that $(\boldsymbol{y}, \boldsymbol{z})$ satisfies (1) through (5). For this purpose the following observations are useful:

$$\sum_{a \in A_{\text{off}}(t) \cup A_{\text{on}}(t)} x_a = 1 \qquad t \in [0, n-1], \tag{11}$$

$$\sum_{a \in A_{\text{out}}(0,t)} x_a = \begin{cases} 1 & \text{if } y_{t-1} = 0 \text{ and } y_t = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad t \in [0, n-1], \tag{12}$$

$$\sum_{a \in A_{\text{out}}(1,t)} x_a = \begin{cases} 1 & \text{if } y_{t-1} = 1 \text{ and } y_t = 0 \\ 0 & \text{otherwise.} \end{cases} \qquad t \in [0, n-1]. \tag{13}$$

These observations can be seen as follows:

(11): Note that

$$(A_{\text{off}}(t) \cup A_{\text{on}}(t)) \setminus (A_{\text{off}}(t-1) \cup A_{\text{on}}(t-1)) = A_{\text{out}}(0, t) \cup A_{\text{out}}(1, t),$$
$$(A_{\text{off}}(t-1) \cup A_{\text{on}}(t-1)) \setminus (A_{\text{off}}(t) \cup A_{\text{on}}(t)) = A_{\text{in}}(0, t) \cup A_{\text{in}}(1, t),$$

and therefore,

$$\sum_{a\in A_{\mathrm{off}}(t)\cup A_{\mathrm{on}}(t)} x_a = \sum_{a\in A_{\mathrm{off}}(t-1)\cup A_{\mathrm{on}}(t-1)} x_a + \sum_{a\in A_{\mathrm{out}}(0,t)\cup A_{\mathrm{out}}(1,t)} x_a - \sum_{a\in A_{\mathrm{in}}(0,t)\cup A_{\mathrm{in}}(1,t)} x_a$$

$$\overset{(7)}{=} \sum_{a\in A_{\mathrm{off}}(t-1)\cup A_{\mathrm{on}}(t-1)} x_a.$$

Together with (6) and induction on t this implies (11).

(12): With $A_{\mathrm{out}}(0,t) = A_{\mathrm{on}}(t) \setminus A_{\mathrm{on}}(t-1)$ we obtain

$$\sum_{a\in A_{\mathrm{out}}(0,t)} x_a = \sum_{a\in A_{\mathrm{on}}(t)\setminus A_{\mathrm{on}}(t-1)} x_a \leqslant \sum_{a\in A_{\mathrm{on}}(t)} x_a \overset{(8)}{=} y_t,$$

which implies $\sum_{a\in A_{\mathrm{out}}(0,t)} x_a = 0$ if $y_t = 0$. If $y_t = 1$ and $y_{t-1} = 0$, then

$$1 = y_t - y_{t-1} \overset{(8)}{=} \sum_{a\in A_{\mathrm{on}}(t)} x_a - \sum_{a\in A_{\mathrm{on}}(t-1)} x_a \leqslant \sum_{a\in A_{\mathrm{on}}(t)\setminus A_{\mathrm{on}}(t-1)} x_a = \sum_{a\in A_{\mathrm{out}}(0,t)} x_a$$

and consequently, $\sum_{a\in A_{\mathrm{out}}(0,t)} x_a = 1$. Finally, for $y_{t-1} = y_t = 1$ we note that $A_{\mathrm{in}}(0,t) \subseteq A_{\mathrm{off}}(t-1)$ and therefore

$$\sum_{a\in A_{\mathrm{out}}(0,t)} x_a \overset{(7)}{=} \sum_{a\in A_{\mathrm{in}}(0,t)} x_a \leqslant \sum_{a\in A_{\mathrm{off}}(t-1)} x_a \overset{(11)}{=} 1 - \sum_{a\in A_{\mathrm{on}}(t-1)} x_a \overset{(8)}{=} 1 - y_{t-1} = 0.$$

(13): With $A_{\mathrm{out}}(1,t) = A_{\mathrm{off}}(t) \setminus A_{\mathrm{off}}(t-1)$ we obtain

$$\sum_{a\in A_{\mathrm{out}}(1,t)} x_a = \sum_{a\in A_{\mathrm{off}}(t)\setminus A_{\mathrm{off}}(t-1)} x_a \leqslant \sum_{a\in A_{\mathrm{off}}(t)} x_a \overset{(11)}{=} 1 - \sum_{a\in A_{\mathrm{on}}(t)} x_a \overset{(8)}{=} 1 - y_t,$$

which implies $\sum_{a\in A_{\mathrm{out}}(1,t)} x_a = 0$ if $y_t = 1$. If $y_t = 0$ and $y_{t-1} = 1$, then

$$1 = y_{t-1} - y_t \overset{(8)}{=} \sum_{a\in A_{\mathrm{on}}(t-1)} x_a - \sum_{a\in A_{\mathrm{on}}(t)} x_a \overset{(11)}{=} \left(1 - \sum_{a\in A_{\mathrm{off}}(t-1)} x_a\right) - \left(1 - \sum_{a\in A_{\mathrm{off}}(t)} x_a\right)$$

$$= \sum_{a\in A_{\mathrm{off}}(t)} x_a - \sum_{a\in A_{\mathrm{off}}(t-1)} x_a \leqslant \sum_{a\in A_{\mathrm{off}}(t)\setminus A_{\mathrm{off}}(t-1)} x_a = \sum_{a\in A_{\mathrm{out}}(1,t)} x_a$$

and consequently, $\sum_{a\in A_{\mathrm{out}}(1,t)} x_a = 1$. Finally, for $y_{t-1} = y_t = 0$ we note that $A_{\mathrm{in}}(1,t) \subseteq A_{\mathrm{on}}(t-1)$ and therefore

$$\sum_{a\in A_{\mathrm{out}}(1,t)} x_a \overset{(7)}{=} \sum_{a\in A_{\mathrm{in}}(1,t)} x_a \leqslant \sum_{a\in A_{\mathrm{on}}(t-1)} x_a \overset{(8)}{=} y_{t-1} = 0.$$

After establishing (11), (12) and (13), we can now proceed to verify (1) through (5).

