# Douglas-Rachford method for the feasibility problem involving a circle and a disc

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**Abstract**

The Douglas-Rachford algorithm is a classical and successful method for solving feasibility problems. Here, we provide a region for global convergence of the algorithm for feasibility problems involving a disc and a circle in the Euclidean space of dimension two.

**Keywords:** Douglas-Rachford algorithm, local convergence, feasibility problem, non-convex, projector, reflector.
**Mathematics Subject Classification 2010:** 90C26

## 1 Introduction

*Douglas-Rachford algorithm (DRA)* is a successful operator splitting technique used in Partial differential equations and optimization problems. This algorithm is also applied successfully in solving convex feasibility problems, i.e., to find a point of intersection of two or more nonempty convex closed subsets in a Hilbert space.

The method was introduced by Douglas and Rachford [2] to find numerical solution of partial differential equations arising in parabolic heat conduction problems. Lions and Mercier [3] extended this method to find a solution of the sum of maximally monotone operators.

Recent computational experiments have demonstrated the surprising ability of DRA method in handling non-convex optimization problems. In the nonconvex setting, it has been successfully used to solve problems related to combinatorial optimization [4, 5, 6], low-rank matrix reconstruction [7], sphere packing [9, 4], matrix completion [6] and image reconstruction [10]. Intriguingly, in non-convex setting, the success is not uniform [5], and also the convergence theory is not fully understood. One approach for proving

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the convergence is to replace the convexity with some regularity properties which are not as convincing as convexity. Also, this type of regularity properties is satisfied locally. Therefore, the results one gets through this type of regularity properties give only the local solutions. Another approach is to investigate the convergence properties for the specific kind of nonconvex problems like nonconvex feasibility problems.

In this direction, the first attempt was made by Borwein and Sims [1]. They have investigated a specific type of nonconvex feasibility problems, i.e., finding intersection points of a sphere and a line in the Euclidean plane. In this sequel, they also investigated the feasibility point problems involving a half space and the finite number of points [11].

In this paper, we analyze the convergence of the DRA method for the feasibility problem involving a circle and a disc in the Euclidean plane. In Section 2, we describe the required notions and available results. In Section 3, we explain the vital points of the algorithm. In Section 4 and appendix, we prove our main results.

## 2 Preliminaries and Notations

Here, we assume that $X$ is a Euclidean space of dimension two with the Euclidean norm. Also, we consider the feasibility problem as

$$\text{Find} \quad (x,y) \in (A \cap B), \tag{2.1}$$

where $A \subset X$ is a unit circle centered at the origin and $B \subset X$ is a closed unit disc centered at $(1,0)$. One may note that $A$ is a nonconvex set. It is convenient to represent $A$ and $B$ in the following form

$$A := \{(x,y) | x^2 + y^2 = 1\} \quad \text{and} \quad B := \{(x,y) | (x-1)^2 + y^2 \leq 1\}. \tag{2.2}$$

For a closed subset $C$ of $X$, the mapping $P_C : D_C \subsetneq X \to C$ is a closest point projection of $D_C$ onto $C$ if $C \subset D_C$, $P_C^2 = P_C$ and

$$\| (x,y) - P_C(x,y) \| = \text{dist}((x,y), C) = \inf \{ \| (x,y) - (a,b) \| : (a,b) \in C \}, \tag{2.3}$$

for all $(x,y) \in D_C$. For a convex set, the closest point projection is unique. For the given $P_C$, we define the reflector of $(x,y)$ corresponding to $C$ as

$$R_C := 2P_C - Id, \tag{2.4}$$

where $Id$ denotes the identity mapping. Here, closest point projection of $(x, y)$ $(\neq (0,0)) \in X$ onto A is calculated as $P_A(x,y) = \frac{(x,y)}{\|(x,y)\|}$. Similarly, closest point projection of $(x, y)$ onto B [13] is

$$P_B(x,y) = (1,0) + \frac{(x-1,y)}{\max\{\|(x-1,y)\|,1\}}. \tag{2.5}$$

Notice that in our case, $A$ is nonconvex and for all $(x,y)$ $(\neq (0,0)) \in X$, $P_A$ is single-valued and

$$\begin{aligned} T_{A,B}(x,y) &= \{\frac{Id + R_A R_B}{2}\}(x,y) = \{Id - P_A + P_B R_A\}(x,y) \\ &= (1-\frac{1}{\rho})(x,y) + P_B\{(\frac{2}{\rho}-1)(x,y)\} \\ &= (1-\frac{1}{\rho})(x,y) + (1,0) + \frac{(\frac{2}{\rho}-1)(x,y)-(1,0)}{\max\{\|(\frac{2}{\rho}-1)(x,y)-(1,0)\|,1\}}. \end{aligned}$$

It implies that

$$T_{A,B}(x,y) = \begin{cases} (\frac{x}{\rho},\frac{y}{\rho}) & \gamma \leq 1, \\ (1-\frac{1}{\rho}+\frac{2}{\rho\gamma}-\frac{1}{\gamma})(x,y) + (1-\frac{1}{\gamma},0) & \gamma > 1, \end{cases} \tag{2.6}$$

where $\rho := \|(x,y)\| = \sqrt{x^2+y^2}$ and $\gamma := \|(\frac{2}{\rho}-1)(x,y)-(1,0)\|$. Here, we will generate sequence of points $(x_{n+1}, y_{n+1})_{n\in N}$ by $(x_{n+1}, y_{n+1}) = T_{A,B}(x_n, y_n)$ for all $n \in N$ and $(x_0, y_0)(\neq (0,0)) \in X$. For the rest of the paper, we denote $T_{A,B}$ by $T$.

# 3 Algorithm

The iteration scheme suggests that we will generate a sequence of points $(x_n)$ through $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$. However, one may observe that the projection operator for origin does not map to a unique point. For avoiding this embarrassing situation, we have the following assumption.

