A Heuristic for the Traveling Salesperson Problem with Forbidden Neighborhoods on Regular 2D and 3D Grids

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Abstract

We examine an extension of the Traveling Salesperson Problem (TSP), the so called TSP with Forbidden Neighborhoods (TSPFN). The TSPFN is asking for a shortest Hamiltonian cycle of a given graph, where vertices traversed successively have a distance larger than a given radius. This problem is motivated by an application in mechanical engineering, more precisely in laser beam melting.

This paper discusses a heuristic for solving the TSPFN when there don't exist closed-form solutions and exact approaches fail. The underlying concept is based on Warnsdorff's Rule but allows arbitrary step sizes and produces a Hamiltonian cycle instead of a Hamiltonian path. We implemented the heuristic and conducted a computational study for various neighborhoods. In particular the heuristic is able to find high quality TSPFN tours on 2D and 3D grids, i.e., it produces optimum and near-optimum solutions and shows a very good scalability also for large instances.

Keywords: Constrained Euclidean Traveling Salesman Problem, Warnsdorff's Rule, Forbidden Neighborhoods, Grid.

1 Introduction

In this work we present a heuristic approach for solving the Traveling Salesperson Problem with Forbidden Neighborhoods (TSPFN) on regular two- and three-dimensional grids. A visualization of regular grids is given in Figure 1. The TSPFN (on regular grids) is an extension of the Traveling Salesperson Problem (TSP) that can be formally defined as follows.

Definition 1 (TSPFN on regular grids)**.** *Given are $mn\ell$ points $V = \{p^0, \dots, p^{mn\ell-1}\}$ on the $m \times n \times \ell$ grid in the Euclidean plane and a radius $r \in \mathbb{R}_0^+$. The task is to find a shortest Hamiltonian cycle $(p^{t_0}, p^{t_1}, \dots, p^{t_{mn\ell-1}}, p^{t_0})$ with $mn\ell$ points, where $\|p^{t_i} - p^{t_{(i+1) \bmod mn\ell}}\| > r$ holds for all $i \in \{0, 1, \dots, mn\ell - 1\}$.*

For a detailed description of the main application of the TSPFN in mechanical engineering we refer to Kordaß [6]. Fischer and Hungerländer [3] determine closed-form TSPFN solutions on two dimensional

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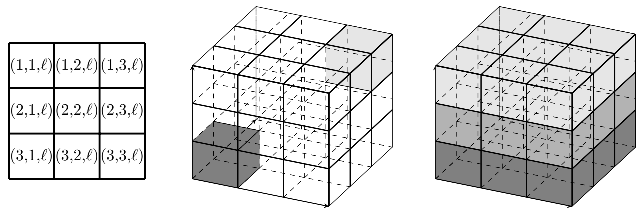

Figure 1: Visualizations of the $3 \times 3 \times 3$ grid. The left picture shows the numbering of layers $\ell \in \{1, 2, 3\}$. In the middle cube, the dark gray cell has the coordinates $(3, 1, 1)$ and the light gray one has the coordinates $(1, 3, 3)$. In the right cube, each layer has one color, where layers with larger numbers are lighter.

regular grids for the smallest reasonable forbidden neighborhoods, i.e., $r \in \{0, 1, \sqrt{2}\}$ and furthermore introduce an Integer Linear Program (ILP) for determining optimal TSPFN solutions. An extension to three dimensions, as well as a study of closed-form solutions for the smallest reasonable forbidden neighborhoods, i.e., $r \in \{0, 1, \sqrt{2}, \sqrt{3}\}$, is provided in [4] and [5]. In [2] the authors investigate the TSPFN with forbidden neighborhood of radius 2, where an optimal solution equals a Knight's Tour if it exists.

A Knight's tour equals a Hamiltonian cycle where each step has length $\sqrt{5}$, i.e., is a Knight's move on a chessboard. Schwenk [7] characterized for which 2D grids Knight's tours exist and DeMaio and Mathew [1] extended these results for 3D grids. Warnsdorff's Rule [9] is a heuristic that aims to find a Knight's path on an $m \times m$ chessboard.