(1): Suppose $y_t - y_{t-1} = 1$, that is, $y_t = 1$ and $y_{t-1} = 0$, and fix $i \in [0, \alpha_t - 1]$. Using $A_{\mathrm{out}}(0,t) \subseteq A_{\mathrm{on}}(t+i)$, we obtain

$$y_{t+i} \overset{(8)}{=} \sum_{a\in A_{\mathrm{on}}(t+i)} x_a \geqslant \sum_{a\in A_{\mathrm{out}}(0,t)} x_a \overset{(12)}{=} 1.$$

Now (11) implies $y_{t+i} \leqslant 1$, and we conclude $y_{t+i} = 1$, as required.

(2): Suppose $y_t - y_{t-1} = 1$, that is, $y_t = 1$ and $y_{t-1} = 0$. Then (12) implies $\sum_{a \in A_{\text{out}}(0,t)} x_a = 1$. In particular, $x_a > 0$ for some $a \in A_{\text{out}}(0,t) = \{((0,t),(1,t+i)) : i \in [\alpha_t, \beta_t]\}$. Fix an $i \in [\alpha_t, \beta_t]$ such that $x_{a^*} > 0$ for $a^* = ((0,t),(1,t+i))$, and note that $a^* \in A_{\text{off}}(t+i)$. Then

$$y_{t+i} \stackrel{(8)}{=} \sum_{a \in A_{\text{on}}(t+i)} x_a \stackrel{(11)}{=} 1 - \sum_{a \in A_{\text{off}}(t+i)} x_a \leqslant 1 - x_{a^*} < 1,$$

and by integrality we conclude $y_{t+i} = 0$, as required.

(3): Suppose $y_{t-1} - y_t = 1$, that is, $y_t = 0$ and $y_{t-1} = 1$, and fix $i \in [0, \gamma_t - 1]$. Using $A_{\text{out}}(1,t) \subseteq A_{\text{off}}(t+i)$, we obtain

$$y_{t+i} \stackrel{(8)}{=} \sum_{a \in A_{\text{on}}(t+i)} x_a \stackrel{(11)}{=} 1 - \sum_{a \in A_{\text{off}}(t+i)} x_a \leqslant 1 - \sum_{a \in A_{\text{out}}(1,t)} x_a \stackrel{(13)}{=} 0.$$

Now $y_{t+i} \geqslant 0$ is a consequence of (10) and (8), and we conclude $y_{t+i} = 0$, as required.

(4): Suppose $y_{t-1} - y_t = 1$, that is, $y_t = 0$ and $y_{t-1} = 1$. Then (12) implies $\sum_{a \in A_{\text{out}}(1,t)} x_a = 1$. In particular, $x_a > 0$ for some $a \in A_{\text{out}}(1,t) = \{((1,t),(0,t+i)) : i \in [\gamma_t, \delta_t]\}$. Fix an $i \in [\gamma_t, \delta_t]$ such that $x_{a^*} > 0$ for $a^* = ((1,t),(0,t+i))$, and note that $a^* \in A_{\text{on}}(t+i)$. Then

$$y_{t+i} \stackrel{(8)}{=} \sum_{a \in A_{\text{on}}(t+i)} x_a \geqslant x_{a^*} > 0,$$

and by integrality we conclude $y_{t+i} = 1$, as required.

(5): This follows immediately from (12) and (9).   □

In the non-cyclic case, the polytope corresponding to Q is integral and and its projection onto the (y, z) space gives $\text{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}))$ ([13, Theorem 1]). Unfortunately, this breaks down in the cyclic case, as the following example shows.

Example 1. Let $n = 6$, $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1,3,1,3)$, for all $t \in [0,5]$. Then Q is not integral, as the cycle $(0,0),(1,3),(0,4),(1,1),(0,2),(1,5),(0,0)$, shown in Figure, with coefficient $1/2$ corresponds to an extreme point of Q. This point projects to $\boldsymbol{y} = (1, 1/2, 1, 1/2, 1, 1/2)$, $\boldsymbol{z} = (1/2, 0, 1/2, 0, 1/2, 0)$, which is not contained in $\text{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}))$.
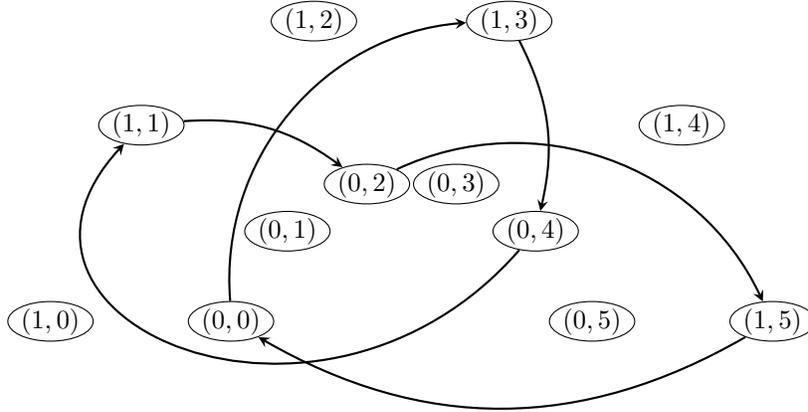

FIGURE 2. A cycle corresponding to a fractional extreme point for $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1,3,1,3)$ for all $t \in [0,5]$.