**Assumption 3.1.** *For the iteration scheme, the starting point $(x_0, y_0)$ satisfies that $(x_0, y_0) \in \mathbb{R}^2 - \Lambda$, where $\Lambda = \{(x,0) | x \in (-\infty,-1] \cup \{0\}\}$.*

In the appendix, we have shown that no points other than the points in $\Lambda$ iterate to $\Lambda$. As our starting point, i.e., $x_0$ is outside $\Lambda$, no points for the subsequent iterations will map to $\Lambda$. One may note that if all the points are outside $(A \cup B)$, then the iterations will converge as a convex feasibility problem [7]. Therefore, it remains to study in the $(A \cup B)$ region only.

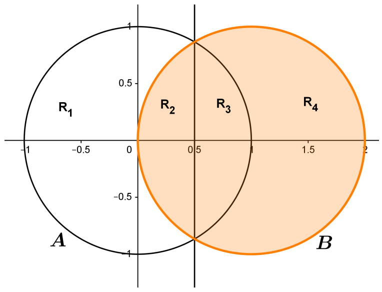


Figure 1: Regions within the circle and the disc.

$R_1 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \text{ and } x \leq 0\}$,
$R_2 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \text{ and } 0 < x < \frac{1}{2}\}$,
$R_3 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \text{ and } \frac{1}{2} \leq x \leq 1\}$ and
$R_4 := \{(x, y) \in \mathbb{R}^2 | (x-1)^2 + y^2 \leq 1 \text{ and } x^2 + y^2 > 1\}$.

In the beginning, we will discuss the following four regions and in the appendix, we will discuss the iterations lie outside separately.

# 4 Main Results

## 4.1 Convergence of the points lying in $R_3$ and $R_4$

In this section, we prove that if any iteration lies in $(R_3 \cup R_4)$ then the next iteration lies in the intersection of A and B. Here in the figure 2 and figure 3, we have shown the convergence of the points in $R_3$ and $R_4$ respectively. Also, the same iterations are mentioned in tabular form in the Table 1.

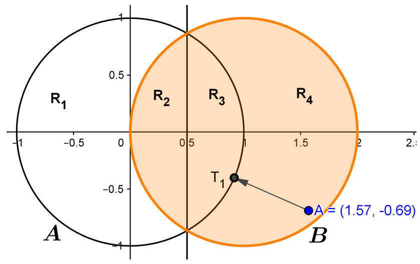


Figure 2: DRA convergence with an initial guess in $R_4$.

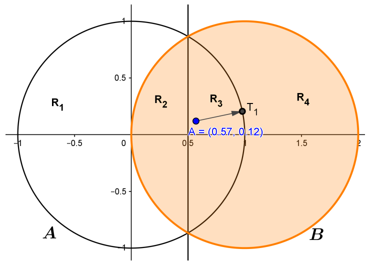


Figure 3: DRA convergence with an initial guess in $R_3$.

**Proposition 4.1.** *If* $x_n \geq \frac{1}{2}$ *and* $\rho_n \leq 2$*, then* $(\frac{2}{\rho_n} - 1)(x_n, y_n) \in B$.

*Proof.* From the assumption of the proposition, we get

$$1 - 2x_n - (\rho_n - 1)^2 \leq 0.$$

Further simplifying and using the fact that $(\frac{2}{\rho_n} - 1) \geq 0$ we get

$$(\frac{2}{\rho_n} - 1)\{(\frac{2}{\rho_n} - 1)\rho_n^2 - 2x_n\} \leq 0.$$

Appealing to the fact $\rho_n^2 = (x_n^2 + y_n^2)$, we get

$$\{(\frac{2}{\rho_n} - 1)x_n - 1\}^2 + \{(\frac{2}{\rho_n} - 1)y_n\}^2 \leq 1. \tag{4.1}$$

which implies that $(\frac{2}{\rho_n} - 1)(x_n, y_n) \in B$. $\blacksquare$ $\blacksquare$

**Proposition 4.2.** *If any iteration* $(x_n, y_n) \in (R_3 \cup R_4)$*, then the next iteration* $(x_{n+1}, y_{n+1}) \in (A \cap B)$.

*Proof.* From the definition of Douglas-Rachford operator, we have

$$(x_{n+1}, y_{n+1}) = T(x_n, y_n) = (Id - P_A + P_B R_A)(x_n, y_n).$$

Using the definition of Reflector operator, we get

$$(x_{n+1}, y_{n+1}) = (1 - \frac{1}{\rho_n})(x_n, y_n) + P_B((\frac{2}{\rho_n} - 1)(x_n, y_n)).$$

As $(x_n, y_n) \in R_3 \cup R_4$, we have $x_n \geq \frac{1}{2}$ and $\rho_n \leq 2$, appealing to the result from the Proposition 4.1, we get

$$(x_{n+1}, y_{n+1}) = (1 - \frac{1}{\rho_n})(x_n, y_n) + (\frac{2}{\rho_n} - 1)(x_n, y_n).$$

On further simplification, we get

$$(x_{n+1}, y_{n+1}) = (\frac{x_n}{\rho_n}, \frac{y_n}{\rho_n}). \tag{4.2}$$

Using the fact $\rho_n^2 = (x_n^2 + y_n^2)$, we get

$$x_{n+1}^2 + y_{n+1}^2 = \frac{x_n^2 + y_n^2}{\rho_n^2} = 1.$$

Or,

$$(x_{n+1}, y_{n+1}) \in A. \tag{4.3}$$

In order to prove that $(x_{n+1}, y_{n+1}) \in B$. We need to show that $(x_{n+1} - 1)^2 + y_{n+1}^2 \leq 1$. Inserting the values of $x_{n+1}$ and $y_{n+1}$ and using Equation (4.2), we get

$$(x_{n+1} - 1)^2 + y_{n+1}^2 = (\frac{x_n}{\rho_n} - 1)^2 + (\frac{y_n}{\rho_n})^2.$$