Warnsdorff's Rule: Choose a starting square arbitrarily. The $(n + 1)^{th}$ square of the path is chosen by considering the three following conditions:

- It is adjacent (reachable) to the n^{th} square.
- It is unvisited (does not appear earlier in the path).
- It has the minimal number of adjacent, unvisited squares.

If there is more than one square that satisfies these conditions, the square is selected randomly. Since randomly choosing is usually not the best strategy, there are many investigations of so called tie-breaking rules, see e.g. Squirrel and Cull [8] for detailed studies.

This paper is structured as follows. We start with a description of our heuristic in Section 2. In Section 3 we conduct a computational study of the performance of the stated TSPFN heuristic. Finally we conclude the paper and give suggestions for future work in Section 4.

2 Solution Approach

A heuristic approach for solving the TSPFN is of interest for different reasons. On the one hand, the TSPFN is well studied only for small forbidden neighborhoods, i.e., $r \leq 2$. To solve unknown instances exact approaches can be applied, i.e., Integer Linear Programs (ILPs), see in [3] and [5] respectively, but the performance on large grids is insufficient. On the other hand, the results of the heuristic are helpful for determining construction schemes and also lower bounds, i.e., in combination closed-form solutions, for further forbidden neighborhoods and grid dimensions. To our knowledge this is the first time a heuristic for the TSPFN is suggested, we depict it in detail in Algorithm 1.

Differences of our heuristic to the original Warnsdorff's Rule:

- The preferred step size depends on the forbidden neighborhood and is not fixed to $\sqrt{5}$.
- The step size is variable, i.e., steps of different sizes are allowed.

- The heuristic applies a tie-breaking rule instead of choosing the next square randomly. Tie-breaking rules have a big impact on the result of the heuristic because many decisions are taken by it. We apply the tie-breaking rule called *Maximal Euclidean Distance*: The square with the maximal distance to the starting square is chosen out of the set of possible squares. If there is more than one square with maximal distance, the next square is chosen randomly.
- It is mandatory to close the Hamiltonian path, otherwise the heuristic fails. Note that our tie-breaking rule aims to increase the chances of our heuristic to succeed.

3 Computational Results

In this section we outline results of our TSPFN heuristic. We implemented the heuristic in Java and ran computational results on a 2.7 GHz Intel Core i7 with 8 GB RAM, running Windows 10. We used Gurobi 8.0.0 as ILP solver.

The first experiment is designed to finetune critical input parameters, i.e., the number of runs and the best starting square. We ran tests on $n \times n$ and $n \times n \times n$ grids for $n \in \{6, 8, 10, 12, 14, 16\}$, for reasons of simplicity, and started the heuristic on each square 1000 times, i.e., $1000n^3$ runs. Squares on the corners and in the middle turn out to be good starting squares and 100 runs transpire to be enough to gain results of high-quality in a short time period. In the following experiments we therefore start the heuristic always for 100 times at the center square, i.e., square $\left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right)$.

In another experiment we consider the the limitations of the ILP, i.e., what are the highest solvable dimensions. Already for grids with size greater or equal to 16×16, respectively $8 \times 8 \times 8$, and $r \geq \sqrt{2}$ a computation time of 10 minutes is not enough to compute at least a feasible solution. We summarize the results in Table 1.

Next we test our TSPFN heuristic on grids up to size 74×74, respectively $18 \times 18 \times 18$. We observe that the lengths of the tours produced are close to optimum, in case the optimum is known. The computation time is still underneath a second for grids where the ILP fails to produce a solution within 10 minutes. The heuristic delivers results on the largest considered grids in approximate one minute. In Tables 2 and 3 we summarize the results obtained.

	$r = \sqrt{2}$		$r = \sqrt{5}$		$r = \sqrt{8}$		$r = 3$	
	time	gap	time	gap	time	gap	time	gap
6×6	< 1 s	0%	< 1 s	0%	300 s	0%	5s	0%
10×10	64 s	0%	85 s	0%	TO	0.11%	TO	0.28%
14×14	562 s	0%	TO	0.12%	TO	0.11%	TO	2.26%
16×16	TO	-	TO	-	TO	-	TO	-
$4 \times 4 \times 4$	37 s	0 %	TO	1.13%	1 s	0 %	-	-
$6 \times 6 \times 6$	TO	-	TO	-	410 s	0%	TO	0.05%
$8 \times 8 \times 8$	TO	-	TO	-	TO	-	TO	-

Table 1: Results of the ILP for the TSPFN with different radii r of the forbidden neighborhood on $n \times n$ and $n \times n \times n$ grids, $n \in \{4, 6, 8, 10, 14, 16\}$. The run time is given in seconds and the gap to optimality in percent. The time limit is set to 600 seconds.