It is still possible to obtain an extended formulation for $\mathrm{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}))$ as a flow problem in a network of size polynomial in n. For this purpose we make copies of the original network: one for every node (i, τ) such that at least one arc in $A_{\mathrm{out}}(i, \tau)$ "wraps around". In other words, there is a copy for node $(0, \tau)$ if $\tau + \beta_\tau \geqslant n$, and there is a copy for node $(1, \tau)$ if $\tau + \delta_\tau \geqslant n$. We also add an origin node O and a destination node D. The underlying idea is that O-D-paths through the copy of the network for node $(0, \tau)$ when $\tau + \beta_\tau \geqslant n$ represent cycles using an arc of the form $((0, \tau), (1, \tau + p - n))$ with $\max\{\alpha_\tau, n - \tau\} \leqslant p \leqslant \beta_\tau$, and O-D-paths through the copy for node $(1, \tau)$ when $\tau + \delta_\tau \geqslant n$ represent cycles using an arc of the form $((1, \tau), (0, \tau + q - n))$ with $\max\{\gamma_\tau, n - \tau\} \leqslant q \leqslant \delta_\tau$. More formally, with $T_0 = \{\tau \in [0, n-1] \ : \ \tau + \beta_\tau \geqslant n\}$ and $T_1 = \{\tau \in [0, n-1] : \tau + \delta_\tau \geqslant n\}$ the extended network has node set

$$V' = \{O, D\} \cup \{(i, t, j, \tau) \ : \ i \in \{0, 1\}, \ t \in [0, n-1], \ j \in \{0, 1\}, \ \tau \in T_j\},$$

and arc set $A' = A'_1 \cup \cdots \cup A'_5$, where

$$A'_1 = \{(O, (1, t, 0, \tau)) \ : \ \tau \in T_0, \ t = \tau + p - n \text{ for some } p \in [\alpha_\tau, \beta_\tau]\},$$
$$A'_2 = \{(O, (0, t, 1, \tau)) \ : \ \tau \in T_1, \ t = \tau + q - n \text{ for some } q \in [\gamma_\tau, \delta_\tau]\},$$
$$A'_3 = \{((0, \tau, 0, \tau), D) \ : \ \tau \in T_0\} \cup \{((1, \tau, 1, \tau), D) \ : \ \tau \in T_1\},$$
$$A'_4 = \{((i, t, 0, \tau), (1 - i, l, 0, \tau)) \ : \ ((i, t), (1 - i, l)) \in A, \ t < l < n, \ \tau \in T_0\},$$
$$A'_5 = \{((i, t, 1, \tau), (1 - i, l, 1, \tau)) \ : \ ((i, t), (1 - i, l)) \in A, \ t < l < n, \ \tau \in T_1\}.$$

The network for $n = 6$, $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1, 2, 1, 2)$ for all $t \in [0, n-1]$, which implies $T_0 = T_1 = \{4, 5\}$, is shown in Figure 3. We define $A'_{\mathrm{on}}(t)$ to be the set of arcs corresponding to $y_t = 1$:
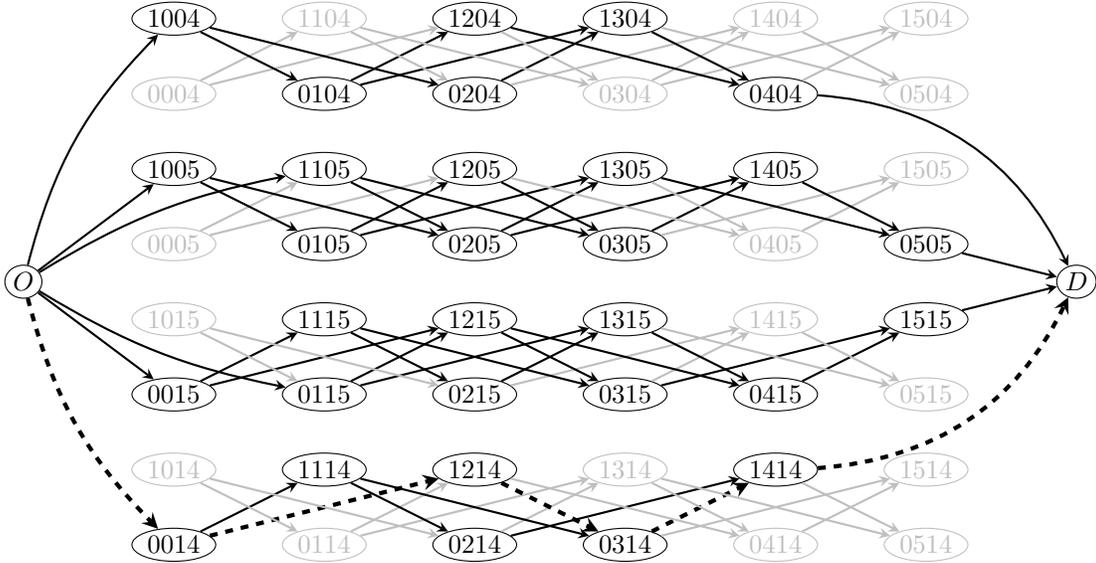

FIGURE 3. The expanded network for $n = 6$ and $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (1, 2, 1, 2)$ for all $t \in [0, 5]$, where we have omitted brackets and commas in the node labels. The light parts do not lie on any O-D-path and so can be eliminated from the network in a preprocessing step. The dashed path corresponds to the cycle in Figure 1.

$$A'_{\mathrm{on}}(t) = \left\{(O, (1, l, 0, \tau)) \in A'_1 \ : \ l > t\right\} \cup \left\{((0, \tau, 0, \tau), D) \in A'_3 \ : \ \tau \leqslant t\right\}$$
$$\cup \left\{((0, k, i, \tau), (1, l, i, \tau)) \in A'_4 \cup A'_5 \ : \ k \leqslant t < l\right\},$$

and then we define the polytope $Q' = Q'(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \subseteq \mathbb{R}^{|A'|+2n}$ by the following constraints:

$$\sum_{a \in A_{\text{out}}(O)} x'_a = 1, \tag{14}$$

$$\sum_{a \in A_{\text{in}}(v)} x'_a - \sum_{a \in A_{\text{out}}(v)} x'_a = 0 \qquad\qquad v \in V' \setminus \{O, D\}, \tag{15}$$

$$y_t = \sum_{a \in A'_{\text{on}}(t)} x'_a \qquad\qquad t \in [0, n-1], \tag{16}$$

$$z_t = \sum_{\tau \in T_1} \sum_{a \in A_{\text{out}}(0,t,0,\tau)} x'_a + \sum_{\tau \in T_2} \sum_{a \in A_{\text{out}}(0,t,1,\tau)} x'_a \qquad t \in [0, n-1], \tag{17}$$

$$x'_a \geqslant 0 \qquad\qquad a \in A'. \tag{18}$$

Proposition 3. *The polytope Q' is integral and* $\text{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})) = \text{proj}_{y,z}(Q')$.