Applying the fact $\rho_n^2 = (x_n^2 + y_n^2)$ and $x_n \geq \frac{1}{2}$, we get

$$(x_{n+1} - 1)^2 + y_{n+1}^2 = 2 - \frac{2x_n}{\rho_n} \leq 2 - \frac{1}{\rho_n}.$$

For $(x_n, y_n)$ in $R_3$, by employing $\rho_n \leq 1$, we get

$$(x_{n+1} - 1)^2 + y_{n+1}^2 \leq 1.$$

For $(x_n, y_n)$ in $R_4$, we have $\rho_n > 1$. Applying triangle's inequality and using the fact that $(x_n, y_n) \in B$, we get

$$\begin{aligned}\|(x_{n+1}, y_{n+1}) - (1,0)\| &\leq \frac{1}{\rho_n}\{\|(x_n - 1, y_n)\| + \|(1-\rho_n, 0)\|\} \\ &\leq \frac{1}{\rho_n}\{1 + \rho_n - 1\} = 1.\end{aligned}$$

Or,

$$(x_{n+1}, y_{n+1}) \in B. \tag{4.4}$$

From equations (4.3) and (4.4), we have

$$(x_{n+1}, y_{n+1}) \in (A \cap B).$$

■ ■

### 4.2 Convergence of $R_1$ and $R_2$ assuming $\gamma_n \leq 1$

In this section, we prove that if any iteration lies in $(R_1 \cup R_2)$ and $\gamma_n \leq 1$ then the next iteration lies in the intersection of A and B. Here, in Figure 4 and Figure 5, we have shown the convergence of points lying in $R_1$ and $R_2$ respectively. Also same iterations are mentioned in tabular form in the Table 1.

**Proposition 4.3.** *If $\gamma_n \leq 1$ and $\rho_n \leq 1$ hold for any iteration $(x_n, y_n) \in (R_1 \cup R_2)$, then the next iteration $(x_{n+1}, y_{n+1}) \in (A \cap B)$.*

*Proof.* Using the definition of $T$ when $\gamma_n \leq 1$ and $\rho_n$ from the Equation (2.6), one can easily see that it $(x_{n+1}, y_{n+1}) \in A$. Now, it remains to show that $(x_{n+1}, y_{n+1}) \in B$. From the assumption, we have

$$\gamma_n^2 \leq 1.$$

Substituting the value of $\gamma_n$ from Equation (2.6), we get

$$(\frac{2}{\rho_n} - 1)\{(\frac{2}{\rho_n} - 1)\rho_n^2 - 2x_n\} \leq 0.$$

Using the fact that $(\frac{2}{\rho_n} - 1) > 0$, we get

$$(\frac{2}{\rho_n} - 1)\rho_n^2 - 2x_n \leq 0.$$

On further simplification, we have

$$\rho_n - 2 \geq -\frac{2x_n}{\rho_n}. \tag{4.5}$$

From the definition of $T$ and Equation (2.6,4.5), it follows

$$\begin{aligned}(x_{n+1}-1)^2 + y_{n+1}^2 &= (\frac{x_n}{\rho_n} - 1)^2 + (\frac{y_n}{\rho_n})^2 \\ &= \frac{x_n^2 + y_n^2}{\rho_n^2} + 1 - \frac{2x_n}{\rho_n} \\ &\leq \rho_n \leq 1.\end{aligned}$$

Hence,

$$(x_{n+1}, y_{n+1}) \in A \cap B.$$

■ ■

In the next section, we prove that if for any $(x_n, y_n) \in (R_1 \cup R_2)$, then the subsequent iterations will lie in $(A \cap B)$ eventually. Therefore, from now onwards assume that $\gamma_n > 1$. Before moving further in order to simplify the notations, we assume

$$k_n := 1 - \frac{1}{\rho_n} + \frac{2}{\rho_n \gamma_n} - \frac{1}{\gamma_n}. \tag{4.6}$$

## 4.3 Convergence of $R_1$ and $R_2$ assuming $\gamma_n > 1$

### 4.3.1 Convergence of points in $R_2$ assuming $\gamma_n > 1$

In this subsection, we prove that if for any $(x_n, y_n) \in R_2$, then the subsequent iterations lie $(R_3 \cup R_4)$ eventually. Before proceeding further, we will find an upper bound for $\gamma_n$.

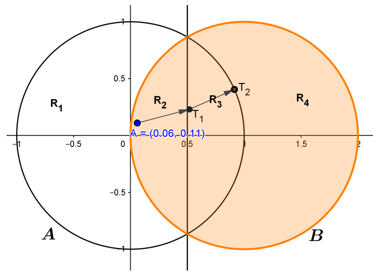


Figure 4: DRA convergence with an initial guess in $R_2$.

**Lemma 4.1.** *If* $\gamma_n > 1,\ \rho_n \leq 1$ *and* $x_n > 0$*, then* $\gamma_n < \sqrt{5}$.

*Proof.* We know that

$$\gamma_n := \|(\frac{2}{\rho_n} - 1)(x_n, y_n) - (1, 0)\|.$$

Squaring both the sides and using $\rho_n^2 = x_n^2 + y_n^2$, we get

$$\gamma_n^2 = (\frac{2}{\rho_n} - 1)^2 \rho_n^2 + 1 - 2(\frac{2}{\rho_n} - 1)x_n.$$

Or,

$$\gamma_n^2 - 1 = (2 - \rho_n)^2 - 2(\frac{2}{\rho_n} - 1)x_n \leq (2 - \rho_n)^2 \leq 4.$$

Thus

$$\gamma_n < \sqrt{5}.$$

■ ■

One may note that the bound is not sharp. But $\sqrt{5}$ is just an upper bound for $\gamma_n$.