Algorithm 1 Algorithmic description of our heuristic for the TSPFN

procedure COMPUTETSPFNTOUR($m \times n \times \ell$ grid, r)
Require: A regular $m \times n \times \ell$ grid and the radius of the forbidden neighborhood r.
Ensure: A feasible TSPFN tour with forbidden neighborhood r.
Initialize a variable `h` as shortest feasible step length $> r$, a variable `s` for the current visited square, an empty list `C` for the TSPFN tour, and an empty set `S`.
Choose a starting square, set `s`= the starting square, and add `s` to `C`.
while there exists an unvisited square **do**
Set `S` = FINDNEXTSQUARES(`s`,`h`).
Set `s`= CHOOSENEXTSQUARE(`S`, `s`, `h`).
Add `s` to `C`.
return `C`
end while
end procedure

procedure FINDNEXTSQUARES($m \times n \times \ell$ grid, `s`, `h`)
Require: A regular $m \times n \times \ell$ grid, the current visited square `s` and the preferred step length `h`.
Ensure: A set `S` of possible reachable squares.
for each unvisited square `v` **do**
if `v` is adjacent (reachable by a step of length `h`) to `s` and if `v` has the minimal number of adjacent, unvisited squares **then**
Add `v` to `S`.
end if
end for
return `S`
end procedure

procedure CHOOSENEXTSQUARE($m \times n \times \ell$ grid, `S`, `s`, `h`)
Require: A regular $m \times n \times \ell$ grid, a set of possible next squares `S`, the current visited square `s`, and the preferred step length `h`.
Ensure: A variable `v` that represents the square which should be chosen next.
if `S` is empty **then**
Initialize a variable `j` as shortest feasible step length >`h` and an empty set `S`.
Set `S` = FINDNEXTSQUARES(`s`,`j`).
Set `v`= CHOOSENEXTSQUARE(`S`, `s`, `j`).
else if `S` contains exactly one element **then**
Set `v`= this square.
else if `S` contains more than one element **then**
Apply a tie-breaking rule to determine `v`.
end if
return `v`
end procedure

		$r=0$	$r=1$	$r=\sqrt{2}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{8}$	$r=3$
8×8	sec	0.109	0.031	0.063	0.297	0.015	0.016	0.031
	min	64.00	96.25	128.94	143.11	195.46	203.29	208.74
	gap	0.00	0.60	0.00	0.00	6.17	4.99	2.69
	opt	64.00	95.67	128.94	143.11	184.11	193.62	203.27
		$r=0$	$r=1$	$r=\sqrt{2}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{8}$	$r=3$
16×16	sec	0.40	0.52	0.62	0.33	0.17	0.22	0.21
	min	256.00	379.35	512.94	572.43	747.45	772.24	810.43
	gap	0.00	2.00	0.00	0.00	-	-	-
	opt	256.00	371.88	512.94	572.43	-	-	-
		$r=0$	$r=1$	$r=\sqrt{2}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{8}$	$r=3$
74×74	sec	61.80	64.37	62.67	74.95	63.34	62.73	73.35
	min	5476.00	7827.46	10953.77	12244.70	15606.97	16431.30	17353.47
	gap	0.00	0.50	0.01	0.00	-	-	-
	opt	5476.00	7788.05	10952.94	12244.70	-	-	-

Table 2: Results of our heurstic for the TSPFN with different radii r of the forbidden neighborhood on $n\times n$ grids, $n\in\{8,16,74\}$. Runtime for 100 starts, calculated minimal lengths, gap to optimality in percent, and optimal lengths are depicted.