Proof. The polytope $\text{proj}_x(Q')$, described by (14), (15) and (18) is integral as the constraint matrix is a network matrix, hence totally unimodular. Constraints (16) and (17) preserve integrality, because they only write the y- and z- variables as linear combinations of the x-variables with integer coefficients. In order to see that the projection of Q' is the convex hull of $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ it is sufficient to note the one-to-one correspondence between elements of $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ and O-D-paths in the network (V', A'). $\qquad\square$

Corollary 1. *The polytope Q' provides a compact extended formulation for* $\text{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}))$ *with $O(n^3)$ variables and $O(n^2)$ constraints. Moreover, if the parameters β_t and δ_t are $O(1)$ for all $t \in [0, n-1]$, then this reduces to $O(n)$ variables and constraints.*

Proof. The original network (V, A) has $O(n)$ nodes, and since (V', A') is constructed from $O(n)$ copies of (V, A), it has $O(n^2)$ nodes and all nodes, except possibly O have degree $O(n)$. This implies $|A'| = O(n^3)$. If the parameters β_t and δ_t are bounded for all $t \in [0, n-1]$ then there are only $O(1)$ copies, and every node has degree $O(1)$. $\qquad\square$

4. TOWARDS A TIGHT FORMULATION IN THE (y, z)-SPACE

Following [13], we now assume that every $\boldsymbol{\varepsilon} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$ satisfies the *weak monotonicity* condition: for every $t \in [0, n-1]$, $\varepsilon_{t+1} \geqslant \varepsilon_t - 1$. This implies that by waiting one period, one cannot be forced to switch on or off earlier. In particular, weak monotonicity guarantees the existence of numbers $s(\boldsymbol{\varepsilon}, t) \in [0, n-1]$ for every $\boldsymbol{\varepsilon} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$ and $t \in [0, n-1]$ such that

$$\{k \in [0, n-1] : t \in [k, k + \varepsilon_k - 1]\} = [s(\boldsymbol{\varepsilon}, t), t].$$

For instance, (a) the interval $[s(\boldsymbol{\alpha}, t), t]$ is the set of time periods k for which $z_k = 1$ implies $y_t = 1$, and (b) $y_t = 1$ implies that $z_k = 1$ for some $k \in [s(\boldsymbol{\beta}, t), t]$.

4.1. **A formulation.** Following the approach taken in [13, Section 3.1] we define a polytope $P = P(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \subseteq \mathbb{R}^{2n}$ by

$$z_t \geqslant y_t - y_{t-1} \qquad\qquad t \in [0, n-1], \tag{19}$$

$$\sum_{k \in [s(\boldsymbol{\alpha}, t), t]} z_k \leqslant y_t \qquad\qquad t \in [0, n-1], \tag{20}$$

$$y_t \leqslant \sum_{k \in [s(\boldsymbol{\beta}, t), t]} z_k \qquad\qquad t \in [0, n-1], \tag{21}$$

$$\sum_{k \in [s(\boldsymbol{\gamma}, t), t]} z_k \leqslant 1 - y_{s(\boldsymbol{\gamma}, t)-1} \qquad\qquad t \in [0, n-1], \tag{22}$$

$$1 - y_{s(\boldsymbol{\delta}, t)-1} \leqslant \sum_{k \in [s(\boldsymbol{\delta}, t), t]} z_k \qquad\qquad t \in [0, n-1], \tag{23}$$

$$0 \leqslant y_t, z_t \leqslant 1 \qquad\qquad t \in [0, n-1]. \tag{24}$$

Analogous to Proposition 2 in [13], we find that this provides a formulation for $\operatorname{conv}(Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}))$.

Proposition 4. *The polytope P is a formulation for $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, that is, $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = P \cap \mathbb{Z}^{2n}$.*

Proof. First, we start with an arbitrary $(\boldsymbol{y}, \boldsymbol{z}) \in Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, and verify that it satisfies (19) through (24).

  (19): This follows immediately from (5).
  (20): Suppose the inequality is violated for some $t \in [0, n-1]$. Then $y_t = 0$ and $z_k = 1$ for some $k \in [s(\boldsymbol{\alpha}, t), t]$. Using (5), this implies $y_k = 1$ and $y_{k-1} = 0$, and then by (1), $y_l = 1$ for all $l \in [k, k + \alpha_k - 1]$, and in particular, $y_t = 1$, which is the required contradiction.
  (21): Let $t \in [0, n-1]$ be an index with $y_t = 1$ and let $k \in [0, n-1]$ be the unique index with $y_{k-1} = 0$ and $y_l = 1$ for all $l \in [k, t]$. Then $z_k = 1$ by (5), and using (2), $t \in [k, k + \beta_k - 1]$, hence $k \in [s(\boldsymbol{\beta}, t), t]$.
  (22): Suppose the inequality is violated for some $t \in [0, n-1]$. Then $y_{s(\boldsymbol{\gamma}, t)-1} = 1$ and $z_k = 1$ for some $k \in [s(\boldsymbol{\gamma}, t), t]$. Using (5), this implies $y_k = 1$ and $y_{k-1} = 0$. Let l be the first index in the sequence $(s(\boldsymbol{\gamma}, t), s(\boldsymbol{\gamma}, t) + 1, \ldots, k-1)$ with $y_l = 0$. By (3), $y_{l+i} = 0$ for all $i \in [0, \gamma_l - 1]$. By weak monotonicity, $k \in [l, l + \gamma_l - 1]$, and this is the required contradiction.
  (23): Let $t \in [0, n-1]$ be a period with $y_{s(\boldsymbol{\delta}, t)-1} = 0$, and set $l = s(\boldsymbol{\delta}, t) - 1$. By (4), $y_{l+i} = 1$ for some $i \in [1, \delta_l]$. Now (5) implies that $z_{l+i} = 1$ for some $i \in [1, \delta_l]$. By definition of $s(\boldsymbol{\delta}, t)$, $\{l + i : i \in [1, \delta_l]\} \subseteq \{l + 1, l + 2, \ldots, t\}$, hence $z_{l+1} + z_{l+2} + \cdots + z_t \geqslant 1$, as required.
  (24): By definition.