**Lemma 4.2.** *If* $0.3 \leq x_n$ *and* $\rho_n \leq 1$ *then* $\gamma_n < 2$.

*Proof.* Suppose that

$$2 - \rho_n - \frac{3}{5\rho_n} \geq \frac{3}{2}.$$

Using the fact that $\rho_n \geq 0$, we get

$$2 - \frac{3}{5\rho_n} \geq \frac{3}{2}.$$

On simplification, we have

$$\rho_n \geq \frac{6}{5}.$$

That is not possible as $\rho_n \leq 1$. Thus

$$2 - \rho_n - \frac{3}{5\rho_n} < \frac{3}{2}. \tag{4.7}$$

We know that

$$\gamma_n := \|(\frac{2}{\rho_n} - 1)(x_n, y_n) - (1, 0)\|.$$

Squaring both the sides, we get

$${\gamma_n}^2 = (2 - \rho_n)^2 + 1 - 2(2 - \rho_n)\frac{x_n}{\rho_n}.$$

Applying $x_n \geq 0.3$, we get

$${\gamma_n}^2 \leq 1 + (2 - \rho_n)(2 - \rho_n - \frac{3}{5\rho_n}).$$

Using Equation (4.7), we have

$${\gamma_n}^2 \leq 1 + (2 - \rho_n)\frac{3}{2} < 4.$$

Or,

$$\gamma_n < 2.$$

■ ■

**Lemma 4.3.** *If $\gamma_n \geq 2$ and $\rho_n \leq 1$, then $x_{n+1} > 0.3$.*

*Proof.* As $0 < \frac{x_n}{\rho_n} \leq 1$ and $0 < x_n < \frac{1}{2}$. From Equation (2.6), we get

$$\begin{aligned} x_{n+1} &= \{\frac{2}{\rho_n\gamma_n} - \frac{1}{\rho_n} - \frac{1}{\gamma_n} + 1\}x_n + 1 - \frac{1}{\gamma_n} \\ &= (\frac{2}{\gamma_n} - 1)\frac{x_n}{\rho_n} + (x_n + 1)\{1 - \frac{1}{\gamma_n}\} \\ &> \frac{2}{\gamma_n} - 1 + \frac{1}{2} \\ &\geq \frac{2}{\gamma_n} - \frac{1}{2} > \frac{2}{\sqrt{5}} - \frac{1}{2} > 0.3. \end{aligned}$$

■ ■

**Lemma 4.4.** *If $(x_n, y_n) \in R_2$ and $\gamma_n > 1$, then either $x_{n+1}^2 + y_{n+1}^2 \leq 1$ or $(x_{n+1} - 1)^2 + y_{n+1}^2 \leq 1$.*

*Proof.* In order to prove this lemma, we divide it into two cases.
**Case 1** $k_n < 0$.
In this case, we show that $(x_{n+1}, y_{n+1}) \in B$. From Equation (2.6), we get

$$(x_{n+1} - 1)^2 + y_{n+1}^2 = (k_n x_n - 1/\gamma_n)^2 + (k_n y_n)^2.$$

Using the fact that $x_n \leq \rho_n$ and $k_n < 0$, we have

$$(x_{n+1} - 1)^2 + y_{n+1}^2 \leq (-k_n \rho_n + 1/\gamma_n)^2.$$

Substituting the value of $k_n$ and appealing to the fact that $\rho_n \leq 1$ and $\gamma_n > 1$, we get

$$(x_{n+1} - 1)^2 + y_{n+1}^2 \leq (1 - \rho_n)^2 (1 - 1/\gamma_n)^2 \leq 1.$$

**Case 2** $k_n \geq 0$.
Here, we require to show that $x_{n+1}^2 + y_{n+1}^2 \leq 1$. Since

$$x_{n+1}^2 + y_{n+1}^2 = (k_n x_n + 1 - 1/\gamma_n)^2 + (k_n y_n)^2.$$

Adding and subtracting the term $2k_n x_n(1 - 1/\gamma_n)$ to the right side of the equation, we get

$$x_{n+1}^2 + y_{n+1}^2 = (k_n \rho_n + 1 - 1/\gamma_n)^2 + 2k_n(x_n - \rho_n)(1 - 1/\gamma_n).$$

Using the fact that $(x_n - \rho_n) \leq 0$ and substituting the value of $k_n$, we get

$${\rho_n}^2 \leq (\rho_n + 1/\gamma_n - \frac{\rho_n}{\gamma_n})^2.$$

Now, we show that $\rho_n + 1/\gamma_n - \frac{\rho_n}{\gamma_n} \leq 1$. Assume on the contrary that

$$\rho_n + 1/\gamma_n - \frac{\rho_n}{\gamma_n} > 1.$$

Rearranging the terms, we have

$$(1 - \rho_n)(1/\gamma_n - 1) > 0.$$

As $\rho_n \leq 1$, it implies that $\frac{1}{\gamma_n} - 1 > 0$ or $\gamma_n < 1$. This is a contradiction to the assumption. Hence $x_{n+1}^2 + y_{n+1}^2 \leq 1$. Also

$$x_{n+1} = k_n x_n + 1 - 1/\gamma_n \geq 0. \tag{4.8}$$

Therefore, $x_{n+1}^2 + y_{n+1}^2 \leq 1$ and $x_{n+1} \geq 0$. ■ ■

**Proposition 4.4.** *If* $(x_n, y_n) \in R_2$ *and* $1 < \gamma_n < 2$*, then the generated sequence eventually lies in* $(R_3 \cup R_4)$*.*