		$r=0$	$r=1$	$r=\sqrt{2}$	$r=\sqrt{3}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{6}$	$r=\sqrt{8}$	$r=3$
$6\times 6\times 6$	sec	0.36	0.28	0.25	0.27	0.29	0.18	0.14	0.20	0.15
	min	216.00	306.11	389.13	435.17	482.99	530.19	621.33	648.00	649.45
	gap	0.00	0.00	-	0.29	0.00	-	-	-	-
	opt	216.00	306.11	-	433.89	482.99	-	-	-	-
		$r=0$	$r=1$	$r=\sqrt{2}$	$r=\sqrt{3}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{6}$	$r=\sqrt{8}$	$r=3$
$10\times 10\times 10$	sec	2.74	2.70	2.375	2.26	3.62	3.38	2.58	3.91	0.429
	min	1000.00	1414.85	1762.98	2004.18	2236.07	2451.30	2834.17	3000.00	3162.58
	gap	0.00	0.00	-	0.11	0.00	-	-	-	-
	opt	1000.00	1414.85	-	2001.88	2236.07	-	-	-	-
		$r=0$	$r=1$	$r=\sqrt{2}$	$r=\sqrt{3}$	$r=2$	$r=\sqrt{5}$	$r=\sqrt{6}$	$r=\sqrt{8}$	$r=3$
$18\times 18\times 18$	sec	75.31	113.23	102.77	85.72	157.70	152.77	103.19	174.47	151.77
	min	5832.83	8248.32	10217.43	11670.80	13040.75	14286.52	16498.13	17496.00	18442.71
	gap	0.01	0.00	-	0.04	0.00	-	-	-	-
	opt	5832.00	8248.32	-	11665.88	13040.75	-	-	-	-

Table 3: Results of our heurstic for the TSPFN with different radii r of the forbidden neighborhood on $n\times n\times n$ grids, $n\in\{6,10,18\}$. Runtime for 100 starts, calculated minimal lengths, gap to optimality in percent, and optimal lengths are depicted.

4 Conclusion and Outlook

In this paper we proposed a heuristic to solve the Traveling Salesperson Problem with Forbidden Neighborhoods (TSPFN) on regular two- and three-dimensional grids for arbitrary radii of the forbidden neighborhood. The TSPFN heuristic is based on the well-known heuristic called Warnsdorff's Rule that aims to find a Knight's path on a chessboard. We conducted a computational study to show that our TSPFN heuristic delivers high-quality solutions, i.e., optimum or near-optimum TSPFN tours. We also showed that the heuristic scales very well, even for large grids and radii. For future work it remains to determine further closed-form solutions for the TSPFN. Additionally we want to extend our heuristic to consider the case that the forbidden neighborhood is time dependent and hence can change over time or be active for more than one step. Finally it would be interesting to examine the TSPFN on grids that are not complete, i.e., there are cells in the grid that cannot be visited.

References

[1] J. DeMaio and B. Mathew. Which chessboards have a closed knight's tour within the rectangular prism? *The Electronic Journal of Combinatorics*, 18, 2011.

[2] M. Firstein, A. Fischer, and P. Hungerländer. Closed almost knights tours on 2d and 3d chessboards. *Operations Research Proceedings*, 2017.

[3] A. Fischer and P. Hungerländer. The traveling salesman problem on grids with forbidden neighborhoods. *Journal of Combinatorial Optimization*, pages 34:(3):891–915, 2017.

[4] A. Fischer, P. Hungerländer, and A. Jellen. The traveling salesperson problem with forbidden neighborhoods on regular 3d grids. *Operations Research Proceedings*, 2017.

[5] A. Jellen. The traveling salesperson problem with forbidden neighborhoods on regular 3d grids. *Masterthesis, Alpen-Adria-Universität Klagenfurt*, 2018.

[6] R. Kordaß. Untersuchungen zum eigenspannungs- und verzugsverhalten beim laserstrahlschmelzen. *Masterthesis, Technische Universität Chemnitz*, 2014.

[7] A. J. Schwenk. Which rectangular chessboards have a knight's tour? *Mathematics Magazine*, pages 64:(5):325–332, 1991.

[8] D. Squirrel and P. Cull. A warnsdorff-rule algorithm for knight's tours on square chessboards. *Oregon State REU Program*, 1996.

[9] H. Warnsdorff. Des rösselsprungs einfachste und allgemeinste lösung. *Schmalkalden*, 1823.