We have shown that $(\boldsymbol{y}, \boldsymbol{z}) \in P$, and therefore $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \subseteq P \cap \mathbb{Z}^{2n}$. For the reverse inclusion, we start with an arbitrary $(\boldsymbol{y}, \boldsymbol{z}) \in P \cap \mathbb{Z}^{2n}$ and verify that it satisfies (1) through (5).

  (5): It follows immediately from (19) that if $y_t = 1$ and $y_{t-1} = 0$ then $z_t = 1$. For the converse, let $t \in [0, n-1]$ be a period with $z_t = 1$, and let $k = s(\boldsymbol{\gamma}, t)$. It follows from (20) that $y_t = 1$. From (22), we obtain $y_{k-1} = 0$. Let l be the first index in $[k, t]$ with $y_l = 1$. Then (19) implies $z_l = 1$, and if $l \neq t$, the left hand side of (22) is at least 2. We conclude $l = t$, hence $y_{t-1} = 0$.
  (1): We use (19) and (20) (note that $t \in [s(\boldsymbol{\alpha}, t+i), t+i]$ for all $i \in [0, \alpha_t - 1]$):

$$y_t - y_{t-1} = 1 \overset{(19)}{\Longrightarrow} z_t = 1 \overset{(20)}{\Longrightarrow} y_{t+i} = 1 \text{ for all } i \in [0, \alpha_t - 1].$$

  (2): If $y_t - y_{t-1} = 1$, then by (1), $y_{t+i} = 1$ for all $i \in [0, \alpha_t - 1]$. For the sake of contradiction, assume $y_{t+i} = 1$ for $i \in [\alpha_t, \beta_t]$. Then (5) implies $z_{t+i} = 0$ for all $i \in [1, \beta_t]$. By weak

monotonicity $[s(\boldsymbol{\beta}, t + \beta_t), t + \beta_t] \subseteq [t + 1, t + \beta_t]$, hence $y_{t+\beta_t} = 0$ by (21), which is the required contradiction.

(3): Suppose there is a pair (t, i) with $t \in [0, n - 1]$ and $i \in [0, \gamma_t - 1]$ with $y_{t-1} = y_{t+i} = 1$, $y_t = 0$, and let i be the smallest possible value. Then (5) implies $z_{t+i} = 1$, and (22) implies $y_{s(\gamma, t+i)-1} = 0$. For $k = s(\boldsymbol{\gamma}, t + i)$, $[k, t + i] \supseteq [t, t + i]$. Let l be the first period in $[k, t - 1]$ with $y_l = 1$. Then $z_l = 1$ by (19), and

$$\sum_{k \in [s(\boldsymbol{\gamma}, t+i), t+i]} z_k \geqslant z_l + z_{t+i} = 2,$$

which contradicts (22).

(4): If $y_{t-1} - y_t = 1$, then by (3), $y_{t+i} = 0$ for all $i \in [0, \gamma_t - 1]$. For the sake of contradiction, assume $y_{t+i} = 0$ for $i \in [\gamma_t, \delta_t]$. Then (5) implies $z_{t+i} = 0$ for all $i \in [0, \delta_t]$. Set $k = s(\boldsymbol{\delta}, t + \delta_t)$. By weak monotonicity $[k, t + \delta_t] \subseteq [t+1, t+\delta_t]$, and (23) implies $y_{k-1} = 1$. This is the required contradiction because $k - 1 \in [t, t + \delta_t - 1]$. $\qquad\square$

4.2. **Valid inequalities.** In contrast to the non-cyclic situation studied in [13], the polytopes P and Q are not integral in general. In this subsection, let $(\alpha_t, \beta_t, \gamma_t, \delta_t) = (\alpha, \beta, \gamma, \delta)$ for all $t \in [0, n - 1]$. Then (20), (21), (22) and (23) can be written as follows:

$$-y_t + \sum_{i \in [0, \alpha-1]} z_{t-i} \leqslant 0 \qquad\qquad t \in [0, n - 1], \tag{25}$$

$$y_t - \sum_{i \in [0, \beta-1]} z_{t-i} \leqslant 0 \qquad\qquad t \in [0, n - 1], \tag{26}$$

$$y_t + \sum_{i \in [1, \gamma]} z_{t+i} \leqslant 1 \qquad\qquad t \in [0, n - 1], \tag{27}$$

$$-y_t - \sum_{i \in [1, \delta]} z_{t+i} \leqslant -1 \qquad\qquad t \in [0, n - 1]. \tag{28}$$

Let P_I denote the integer hull of P, that is $P_I = \operatorname{conv}(P \cap \mathbb{Z}^{2n})$.