*Proof.* Note that from Lemma 4.4, it is clear that $(x_{n+1}, y_{n+1}) \in A \cup B$. Since, $(x_n, y_n) \in R_2$, we get $0 < x_n < \frac{1}{2}$. For proving points are eventually in $(R_3 \cup R_4)$, we require to show that for some $k \in \mathbb{N}$, $x_{n+k} \geq \frac{1}{2}$. If $\rho_{n+k} \geq 1$ for some $k > 1$, the point lies in $(R_3 \cup R_4)$. Therefore, on the contrary, we assume $\rho_{n+k} \leq 1$ for all $k > 1$. Also, assume that $0 < x_{n+k} < \frac{1}{2}\ \forall\ k \geq 1$. We first show that $x_{n+k}$ is strictly increasing. It is sufficient to show that $x_{n+1} > x_n$ or $x_{n+1} - x_n > 0$ (as it will follow by induction on n). Remember $\rho_{n+k} \leq 1\ \forall\ k \geq 1$.

$$\begin{aligned} x_{n+1} - x_n &= (\frac{2}{\rho_n \gamma_n} - \frac{1}{\gamma_n} - \frac{1}{\rho_n})x_n + 1 - \frac{1}{\gamma_n} \\ &= (1 - \frac{1}{\gamma_n})(1 - \frac{x_n}{\rho_n}) + \frac{1}{\gamma_n}(\frac{1}{\rho_n} - 1)x_n \\ &> 0. \end{aligned}$$

Hence, $(x_{n+k})_{k=1}^{\infty}$ is a strictly increasing and bounded sequence of real number. Hence, the sequence converges. Also without loss of generality assume that $x, \rho,$ and $\gamma$ is the limit of the sequence $(x_{n+k})_{k\geq 1}, (\rho_{n+k})_{k\geq 1}$ and $(\gamma_{n+k})_{k\geq 1}$ respectively.

$$x_{n+1} = \{\frac{2}{\rho_n \gamma_n} - \frac{1}{\rho_n} - \frac{1}{\gamma_n} + 1\}x_n + 1 - \frac{1}{\gamma_n}.$$

Applying the limits, we have

$$x = \{\frac{2}{\rho\gamma} - \frac{1}{\rho} - \frac{1}{\gamma} + 1\}x + 1 - \frac{1}{\gamma}$$

or

$$\frac{1}{\gamma} - 1 = \{\frac{2}{\rho\gamma} - \frac{1}{\rho} - \frac{1}{\gamma}\}x \geq (\frac{1}{\gamma} - \frac{1}{\rho})x.$$

Using the fact that $\rho \leq 1$ and $\gamma < 2$, we get

$$\frac{1}{\gamma} - 1 \geq \{\frac{1}{\gamma} - 1\}x.$$

Using the fact that $(\frac{1}{\gamma} - 1) < 0$, we get

$$x \geq 1.$$

This is a contradiction to that assumption $0 < x_{n+k} < \frac{1}{2}\ \forall\ k \geq 1$. Thus $x_{n+k} \geq \frac{1}{2}$. ■ ■

**Remark 4.1.** *Observe that if $\gamma_n \geq 2$ then $x_{n+1} > 0.3$ by Lemma 4.3 and for $0.3 \leq x_n$ we have $\gamma_n < 2$ by Lemma 4.2. Therefore, by using proposition 4.4, we get that eventually points are in $R_3 \cup R_4$.*

### 4.3.2 Convergence of points in $R_1$ assuming $\gamma_n > 1$

In this subsection, we prove that if any iteration lies in $R_1$ then the next iterate lies in $(R_2 \cup R_3)$.

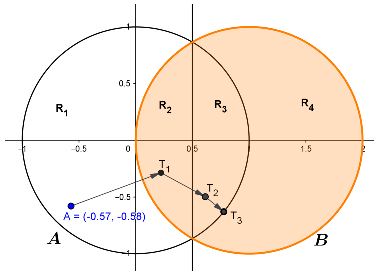


Figure 5: DRA convergence with an initial guess in $R_1$.

**Lemma 4.5.** *If $(x_n, y_n) \in R_1$, then $\sqrt{2} \leq \gamma_n \leq 3$.*

*Proof.* We prove this by contradiction. Assume that $\gamma_n < \sqrt{2}$. Squaring both the sides and substituting the value of $\gamma_n$, we get

$$(2-\rho_n)^2 - 2(\frac{2}{\rho_n} - 1)x_n + 1 < 2.$$

Using the fact that $\rho_n \leq 1$ and $x_n < 0$, we have

$$(2-\rho_n)^2 - 1 < 2(\frac{2}{\rho_n} - 1)x_n < 0.$$

Taking square root both sides, we get

$$2 - \rho_n < 1.$$

or,

$$1 < \rho_n.$$

It is a contradiction. Now, we prove the other inequality. Note that

$$\gamma_n = \|(\frac{2}{\rho_n} - 1)(x_n, y_n) - (1, 0)\|.$$

Applying Triangle inequality to the right side of the above equation, we get

$$\gamma_n \leq \mid (\frac{2}{\rho_n} - 1) \mid \rho_n + 1 \leq 3.$$

■ ■

Here we remind that inequalities in Lemma 4.5 are not sharp.

**Lemma 4.6.** *If* $(x_n, y_n) \in R_1$ *and* $k_n < 0$ *then* $k_n \geq -\frac{1}{3\rho_n}$. *Moreover* $(x_{n+1}, y_{n+1}) \in B$.