Proposition 5. *The inequalities*

$$\sum_{t \in [0, n-1]} z_t \leqslant \lfloor n/(\alpha + \gamma) \rfloor, \tag{29}$$

$$\sum_{t \in [0, n-1]} z_t \geqslant \lceil n/(\beta + \delta) \rceil \tag{30}$$

are valid for P_I. If $\alpha < \beta$, $\gamma < \delta$ and $\lfloor n/(\alpha + \gamma) \rfloor > \lceil n/(\beta + \delta) \rceil$, then the following statements are true:

(i) If n is not divisible by $\alpha + \gamma$, then $\dim P_I = 2n$ and (29) is a facet of P_I.
(ii) If n is not divisible by $\beta + \delta$, then $\dim P_I = 2n$ and (30) is a facet of P_I.

Proof. The upper bound (29) comes from summing constraints (25) and (27) over all t, and then using integrality to round the RHS. For the lower bound (30) we do the same with constraints (26) and (28).

In order to prove (i), we write $n = q(\alpha + \gamma) + r$ with $r \in [1, \alpha + \gamma - 1]$, and set

$$X = \{(\boldsymbol{y}, \boldsymbol{z}) \in P_I \,:\, z_0 + z_1 + \cdots + z_{n-1} = q\}.$$

The claim follows if we can show that $\dim X = 2n - 1$, or equivalently, the affine hull of X is $\{(\boldsymbol{y}, \boldsymbol{z}) : z_0 + \cdots + z_{n-1} = q\}$. For this purpose, suppose X lies in the affine subspace defined by

$$\sum_{t \in [0, n-1]} a_t y_t + \sum_{t \in [0, n-1]} b_t z_t = c. \tag{31}$$

By assumption, there are vectors $(\boldsymbol{y}, \boldsymbol{z}), (\boldsymbol{y}', \boldsymbol{z}) \in X$, where

$$\boldsymbol{y} = 0 \underbrace{00\ldots0}_{d_1} \underbrace{11\ldots1}_{d_1'} \underbrace{00\ldots0}_{d_2} \underbrace{11\ldots1}_{d_2'} \ldots \underbrace{00\ldots0}_{d_q} \underbrace{11\ldots1}_{d_q'},$$

$$\boldsymbol{y}' = 1 \underbrace{00\ldots0}_{d_1} \underbrace{11\ldots1}_{d_1'} \underbrace{00\ldots0}_{d_2} \underbrace{11\ldots1}_{d_2'} \ldots \underbrace{00\ldots0}_{d_q} \underbrace{11\ldots1}_{d_q'},$$

$$\boldsymbol{z} = \underbrace{00\ldots0}_{d_1+1} 1 \underbrace{00\ldots0}_{d_1'+d_2-1} 1 \ldots \underbrace{00\ldots0}_{d_{q-1}'+d_q-1} 1 \underbrace{00\ldots0}_{d_q'-1}.$$

Taking the difference between the two equations obtained from substituting $(\boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{y}', \boldsymbol{z})$ into (31), we conclude $a_0 = 0$, and applying the same argument to the cyclic shifts of $(\boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{y}', \boldsymbol{z})$, $a_t = 0$ for all $t \in [0, n-1]$. Applying a similar argument to vectors $(\boldsymbol{y}, \boldsymbol{z}), (\boldsymbol{y}', \boldsymbol{z}') \in X$ with

$$\boldsymbol{y} = 0 \underbrace{11\ldots1}_{d_1} \underbrace{00\ldots0}_{d_1'} \underbrace{11\ldots1}_{d_2} \underbrace{00\ldots0}_{d_2'} \ldots \underbrace{11\ldots1}_{d_q} \underbrace{00\ldots0}_{d_q'},$$

$$\boldsymbol{y}' = 1 \underbrace{11\ldots1}_{d_1} \underbrace{00\ldots0}_{d_1'} \underbrace{11\ldots1}_{d_2} \underbrace{00\ldots0}_{d_2'} \ldots \underbrace{11\ldots1}_{d_q} \underbrace{00\ldots0}_{d_q'},$$

$$\boldsymbol{z} = 01 \underbrace{00\ldots0}_{d_1+d_1'-1} 1 \underbrace{00\ldots0}_{d_2+d_2'-1} \ldots 1 \underbrace{00\ldots0}_{d_q+d_q'-1},$$

$$\boldsymbol{z}' = 10 \underbrace{00\ldots0}_{d_1+d_1'-1} 1 \underbrace{00\ldots0}_{d_2+d_2'-1} \ldots 1 \underbrace{00\ldots0}_{d_q+d_q'-1},$$

and their cyclic shifts, we obtain $b_{t+1} - b_t = a_t = 0$ for all $t \in [0, n-1]$. As a consequence, (31) is a multiple of the relation $z_0 + \cdots + z_{n-1} = q$, and this concludes the proof of (i). The proof of (ii) is similar. □

Example 2. Let $(\alpha, \beta, \gamma, \delta) = (1, 2, 1, 2)$. For $n \in \{4, 5\}$, P_I is completely described by constraints (19) through (24), together with (29) and (30). Constraint (30) is a facet of P_I unless n is a multiple of 4, and (29) is a facet whenever n is odd.

Proposition 6. *Let $n = q_1(\alpha + \delta) + r_1 = q_2(\beta + \gamma) - r_2$ with $r_1 \in [0, \alpha + \delta - 1]$ and $r_2 \in [0, \beta + \gamma - 1]$. The inequalities*

$$\sum_{t \in [0, n-1]} y_t \geqslant q_1 \alpha + \min\{r_1, \alpha\} \tag{32}$$

$$\sum_{t \in [0, n-1]} y_t \leqslant q_2 \beta - \min\{r_2, \beta\} \tag{33}$$

are valid for P_I.