*Proof.* We proceed this by contradiction. So assume that

$$k_n < -\frac{1}{3\rho_n}.$$

Substituting the value of $k_n$ from Equation (4.6), we get

$$1 - \frac{1}{\rho_n} + \frac{2}{\rho_n \gamma_n} - \frac{1}{\gamma_n} < -\frac{1}{3\rho_n}.$$

Adding $\frac{-1}{\rho_n}$ both sides and rearranging the terms on the left side of the equation we get

$$(1 - \frac{1}{\gamma_n})(1 - \frac{2}{\rho_n}) < -\frac{4}{3\rho_n}.$$

or,

$$(\gamma_n - 1)(2 - \rho_n) > \frac{4}{3}\gamma_n. \tag{4.9}$$

Also

$$2(\gamma_n - 1) > (\gamma_n - 1)(2 - \rho_n). \tag{4.10}$$

Using equations 4.9 and 4.10, we get $\gamma_n > 3$. This contradicts to Lemma 4.5. So $k_n \geq -\frac{1}{3\rho_n}$. Now

$$(x_{n+1} - 1)^2 + y_{n+1}^2 = (k_n x_n - \frac{1}{\gamma_n})^2 + k_n^2 y_n^2$$

or,

$$(x_{n+1} - 1)^2 + y_{n+1}^2 = k_n^2 \rho_n^2 + \frac{1}{\gamma_n^2} - \frac{2k_n x_n}{\gamma_n}.$$

Using $-\frac{1}{3\rho_n} \leq k_n < 0$ and $\frac{k_n x_n}{\gamma_n}$, we have

$$(x_{n+1}-1)^2 + y_{n+1}^2 < \frac{1}{9} + \frac{1}{\gamma_n^2} < \frac{1}{9} + \frac{1}{2} < 1.$$

Hence,

$$(x_{n+1}, y_{n+1}) \in B.$$

■ ■

Note that Lemma 4.6 implies that if $(x_n, y_n) \in R_1$ and $k_n < 0$, then the next iteration $(x_{n+1}, y_{n+1}) \in (R_2 \cup R_3 \cup R_4)$ and for the above region convergence is already proved. Now, we will study the case when $k_n \geq 0$.

**Lemma 4.7.** *If $(x_n, y_n) \in R_1$ and $k_n \geq 0$, then $x_{n+1}^2 + y_{n+1}^2 < 1$.*

*Proof.* In order to prove that $(x_n, y_n) \in B$, we divide into two cases.
**Case 1.**$\gamma_n > 2$
Here, we will calculate the upper bound for $k_n$. From the definition of $k_n$, we have

$$k_n = 1 - \frac{1}{\rho_n} + \frac{1}{\gamma_n}(\frac{2}{\rho_n} - 1).$$

Since $\rho_n < 1$ and $\frac{1}{\gamma_n} < \frac{1}{2}$, we get

$$k_n \leq 1 - \frac{1}{\rho_n} + \frac{1}{2}(\frac{2}{\rho_n} - 1) = \frac{1}{2}$$

or,

$$0 \leq k_n < \frac{1}{2}. \tag{4.11}$$

Multiplying $x_n$ to Equation (4.11) and using $-1 \leq x_n < 0$, we get

$$\frac{-1}{2} + \frac{1}{2} < k_n x_n + 1 - \frac{1}{\gamma_n} = x_{n+1} \leq 1 - \frac{1}{\gamma_n}.$$

Using $\gamma_n < 3$ and Lemma 4.5, we get

$$0 < x_{n+1} \leq \frac{2}{3}. \tag{4.12}$$

Again using Equation (4.11) and $-1 \leq y_n \leq 1$, we get

$$0 \leq y_{n+1}^2 \leq \frac{1}{4}. \tag{4.13}$$

From the Equations (4.12 and 4.13), it follows that

$$x_{n+1}^2 + y_{n+1}^2 + \leq \frac{4}{9} + \frac{1}{4} < 1.$$

**Case 2.**$\sqrt{2} \leq \gamma_n \leq 2$
We first find out the bounds for $k_n$. In order to find lower bound of $k_n$, we use $\rho_n \geq 1$.

$$\begin{aligned} k_n &= 1 - \frac{1}{\gamma_n} + \frac{1}{\rho_n}(\frac{2}{\gamma_n} - 1) \\ &\geq 1 - \frac{1}{\gamma_n} + \frac{2}{\gamma_n} - 1 \\ &\geq \frac{1}{2}. \end{aligned}$$

For finding the upper bound of $k_n$, we use the fact $\frac{1}{\gamma_n} \leq \frac{1}{\sqrt{2}}$.

$$\begin{aligned} k_n &= 1 - \frac{1}{\rho_n} + \frac{1}{\gamma_n}(\frac{2}{\rho_n} - 1) \\ &\leq 1 - \frac{1}{\rho_n} + \frac{1}{\sqrt{2}}(\frac{2}{\rho_n} - 1) \\ &\leq 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2} - 1}{\rho_n}. \end{aligned}$$

Thus, the bound to $k_n$ is

$$\frac{1}{2} \leq k_n \leq 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2} - 1}{\rho_n}.$$

Multiplying the above inequality by $x_n$ with remembering $x_n$ is negative, we have

$$(1 - \frac{1}{\sqrt{2}})x_n + \frac{\sqrt{2} - 1}{\rho_n}x_n \leq k_n x_n \leq \frac{x_n}{2}.$$