Proof. Taking the sums of (25) through (28) over all $t \in [0, n-1]$, we obtain

$$\alpha \sum_{t \in [0,n-1]} z_t \overset{(25)}{\leqslant} \sum_{t \in [0,n-1]} y_t \overset{(26)}{\leqslant} \beta \sum_{t \in [0,n-1]} z_t,$$

$$n - \delta \sum_{t \in [0,n-1]} z_t \overset{(28)}{\leqslant} \sum_{t \in [0,n-1]} y_t \overset{(27)}{\leqslant} n - \gamma \sum_{t \in [0,n-1]} z_t.$$

As a consequence,

$$\sum_{t \in [0,n-1]} y_t \geqslant \begin{cases} q_1 \alpha + \alpha & \text{if } z_0 + \cdots + z_{n-1} \geqslant q_1 + 1, \\ q_1 \alpha + r_1 & \text{if } z_0 + \cdots + z_{n-1} \leqslant q_1, \end{cases}$$

$$\sum_{t \in [0,n-1]} y_t \leqslant \begin{cases} q_2 \beta - \beta & \text{if } z_0 + \cdots + z_{n-1} \leqslant q_2 - 1, \\ q_2 \beta - r_2 & \text{if } z_0 + \cdots + z_{n-1} \geqslant q_2. \end{cases} \qquad \square$$

Example 3. For $n = 7$ and $(\alpha, \beta, \gamma, \delta) = (1, 2, 1, 2)$, (32) and (33) become $\sum y_t \geqslant 3$ and $\sum y_t \leqslant 4$, respectively, and both of them are facets of P_I, as can be verified by hand or using software such as `polymake`[1].

Example 4. For $(\alpha, \beta, \gamma, \delta) = (1, 2, 1, 2)$ it can be checked that the following are valid inequalities for P_I:

$$y_t + y_{t+1} - z_t + \sum_{i \in [2,n-2]} z_{t+i} \leqslant \lfloor (n-1)/2 \rfloor \qquad\qquad t \in [0, n-1], \qquad (34)$$

$$y_t - y_{t-2} - z_t - z_{t-1} + \sum_{i \in [1,n-2]} z_{t+i} \leqslant \lfloor (n-3)/2 \rfloor \qquad\qquad t \in [0, n-1], \qquad (35)$$

$$-y_t - y_{t+1} - z_{t+2} + \sum_{i \in [0,n-4]} z_{t-i} \leqslant \lfloor (n-5)/2 \rfloor \qquad\qquad t \in [0, n-1]. \qquad (36)$$

For $n = 6$, these are facets, and P_I is completely described by (19) through (24), (30), (34), (35), (36).

4.3. **An extended formulation.** In Section 3 we defined the sets $T_0 = \{\tau \in [0, n-1] : \tau + \beta_\tau \geqslant n\}$ and $T_1 = \{\tau \in [0, n-1] : \tau + \delta_\tau \geqslant n\}$ in order to classify the feasible solutions according to the last switching period in the time horizon. More precisely, we have a partition

$$Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = \bigcup_{i \in \{0,1\}} \bigcup_{\tau \in T_i} Z^{(i,\tau)}(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}),$$

with

$$Z^{(0,\tau)}(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \cap \{(\boldsymbol{y}, \boldsymbol{z}) : y_{\tau-1} = 0, \, y_\tau = y_{\tau+1} = \cdots = y_{n-1} = 1\}$$

$$Z^{(1,\tau)}(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \cap \{(\boldsymbol{y}, \boldsymbol{z}) : y_{\tau-1} = 1, \, y_\tau = y_{\tau+1} = \cdots = y_{n-1} = 0\}$$

We will describe the convex hulls of the sets $Z^{(i,\tau)}(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ following [13], and then a result from disjunctive programming implies an extended formulation for $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$. For the rest of this subsection we fix $(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ and omit them from the notation, writing for instance Z instead of $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$. In order to describe the convex hulls of the sets $Z^{(i,\tau)}$, we need additional parameters. The underlying idea is that the elements of a set $Z^{(i,\tau)}$ correspond to on-off-sequences in the non-cyclic setting with known initial state as described in [13, Section 2]. In order to capture the initial state, which is determined by the pair (i, τ), we introduce an additional time period -1, which is essentially a copy of period $n-1$. For instance, an element of $Z^{(0,n-3)}$ with $\alpha_{n-3} = 5$ and

$\beta_{n-3} = 8$, corresponds to a non-cyclic sequence with an on-switch in period -1, starting with at least 3 and at most 6 on-periods. This is enforced by setting $\alpha_{-1}^{(0,n-3)} = 3$ and $\beta^{(0,n-3)} = 6$. In general, we introduce the following parameters:

$$\alpha_{-1}^{(0,\tau)} = \max\{1, \tau + \alpha_\tau - (n-1)\} \qquad\qquad \tau \in T_0,$$

$$\beta_{-1}^{(0,\tau)} = \tau + \beta_\tau - (n-1) \qquad\qquad \tau \in T_0,$$

$$\left(\gamma_{-1}^{(0,\tau)}, \delta_{-1}^{(0,\tau)}\right) = (\gamma_{n-1}, \delta_{n-1}) \qquad\qquad \tau \in T_0,$$

$$\left(\alpha_{-1}^{(1,\tau)}, \beta_{-1}^{(1,\tau)}\right) = (\alpha_{n-1}, \beta_{n-1}) \qquad\qquad \tau \in T_1,$$

$$\gamma_{-1}^{(1,\tau)} = \max\{1, \tau + \gamma_\tau - (n-1)\} \qquad\qquad \tau \in T_1,$$

$$\delta_{-1}^{(1,\tau)} = \tau + \delta_\tau - (n-1) \qquad\qquad \tau \in T_1.$$