Applying the bound of $x_n$, we have

$$(1 - \frac{1}{\sqrt{2}})(-1) + (\sqrt{2} - 1)(-1) \leq k_n x_n \leq 0.$$

Adding $1 - \frac{1}{\gamma_n}$ to the above inequality and using the bound of $\gamma_n$, we have

$$-\sqrt{2} + 1 \leq x_{n+1} \leq \frac{1}{2}.$$

Applying $(\sqrt{2}-1) < \frac{1}{2}$, we get

$$(\sqrt{2}-1)^2 \leq x_{n+1}^2 < \frac{1}{4}.$$

Following the similar arguments and $-1 \leq y_n \leq 1$, we get

$$y_{n+1}^2 \leq \frac{1}{2}.$$

Using the previous two inequalities, we will get

$$x_{n+1}^2 + y_{n+1}^2 \leq \frac{1}{4} + \frac{1}{2} < 1. \tag{4.14}$$

■ ■

**Lemma 4.8.** *If $(x_n, y_n) \in R_1$ and $k_n \geq 0$, then $(x_{n+1}-1)^2 + y_{n+1}^2 \leq \frac{4}{\gamma_n^2}$.*

*Proof.* Putting the values of $x_{n+1}$ and $y_{n+1}$, we have

$$(x_{n+1}-1)^2 + y_{n+1}^2 = (k_n x_n - \frac{1}{\gamma_n})^2 + (k_n y_n)^2.$$

Using the fact that $-x_n \leq \rho_n$, we get

$$(x_{n+1}-1)^2 + y_{n+1}^2 \leq (k_n \rho_n)^2 + \frac{1}{\gamma_n^2} + \frac{2k_n\rho_n}{\gamma_n} = (k_n\rho_n + \frac{1}{\gamma_n})^2.$$

Now, estimating the bounds of $k_n\rho_n + \frac{1}{\gamma_n}$, we get $k_n\rho_n + \frac{1}{\gamma_n} \geq 0$. Calculating the upper bound by using $\rho_n \leq 1$ and $\gamma_n \geq \sqrt{2}$, we get

$$\begin{aligned} k_n\rho_n + \frac{1}{\gamma_n} &= \frac{2}{\gamma_n} - 1 - \frac{\rho_n}{\gamma_n} + \rho_n + \frac{1}{\gamma_n} \\ &= (1-\gamma_n)(\frac{1}{\gamma_n} - 1) \\ &\leq \frac{2}{\gamma_n}. \end{aligned}$$

Or,

$$(x_{n+1}-1)^2 + y_{n+1}^2 \leq \frac{4}{\gamma_n^2}. \tag{4.15}$$

■ ■

Table 1: Table shows first four iterates of DRA with an initial guess $T_0$

| Initial guess | Outside | $R_4$ | $R_3$ | $R_2$ | $R_1$ |
|---|---|---|---|---|---|
| $T_0$ | (0.88, 2.52) | (1.57, -0.69) | (0.57, 0.12) | (0.06, 0.11) | (-0.57, -0 |
| $T_1$ | (0.6622, 1.1162) | (0.9155, -0.4023) | (0.9785, 0.2060) | (0.5191, 0.2302) | (0.2232, |
| $T_2$ | (0.5102, 0.8600) | (0.9155, -0.4023) | (0.9785, 0.2060) | (0.9142, 0.4053) | (0.6134, |
| $T_3$ | (0.5102, 0.8600) | (0.9155, -0.4023) | (0.9785, 0.2060) | (0.9142, 0.4053) | (0.7766, |
| $T_4$ | (0.5102, 0.8600) | (0.9155, -0.4023) | (0.9785, 0.2060) | (0.9142, 0.4053) | (0.7766, |

**Remark 4.2.** *For $\gamma_n > 2$ and $k_n \geq 0$, Using the case 1 of Lemmas 4.7 and 4.8, we conclude that $(x_{n+1}, y_{n+1})$ lies in the intersection of $(x_{n+1} - 1)^2 + y_{n+1}^2 \leq 1$ and $x_{n+1}^2 + y_{n+1}^2 \leq 1$. It implies that $(x_{n+1}, y_{n+1}) \in (R_2 \cup R_3)$. However, for $\sqrt{2} \leq \gamma_n \leq 2$, we get from Lemmas 4.7 and 4.8 that $(x_{n+1}, y_{n+1})$ lies in the intersection of $(x_{n+1} - 1)^2 + y_{n+1}^2 \leq 2$ and $x_{n+1}^2 + y_{n+1}^2 \leq 1$, i.e., $(x_{n+1}, y_{n+1}) \in R_1$ and $x_{n+1} > 1 - \sqrt{2}$.*

**Proposition 4.5.** *If $(x_n, y_n) \in R_1$ and $\sqrt{2} \leq \gamma_n \leq 2$ and $x_{n+1} > 1 - \sqrt{2}$, then $(x_{n+1}, y_{n+1}) \in (R_2 \cup R_3)$.*

*Proof.* It is sufficient to prove that $x_{n+1} > 0$. We prove this by contradiction. Suppose

$$x_{n+1} \leq 0.$$

Substituting the value of $x_{n+1}$ and simplifying using $\frac{1}{\gamma_n} \leq \frac{1}{\sqrt{2}}$, we get

$$k_n x_n \leq \frac{1}{\gamma_n} - 1 \leq \frac{1}{\sqrt{2}} - 1.$$

Appealing to the case 2 of Lemma 4.7 and $\frac{1}{2} < k_n$, we get

$$x_n \leq \{\frac{1}{\sqrt{2}} - 1\}\frac{1}{k_n} \leq \{\frac{1}{\sqrt{2}} - 1\}(2) = \sqrt{2} - 2.$$

But $x_n \geq 1 - \sqrt{2}$, it is a contradiction. So $x_{n+1} > 0$. ■ ■

From the remark after Lemma 4.8 and the above proposition 4.5, it is clear that if $(x_n, y_n) \in R_1$, then $(x_{n+2}, y_{n+2}) \in (R_2 \cup R_3)$.

## 5 Appendix

Here, in Figure 6, we have shown the convergence of points lying outside $(A \cup B)$. Also, same iterations are mentioned in tabular form in Table 1.

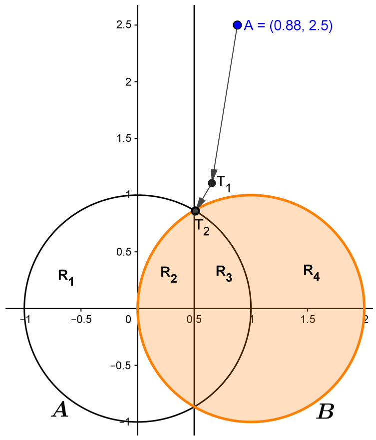


Figure 6: DRA convergence with an initial guess is outside (using Geogebra).

In this section, we want to find all those points for which the (n+1)th iteration lies in the origin that is $T(x_n, y_n) = (0, 0)$. It may be noted that $(x_n, y_n) \in \mathbb{R}^2 - \Lambda$. we consider two cases $\gamma_n \leq 1$ and $\gamma_n > 1$.

**Case 1.** $\gamma_n \leq 1$

$$T(x_{n+1}, y_{n+1}) = (0, 0).$$

Substituting the value of $T$ from Equation (2.6), we have

$$(\frac{x_n}{\rho_n}, \frac{y_n}{\rho_n}) = (0, 0).$$

This implies

$$x_n = 0 \text{ and } y_n = 0.$$

Therefore, only the origin maps to the origin in this case.

**Case 2.** $\gamma_n > 1$

$$T(x_{n+1}, y_{n+1}) = (0, 0).$$

Substituting the value of $T$ from Equation (2.6), we have

$$(k_n x_n + 1 - \frac{1}{\gamma_n}, k_n y_n) = (0, 0).$$

Equating the terms component wise to zero, we get

$$k_n x_n + 1 - \frac{1}{\gamma_n} = 0 \quad \text{and} \quad k_n y_n = 0.$$

Suppose that $k_n = 0$, then from the above equation, we get

$$\gamma_n = 1.$$

This contradicts the assumption of the case2. Therefore, we get $y_n = 0$. In the other words, only points on the X-axis will map to (0,0). Now, assume $k_n \neq 0$. It implies $y_n = 0$ ($x_n \neq 0$). Then $\gamma_n = \|(\frac{2}{\|(x_n,0)\|} - 1)(x_n, 0) - (1, 0)\|$. Assume$x_n > 0$. Simplifying $\gamma_n$, we get $\gamma_n = \|(1 - x_n, 0)\|$

$$\gamma_n = \begin{cases} 1 - x_n & \text{for} \quad x_n \leq 1, \\ x_n - 1 & \text{for} \quad x_n > 1. \end{cases}$$

When $0 < x_n \leq 1$ then $\gamma_n \leq 1$. Thus it belongs to the case 1. Therefore, assuming that $x_n > 1$, from Equation (4.6), we get

$$(1 - \frac{1}{\gamma_n})(x_n + 1) + (\frac{2}{\rho_n \gamma_n} - \frac{1}{\rho_n})x_n = 0.$$

Substituting the value of $\rho_n = x_n$ and $\gamma_n = x_n - 1$, we get

$$(x_n - 2)(x_n + 1) + 2 - (x_n - 1) = 0.$$

or,

$$x_n = 1.$$

This contradicts the assumption that $x_n > 1$. In the other words no points in positive X-axis will map to (0,0).

**Remark 5.1.** *From the Proposition 3.7, we can conclude that the points whose initial point lies in* $\Lambda = \{(x, 0)|x \in (-\infty, -1] \cup \{0\}\}$ *may eventually converge to* $(0, 0)$. *Also note that no points other than the points in* $\Lambda$ *iterate to* $\Lambda$.

# References


[1] Borwein, J.M., Sims, B.: The Douglas-Rachford algorithm in the absence of convexity. Fixed-point Algorithms for Inverse Problems in Science and Engineering. 49, 93-109 (2011)

[2] Douglas, J., Rachford, H.H.: On the numerical solution of heat conduction problems in two and three space variables. Transactions of the AMS. 82, 421-439 (1956)

[3] Lions, P.L., Mercier,B.: Splitting algorithms for the sum of two nonlinear operators. SIAM Journal on Numerical Analysis. 16, 964-979(1979).

[4] Elser, V., Rankenburg I., Thibault, P.: Searching with iterated maps. Proceedings of the National Academy of Sciences. 104(2), 418-423 (2007).

[5] Aragn Artacho, F.J., Borwein J.M., Tam, M.K.: Recent results on Douglas-Rachford methods for combinatorial optimization problem. J. Optim. Theory Appl. 163(1), 1-30 (2014).

[6] Aragn Artacho, F.J., Borwein J.M., Tam, M.K.: Douglas-Rachford feasibility methods for matrix completion problems. ANZIAM J. 55(4), 299-326 (2014).

[7] Borwein, J.M., Tam, M.K.: Reflection methods for inverse problems with applications to protein conformation determination. In: Springer Volume on the CIMPA school Generalized Nash Equilibrium Problems, Bilevel Programming and MPEC, New Delhi (2012).

[8] Benoist, J.: The DouglasRachford algorithm for the case of the sphere and the line. J. Global Optimization. 63, 363-380 (2015).

[9] Gravel, S., Elser, V.: Divide and concur: A general approach constraint satisfaction. Phys. Rev. E. 78, 036706, 15 (2008).

[10] Bauschke, H.H., Combettes, P.L., Luke, D.R.: Phase retrieval, error reduction algorithm, and Fienup variants: A view from convex optimization. J. Opt. Soc. Amer. A. 19, 1334-1345 (2002).

[11] Aragn Artacho, F.J., Borwein, J.M., Tam, M.K.: Global behavior of the Douglas-Rachford method for a nonconvex feasibility problem. J. Global Optimization. 65(2), 309-327 (2016)

[12] Svaiter, B.F.: On weak convergence of the DouglasRachford method. SIAM J. Control Optimization. 49(1), 280-287 (2011).

[13] Bauschke, H.H., Dao, M.N.: On the finite convergence of the Douglas-Rachford algorithm for solving (not necessarily convex) feasibility problems in Euclidean spaces. SIAM Journal on Optimization. 27 (1), 507-537 (2017).