For $t \in [0, n-1]$, we set $\varepsilon_t^{(i,\tau)} = \varepsilon_t$ for all $i \in \{0,1\}$, $\tau \in T_i$. Every vector $\boldsymbol{\varepsilon} \in \{\boldsymbol{\alpha}^{(i,\tau)}, \boldsymbol{\beta}^{(i,\tau)}, \boldsymbol{\gamma}^{(i,\tau)}, \boldsymbol{\delta}^{(i,\tau)} : i \in \{0,1\}, \tau \in T_i\}$ satisfies weak monotonicity, that is, $\varepsilon_{t+1} \geqslant \varepsilon_t - 1$ for all $t \in [-1, n-2]$, and therefore we can apply the results of [13, Section 3.1]. We set

$$s'(\boldsymbol{\varepsilon}, t) = \min\{k \in [-1, t] : k + \varepsilon_k \geqslant t+1\}$$

and define polytopes $\tilde{P}^{(i,\tau)} \subseteq \mathbb{R}^{2n+3}$ for $i \in \{0,1\}$, $\tau \in T_i$ by the following constraints:

$$z_{-1} = y_{-1},$$

$$z_t \geqslant y_t - y_{t-1} \qquad\qquad t \in [0, n-1],$$

$$\sum_{k \in [s'(\boldsymbol{\alpha}^{(i,\tau)}, t), t]} z_k \leqslant y_t \qquad\qquad t \in [0, n-1],$$

$$y_t \leqslant \sum_{k \in [s'(\boldsymbol{\beta}^{(i,\tau)}, t), t]} z_k \qquad\qquad t \in [0, n-1],$$

$$\sum_{k \in [s'(\boldsymbol{\gamma}^{(i,\tau)}, t), t]} z_k \leqslant \lambda - y_{s'(\boldsymbol{\gamma}^{(i,\tau)}, t)-1} \qquad\qquad t \in [0, n-1] : s'(\boldsymbol{\gamma}^{(i,\tau)}, t) \geqslant 0,$$

$$\lambda - y_{s'(\boldsymbol{\delta}^{(i,\tau)}, t)-1} \leqslant \sum_{k \in [s'(\boldsymbol{\delta}^{(i,\tau)}, t), t]} z_k \qquad\qquad t \in [0, n-1] : s'(\boldsymbol{\delta}^{(i,\tau)}, t) \geqslant 0,$$

$$0 \leqslant y_t,\ z_t \leqslant \lambda \leqslant 1 \qquad\qquad t \in [0, n-1].$$

The polytopes $\hat{P}^{(i,\tau)} \subseteq \mathbb{R}^{2n+3}$ are defined as follows:

$$\hat{P}^{(0,\tau)} = \tilde{P}^{(0,\tau)} \cap \{(\boldsymbol{y}, \boldsymbol{z}, \lambda) : y_{\tau-1} = 0,\ y_\tau = \cdots = y_{n-1} = y_{-1} = \lambda\},$$

$$\hat{P}^{(1,\tau)} = \tilde{P}^{(1,\tau)} \cap \{(\boldsymbol{y}, \boldsymbol{z}, \lambda) : y_{\tau-1} = \lambda,\ y_\tau = \cdots = y_{n-1} = y_{-1} = 0\}.$$

For $\lambda^* \in \mathbb{R}$, let $\tilde{P}^{(i,\tau)}(\lambda^*)$ and $\hat{P}^{(i,\tau)}(\lambda^*)$ be the slices of $\tilde{P}^{(i,\tau)}$ and $\hat{P}^{(i,\tau)}$, respectively, obtained by fixing $\lambda = \lambda^*$.

Lemma 1. *For every $i \in \{0,1\}$ and $\tau \in T_i$, $\mathrm{conv}(Z^{(i,\tau)}) = f(\hat{P}^{(i,\tau)}(1))$, where $f : \mathbb{R}^{2n+3} \to \mathbb{R}^{2n}$ is the projection $(y_{-1}, y_0, \ldots, y_{n-1}, z_{-1}, z_0, \ldots, z_{n-1}, \lambda) \mapsto (y_0, y_1, \ldots, y_{n-1}, z_0, z_1, \ldots, z_{n-1})$.*

Proof. The polytopes $\tilde{P}^{(i,\tau)}(1)$ are integral by [13, Theorem 2], and since fixing some binary variables does not destroy integrality, the polytopes $\hat{P}^{(i,\tau)}(1)$ are integral. The result follows since $f(\hat{P}^{(i,\tau)}(1))$ is a formulation for $Z^{(i,\tau)}$. $\square$

We can now apply a result from disjunctive programming (see [2, 3, 8]) to obtain an extended formulation.

Proposition 7. *The polytope $\hat{P} \subseteq \mathbb{R}^{2n+(2n+3)(|T_0|+|T_1|)}$ defined by the constraints*

$$\sum_{i \in \{0,1\}} \sum_{\tau \in T_i} \lambda^{(i,\tau)} = 1,$$

$$\left(\boldsymbol{y}^{(i,\tau)}, \boldsymbol{z}^{(i,\tau)}, \lambda^{(i,\tau)} \right) \in \hat{P}^{(i,\tau)} \qquad\qquad i \in \{0,1\}, \tau \in T_i,$$

$$\sum_{i \in \{0,1\}} \sum_{\tau \in T_i} y_t^{(i,\tau)} = y_t \qquad\qquad t \in [0, n-1],$$

$$\sum_{i \in \{0,1\}} \sum_{\tau \in T_i} z_t^{(i,\tau)} = z_t \qquad\qquad t \in [0, n-1].$$

provides an extended formulation for Z.

5. Open problems

We conclude with some open problems. Trying to proceed along the lines of [13], it is natural to consider the following two problems.

Problem 1. Characterize the convex hull of $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ in terms of the x-variables, that is, determine the integer hull of the polytope Q.

Problem 2. Characterize the convex hull of $Z(n, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ in terms of the original y- and z-variables, that is, determine the integer hull of the polytope P.

In particular, in both cases we would like to know if the number of facets is polynomial. In the small cases we have analyzed with `polymake` [1] we observed that $\text{proj}_{y,z}(Q) = P$. This motivates the following question.

Problem 3. Is it true that $\text{proj}_{y,z}(Q) = P$ in general?

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